QUARDATIC EQUATION

POLYNOMIAL

A function f defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial of degree n with real coefficients ($a_n \neq 0$, $n \in W$). If $a_0, a_1, a_2, ..., a_n \in C$, it is called a polynomial with complex coefficients.

Introduction of Quadratic Expression and Quadratic Equation

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2+bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

Difference Between Equation & Identity

If a statement is true for all the values of the variable, such statements are called as identities. If the statement is true for some or no values of the variable, such statements are called as equations.

Ex. **(i)** $(x + 3)^2 = x^2 + 6x + 9$ is an identity

(ii) $(x + 3)^2 = x^2 + 6x + 8$, is an equation having no root.

(iii) $(x + 3)^2 = x^2 + 5x + 8$, is an equation having – 1 as its root.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

$ax^2 + bx + c = 0$ is:

- ❖ a quadratic equation if — $a \neq 0$ — Two Roots
- ❖ a linear equation if — $a = 0, b \neq 0$ — One Root
- ❖ a contradiction if — $a = b = 0, c \neq 0$ — No Root
- ❖ an identity if — $a = b = c = 0$ — Infinite Roots

If $ax^2 + bx + c = 0$ is satisfied by three distinct values of 'x', then it is an identity.

Ex. **(i)** $3x^2 + 2x - 1 = 0$ is a quadratic equation here $a = 3$. **(ii)** $(x + 1)^2 = x^2 + 2x + 1$ is an identity in x.

Sol. Here highest power of x in the given relation is 2 and this relation is satisfied by three different values $x = 0$, $x = 1$ and $x = -1$ and hence it is an identity because a polynomial equation of n^{th} degree cannot have more than n distinct roots.

Solution of Quadratic Equation & Relation Between Roots & Co-efficients

(A) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(B) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

Relation Between Roots & Co-efficients

(i) The solutions of quadratic equation, $a x^2 + b x + c = 0$, $(a \neq 0)$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression, $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

(ii) If α, β are the roots of quadratic equation,

$a x^2 + b x + c = 0$ ....**(i)**

then equation **(i)** can be written as

$a(x - \alpha)(x - \beta) = 0$

or $ax^2 - a(\alpha + \beta)x + a\alpha\beta = 0$ ......**(ii)**

equations **(i)** and **(ii)** are identical,

$\therefore$ by comparing the coefficients sum of the roots, $\alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and product of the roots, $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

(iii) Dividing the equation **(i)** by a, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\Rightarrow$ $x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0$ $\Rightarrow$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow$ x^2 – (sum of the roots) x + (product of the roots) = 0

Hence we conclude that the quadratic equation whose roots are α & β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Ex. If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha + 2$ and $\beta + 2$.

Sol. Replacing x by $x - 2$ in the given equation, the required equation is

$a(x - 2)^2 + b(x - 2) + c = 0$ i.e., $ax^2 - (4a - b)x + (4a - 2b + c) = 0$.

Ex. If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

Sol. Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$, $\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$, $\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ....**(i)**

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is $x^2 - 4x + 4 = 0$.

Nature of Roots

(A) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ R & a $\neq$ 0 then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. $(p, q \in R \;\&\; i = \sqrt{-1})$.

(B) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ Q & a $\neq$ 0 then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

Ex. Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Sol. Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From **(i)** $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$

Ex. For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Sol. Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ ....**(i)**

Let D be the discriminant of equation **(i)**.

Roots of equation **(i)** will be equal if D = 0.

or $\quad 4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

or $\quad 4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

or $\quad m^2 - 3m = 0 \quad$ or, $\quad m(m - 3) = 0$

$\therefore \quad m = 0, 3.$

Ex. Determine 'a' such that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

Sol. Let $x - \alpha$ be a common factor of $x^2 - 11x + a$ and $x^2 - 14x + 2a$.

Then $x = \alpha$ will satisfy the equations $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$.

$\therefore \quad \alpha^2 - 11\alpha + a = 0 \quad$ and $\quad \alpha^2 - 14\alpha + 2a = 0$

Solving **(i)** and **(ii)** by cross multiplication method, we get a = 0, 24.

