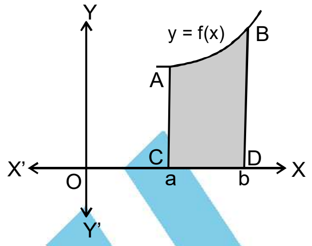
AREAS OF BOUNDED REGIONS

6.1 AREA UNDER SIMPLE CURVE

1. The area of the region bounded by a curve $y = f(x)$, $x = a$, $x = b$, and the x - axis.

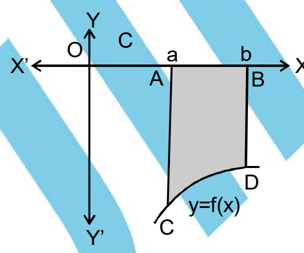
Case – I When the curve $y = f(x)$ lies above the x-axis.
The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by $\int_a^b y\,dx$



Case – II When the curve $y = f(x)$ lies below the x-axis.

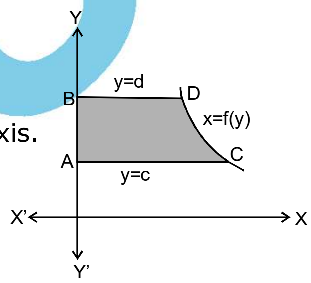
The area bounded by the curve $y = f(x)$, the x-axis,

and the ordinaes $x = a$ and $x = b$ is given by $\int_a^b (-y)\,dx$

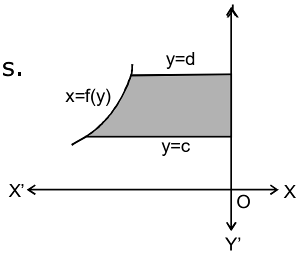


2. The Area of the Region Bounded by a Curve $x = f(y)$,
The Abscissae $y = c$, $y = d$ and the y -axis.

Case – I When the curve $x = f(y)$ lies to the right of the y -axis.
The area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = c$ and $y = d$ is given by $\int_c^d x\,dy$

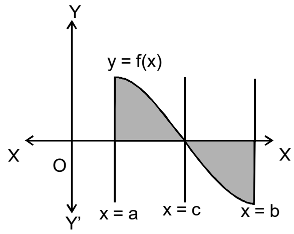


Case – II When the curve $x = f(y)$ lies to the left of the y-axis.
The area bounded by the curve $x = f(y)$, the y-axis and the absissa $y = c$ and $y = d$ is given by $\int_c^d (-x)\,dy$

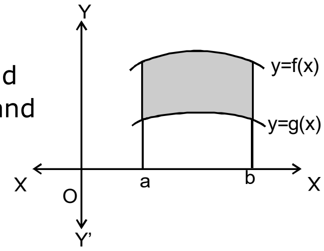


3. The Area Bounded by Curve $y = f(x)$, Ordinates $x = a$ and $x = b$ and x -axis is given by

$$A = \left|\int_a^c y\,dx\right| + \left|\int_c^b (-y)\,dx\right|, \text{ where } a < c < b.$$

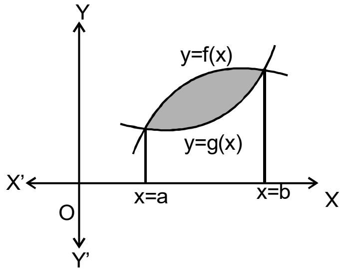


4. The Area Between Two Curves
Case – I The area bounded by two curves $y = f(x)$ and $y = g(x)$, which are intersected by the ordinates $x = a$ and $x = b$ is given by $\int_a^b [f(x) - g(x)]\,dx$



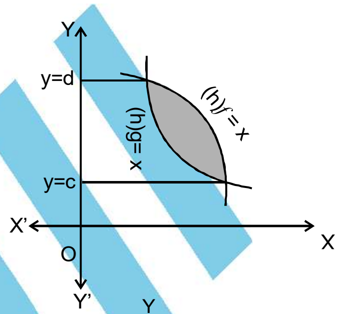
Case – II The area bounded by two curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b \{f(x) - g(x)\}dx$,

where a and b are the abscissae of the points of intersection of the two curves.



Case – III The area bounded by two curves $x = f(y)$ and $x = g(y)$ is given by $\int_c^d \{f(y) - g(y)\}dx$,

where c and d are ordinates of the points of intersection of the two curves.



