CIRCLES

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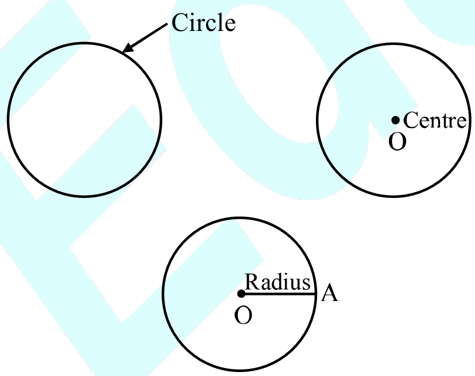
TERM & DEFINITIONS

1. **Circle.** A circle is a collection of all those points in a plane that are at a given constant distance from a given fixed point in the plane.

2. **Centre.** The fixed point is called the centre of the circle. In the figure O is the centre.

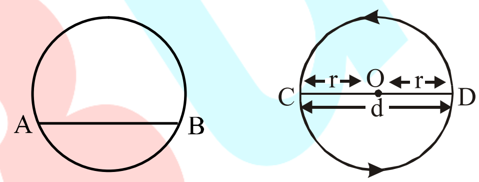
3. **Radius.** The constant distance from its centre is called the radius of the circle. In the figure, OA is radius-



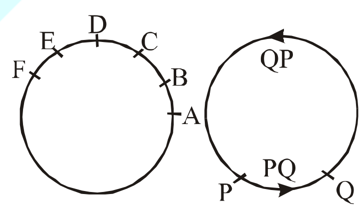
4. **Chord.** A line segment joining two points on a circle is called a chord of the circle. In the figure, AB is a chord of the circle. If a chord passes through centre then it is longest chord.

5. **Diameter.** A chord passing through the centre of a circle is called the diameter of the circle. A circle has an infinite number of diameters. CD is the diameter of the circle as shown in the figure. If d is the diameter of the circle then d = 2r. where r is the radius. or the longest chord is called diameter.

In the figure, AB is the diameter and the arcs $\overset{\frown}{CD}$ and $\overset{\frown}{DC}$ are semicircles.



6. **Arc.** A continuous piece of a circle is called an arc. Let A,B,C,D,E,F be the points on the circle. The circle is divided into different pieces. Then, the pieces AB, BC, CD, DE, EF etc. are all arcs of the circle.



Let P,Q be two points on the circle. These P, Q divide the circle into two parts. Each part is an arc. These arcs are denoted in anti-clockwise direction from P to Q as $\overset{\frown}{PQ}$ and form Q to P as $\overset{\frown}{QP}$. The counter clockwise direction distinguishes between these two arcs $\overset{\frown}{PQ}$ and $\overset{\frown}{QP}$.

The length of arc $\overset{\frown}{PQ}$ can be less than, equal to or greater than the length of the arc $\overset{\frown}{QP}$

i.e., (i) $\ell(\overset{\frown}{PQ}) < \ell(\overset{\frown}{QP})$ (ii) $\ell(\overset{\frown}{PQ}) = \ell(\overset{\frown}{QP})$
(iii) $\ell(\overset{\frown}{PQ}) > \ell(\overset{\frown}{QP})$

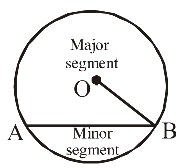
when $\ell(\overset{\frown}{PQ}) < \ell(\overset{\frown}{QP})$, then the arc $(\overset{\frown}{PQ})$ is called a minor arc.

If $\ell(\overset{\frown}{PQ}) = \ell(\overset{\frown}{QP})$, then the arc $\overset{\frown}{PQ}$ and $\overset{\frown}{QP}$ are called semi circle. At this time points of arc at end of diameter.

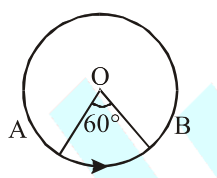
And when $\ell(\overset{\frown}{PQ}) > \ell(\overset{\frown}{QP})$, then the arc $\overset{\frown}{PQ}$ is called a major arc.

7. **Circumference of a circle.** The perimeter of a circle is called its circumference. The circumference of a circle of radius r is $2\pi r$.

8. **Segment.** Let AB be a chord of the circle. Then, AB divides the region enclosed by the circle (i.e., the circular disc) into two parts. Each of the parts is called a segment of the circle. The segment, containing the minor arc is called minor segment and the segment, containing the major arc, is called the major segment and segment of a circle is the region between an arc and chord of the circle.



9. **Central Angles.** Consider a circle. The angle subtended by an arc at the centre O is called the central angle. The vertex of the central angle is always at the centre O.



Degree measure of an arc : Degree measure of a minor arc is the measure of the central angle subtended by the arc.

In the figure, the measure of the arc $\overset{\frown}{PQ}$ is 60° i.e., $m\overset{\frown}{PQ} = 60°$. The measure of a major arc is 360° – $m\overset{\frown}{PQ}$ the degree measure of the corresponding minor arc.

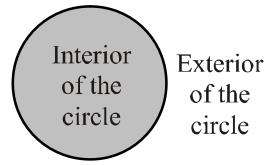
The degree measure of the major arc is 360° – 60° = 300°

$\therefore\ m\overset{\frown}{QP} = 300°$.

The degree measure of the circumference of the circle is always 360°.

10. **Interior and Exterior of Circle.**

A circle divides the plane on which lies into three parts.



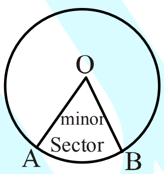
(i) Inside the circle. which is called the interior of the circle

(ii) Circle

(iii) Outside the circle, which is called the exterior of the circle.

The circle and its interior make up the circular region.

11. **Sector :**

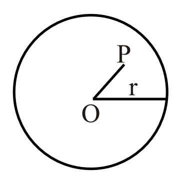


A sector is that region of a circular disc which lies between an arc and the two radii joining the extremities of the arc and the centre. OAB is a sector as shown in the figure.

Quadrant. One fourth of a circular disc is **called a quadrant**.

12. **Position of a point :**

Point Inside the circle. A point P, such that OP < r, is said to lie inside the circle.