Ex. If roots of the equation $(a - b)x^2 + (c - a)x + (b - c) = 0$ are equal, then a, b, c are in

(A) A.P. **(B)** H.P. **(C)** G.P. **(D)** none of these

Sol. $(a - b)x^2 + (c - a)x + (b - c) = 0$

As roots are equal so

$B^2 - 4AC = 0 \Rightarrow (c - a)^2 - 4(a - b)(b - c) = 0 \Rightarrow (c - a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow \quad (c - a)^2 + 4ac - 4b(c + a) + 4b^2 = 0 \quad \Rightarrow \quad (c + a)^2 - 2 . (2b)(c + a) + (2b)^2 = 0$

$\Rightarrow \quad [c + a - 2b]^2 = 0 \quad \Rightarrow \quad c + a - 2b = 0 \Rightarrow c + a = 2b$

Hence a, b, c are in A. P.

Alternative method

✈ Sum of the coefficients = 0

Hence one root is 1 and other root is $\dfrac{b - c}{a - b}$.

Given that both roots are equal, so

$1 = \dfrac{b - c}{a - b} \Rightarrow a - b = b - c \Rightarrow 2b = a + c$

Hence a, b, c are in A.P.

COMMON ROOTS OF TWO QUADRATIC EQUATIONS

(A) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(B) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

Ex. Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Sol. $a_1 = p, b_1 = 5, c_1 = 2$

$a_2 = 3, b_2 = 10, c_2 = q$

We know that :

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q} \Rightarrow p = \frac{3}{2};\ q = 4$$

Ex. If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots, show that $b + q = \dfrac{ap}{2}$.

Sol. Given equations are : $x^2 - ax + b = 0$ .....(i)

and $x^2 - px + q = 0$. .....(ii)

Let α be the common root. Then roots of equation (ii) will be α and α. Let β be the other root of equation (i). Thus roots of equation (i) are α, β and those of equation (ii) are α, α.

Now $\alpha + \beta = a$ .....(iii)

$\alpha\beta = b$ .....(iv)

$2\alpha = p$ .....(v)

$\alpha^2 = q$ .....(vi)

L.H.S. $= b + q = \alpha\beta + \alpha^2 = \alpha(\alpha + \beta)$ .....(vii)

and R.H.S. $= \dfrac{ap}{2} = \dfrac{(\alpha+\beta)\,2\alpha}{2} = \alpha(\alpha + \beta)$ .....(viii)

from (vii) and (viii), L.H.S. = R.H.S.

Remainder Theorem

If we divide a polynomial f(x) by $(x - \alpha)$ the remainder obtained is $f(\alpha)$. If $f(\alpha)$ is 0 then $(x - \alpha)$ is a factor of f(x).

Consider $f(x) = x^3 - 9x^2 + 23x - 15$

$f(1) = 0 \Rightarrow (x - 1)$ is a factor of f(x).

$f(x) = (x - 2)(x^2 - 7x + 9) + 3$. Hence $f(2) = 3$ is remainder when f(x) is divided by $(x - 2)$.

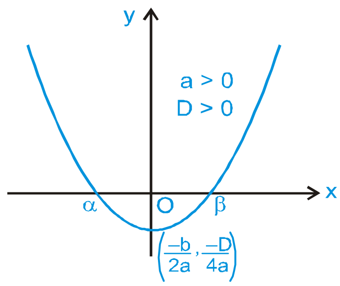
GRAPH OF QUADRATIC EXPRESSION

Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(A) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

(B) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows :

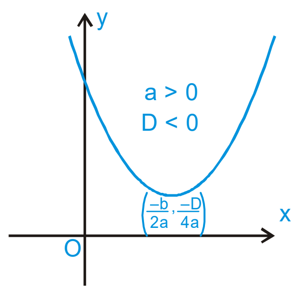
(Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \le \beta$).



Roots are real & distinct

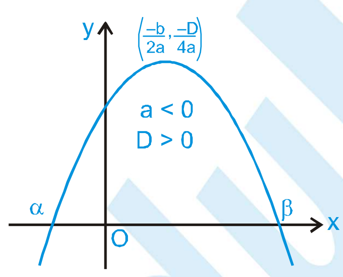
$ax^2 + bx + c > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \ \forall \ x \in (\alpha, \beta)$



Roots are complex conjugate

$ax^2 + bx + c > 0 \ \forall \ x \in R$



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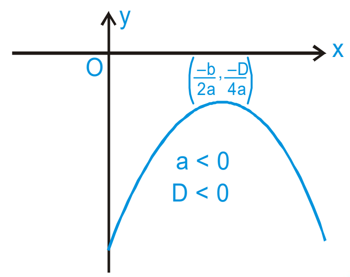
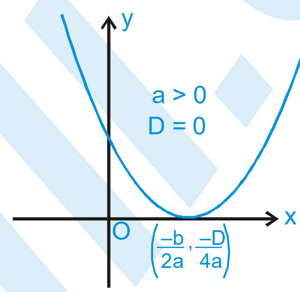