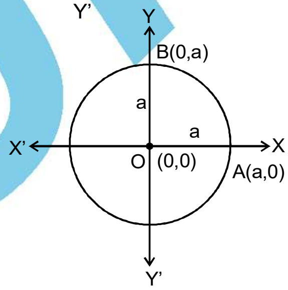
6.2 SOME STANDARD CURVES

1. Circle

(i) $x^2 + y^2 = a^2$ represents a circle with centre at (0,0) and radius = a. It is symmetric about both the axes.

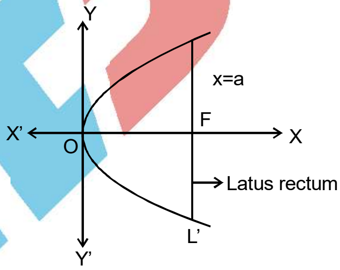
(ii) $(x-\alpha)^2 + (y - \beta)^2 = a^2$ represents a circle with centre at (α, β) and radius = a.

(iii) $x^2 + y^2 + 2gx + 2fy + c = 0$ represents the general equation of a circle with centre at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

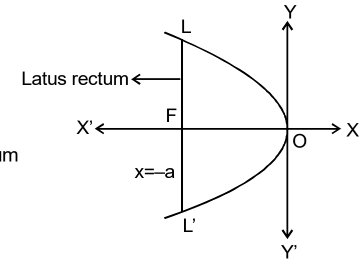


2. Parabola

(i) $y^2 = 4ax$, where $a > 0$ or $y^2 = 4ax$, whre $a < 0$ (standard equation)
It is symmetric about x -axis,
where O is the vertex,
F is the focus, LL'
is the latus rectum :
$LL' \perp X'X; X'X$ being the axis of the parabola.



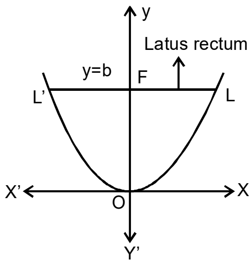
$y^2 = 4ax,\ a > 0$



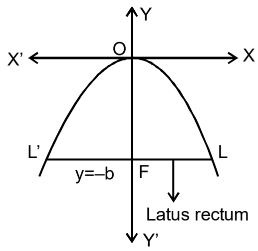
(ii) $x^2 = 4by$, where $b > 0$ or $x^2 = 4by$, where $b < 0$

It is symmetric about y-axis,
where LL' is the latus rectum.

O is the vertex, F is the focus,
LL' is the latus rectum, $LL' \perp YY'$.



$x^2 = 4by,\ b > 0$

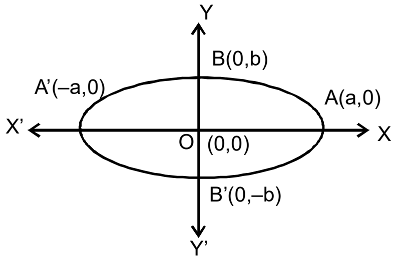


$x^2 = 4by,\ b < 0$

3. Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an ellipse with centre at (0, 0).

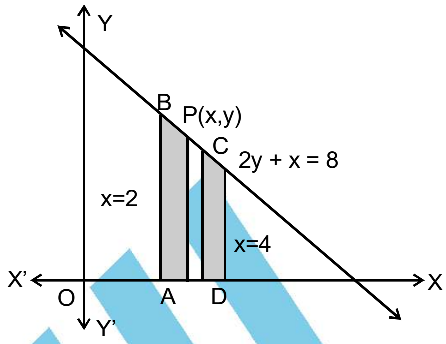
It is symmetric about both the axes, meeting x-axis at $(\pm a, 0)$ and y-axis at $(0, \pm b)$ and $a > b$. Here, $AA' = 2a$ = length of major axis. $BB' = 2b$ = length of minor axis.