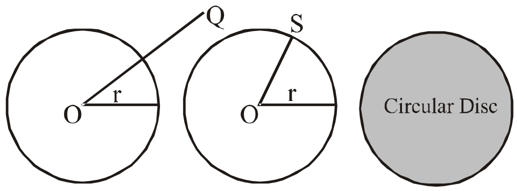
The point inside the circle is also called interior point. (Example : Centre of cirle)

Point outside the circle, A point Q, such that OQ > r, is said to lie outside the circle C (O, r) = {X, OX = r}

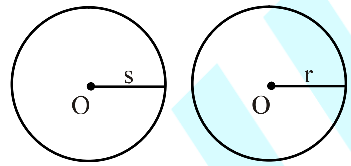
The point outside the circle is also called exterior point.

Point on the circle. A point S, such that OS = r is said to lie on the circle C(O, r) = {X ,OX = r}.

Circular Disc. It is defined as a set of interior points and points on the circle. In set notation, it is written as : C(O, r) = {X : P OX ≤ r}



Congruent circles. Two circles are said to be congruent if and only if, one of them can be superposed on the other, so as the cover it exactly. It means two circles are congruent if and only if, their radii are equal. i.e., C (O, r) and C (O′ , r) are congruent if only if r = s.

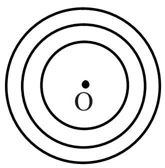


Congruent circles if r = s

Congruent arcs : Two arcs of a circle are congruent, if either of them can be superposed on the other, so as to cover it exactly. It is only possible, if degree measure of two arcs are the same.

13. Concentric Circles.

Circles having the same centre and different radius are said to be concentric circles.



Concentric Circles

Remark. The word 'radius' is used for a line segment joining the centre to any point on the circle and also for its length.

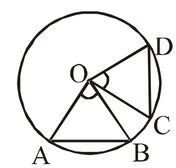
14. Congruence of Circles & Arcs

➢ **IMPORTANT POINTS**

1. Equal chords of a circle subtend equal angles at the centre

Given. Chord AB = chord CD in a circle with centre O.

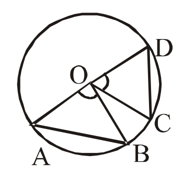
To prove. ∠ AOB = ∠ COD



2. Conversely, if the angles subtended the by chords at the centre of a circle are equal, then the chord are equal.

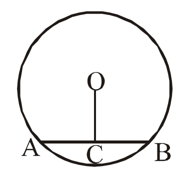
Given : Two chords AB and CD subtend equal angles ∠AOB and ∠COD at the centre O.

To prove. AC = CD



3. The perpendicular from the centre of a circle to a chord bisects the chord.

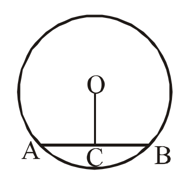
Given : OC is perpendicular to a chord AB in a circle with centre O.
To prove. AC = CB



4. Conversely, the line draw through the centre of a circle to bisect a chord is perpendicular to the chord.

Given : AB is a chord and C is the mid point of AB. O is the centre of the circle.

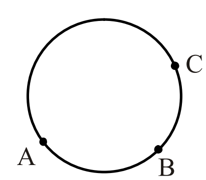
To prove. OC is ⊥ to AB



5. There is one and only one circle passing through three given non-collinear points.

Given : There are three non collinear points A,B and C.

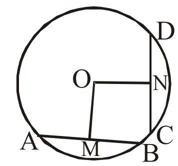
To prove. Only one circle will pass through the points A, B and C.



6. Equal chords of a circle (or of congruent circles) are equidistant from the centre(s).

Given : Two chords AB and CD are equal in a circle with centre O.

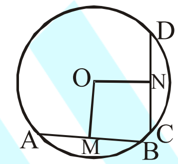
To prove. $OM \perp AB = ON \perp CD$



7. Conversely, chords of a circle (or of congruent circles) that are equidistant from the centre(s) are equal.

Given : Two chords AB and CD are equidistant from the centre O of a circle, i.e., OM ($\perp$ AB) = ON ($\perp$ AB).

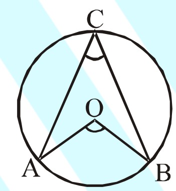
To prove. AB = CD



8. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Given : Let $\overset{\frown}{AB}$ an arc in a circle with centre O and there is a point C in the alternate segment.

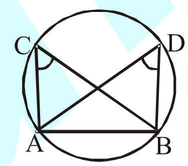
To prove. $\angle AOB = 2\angle ACB$



9. Angles in the same segment of a circle are equal.

Given : Two angles $\angle ACB$ and $\angle ADB$ subtended in the same segment AB.

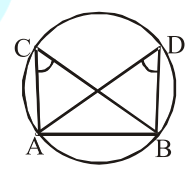
To prove. $\angle ACB = \angle ADB$.



10. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on a circle.

Given : Two angles $\angle ACB$ and $\angle ADB$ are subtended by the line segment AB are equal $\angle ACB = \angle ADB$.

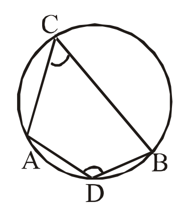
To prove. A,B,C,D lie on a circle.



11. The sum of the either pair of the opposite angles of a cyclic quadrilateral is 180°.

Given : $\angle ACB$ and $\angle ADB$ are in the alternate segments of a circle.

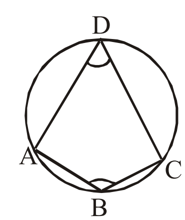
To prove. $\angle ACB + \angle ADB = 180^\circ$



12. If a pair of opposite angles of a quadrilateral is supplementary then the quadrilateral is cyclic.

Given : The sum of the angles in the alternate segments is 180 i.e., $\angle ABC + \angle ADC = 180^\circ$.

To prove. A, B, C, D is a cyclic quadrilateral.

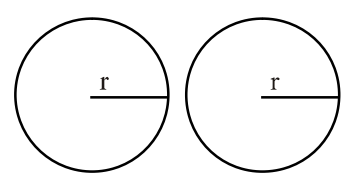


Properties :

13. Two circles are congruent, if and only if they have equal radii

Given : Two circles of equal radii.

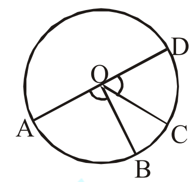
To prove. Given circles are congruent.



14. Two arcs of a circle are congruent if the angles subtended by them at the centre are equal.

Given : Two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ subtend equal angles $\angle AOB$ and $\angle COD$.

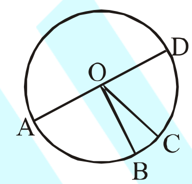
To prove. Arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent.



