Real Numbers

INTRODUCTION OF NUMBER SYSTEM

INTRODUCTION :

Natural numbers :

The counting numbers 1,2,3..... are called natural numbers. It is denoted by N.

N = {1,2.3..........}

Whole numbers :

In the set of natural number if we include the number 0, the resulting set is known as the set of whole numbers. It is represented by W.

W = {0,1,2.......}

Integers :

Natural numbers along with 0 and their negatives are called integers and the set of integers is denoted by I

I = {.........–4, –3, –2, –1, 0, 1,2,3..........}

Rational numbers :

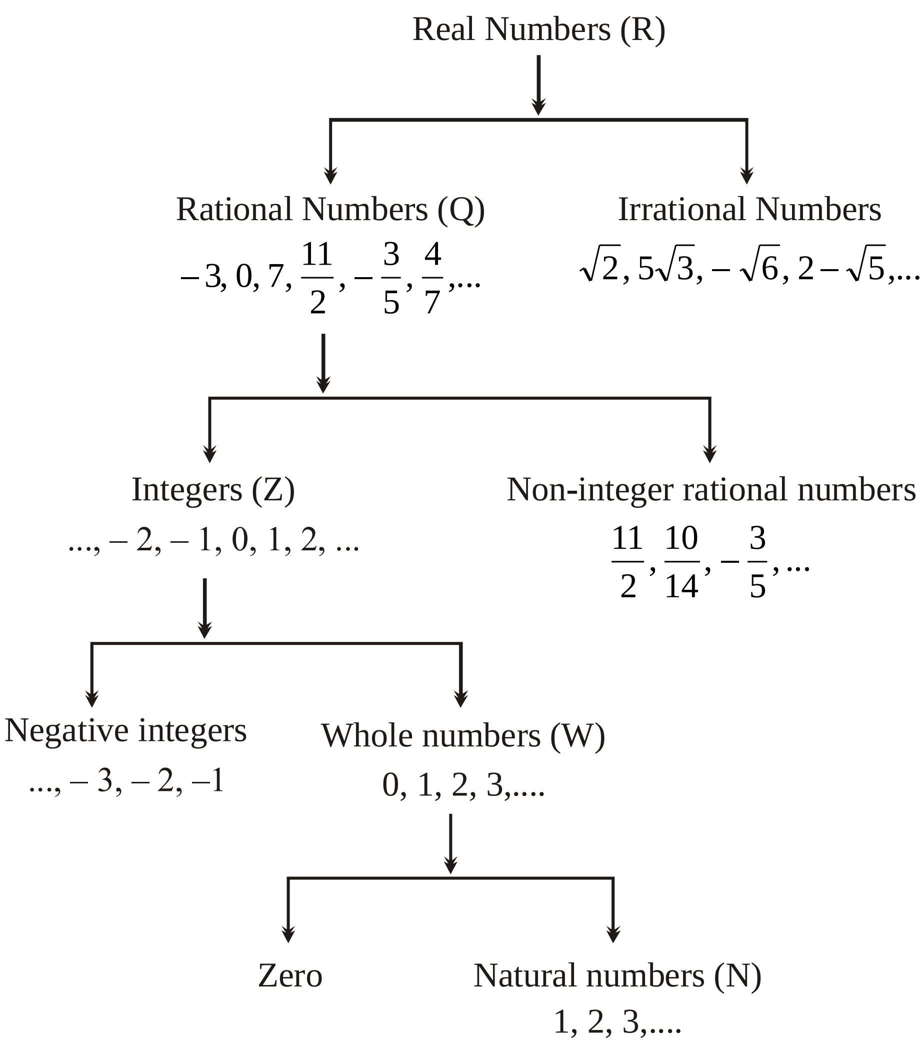
A rational number is a number which can be expressed in the form of p/q, where p and q are integers and q is not zero.

Irrational numbers :

A number is called irrational if it can not be written in the formof p/q, where p and q are integers and q ^{1} 0

The system R of real numbers includes rational as well irrational numbers.

In this chapter we will begin with a brief recall of divisibility of integers as well state some important properties of integers.



RATIONAL NUMBERS

Decimal Representation of Rational Numbers :

i) Finite or Terminating Decimal :

Every fraction p/q can be expressed as a decimal, if the decimal expression of p/q terminates, i.e. comes to an end, then the decimal so obtained is called a terminating decimal.

e.g,. $1/4 = 0.25$, $5/8 = 0.625$, $2\frac{3}{5} = \frac{13}{5} = 2.6$

Thus, each of the numbers $\frac{1}{4}, \frac{5}{8}$ and $2\frac{3}{5}$ can be expressed in the form of a terminating decimal.