SOLVED PROBLEMS

Ex.1 ***Using integration, find the region bounded by the line 2y + x = 8 or, (2y = –x + 8) the x-axis and the lines x = 2 and x = 4.***

Sol. Here, the given line is



$2y + x = 8 \qquad \Rightarrow y = \frac{1}{2}(8 - x)$ ....(1)

$\therefore$ the required area (shaded region) bounded by the line, the x-axis and the line x = 2 and x = 4

$$= \int_2^4 y\,dx = \frac{1}{2}\int_2^4 (8 - x)dx \quad \text{[From (1)]}$$

$$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4 = = \frac{1}{2}[(32 - 8) - (16 - 2)] = 5 \text{ sq. units.}$$

Ex.2 ***Find the area of the region bounded by the parabola $y^2 = 4x$ and y = 2x***

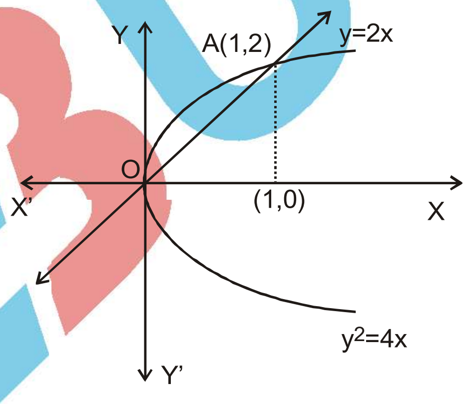
Sol. The points of intersection of $y^2 = 4x$ and y = 2x and O(0, 0) and A(1, 2)

Here, we take $y^2 = 4x$ or $y = 2\sqrt{x}$ as f(x)

and y = 2x as g(x), where f(x) ≥ g(x) in [0, 1]

Therefore,

Area of the shaded region



$$= \int_0^1 [f(x) - g(x)]dx$$

$$= \int_0^1 (2\sqrt{x} - 2x)dx$$

$$= 2\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = 2\left[\frac{2}{3} - \frac{1}{2}\right] = \frac{1}{3}\text{sq.units}$$

Ex.3 ***Find the area of the smaller region enclosed between the circle $x^2 + y^2 = 4$ and the line x + y = 2.***

Sol. The points of intersection of $x^2 + y^2 = 4$ and the line x + y = 2 are A(2, 0) and B(0, 2)

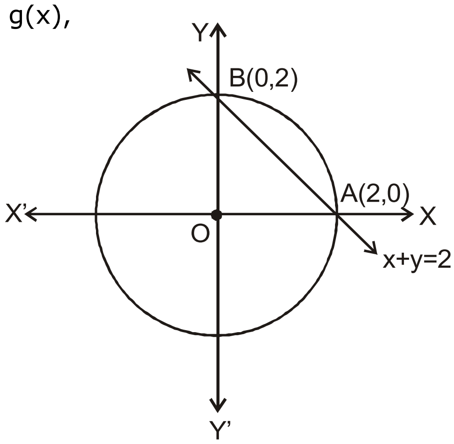
Here, we take $x^2 + y^2 = 4$ i.e., $y = \sqrt{4 - x^2}$

as f(x) and x + y = 2 i.e., y = 2 – x as g(x),

where f(x) ≥ g (x) in [0, 2].

Therefore,

Area of the shaded region



$$= \int_0^2 [f(x) - g(x)]dx$$

$$= \int_0^2 \left[\sqrt{4 - x^2} - (2 - x)\right]dx$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$= 2\sin^{-1}(1) - 4 + 2 = (\pi - 2)$ sq. units

Ex.4 ***Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.***

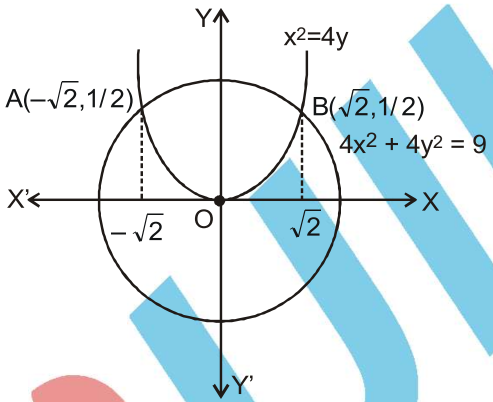
Sol. Given equations to the curves are

$$x^2 + y^2 = \frac{9}{4} \quad ...(1) \qquad x^2 = 4y \quad ...(2)$$

Equation (1) represents a circle whose centre is (0, 0) and radius $\frac{3}{2}$.

Equation (2) represents a parabola whose vertex is O(0, 0) and x-axis is y-axis.