15. Converse : Two arcs subtend equal angles at the centre, if the arcs are congruent.

Given : Two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent in a circle with centre O.

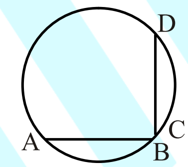
To prove. $\angle AOB = \angle COD$



16. If two arcs of a circle are congruent, their corresponding chords are equal.

Given : Two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent in a circle.

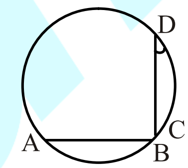
To prove. chord AB = chord CD



17. **Converse**. If two chords of a circle are equal , their corresponding arcs are equal.

Given : Two chords AB and CD are equal in a circle.

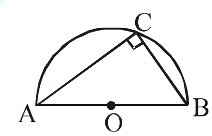
To prove. $\overset{\frown}{AB} = \overset{\frown}{CD}$



18. The angle in a semi-circle is a right angle.

Given : ABC is a semi circle with centre O.

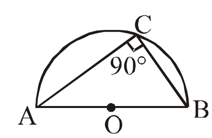
To prove. $\angle ACB = 90°$



19. **Converse.** The arc of a circle subtending a right angle at any point of the circle in its alternate segment is a semicircle.

Given : $\angle ACB = 90°$

To prove. $\overset{\frown}{ACB}$ is a semicircle

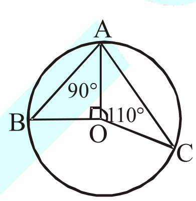


❖ EXAMPLES ❖

Ex.1 O is the centre of the circle. If $\angle BOA = 90°$ and $\angle COA = 110°$, find $\angle BAC$.

Sol. **Given** : A circle with centre O and $\angle BOA = 90°$, $\angle AOC = 110°$.

To find : $\angle BAC = ?$



Procedure, $\angle AOB + \angle AOC + \angle BOC = 360°$

$\Rightarrow$ $90° + 110° + \angle BOC = 360°$

$\Rightarrow$ $\angle BOC = 360° - 90° - 110°$

$\Rightarrow$ $\angle BOC = 160°$

But, arc $\overset{\frown}{BC}$ subtends $\angle BOC$ at the centre and $\angle BAC$ at the remaining part of the circle.

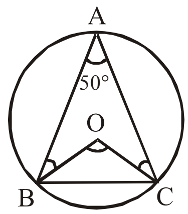
$\Rightarrow$ $\angle BAC = \frac{1}{2}\angle BOC$

$\therefore$ $\angle BAC = \frac{1}{2}(160°) = 80°$

Ex.2 O is the centre of the circle. If $\angle BAC = 50°$, find $\angle OBC$.

Sol. **Given :** In a circle with centre at O $\angle BAC = 50°$.

To find: $\angle OBC = ?$



Procedure. $\angle BAC = 50°$

$\angle BOC = 2\angle BAC = 2(50°) = 100°$

[Arc BC subtends $\angle BOC$ at the centre and $\angle BAC$ at remaining part of c]

In ΔOBC, OB = OC = radius

$\Rightarrow$ $\angle OBC = \angle OCB$

(Opposite angles of equal sides of a Δ)

Now, $\angle OBC + \angle OCB + \angle BOC = 180°$

[Sum of angles of a triangle]

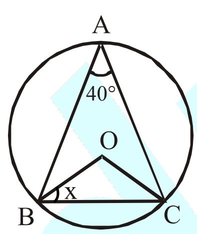
$\Rightarrow$ $\angle OBC + \angle OCB + 100° = 180°$

$\Rightarrow$ $\angle OBC + \angle OBC = 180° - 100°$

$2\angle OBC = 80°$

$\therefore$ $\angle OBC = 40°$.

Ex.3 Find the value of x from the given figure, in which O is the centre of the circle.



Sol. **Given.** $\angle BAC$ in a circle with centre O is 40°.

To find. $\angle OBC$ = (say x)

Procedure, $\angle BOC = 2\angle BAC$

$= 2 \times 40° = 80°$

In ΔBOC,

BO = OC (Radii of the same circle)

$\Rightarrow$ $\angle B = \angle C = x$

$\therefore$ $x + \angle BOC + x = 180°$ [Sum of $\angle$s of a Δ]

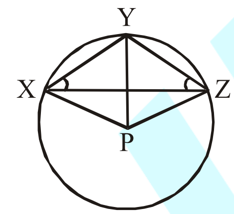
$\Rightarrow$ $2x = 180° - \angle BOC$

$\Rightarrow$ $2x = 180° - 80°$

$\Rightarrow$ $2x = 100°$

$\Rightarrow$ $x = 50°$.

Ex.4 P is the centre of the circle . Prove that $\angle XPZ = 2(\angle XZY + \angle YXZ)$.



Sol. **Given.** A circle with centre P, XY and YZ are two chords.

To prove. $\angle XPZ = 2(\angle XZY + \angle YXZ)$

Proof. In a circle with centre P, arc XY subtends $\angle XPY$ at the centre and $\angle XZY$ at remaining part of the circle.

$\Rightarrow$ $\angle XPY = 2\angle XZY$ ....(1)

Similarly, arc YZ subtends $\angle YPZ$ at the centre and $\angle YXZ$ at remaining part

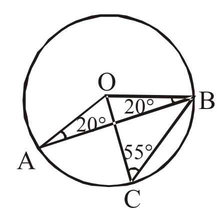
$\therefore$ $\angle YPZ = 2\angle YXZ$ ....(2)

Adding (1), and (2), we get

$\angle XPY + \angle YPZ = 2\angle XZY + 2\angle YXZ$

$\Rightarrow$ $\angle XPZ = 2(\angle XZY + \angle YXZ)$.

Ex.5 O is the centre of the circle. $\angle OAB = 20°$, $\angle OCB = 55°$. Find $\angle BOC$ and $\angle AOC$.



Sol. **Given.** $\angle OAB = 20°$, $\angle OCB = \angle 55°$

To find. $\angle BOC$ = ? and $\angle AOC$ = ?