Roots are complex conjugate

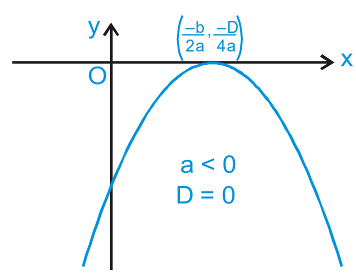
$ax^2 + bx + c < 0 \ \forall \ x \in R$



Roots are coincident

$ax^2 + bx + c > 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0 \text{ for } x = \alpha = \beta$



Roots are coincident

$ax^2 + bx + c = 0 \text{ for } x = \alpha = \beta$

$ax^2 + bx + c < 0 \ \forall \ x \in R - \{\alpha\}$

Range of Quadratic Expressions $y = ax^2 + bx + c$

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{(b^2-4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

Ex. If f(x) is a quadratic expression such that $f(x) > 0\ \forall\ x \in R$, and if $g(x) = f(x) + f'(x) + f''(x)$, then prove that $g(x) > 0\ \forall\ x \in R$.

Sol. Let $f(x) = ax^2 + bx + c$

Given that $f(x) > 0$ so $a > 0$, $b^2 - 4ac < 0$

Now $g(x) = ax^2 + bx + c + 2ax + b + 2a = ax^2 + (b + 2a)x + (b + c + 2a)$

For this quadratic expression $a > 0$ and discriminant

$$D = (b + 2a)^2 - 4a(b + c + 2a) = b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2 = b^2 - 4ac - 4a^2 < 0$$

So $g(x) > 0\ \forall\ x \in R$.

Ex. If $c < 0$ and $ax^2 + bx + c = 0$ does not have any real roots then prove that

(i) $a - b + c < 0$ **(ii)** $9a + 3b + c < 0$.

Sol. $c < 0$ and $D < 0$ $\Rightarrow$ $f(x) = ax^2 + bx + c < 0$ for all $x \in R$

$\Rightarrow$ $f(-1) = a - b + c < 0$

and $f(3) = 9a + 3b + c < 0$

Ex. Find the minimum value of the expression $4x^2 + 2x + 1$.

Sol. Since $a = 4 > 0$ therefore its minimum value is $= \frac{4(4)(1)-(2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$

Maximum & Minimum Values of Rational Algebraic Expressions

$$y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}, \frac{1}{ax^2+bx+c}, \frac{a_1x+b_1}{a_2x^2+b_2x+c_2}, \frac{a_1x^2+b_1x+c_1}{a_2x+b_2} :$$

Sometime we have to find range of expression of form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$. The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.

$$(a_1 - a_2y)x^2 + (b_1 - b_2y)x + (c_1 - c_2y) = 0$$

Step 3 : Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Ex. For $x \in R$, find the set of values attainable by $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$.

Sol. Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$x^2(y-1) + 3x(y+1) + 4(y-1) = 0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R, D \geq 0$

$\Rightarrow \quad 9(y+1)^2 - 16(y-1)^2 \geq 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y-1)(y-7) \leq 0 \quad \Rightarrow \quad y \in \left[\frac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1 = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow \quad x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

Ex. Find the range of $y = \frac{x+2}{2x^2 + 3x + 6}$, if x is real.

Sol. $y = \frac{x+2}{2x^2 + 3x + 6}$

$\Rightarrow \quad 2yx^2 + 3yx + 6y = x + 2$

$\Rightarrow \quad 2yx^2 + (3y-1)x + 6y - 2 = 0 \quad$ ....**(i)**

Case I : if $y \neq 0$, then equation **(i)** is quadratic in x

$\because \quad$ x is real

$\therefore \quad D \geq 0$

$\Rightarrow \quad (3y-1)^2 - 8y(6y-2) \geq 0$

$\Rightarrow \quad (3y-1)(13y+1) \leq 0$

$y \in \left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$

Case II : if $y = 0$, then equation becomes

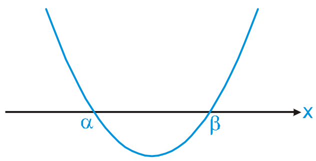
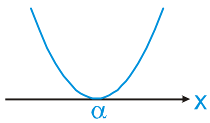
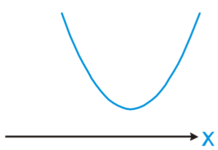
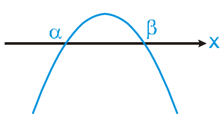
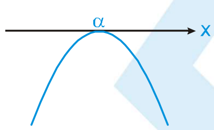
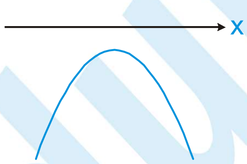
$x = -2$ which is possible as x is real