From (2), we have $y = \frac{x^2}{4}$



Put $y = \frac{x^2}{4}$ in (1), we have

$$x^2 + \frac{x^4}{16} = \frac{9}{4}$$

$\Rightarrow \quad x^4 + 16x^2 - 36 = 0$

$\Rightarrow \quad (x^2 + 18)(x^2 - 2) = 0$

$\Rightarrow \quad x^2 = 2 \text{ or } -18$

$\Rightarrow \quad x = \pm\sqrt{2}$

$(\because x^2 \text{ cannot be } -18)$

$\therefore \quad y = \frac{x^2}{4} = \frac{1}{2}$

Taking $y = \sqrt{\frac{9}{4} - x^2}$ on f(x) and $y = \frac{x^2}{4}$ as g(x), where $f(x) \geq g(x)$ in $[-\sqrt{2}, \sqrt{2}]$ we have

Required area $= \int_{-\sqrt{2}}^{\sqrt{2}} [f(x) - g(x)]dx = 2\int_0^{\sqrt{2}} [f(x) - g(x)]dx = 2\int_0^{\sqrt{2}} \left[\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\right]dx$

$$= 2\left[\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{8}\right) - \frac{\sqrt{2}}{6}\right]_0^{\sqrt{2}} = 2\left[\frac{\sqrt{2}\sqrt{\frac{9}{4} - 2}}{2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{8}\right) - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] = 2\left[\frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] = \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ sq. units}$$

Ex.5 ***Find the area bounded by curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.***

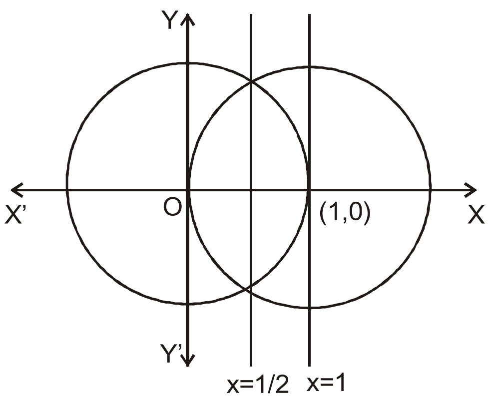
Sol. Given equations to the curves are

$x^2 + y^2 = 1 \quad ...(1)$

$(x - 1)^2 + y^2 = 1 \quad ...(2)$

$(x - 1)^2 + (1 - x^2) = 1$

$\Rightarrow \quad x^2 - 2x + 1 + 1 - x^2 = 1 \quad \Rightarrow \quad 2x = 1 \Rightarrow x = \frac{1}{2}$



Requried area

$$= 2\left[\int_0^{1/2} y \text{ of circle (2) } dx + \int_{1/2}^{1} y \text{ of circle (1) } dx\right]$$

$$= 2\left[\int_0^{1/2} \sqrt{1 - (x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1 - x^2}\,dx\right]$$

$$= 2\left\{\left[\frac{(x-1)\sqrt{1-(x-1)^2}}{2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2}+\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\left[-\frac{1}{4}\sqrt{1-\frac{1}{4}}+\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)-0-\frac{1}{2}\sin^{-1}(-1)\right]+\left[0+\frac{1}{2}\sin^{-1}(1)-\frac{1}{4}\sqrt{1-\frac{1}{4}}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]\right\}$$

$$= -\frac{1}{2}\sqrt{\frac{3}{4}}+\sin^{-1}\left(-\frac{1}{2}\right)-\sin^{-1}(-1)+\sin^{-1}(1)-\frac{1}{2}\sqrt{\frac{3}{4}}-\sin^{-1}\left(\frac{1}{2}\right) \quad = -\sqrt{\frac{3}{4}}+\left(-\frac{\pi}{6}\right)-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}$$

$$= \left(-\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right) \text{ sq. units.}$$

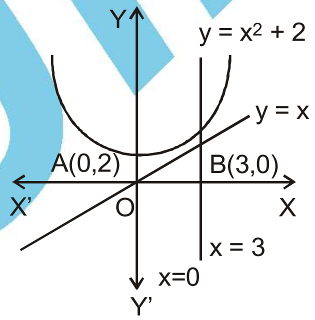
Ex.6 ***Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.***

Sol. Given equations are

$y = x^2 + 2$ ...(1) $\quad y = x$ ...(2)

$x = 0$ ...(3) $\quad x = 3$ ...(4)

Equation (1) represetns a parabola with vertex A(0, 2) and axis is y-axis.