Procedure. Let $\angle AOC = y°$ and $\angle BOC = x°$

$\angle OBA = \angle OAB$ [As OA = OB = radius]

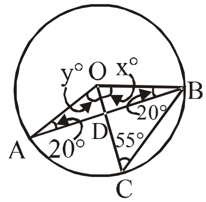
$\therefore$ $\angle OBA = 20°$

In ΔOAD and ΔOBD,

OA = OB ...[Radii of circle]

$\angle OAD = \angle OBD = 20°$ ...[Proved]

OD = OD ...[Common]



$\therefore$ $\Delta OAD \cong \Delta OBD$

(SAS theorem of congruence)

$\Rightarrow$ $x° = y°$ ...(C.P.C.T)

Also, $\angle ODA = \angle ODB$ ...(C.P.C.T)

$\angle ODA + \angle ODB = 180°$ ...[Linear pair]

$2\angle ODA = 180°$

$\Rightarrow$ $\angle ODA = 90°$

$\therefore$ $\angle ODB = 90°$ ...[$\because$ $\angle ODA = 90°$]

So in ΔODA,

$\angle AOD + \angle OAD + \angle ODA = 180°$

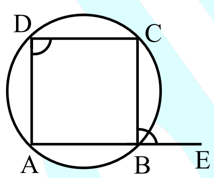
$y° + 20° + 90° = 180°$

$y° = 180° - 110° = 70°$

$\therefore$ $x° = y° = 70°$.

Ex.6 If a side of a cyclic quadrilateral is produced, then prove that the exterior angle is equal to the interior opposite angle.

Sol. **Given.** A cyclic quadrilateral ABCD. Side AB is produced to E.



To prove. $\angle CBE = \angle ADC$

Proof. $\angle ABC + \angle ADC = 180°$ ....(1)

[Sum of opposite pairs of angles in a cyclic quadrilateral.]

But, $\angle ABC + \angle CBE = 180°$ ....(2)

[$\because$ $\angle ABC$ and $\angle CBE$ are linear pairs]

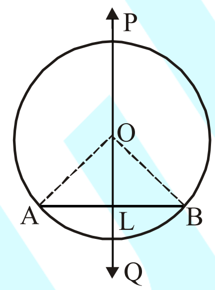
From (1) and (2)

$\angle ABC + \angle ADC = \angle ABC + \angle CBE$

$\Rightarrow$ $\angle ADC = \angle CBE$ or $\angle CBE = \angle ADC$.

Ex.7 Prove that the right bisector of a chord of a circle, bisects the corresponding arc of the circle.

Sol. Let AB be a chord of a circle having its centre at O. Let PQ be the right bisector of the chord AB, intersecting AB at L and the circle at Q. Since the right bisector of a chord always passes through the centre, so PQ must pass through the centre O. Join OA and OB. In triangles OAL and OBL we have



OA = OB [Each equal to the radius]

$\angle ALO = \angle BLO$ [Each equal to 90º]

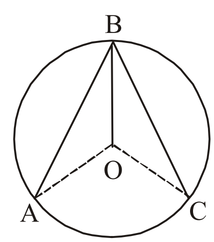
OL = OL [Common]

$\therefore$ $\Delta OAL \cong \Delta OBL$

$\Rightarrow$ $\angle AOL = \angle BOL$

$\Rightarrow$ AQ = BQ

Ex.8 In figure AB = CB and O is the centre of the circle. Prove that BO bisects $\angle ABC$.



Sol. In Δ´s AOB and COB, we have

AB = CB [Given]

OB = OB [Common]

and, OA = OC [Each equal to radius]

So, by SSS criterion of congruence

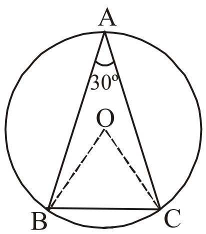
$\Delta AOB \cong \Delta COB$

$\Rightarrow$ $\angle OBA = \angle OBC$

$\Rightarrow$ OB bisects $\angle ABC$.

Ex.9 In fig. ABC is a triangle in which $\angle BAC = 30^o$. Show that BC is the radius of the circumcircle of ΔABC, whose centre is O.

Sol. Join OB and OC. Since the angle subtended by an arc of a circle at its centre is twice the angle subtended by the same arc at a point on the circumference.



$\therefore \quad \angle BOC = 2\angle BAC$

$\Rightarrow \quad \angle BOC = 2 \times 30^o = 60^o$

Now, in ΔBOC, we have

OB = OC [Each equal to radius]

$\Rightarrow \quad \angle OBC = \angle OCB$

[∵ Angles opposite to equal sides of a triangle are equal]

But, $\angle OBC + \angle OCB + \angle BOC = 180^o$

$\Rightarrow \quad 2\angle OBC + 60^o = 180^o$

$\Rightarrow \quad 2\angle OBC = 120^o$

$\Rightarrow \angle OBC = 60^o$

Thus, we have

$\angle OBC = \angle OCB$

$= \angle BOC = 60^o$

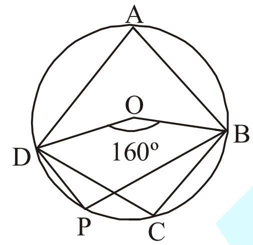
$\Rightarrow \quad \Delta OBC$ is equilateral

$\Rightarrow \quad OB = BC$

$\Rightarrow \quad$ BC is the radius of the circumcircle of ΔABC.

Ex.10 In figure ABCD is a cyclic quadrilateral; O is the centre of the circle. If $\angle BOD = 160^o$, find the measure of $\angle BPD$.

Sol. Consider the arc BCD of the circle. This arc makes angle $\angle BOD = 160^o$ at the centre of the circle and $\angle BAD$ at a point A on the circumference.



$\therefore \quad \angle BAD = \frac{1}{2}\angle BOD = 80^o$

Now, ABPD is a cyclic quadrilateral.

$\Rightarrow \quad \angle BAD + \angle BPD = 180^o$

$80^o + \angle BPD = 180^o$

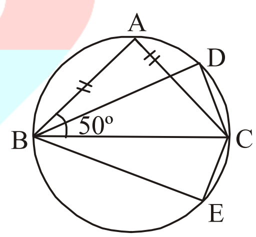
$\Rightarrow \quad \angle BPD = 100^o$

$\Rightarrow \quad \angle BCD = 100^o$

[∵ $\angle BPD$ and $\angle BCD$ are angles in the same segment ∴ $\angle BCD = \angle BPD$]

Ex.11 In figure ΔABC is an isosceles triangle with AB = AC and m $\angle ABC = 50^o$. Find m $\angle BDC$ and m $\angle BEC$

Sol.