Important : A fraction p/q is a terminating decimal only, when prime factors of q are 2 and 5 only.

e.g. Each one of the fractions $\frac{1}{2}, \frac{3}{4}, \frac{7}{20}, \frac{13}{25}$ is a terminating decimal, since the denominator of each has no prime factor other than 2 and 5.

(ii) Repeating (or Recurring) Decimals:

A decimal in which a digit or a set of digits repeats periodically, is called a repeating or a recurring decimal.

In a recurring decimal, we place a bar over the first block of the repeating part and omit the other repeating blocks.

e.g. (i) $\frac{2}{3} = 0.666 \ldots\ldots\ldots\ldots = 0.\overline{6}$

(ii) $\frac{15}{7} = 2.142857142857 \ldots\ldots\ldots = 2.\overline{142857}$

Real Numbers in decimal form

- Rational numbers
  All terminating or non-terminating repeating decimals
  - Terminating decimals
    $x = \frac{p}{q}$, q can be expressed in the form 2^m5^n, where m, n are non-negative integers
  - Non-terminating decimals
    $x = \frac{p}{q}$, q can not be expressed in the form 2^m5^n, where m, n are non-negative integers
- Irrational numbers
  All non-terminating and non-repeating decimals

Special Characteristics of Rational Numbers :

(i) Every rational number is expressible either as a terminating decimal or as a repeating decimal.

(ii) Every terminating decimal is a rational number.

(iii) Every repeating decimal is a rational number.

Prime numbers :

All natural numbers that have one and itself only as their factors are called prime numbers i.e. prime numbers are exactly divisible by 1 and themselves.

Example : 2, 3, 5, 7, 11, 13, 17, 19, 23 ....etc.

Twin Primes :

The term twin primes is used for a pair of odd prime numbers that differ by two.

Example : 3 and 5 are twin primes.

Co-prime numbers :

If the H.C.F. of the given numbers (not necessarily prime) is 1 then they are known as co-prime numbers.

Example : 5, 6, are co-prime as H.C.F. of (5, 6) = 1.

Note : (i) 1 is neither prime nor composite number.

(ii) 2 is the only prime number which is even.

(iii) Any two consecutive numbers will always be co-prime.

Composite numbers :

All natural numbers that have more than two different factors are called composite numbers. If C is the set of composite numbers then C = {4,6,8,9,10,12,.....}.

Perfect Number :

If the sum of all factors of a number is twice the number then this number is called perfect number.

If $2^k - 1$ = Prime number, then $(2^k - 1)\ (2^{k-1})$ is a perfect number.

Example : 6, 28, etc.

Imaginary Numbers:

All the numbers whose square is negative are called imaginary numbers.

Example : 2i, – 7i, i, ... where $i = \sqrt{-1}$ ($i^2 = -1$) .

Complex Numbers :

The combined form of real and imaginary numbers is known as complex numbers. It is denoted by $Z = a + ib$ where a is real part and b is imaginary part of Z and a, b Î R.

The set of complex numbers is the super set of all the sets of numbers.

Ex. 1: Express $\frac{2157}{625}$ in the decimal form.

Sol. We have,

$$\begin{array}{r|l} 625 & 2157.0000 \; (3.4512 \\ & \underline{1875} \\ & \;\;2820 \\ & \;\;\underline{2500} \\ & \;\;\;\;3200 \\ & \;\;\;\;\underline{3125} \\ & \;\;\;\;\;\;750 \\ & \;\;\;\;\;\;\underline{625} \\ & \;\;\;\;\;\;\;\;1250 \\ & \;\;\;\;\;\;\;\;\underline{1250} \\ & \;\;\;\;\;\;\;\;\;\;\;\;0 \end{array}$$

$\therefore \frac{2157}{625} = 3.4512$ **Ans.**

Ex. 2: Find the decimal representation of $\frac{-16}{45}$.