$\therefore \quad$ Range $y \in \left[-\frac{1}{13}, \frac{1}{3}\right]$

SIGN OF QUADRATIC EXPRESSIONS

The value of expression $f(x) = ax^2 + bx + c$ at $x = x_0$ is equal to y-co-ordinate of the point on parabola $y = ax^2 + bx + c$ whose x–co–ordinate is x_0. Hence if the point lies above the x–axis for some $x = x_0$, then $f(x_0) > 0$ and vice–versa.

We get six different positions of the graph with respect to x–axis as shown.

	Conclusions
(i) 	**(a)** $a > 0$ **(b)** $D > 0$ **(c)** Roots are real & distinct. **(d)** $f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$ **(e)** $f(x) < 0$ in $x \in (\alpha, \beta)$
(ii) 	**(a)** $a > 0$ **(b)** $D = 0$ **(c)** Roots are real & equal. **(d)** $f(x) > 0$ in $x \in R - \{\alpha\}$
(iii) 	**(a)** $a > 0$ **(b)** $D < 0$ **(c)** Roots are imaginary. **(d)** $f(x) > 0 \ \forall\ x \in R$.
(iv) 	**(a)** $a < 0$ **(b)** $D > 0$ **(c)** Roots are real & distinct. **(d)** $f(x) < 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$ **(e)** $f(x) > 0$ in $x \in (\alpha, \beta)$
(v) 	**(a)** $a < 0$ **(b)** $D = 0$ **(c)** Roots are real & equal. **(d)** $f(x) < 0$ in $x \in R - \{\alpha\}$
(vi) 	**(a)** $a < 0$ **(b)** $D < 0$ **(c)** Roots are imaginary. **(d)** $f(x) < 0 \ \forall\ x \in R$.

LOCATION OF ROOTS

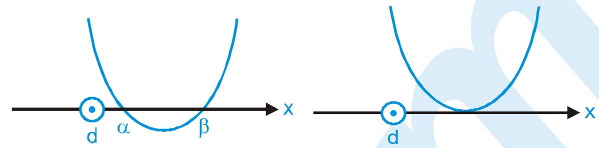
This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation **$ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$**

Type-1 : Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \ge 0$; **(ii)** $f(d) > 0$; **(iii)** $-\dfrac{b}{2a} > d$

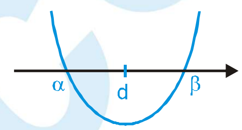


❖ When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \ge 0$; **(ii)** $f(d) > 0$; **(iii)** $-\dfrac{b}{2a} < d$

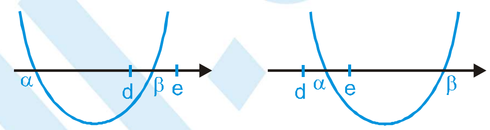
Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.
The necessary and sufficient condition for this are : **$f(d) < 0$**



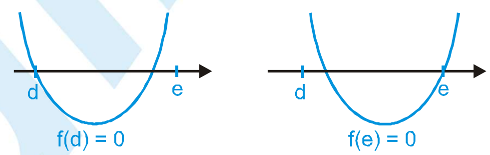
Type-3 :

Exactly one root lies in the interval (d, e).
The necessary and sufficient condition for this are :
$f(d) \cdot f(e) < 0$



❖ The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation.

Hence in this case also check for end points.

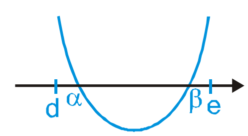


Type-4 :

When both roots are confined between the number d and e (d < e).
The necessary and sufficient condition for this are :

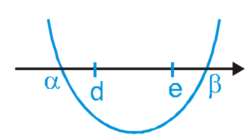
(i) $D \ge 0$; **(ii)** $f(d) > 0$; **(iii)** $f(e) > 0$

(iv) $d < -\dfrac{b}{2a} < e$



Type-5 :

One root is greater than e and the other roots is less than d (d < e).
The necessary and sufficient condition for this are : **$f(d) < 0$** and **$f(e) < 0$**



❖ If $a < 0$ in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Ex. Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Sol. $4 - 2(k+1) + k^2 + k - 8 < 0 \quad \Rightarrow \quad k^2 - k - 6 < 0$

$(k-3)(k+2) < 0 \quad \Rightarrow \quad -2 < k < 3$

Taking intersection, $k \in (-2, 3)$.