In ΔABC, we have

AB = AC

$\Rightarrow \quad \angle ACB = \angle ABC$

$\Rightarrow \quad \angle ACB = 50^o \qquad [\because \angle ABC = 50^o]$

$\therefore \quad \angle BAC = 180^o - (\angle ABC + \angle ACB)$

$\Rightarrow \quad \angle BAC = 180^o - (50^o + 50^o) = 80^o$

Since $\angle BAC$ and $\angle BDC$ are angles in the same segment.

$\therefore \quad \angle BDC = \angle BAC \Rightarrow \angle BDC = 80^o$

Now, BDCE is a cyclic quadrilateral.

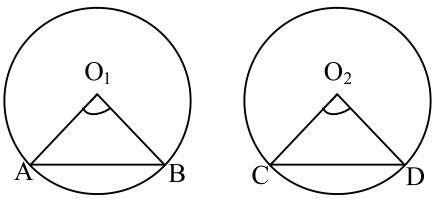
$\therefore \quad \angle BDC + \angle BEC = 180^o$

$\Rightarrow \quad 80^o + \angle BEC = 180^o \Rightarrow \angle BEC = 100^o$

Hence, $m\angle BDC = 80^o$ and $m\angle BEC = 100^o$

Q.12 Recall that two circles are congruent if they have the same radii. Prove that equal chords of congruent circles subtend equal angles at their centres. **[NCERT]**

Sol. Given : $C(O_1, r)$ & $C'(O_2, r)$ are congument & chord $\overline{AB}$ = chord $\overline{CD}$



To Prove : $\angle AO_1 B = \angle CO_2D$

Construction : Join O_1 to A & B and O_2 to C & D.

Proof : In $\Delta\ AO_1B$ & ΔCO_2D

$AO_1 = CO_2 = r$

$BO_1 = DO_2 = r$

$AB = CD$ = given

$\therefore$ By SSS $\Delta AO_1B \cong \Delta CO_2D$

$\therefore \angle AO_1B = \angle CO_2 D$ (CPCT)

Theroem 1 : There is one and only one circle passing through three given non-collinear points. **[NCERT]**

Proof : Take three non collinear points A, B, C & we draw perpendicular bisectors of lines BA & BC, which are intersect at point O.

Now O is on $\perp$ bisectors of AB $\therefore$ OA = OB ....(1)

also O is on $\perp$ bisector of BC $\therefore$ OB = OC .....(2)

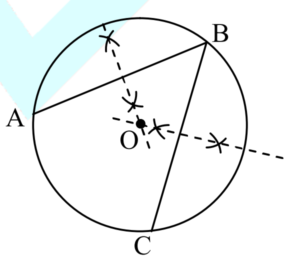
($\because$ Each point, on $\perp$ bisector of a line, is equidistant from both vertices of that line)

$\therefore$ By (1) & (2) OA = OB = OC

Now taking O as centre & OA as radius & draw a circle which passes throguh points A, B, C.

Uniqueness :

This circle is unique



$\because$ O is intersection point of two lines & lines can intersect only at a point.

$\therefore$ We can not get any other point (O′) which is equidistant from A, B & C.

Ex.13 Suppose you are given a circle. Give a construction to find its centre. **[NCERT]**

Sol. (i) Take three points A, B, C on given circle.

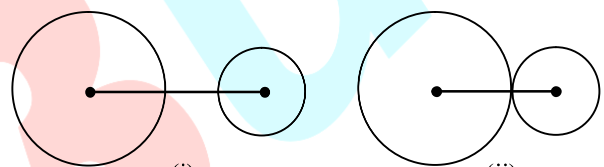
(ii) Join B to A & C.

(iii) Draw $\perp$ bisectors of BA & BC.

(iv) The intersection point of $\perp$ bisecteros is centre.

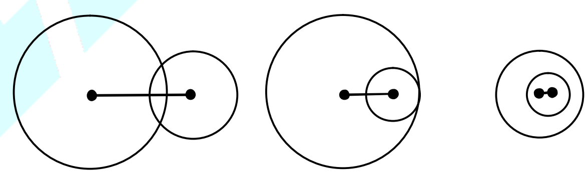
➢ INTERSECTION OF CIRCLES

If centre and radius of circles are c_1, r_1 & c_2, r_2



(i) not intersecting $c_1c_2 > r_1r_2$

(ii) Touch each other externally $c_1c_2 = r_1 + r_2$



(iii) Intersection $r_1 - r_2 < c_1c_2 < r_1 + r_2$

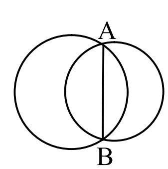
(iv) Touch internally $c_1c_2 = r_1 - r_2$

(v) not intersecting $c_1c_2 < (r_1 - r_2)$

So we can say two circles can intersect at most two points and these are called common points for both circles.

➢ COMMON CHORD

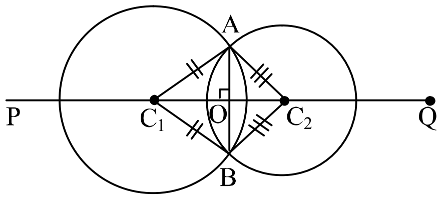
A line joining common points of two intersecting circles is called common chord.



AB is common chord.

Ex.14 If two circles intersect at two points, prove that their centres lie on the perpendicular bisector of the common chord. **[NCERT]**

Sol. I[st] Method



Given : Two circles of radius r_1 & r_2 intersect at two different points A & B. and PQ is $\perp$ bisector of AB. $\therefore$ AO = OB & $\angle O = 90^o$

To prove : Centres of circles C_1 & C_2 lie on PQ.

Construction : Join A to C_1, C_2 and also B to C_1, C_2.

Proof : $\because$ $C_1A = C_1B = r_1$ and $C_2A = C_2B = r_2$

$\therefore$ quadrilateral C_1AC_2B is kite.

$\because$ PQ is $\perp$ & bisector of AB

$\therefore$ AB is shorter digagonal.

$\therefore$ C_1, C_2 are on PQ .

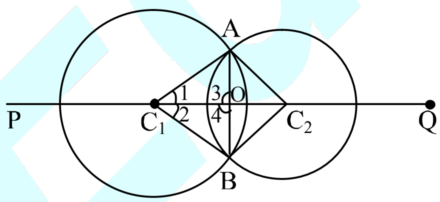
II[nd] method

Let PC_1C_2 Q is a line. We will prove that line PQ is $\perp$ bisector of common chord AB.