Sol. By long division, we have

```
45 ) 160 ( 0.3555
     135
     ----
      250
      225
     ----
       250
       225
     ----
        250
        225
     ----
         25
```

$\therefore \frac{16}{45} = 0.355 \ldots = 0.3\overline{5}$

Hence, $\frac{-16}{45} = -0.3\overline{5}$. **Ans.**

Conversion of Decimal Numbers into Rational Numbers of the form p/q :

(i) Procedure for terminating decimal :

Step. 1 : Count the number of numerals to the right of the decimal point. Let it be m.

Step. 2 : Drop the decimal point and in the denominator write 1 followed by m zeros.

Step. 3 : Simplify the fraction.

Ex. 3: Convert 6.225 to the form p/q.

Sol. 1. Number of numerals to the right of decimal is 3 i.e. m = 3.

2. Write $6.225 = \frac{6225}{1000}$

3. Simplify (divide the numerator and denominator by 25) = $6.225 = \frac{249}{40}$

(ii) Conversion of Pure Recurring Decimal to the form p/q.

Step.1 : Obtain the repeating decimal and put it equal to x.

Step 2 : Write the number in decimal form by removing bar from the top of repeating digits and listing repeating digits at least twice :

e.g. write $x = 0.\overline{8}$ as $x = 0.888$ ....

Step 3 : Determine the no. of digits having bar on their heads.

Step 4 : If the repeating decimal has 1 place repetition, multiply by 10, a two place repetition, multiply by 100, a three place repetition, multiply by 1000 and so on.

Step 5 : Subtract the number in step II from the numbers obtained in step IV.

Step 6 : Divide both sides of the equation by the coefficient of x.

Step 7 : Write the rational number in its simplest form.

Ex. 4 : Express $0.\overline{585}$ in the form p/q.

Sol. Let x = $0.\overline{585}$

x = 0.585585585 ....(i)

Here, we have 3 repeating digits after the decimal point. So, we multiply both sides of (i) by 10^3=1000 to get

1000 x = 585.585585 ....(ii)

Subtracting (i) from (ii), we get

1000 x – x = (585.585585 ....) - (0.585585...)

999 x = 585 $\Rightarrow$ $x = \frac{585}{999}$

(iii) Conversion of a Mixed Recurring Decimal to the form p/q.

Step 1 : Obtain the mixed recurring decimal and write it equal to x.

Step 2 : Determine the number of digits after the decimal point which do not have bar on them. Let there be n digits without bar just after the decimal point.

Step 3 : Multiply both sides of x by 10^n, so that only the repeating decimal is on the right side of the decimal point.

Step 4 : Use the method of converting pure recurring decimal to the form p/q and obtain the value of x.

Ex. 5: Express $0.22\overline{5}$ in the form p/q.

Sol. Let x = $0.22\overline{5}$ ....(i)

The no of digits after the decimal point which do not have bar on them is 2.

Multiply both sides of x by 10^2.

$100\,x = 22.\overline{5}$ ....(ii)

Here, we have 1 repeating digit after the decimal point. So, multiply both sides of (ii) by 10 to get.

$1000\,x = 225.55$ ....(iii)

Subtracting (ii) from (iii)

$1000\,x - 100\,x = (225.55......) - (22.55......)$

$900\,x = 203 \quad \Rightarrow x = \frac{203}{900}$.

IRRATIONAL NUMBERS :

A number is an irrational number, if it has a non terminating and non-repeating decimal representations. A number that cannot be put in the form p/q where p, q are integers and $q^{1}0$ is called irrational number.

e.g. $\sqrt{2}, \sqrt{3}, \sqrt{11}$ p etc.

Ex. 6 : Prove that $\sqrt{5}$ is an irrational number.

Sol. Let us assume on the contrary that $\sqrt{5}$ is a rational number. Then, there exist co-prime positive integers a and b such that

$=\sqrt{5} \quad \Rightarrow 5b^2 = a^2$

$\Rightarrow \quad 5 \mid a^2 \qquad [\because 5 \mid 5b^2]$

$\Rightarrow \quad 5 \mid a \qquad$ ......(i)

$\Rightarrow \quad a = 5c$ for some positive integer c

$\Rightarrow \quad a^2 = 25c^2$

$\Rightarrow \quad 5b^2 = 25c^2 \qquad [\because a^2 = 5c^2]$

$\Rightarrow \quad b^2 = 5c^2$

$\Rightarrow \quad 5 \mid b^2 \qquad [\because 5 \mid 5c^2]$

$\Rightarrow \quad 5 \mid b \qquad$ ....(ii)

From (i) and (ii), we find that a and b have at least 5 as a common factor. This contradicts the fact that a and b are co-prime.