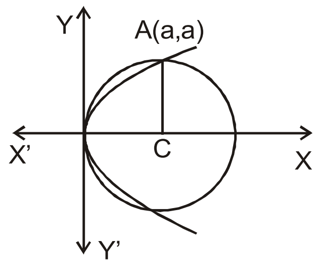


Equation (2) represents a line passing through (0, 0) and (1, 1)
Equation (3) represents y-axis
Equation (4) represent a line parallel to y-axis and passing through B(3, 0)

$$\text{Required area} = \int_0^3 y(\text{of parabola})dx - \int_0^3 y(\text{of line } y = x)dx$$

$$= \int_0^3 (x^2+2)dx - \int_0^3 x dx = \left[\frac{x^3}{3}+2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3 = (9+6)-\left(\frac{9}{2}+0\right) = \frac{21}{2} \text{ sq. units.}$$

Ex.7 ***Find the area of the region***
$\{(x,y) : x^2 + y^2 \le 2ax,\ y^2 \le 2ax,\ a > 0,\ x \ge 0,\ y \ge 0\}$



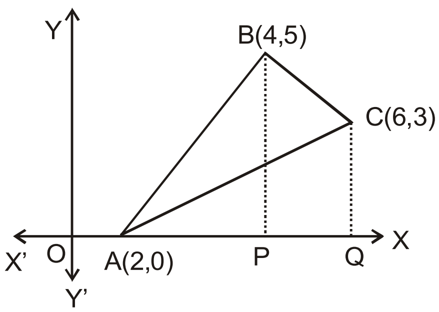
Sol. Required area $= \int_0^a \left[\sqrt{a^2-(x-a)^2}-\sqrt{ax}\right]dx$

$$= \int_0^a \left[\sqrt{a^2-(x-a)^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right)-\frac{2\sqrt{a}}{3}x^{3/2}\right]_0^a$$

$$= \left[\frac{1}{2}a^2\sin^{-1}(0)-\frac{2}{3}a^2\right]-\left[\frac{1}{2}a^2\sin^{-1}(-1)\right] = \frac{-2}{3}a^2-\frac{1}{2}a^2\left(\frac{-\pi}{2}\right) = \left(\frac{\pi a^2}{4}-\frac{2}{3}a^2\right) \text{ sq. units.}$$

Ex.8 ***Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (2, 0), B(4, 5) and C (6, 3).***

Sol. Required area = area of the region ABC
= area of the region APB + area of the region PQCB – Area of the region AQC



$$= \int_2^4 \frac{5x-10}{2}dx + \int_4^6 (9-x)dx - \int_2^6 \left(\frac{3x-6}{4}\right)dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4 + \left[9x-\frac{x^2}{2}\right]_4^6 - \frac{3}{4}\left[\frac{x^2}{2}-2x\right]_2^6$$

= 7 sq. units.

EXERCISE – I UNSOLVED PROBLEMS

Q.1 Using integration, find the area of the region bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y - x = 5$.

Q.2 Find the area of the region bounded by the curve $x^2 = 4y$ and the lines $x = 4y - 2$

Q.3 Find the area of the region bounded by the curve $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$

Q.4 Find the area of the region enclosed between the parabola $y^2 = x$ and the line $x + y = 2$ in the first quadrant.

Q.5 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

Q.6 Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

Q.7 Find the area of the region bounded by the curve $y = \sqrt{1-x^2}$, the line $y = x$ and the positive x-axis.

Q.8 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

Q.9 Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$

Q.10 Draw the rough sketch of $y^2 = x + 1$ and $y^2 = -x + 1$ and determine the area enclosed by the two curves.

Q.11 Calculate the area of the region enclosed between the circle $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$

Q.12 Using integration, find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$

Q.13 Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Also find the area of the region, using integration

Q.14 Sketch the graph of $y = |x + 3|$. Evaluate $\int_{-6}^{0}|x+3|dx$ What does the value of the integral represent on the graphs ?