Proof : $\Delta AC_1C_2 \cong \Delta BC_1C_2$

$$\left\{\begin{matrix} C_1A = C_1B = r_1 \\ C_1C_2 = C_1C_2 = \text{common} \\ C_2A = C_2B = r_2 \end{matrix}\right\}_{(SSS)}$$

$\therefore$ $\angle 1 = \angle 2$ (CPCT) ....(1)



Now, $\Delta AC_1O \cong \Delta BC_1O$

$$\left\{\begin{matrix} C_1A = C_2B = r \\ \angle 1 = \angle 2 \ \ (\text{by } 1) \\ C_1O = C_1O \text{ common} \end{matrix}\right\}_{(SAS)}$$

$\therefore$ AO = OC (CPCT) ....(2)

i.e PQ is bisector of line AB.

also $\angle 3 = \angle 4$ (CPCT).

But $\angle 3 + \angle 4 = 180^o$ (ℓ.p.)

also $\angle 3 = \angle 4 = 90^o$ ...(3)

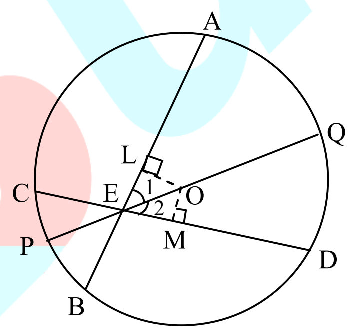
$\therefore$ PQ $\perp$ AB

Hence PQ is $\perp$ bisector of common chord AB

$\therefore$ C_1 & C_2 lie on $\perp$ bisector of common chord.

Ex.15 If two intersecting chords of a circle make equal angles with the diameter passing through their point of intersection, prove that the chords are equal. **[NCERT]**

Sol. Given that AB and CD are two chords of a circle, with centre O intersecting at a point E. PQ is a diameter through E, such that $\angle AEQ = \angle DEQ$. We have to prove that AB = CD. Draw perpendiculars OL and OM on chords AB and CD, respectively. Now



In ΔOLE and ΔOME

$\angle OLE = \angle OME = 90^o$

$\angle 1 = \angle 2$ given

EO = EO common

Therefore, $\Delta OLE \cong \Delta OME$ (AAS)

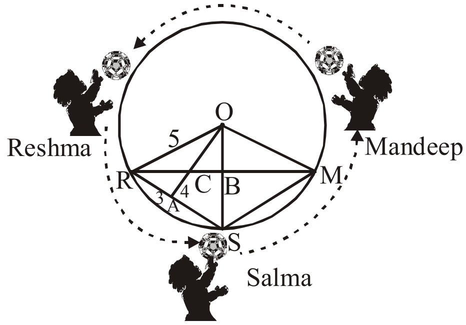
This gives OL = OM (CPCT)

So, AB = CD

{Chords are equidistant from centre are equal.}

Ex.16 Three girls Reshma, Salma and Mandeep are playing a game by standing on a circle of radius 5 m drawn in a pack. Reshma throws a ball to Salma, Salma to Mandeep. Mandeep to Reshma. If the distance between Reshma and Salma and between Salma and Mandeep is 6 m each. What is the distance between Reshma and Mandeep ? **[NCERT]**

Sol. Let the position of Reshma, Salma and Mandeep be at R, S and M on the circumference of the circular park.



RS = SM = 6 m

Radius = OR = OS = 5m.

In rt angled ΔRBO

$RB^2 = OR^2 - OB^2$ ....(1)

In rt angled ΔSBR

$RB^2 = RS^2 - SB^2$ ....(2)

From (1) and (2), we get

$OR^2 - OB^2 = RS^2 - SB^2$

$(5)^2 - x^2 = (6)^2 - (5 - x)^2$

(Let OB = x)

$25 - x^2 = 36 - x^2 - 25 + 10x$

$10x = -36 + 25 + 25$

$10x = 14 \Rightarrow x = \frac{14}{10}$

Distance between Reshma and Mandeep = RM

$= 2RB = 2\sqrt{OR^2 - OB^2}$

$= 2\sqrt{25 - x^2} \;\; = 2\sqrt{25 - \left(\frac{14}{10}\right)^2}$

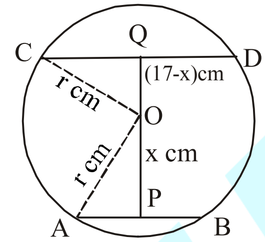
$= 2\sqrt{\frac{2500 - 196}{100}} = \frac{2\sqrt{2304}}{10} = 2 \times \frac{48}{10} = \frac{48}{5}$ m

Ex.17 AB and CD are two parallel chords of a circle such that AB = 10 cm and CD = 24 cm. If the chords are on the opposite sides of the centre and the distance between them is 17 cm, find the radius of the circle.

Sol. let O be the centre of the given circle and let its radius be r cm. Draw $OP \perp AB$ and $OQ \perp CD$. Since $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$. Therefore, points P, O and Q are collinear. So, PQ = 17 cm.

Let OP = x cm. Then, OQ = (17 – x) cm.

Join OA and OC. Then, OA = OC = r.



Since the perpendicular from the centre to a chord of the circle bisects the chord.

$\therefore$ AP = PB = 5 cm and CQ = QD = 12 cm.

In right triangles OAP and OCQ, we have

$OA^2 = OP^2 + AP^2$ and $OC^2 = OQ^2 + CQ^2$

$\Rightarrow \; r^2 = x^2 + 5^2$ ....(i)

and, $r^2 = (17 - x)^2 + 12^2$ ....(ii)

$\Rightarrow \; x^2 + 5^2 = (17 - x)^2 + 12^2$

[On equating the values of r^2]

$\Rightarrow \; x^2 + 25 = 289 - 34x + x^2 + 144$

$\Rightarrow \; 34x = 408 \Rightarrow x = 12$ cm.

Putting x = 12 cm in equation (i), we get

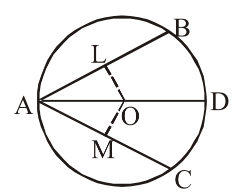
$r^2 = 12^2 + 5^2 = 169$

$\Rightarrow \; r = 13$ cm.

Hence, the radius of the circle is 13 cm.

Ex.18 If two chords of a circle are equally inclined to the diameter through their point of intersection, prove that the chords are equal.