Ex. Find all possible values of a for which exactly one root of $x^2-(a+1)x+2a=0$ lies in interval (0,3).

Sol. $f(0) . f(3) < 0$

$\Rightarrow \quad 2a(9-3(a+1)+2a) < 0 \quad \Rightarrow \quad 2a(-a+6) < 0$

$\Rightarrow \quad a(a-6) > 0 \quad \Rightarrow \quad a < 0$ or $a > 6$

Checking the extremes.

If $a = 0$, $x^2 - x = 0$

$x = 0, 1$ $\quad 1 \in (0, 3)$

If $a = 6$, $x^2 - 7x + 12 = 0$

$x = 3, 4$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty, 0] \cup (6, \infty)$

General Quadratic Expression in Two Variables

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Ex. If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

Sol. Here $a = 1, h = 1, b = 0, g = 1, f = m/2, c = -3$

So

$$\Delta = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$\Rightarrow \quad -\dfrac{m^2}{4} - (-3 - m/2) + m/2 = 0 \quad \Rightarrow \quad -\dfrac{m^2}{4} + m + 3 = 0$

$\Rightarrow \quad m^2 - 4m - 12 = 0 \quad \Rightarrow \quad m = -2, 6$

THEORY OF EQUATIONS

Let $\alpha_1, \alpha_2, \alpha_3, ... \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... a_{n-1} x + a_n = 0$,

where $a_0, a_1, ... a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) .... (x-\alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + ... a_{n-1}x + a_n = a_0(x-\alpha_1)(x-\alpha_2) ..... (x-\alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \quad \text{(say)}$$

or $$S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$S_n = \alpha_1\alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$

where S_k denotes the sum of the product of root taken k at a time.

I **Quadratic Equation :** If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

II **Cubic Equation :** If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$ or $(x-\alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x)=0 must have atleast one real root between 'a' and 'b'.

(vi) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

Descartes Rule of Signs

The maximum number of positive real roots of polynomial equation $f(x) = 0$ is the number of changes of signs in $f(x)$.

Consider $x^3 + 6x^2 + 11x - 6 = 0$, The signs are : + + + –

As there is only one change of sign, the equation has atmost one positive real root.

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs in $f(-x)$

Consider $f(x) = x^4 + x^3 + x^2 - x - 1 = 0$

$f(-x) = x^4 - x^3 + x^2 + x - 1 = 0$

3 sign changes, hence atmost 3 negative real roots.

Ex. If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ **(ii)** $\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)$

Sol. We know that $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) = \alpha^2(p - \alpha) + \beta^2(p - \beta) + \gamma^2(p - \gamma)$

$= p(\alpha^2 + \beta^2 + \gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Ex. If $b^2 < 2ac$ and a, b, c, d $\in R$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Sol. Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha + \beta + \gamma = -\dfrac{b}{a}$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$\Rightarrow$ $\alpha^2 + \beta^2 + \gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Transformation of the Equation

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$y = f(\alpha)$

By transformation, $\alpha = g(y)$

$a(g(y))^2 + b(g(y)) + c = 0$

This is the required equation in y.

Ex. If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\dfrac{-2}{\alpha}, \dfrac{-2}{\beta}$ **(b)** $\dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$ **(c)** α^2, β^2

Sol. **(a)** $\dfrac{-2}{\alpha}, \dfrac{-2}{\beta}$

put, $y = \dfrac{-2}{\alpha}$ $\Rightarrow$ $\alpha = \dfrac{-2}{y}$

$a\left(-\dfrac{2}{y}\right)^2 + b\left(\dfrac{-2}{y}\right) + c = 0$ $\Rightarrow$ $cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$

put, $y = \dfrac{\alpha}{\alpha+1}$ $\Rightarrow$ $\alpha = \dfrac{y}{1-y}$

$\Rightarrow$ $a\left(\dfrac{y}{1-y}\right)^2 + b\left(\dfrac{y}{1-y}\right) + c = 0$ $\Rightarrow$ $(a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)x^2 + (b - 2c)x + c = 0$

(c) α^2, β^2

put $y = \alpha^2$

$\Rightarrow$ $\alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$ $\Rightarrow$ $a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

Solution of Quadratic Inequalities

(a) The values of 'x' satisfying the inequality $ax^2 + bx + c > 0$ $(a \neq 0)$ are:

(i) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots α & β such that $\alpha < \beta$

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

$a < 0 \Rightarrow x \in (\alpha, \beta)$

(ii) If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$

$a < 0 \Rightarrow x \in \phi$

(iii) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real root.