Hence, $\sqrt{5}$ is an irrational number.

Ex 7: Prove that $\sqrt{3}-\sqrt{2}$ is an irrational number.

Sol. If possible, let be a rational number equal to x.

Then, $x = \sqrt{3}-\sqrt{2}$

$\Rightarrow \quad x^2 = \left(\sqrt{3}-\sqrt{2}\right)^2$

$\Rightarrow \quad x^2 = 3 + 2 - 2\sqrt{3}\sqrt{2}$

$\Rightarrow \quad x^2 = 5 - 2\sqrt{6}$

$\Rightarrow \quad x^2 - 5 = -2\sqrt{6}$

$$\Rightarrow \quad \frac{5-x^2}{2} = \sqrt{6}$$

Now, x is rational

$\Rightarrow$ x^2 is rational

$\Rightarrow$ $\frac{5-x^2}{2}$ is rational

$\Rightarrow$ $\sqrt{6}$ is rational.

But, $\sqrt{6}$ is irrational.

Thus, we arrive at a contradiction. So, our supposition that $\sqrt{3}-\sqrt{2}$ is rational, is wrong.

Hence, $\sqrt{3}-\sqrt{2}$ is an irrational number. **Ans.**

Some Properties of irrational numbers :

(a) The –ve of an irrational number is an irrational number.

(b) The sum of a rational and an irrational number is an irrational number.

(c) The product of a non–zero rational number with an irrational number is always an irrational number.

REAL NUMBERS :

The collection of real numbers consists of all the rational and irrational numbers and is denoted by R.

Every real number corresponds to a point on the line and conversely, every point on the number line represents a real number.

Ex.8 Insert a rational and an irrational number between 2 and 3.

Sol. If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b. Also, if a,b are rational numbers, then $\frac{a+b}{2}$ is a rational number between them.

$\therefore$ A rational number between 2 and 3 is

$\frac{2+3}{2} = 2.5$

An irrational number between 2 and 3 is

$\sqrt{2\times3} = \sqrt{6}$

Ex.9 Find two irrational numbers between 2 and 2.5.

Sol. If a and b are two distinct positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between 2 and 2.5 is

$\sqrt{2\times2.5} = \sqrt{5}$

Similarly, irrational number between 2 and $\sqrt{5}$ is $\sqrt{2\times\sqrt{5}}$

So, required numbers are $\sqrt{5}$ and $\sqrt{2\times\sqrt{5}}$.

Ex.3 Find two irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$.

Sol. We know that, if a and b are two distinct positive irrational numbers, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between $\sqrt{2}$ and $\sqrt{3}$ is $\sqrt{\sqrt{2}\times\sqrt{3}} = \sqrt{\sqrt{6}} = 6^{1/4}$

Irrational number between $\sqrt{2}$ and $6^{1/4}$ is $\sqrt{\sqrt{2}\times 6^{1/4}} = 2^{1/4} \times 6^{1/8}$.

Hence required irrational number are $6^{1/4}$ and

$2^{1/4} \times 6^{1/8}$.

Ex.5 Prove that

(i) $\sqrt{2}$ is irrational number

(ii) $\sqrt{3}$ is irrational number

Similarly $\sqrt{5}, \sqrt{7}, \sqrt{11}$...... are irrational numbers.

Sol. (i) Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s not having a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by Theorem it following that 2 divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is,

$b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational

(ii) Let us assume, to contrary, that $\sqrt{3}$ is rational. That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3}=\frac{a}{b}$.

Suppose a and b not having a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3}=a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational

Ex.6 Prove that $7-\sqrt{3}$ is irrational

Sol. Let $7-\sqrt{3}$ is rational number

$\therefore\ 7-\sqrt{3}=\frac{p}{q}$ (p, q are integers, $q \neq 0$)

$\therefore\ 7-\frac{p}{q}=\sqrt{3}$

$\Rightarrow\ \sqrt{3}=\frac{7q-p}{q}$

Here p, q are integers

$\therefore\ \frac{7q-p}{q}$ is also integer

$\therefore$ LHS = $\sqrt{3}$ is also integer but this is contradiction that $\sqrt{3}$ is irrational so our assumption is wrong that $(7-\sqrt{3})$ is rational

$\therefore\ 7-\sqrt{3}$ is irrational proved.