Sol. **Given.** Two chords AB and AC of a circle C(O, r), such that AB and AC are equally inclined to diameter AOD.



To prove. AB = AC

Construction. Draw $OL \perp AB$ and $OM \perp AC$.

Proof. In ΔOLA & ΔOMA,

$\angle OLA = \angle OMA$ [Each equal to 90°]

$\angle OAL = \angle OAM$ [Given]

and OA = OA [Common]

$\therefore \quad \Delta OLA \cong \Delta OMA$

by AAS criteria

$\Rightarrow \quad OL = OM$

$\Rightarrow$ Chords AB and AC are equidistant from O.

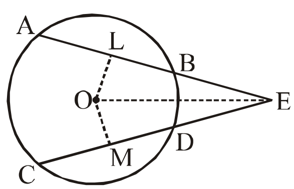
$\Rightarrow \quad AB = AC$

Ex.19 Two equal chords AB and CD of a circle with centre O, when produced meet at a point E. Prove that BE = DE and AE = CE.

Sol. **Given.** Two equal chords AB and CD intersecting at a point E.

To prove. BE = DE and AE = CE.

Construction. Join OE, Draw $OL \perp AB$ and $OM \perp CD$



Proof.

In triangles OLE and OME, $(\because \overline{AB} = \overline{CD})$

$OL = OM$

$\angle OLE = \angle OME$ [Each equal to 90°]

and $OE = OE$ [Common]

$\therefore \quad \Delta OLE \cong \Delta OME$ [By RHS criteria]

$\Rightarrow \quad LE = ME$ ....(1) [C.P.C.T]

$AB = CD$ [Given]

Now, $\frac{1}{2}AB = \frac{1}{2}CD$

$\Rightarrow \quad BL = DM$ ….(2)

Subtracting (2) from (1), we get

$LE - BL = ME - DM$

$BE = DE.$

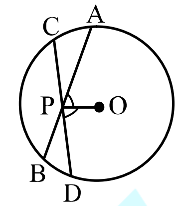
Again, AB = CD and BE = DE

$\Rightarrow \quad AB + BE = CD + DE$

$\Rightarrow \quad AE = CE$

Hence, BE = DE and AE = CE.

Ex.20 O is the centre of the circle and PO bisects the angle APD. Prove that AB = CD.

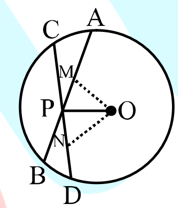


Sol. **Given.** A circle with centre O. Chords AB and CD meet at point P. PO bisects the angle APD.

To prove. AB = CD

Construction. Draw $OM \perp AB$ and $ON \perp CD$.

Proof. In ΔOMP and ΔONP,



$\angle OMP = \angle ONP$ ...(Each 90°)

$OP = OP$ ...(Common)

$\angle OPM = \angle OPN$ ...(Given)

$\Rightarrow \quad \Delta OMP \cong \Delta ONP$ (AAS congruency)

$\Rightarrow \quad OM = ON$ (CPCT)

$\Rightarrow$ Chords AB and CD are equidistant from centre.

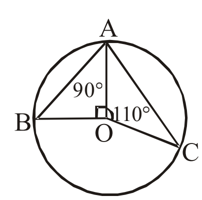
We know that chords, which are equidistant from the centre of a circle, are also equal.

$\therefore \quad AB = CD.$

Ex.21 O is the centre of the circle. If $\angle BOA = 90°$ and $\angle COA = 110°$, find $\angle BAC$.

Sol. **Given** : A circle with centre O and $\angle AOB = 90°$, $\angle AOC = 110°$.

To find : $\angle BAC = ?$



Sol. $\angle AOB + \angle AOC + \angle BOC = 360°$

$\Rightarrow \quad 90° + 110° + \angle BOC = 360°$

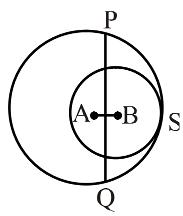
$\Rightarrow \angle BOC = 360° - 90° - 110° \Rightarrow \angle BOC = 160°$

But, arc $\overset{\frown}{BC}$ subtends $\angle BOC$ at the centre and $\angle BAC$ at the remaining part of the circle.

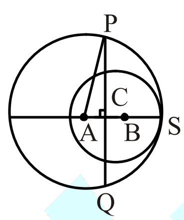
$$\Rightarrow \angle BAC = \frac{1}{2}\angle BOC$$

$$\therefore \angle BAC = \frac{1}{2}(160°) = 80°$$

Ex.22 Two circles with centres, A and B and of radii 5 cm and 3 cm respectively touch each other internally. If the perpendicular bisector of segment AB meets the bigger circle in P and Q. Find the length of PQ.



Sol. **Given.** Two circles touch internally at S, A and B be the centres of the bigger and smaller circle respectively. The perpendicular bisector PQ bisects AB and meets the circle at P and Q.



To find. PQ

Construction. Join PA, ABS

Procedure. With given radii, we find

AS = 5 cm

BS = 3 cm

AB = 5 – 3 = 2 cm and AC = 1 cm

[$\perp$ bisector bisects the chord]

PA = radius of bigger circle = 5 cm

In right triangle ACP,

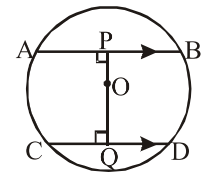
$PC^2 = PA^2 - AC^2$ [By Pythagoras Theorem]

$\Rightarrow PC^2 = (5)^2 - (1)^2 \Rightarrow PC^2 = 25 - 1 = 24$

$\Rightarrow PC = \sqrt{24} \qquad \Rightarrow PC = 2\sqrt{6}$

$PQ = 2PC = 4\sqrt{6}$ cm.

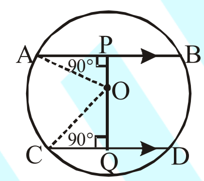
Ex.23 O is the centre of the circle with radius 5 cm. $OP \perp AB$, $OQ \perp CD$, $AB \parallel CD$, AB = 8 cm and CD = 6 cm. Determine PQ.



Sol. **Given.** AB and CD are two parallel chords. AB = 8 cm, CD = 6 cm, radius = 5 cm.

To find. PQ

Construction. Join OA, OC where O is the centre of the circle.



Procedure.