Then $a > 0 \Rightarrow x \in R$

$a < 0 \Rightarrow x \in \phi$

(b) Inequalities of the form $\frac{P(x)\ Q(x)\ R(x)\ldots\ldots\ldots}{A(x)\ B(x)\ C(x)\ldots\ldots\ldots} \begin{cases} < \\ = \\ > \end{cases} 0$ can be quickly solved using the method of intervals, where A, B, C........, P, Q, R......... are linear functions of 'x'.

Ex. Find x such that $3x^2 - 7x + 6 < 0$

Sol. $D = 49 - 72 < 0$

As $D < 0$, $3x^2 - 7x + 6$ will always be positive. Hence $x \in \phi$.

Solution of Rational Inequalities

Let $y = \frac{f(x)}{g(x)}$ be an expression in x where f(x) & g(x) are polynomials in x. Now, if it is given that $y > 0$ (or < 0 or ≥ 0 or ≤ 0), this calls for all the values of x for which y satisfies the constraint. This solution set can be found by following steps :

Step I Factorize f(x) & g(x) and generate the form :

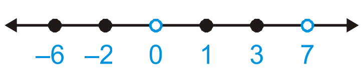
$$y = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\ldots\ldots\ldots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots\ldots\ldots(x-b_p)^{m_p}}$$

where $n_1 n_2 \ldots\ldots\ldots n_k$, $m_1, m_2 \ldots\ldots\ldots m_p$ are natural numbers and $a_1, a_2 \ldots\ldots\ldots a_k$, $b_1, b_2 \ldots\ldots b_p$ are real numbers. Clearly, here $a_1, a_2 \ldots\ldots a_k$ are roots of f(x)=0 & $b_1, b_2 \ldots\ldots b_p$ are roots of g(x) = 0.

Step II Here y vanishes (becomes zero) for $a_1, a_2, \ldots\ldots\ldots a_k$. These points are marked on the number line with a black dot. They are solution of y=0.

If g(x)=0, $y = \frac{f(x)}{g(x)}$ attains an undefined form, hence $b_1, b_2 \ldots\ldots b_k$ are excluded from the solution. These points are marked with white dots.

e.g. $f(x) = \frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$



Step III Check the value of y for any real number greater than the right most marked number on the number line. If it is positive, then y is positive for all the real numbers greater than the right most marked number and vice versa.

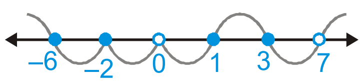
Step IV If the exponent of a factor is odd, then the point is called simple point and if the exponent of a factor is even, then the point is called double point

$$\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$

Here 1,3,–6 and 7 are simple points and –2 & 0 are double points.

From right to left, beginning above the number line (if y is positive in step 3 otherwise from below the line), a wavy curve should be drawn which passes through all the marked points so that when passing through a simple point, the curve intersects the number line and when passing through a double point, the curve remains on the same side of number line.

$$f(x)=\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$



As exponents of (x + 2) and x are even, the curve does not cross the number line. This method is called wavy curve method.

Step V The intervals where the curve is above number line, y will be positive and the intervals where the curve is below the number line, y will be negative. The appropriate intervals are chosen in accordance with the sign of inequality & their union represents the solution of inequality.

(i) Points where denominator is zero will never be included in the answer.

(ii) If you are asked to find the intervals where f(x) is non-negative or non-positive then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

(iii) Normally we cannot cross-multiply in inequalities. But we cross multiply if we are sure that quantity in denominator is always positive.

(iv) Normally we cannot square in inequalities. But we can square if we are sure that both sides are non negative.

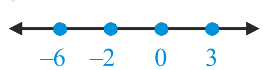
(v) We can multiply both sides with a negative number by changing the sign of inequality.

(vi) We can add or subtract equal quantity to both sides of inequalities without changing the sign of inequality.