Q.15 Using integration find the area of the region bounded by the following curves after making a rough sketch :

$$y = 1 + |x - 1|,\ x = -3,\ x = 3 \text{ and } y = 0$$

Q.16 Find the area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$

Q.17 Find the area of the region $\{(x, y) : x^2 \le y \le |x|\}$

Q.18 Find the area of the region : $\{(x, y) : 0 \le y \le (x^2 + 1), 0 \le y \le (x + 1)\}, 0 \le x \le 2\}$

Q.19 Find the area of the region : $\{(x,y) : x^2 + y^2 \le 2ax\}, y^2 \ge ax, x \ge 0, y \ge 0$

Answers

1. $\frac{15}{2}$ **2.** $\frac{9}{8}$ **3.** $\frac{21}{2}$ **4.** $\frac{7}{6}$ **5.** $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$ **6.** 4π **7.** $\frac{\pi}{8}$ **8.** $\frac{ab\pi}{4} - \frac{ab}{2}$

9. $\frac{16a^2}{3}$ **10.** $\frac{8}{3}$ **11.** $\left(\frac{8\pi}{3}-2\sqrt{3}\right)$ **12.** $\frac{4\pi+\sqrt{3}}{8\pi-\sqrt{3}}$ **13.** $\frac{16\pi+4\sqrt{3}}{3}$ **14.** 9

15. 16 **16.** $\left(\frac{\pi}{4}-\frac{1}{2}\right)$ **17.** $\frac{1}{3}$ **18.** $\frac{23}{6}$ **19.** $\left(\frac{\pi a^2}{4}-\frac{2a^2}{3}\right)$ **20.** 4

EXERCISE – II BOARD PROBLEMS

Q.1 Using integration, find the area of the region, given : $\{(x, y) : 0 \le y \le x^2 + 1 ; 0 \le y \le x + 1, 0 \le x \le 2\}$

Q.2 Using integration, find the area of ΔPQR where P is (2, 1) Q is (3, 4) and R is (5, 2)

Q.3 Using integration, find the area of the region bounded by the line 2y + x = 8, x-axis and the line x = 2 sand x = 4.

Q.4 Find the area of the region given by $\{(x, y) : x^2 \le y \le |x|\}$.

Q.5 Using integration, find the area of triangle ABC, whose vertices are A(3,0), B(4, 5) and C(5, 1)

Q.6 Find the area of the following region : $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$.

Q.7 Using integration, find the area of the region in the first quadrant enclosed by the x-axis and the line y = x and the circle $x^2 + y^2 = 32$.

Q.8 Find the area of the region lying between the parabola $y^2 = 4ax$ and $x^2 = 4ay$, where a > 0

Q.9 Using integration, find the area of the smaller region bounded by the curve $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the straight line $\frac{x}{4}+\frac{y}{3}=1$

Q.10 Find the area enclosed by the parabola $y^2 = x$ and the line x + y = 2 and the x-axis.

Q.11 Using integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the straight line $\frac{x}{a}+\frac{y}{b}=1$

Q.12 Using integration, find the area of the region enclosed between two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 =1$.

Q.13 Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

Q.14 Find the area of that part of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.

Q.15 Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Q.16 Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Q.17 Find the area of the region included between the parabola $y^2 = x$ and the line x + y = 2.

Q.18 Using the method of integration, find the area of the region bounded by the lines 2x + y = 4, 3x – 2y = 6 and x – 3y + 5 = 0

Q.19 Find the are of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Q.20 Using integration, find the area of the triangle ABC, coordinates of whose vertices are A(4, 1), B(6, 6) and C(8, 4).

Q.21 Sketch the graph of y = |x + 3| and evaluate the area under the curve y = |x + 3| above x-axis and between x = – 6 to x = 0.

Q.22 Find the area of the region : $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$.

Q.22 Find the area of the region bourded by the parabola $y = x^2$ and y = |x|

Answers

1. $\frac{23}{6}$ **2.** 4 **3.** 5 **4.** $\frac{1}{3}$ **5.** 9/2 **6.** $\frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$

7. 4π **8.** 16/3 **9.** $3(\pi - 2)$ **10.** 7/6 **11.** $\frac{ab}{4}(\pi-2)$ **12.** $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

13. $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ **14.** $\frac{4}{3}(8\pi - \sqrt{3})$ **15.** $\frac{16}{3}a^2$ **16.** $\left(\frac{8\pi}{3}-2\sqrt{3}\right)$ **21.** 9 **22.** $(\pi - 2)$