AP = PB = 4 cm ...[$\because$ AB = 8 cm]

CQ = QD = 3cm ...[$\because$ CD = 6 cm]

[$\because$ $\perp$ from the centre bisects the chord]

In rt. ΔOAP,

OA = OC = radii of the circle = 5 cm

$OP^2 = OA^2 - AP^2$ [By pythagoras theorem]

$= (5)^2 - (4)^2 = 25 - 16 = 9 \Rightarrow OP = 3$

In rt. ΔOCQ, $OQ^2 = OC^2 - CQ^2$

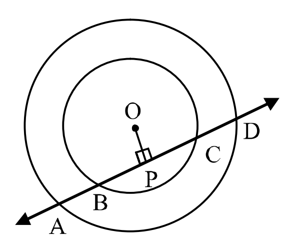
[By pythagoras theorem]

$(5)^2 - (3)^2 = 25 - 9 = 16 \Rightarrow OQ = 4$

$\therefore$ PQ = PO + OQ = 3 + 4 = 7 cm.

Ex. 24 If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, Prove that AB = CD (figure)

[NCERT]



Sol. Draw $OP \perp AD$

For outer circle, AD is chord

$\therefore$ AP = PD .....(1)

($\because$ $\perp$ from centre bisect the chord)

& for inner circle, BC is chord

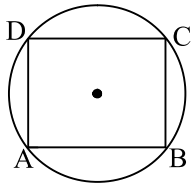
$\therefore$ BP = PC .....(2)

Subtrtact equation (2) from eqation (1)

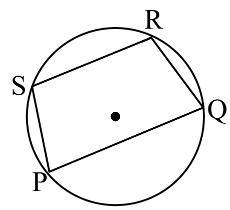
$\Rightarrow$ AP – BP = PD – PC $\Rightarrow$ AB = CD

CYCLIC QUADRILATERALS

If all four points of a quadrilateral are on circle then it is called cyclic Quadrilateral.



or

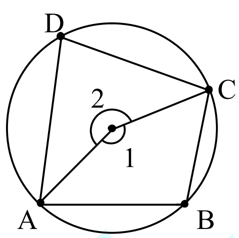


Properties :

1. Sum of opposite angles is 180º (or opposite angles of cyclic quadrilateral is supplementary)'

Proof : For Arc $\overset{\frown}{ABC}$ $\angle D = \frac{1}{2}\angle 1$ .....(1)

& for Arc $\overset{\frown}{ADC}$ $\angle B = \frac{1}{2}\angle 2$ ......(2)



{angle at circumference is half of the angle at the centre}

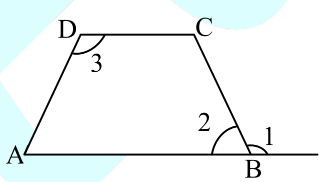
Now, adding equation (1) & (2)

$\angle B + \angle D = \frac{1}{2}(\angle 1 + \angle 2) = \frac{1}{2}(360) = 180º$

2. Exterior angle **:** Exterior angle of cyclic quadrilateral is equal to opposite interior angle.

Proof : Let ABCD is cyclic quadrilateral.

Then $\angle 3 + \angle 2 = 180º$ ....(1)



($\because$ opposite angle are supplementary)

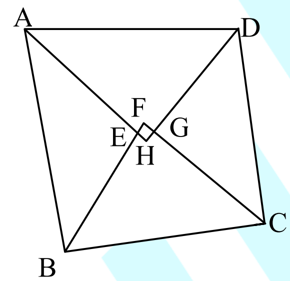
but $\angle 1 + \angle 2 = 180º$ ....(2) (L.P)

$\therefore$ by (1) & (2)

$\angle 3 + \angle 2 = \angle 1 + \angle 2 \Rightarrow \angle 3 = \angle 1$

Ex.25 Prove that the quadrilateral formed by the internal angle bisectors of any quadrilateral is cyclic. **[NCERT]**

Sol. In fig. ABCD is a quadrilateral in which the angle bisectors AH, BF, CF and DH of internal angles A, B, C and D respectively form a quadrilateral EFGH.



Now,

$\angle FEH = \angle AEB = 180º - \angle EAB - \angle EBA$

$= 180º - \frac{1}{2}(\angle A + \angle B)$

and $\angle FGH = \angle CGD = 180º - \angle GCD - \angle GDC$

$= 180º - \frac{1}{2}(\angle C + \angle D)$

Therefore, $\angle FEH + \angle FGH$

$= 180º - \frac{1}{2}(\angle A + \angle B) + 180º - \frac{1}{2}(\angle C + \angle D)$

$= 360º - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$

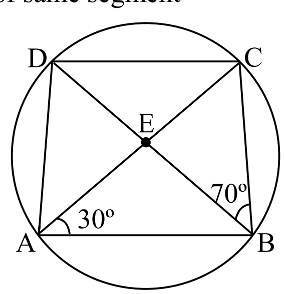
$= 360º - \frac{1}{2} \times 360º = 360º - 180º = 180º$

Therefore, by therorem the quadrilateral EFGH is cyclic.

Ex.26 ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70º$, $\angle BAC$ is 30º, Find $\angle BCD$. Further, if AB = BC, find $\angle ECD$. **[NCERT]**

Sol. $\angle BDC = \angle BAC = 30º$

angle of same segment



$\therefore$ In ΔBCD

$\angle BCD = 180^\circ - (\angle CBD + \angle BDC)$

$= 180^\circ - (70^\circ + 30^\circ) = 180^\circ - 100^\circ = 80^\circ$

If AB = BC

$\therefore$ ΔABC is an isosceles Δ.

$\therefore$ $\angle BCA = \angle BAC = 30^\circ$

so $\angle ECD = \angle BCD - \angle BCA = 80^\circ - 30^\circ = 50^\circ$

Ex.27 Prove that a cyclic parallelogram is a rectangle.

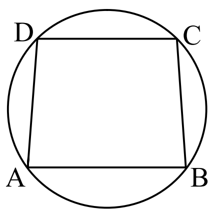
(NCERT)

Sol. ABCD is cyclic

$\therefore$ $\angle A + \angle C = 180^\circ$ ...(1)

But ABCD is $||^{gm}$

$\therefore$ $\angle A = \angle C$ ....(2)



By (1) & (2)

$\angle A + \angle A = 180^\circ$

$2\angle A = 180^\circ$

$\angle A = 90^\circ$

$\therefore$ $||^{gm}$ ABCD is rectangle .