Ex. $(x^2-x-6)(x^2+6x)\geq 0$

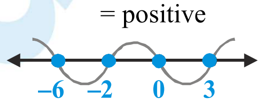
Sol. $(x-3)(x+2)(x)(x+6)\geq 0$

Consider $E=x(x-3)(x+2)(x+6)$, $E=0$ $\Rightarrow$ $x=0,3,-2,-6$ (all are simple points)



For $x\geq 3$ $E=\underbrace{x}_{+ve}\ \underbrace{(x-3)}_{+ve}\ \underbrace{(x+2)}_{+ve}\ \underbrace{(x+6)}_{+ve}$

= positive



Hence for $x(x-3)(x+2)(x+6)\geq 0$

$x\in(-\infty,-6]\cup[-2,0]\cup[3,\infty)$

Inequalities Involving Modulus Function

Properties of Modulus Function

(i) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.

(ii) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$

(v) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Ex. If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then find range of x.

Sol. **Case I :** $x \leq 1$, then

$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$

Hence $x \leq 0$ ...(i)

Case II : $1 < x \leq 2$, then

$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$

But $1 < x \leq 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \leq 3$, then

$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$

But $2 < x \leq 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$.

Irrational Inequalities

Ex. Solve for x, if $\sqrt{x^2 - 3x + 2} > x - 2$

Sol.

$$\left[\begin{array}{l} \begin{cases} x^2 - 3x + 2 \geq 0 \\ x - 2 \geq 0 \\ (x^2 - 3x + 2) > (x - 2)^2 \end{cases} \\ \quad or \\ \begin{cases} x^2 - 3x + 2 \geq 0 \\ x - 2 < 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} (x-1)(x-2) \geq 0 \\ (x-2) \geq 0 \\ x - 2 > 0 \end{cases} \Rightarrow x > 2 \\ \quad or \\ \begin{cases} (x-1)(x-2) \geq 0 \\ x - 2 < 0 \end{cases} \Rightarrow x \leq 1 \end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Logarithmic Inequalities

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$

(ii) If $a > 1$, then

(a) $\log_a x < p \Rightarrow 0 < x < a^p$ **(b)** $\log_a x > p \Rightarrow x > a^p$

(iii) If $0 < a < 1$, then

(a) $\log_a x < p \Rightarrow x > a^p$ **(b)** $\log_a x > p \Rightarrow 0 < x < a^p$

Ex. Solve for x : $\log_2 x \le \dfrac{2}{\log_2 x - 1}$.

Sol. Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

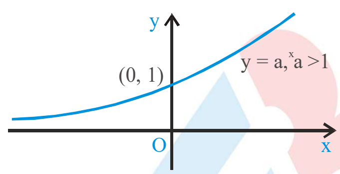
$$\Rightarrow \frac{t^2 - t - 2}{t-1} \le 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \le 0$$

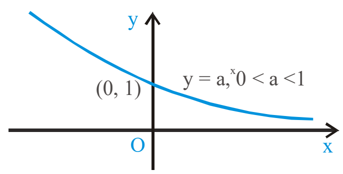
$\Rightarrow$ $t \in (-\infty, -1] \cup (1, 2]$

or $\log_2 x \in (-\infty, -1] \cup (1, 2]$

or $x \in \left(0, \frac{1}{2}\right] \cup (2, 4]$

Exponential Inequations





If $a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b & \textit{when} \quad a > 1 \\ f(x) < \log_a b & \textit{when} \quad 0 < a < 1 \end{cases}$

Ex. Solve for x : $2^{x+2} > \left(\dfrac{1}{4}\right)^{\frac{1}{x}}$

Sol. We have $2^{x+2} > 2^{-2/x}$. Since the base $2 > 1$, we have $x + 2 > -\dfrac{2}{x}$ (the sign of the inequality is retained).

Now $x + 2 + \dfrac{2}{x} > 0$

$$\Rightarrow \frac{x^2 + 2x + 2}{x} > 0$$

$$\Rightarrow \frac{(x+1)^2 + 1}{x} > 0 \Rightarrow x \in (0, \infty)$$

1. Solution of Quadratic Equation & Relation Between Roots & Co-Efficients

(a) The solution of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $|\alpha - \beta| = \sqrt{D}/|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 -$ (sum of roots) x + product of roots $= 0$

2. Nature of Roots

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. $(p, q \in R$ & $i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$
where $a, b, c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $a = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3. Common roots of two Quadratic Equations

(a) Only one common root.
Let α be the common root of $ax^2 + bx + c = 0$
then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha^2 + c' = 0$. By Cramer`s

Rule $\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is
$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(b) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

4. Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are equal magnitude but of opposite sign

(b) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $\left.\begin{matrix} a > 0 \; c < 0 \\ a < 0 \; c > 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(e) If $\left.\begin{matrix} a > 0, \; b > 0, c > 0 \\ a < 0, \; b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

5. Maximum & Minimum Value of Quadratic Expression

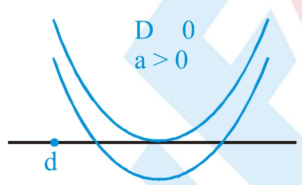
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as a $a < 0$ or $a > 0$.

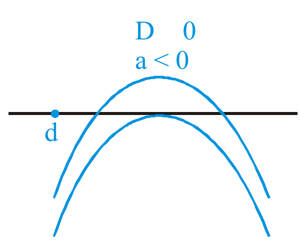
$y \in \left[\frac{-D}{4a}, \infty\right)$ if $a > 0$ & $y \in \left[-\infty, \frac{-D}{4a}\right)$ if $a < 0$.

6. Location of Roots

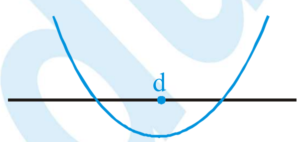
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

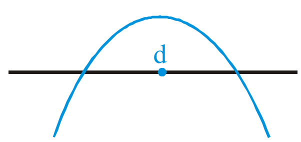
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are
$D \geq 0$; $a.f(d) > 0$ & $(-b/2a) > d$.



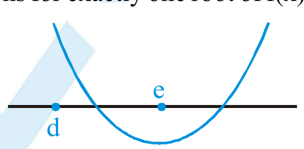


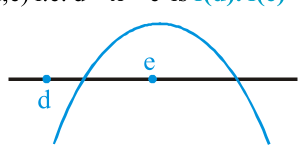
(b) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is **$a.f(d) < 0$.**





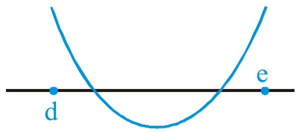
(c) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) i.e. $d < x < e$ is **$f(d).\, f(e) < 0$**

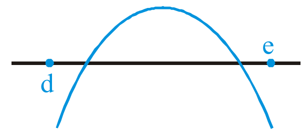




(d) Conditions that both roots of f(x) = 0 to be confined between the numbers d & e are (here d < e).

$D \geq 0$; a.f(d) > 0 & af(e) > 0 ; d < (–/2a) < e





7. General Quadratic Expression in two Variables

$f(x, y) = ax^2 + 2hy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if :

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ OR $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

8. Theory of Equations

If $\alpha_1, \alpha_2, \alpha_3$ ....α_n are roots of the equation :

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n = 0$ where a_0, a_1.........a_n are constant $a_0 \neq 0$ then.

a_1,..........a_n are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \alpha_1\alpha_2\alpha_3$$

$$= -\frac{a_3}{a_0}, ..., \alpha_1\alpha_2\alpha_3.....\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

9. Solution of Quadratic Inequalities

(a) The values of 'x' satisfying the inequality $ax^2 + bx + c > 0$ $(a \neq 0)$ are:

(i) If D > 0, i.e. the equation $ax^2 + bx + c = 0$ has two different roots α & β such that $\alpha < \beta$

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

$a < 0 \Rightarrow x \in (\alpha, \beta)$

(ii) If D = 0, i.e. roots are equal, i.e. $\alpha = \beta$.

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$

$a < 0 \Rightarrow x \in \phi$

(iii) If D < 0, i.e. the equation $ax^2 + bx + c = 0$ has no real root.

Then $a > 0 \Rightarrow x \in R$

$a < 0 \Rightarrow x \in \phi$

(b) Inequalities of the form $\frac{P(x)\ Q(x)\ R(x)........}{A(x)\ B(x)\ C(x)........}\left\{\begin{matrix}\le\\=\\>\end{matrix}\right. 0$ can be quickly solved using the method of intervals, where A, B, C........, P, Q, R......... are linear functions of 'x'.

10. Rational Algebraic Inequations

(i) Values of Rational Expression $\frac{P(x)}{Q(x)}$ for Real Values of x, where P(x) and Q(x) are Quadratic Expressions. To find the values attained by rational expression of the form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$ for real values of x.

(a) Equate the given rational expression to y.

(b) Obtain a quadratic equation in x by simplifying the expression.

(c) Obtain the discriminant of the quadratic equation.

(d) Put discriminant ≥ 0 and solve the inequation for y.

The values of y, so obtained determines the set of values attained by the given rational expression.

(ii) Solution of Rational Algebraic Inequation

If P(x) and Q(x) are polynomials in x, then the inequation

$$\frac{P(x)}{Q(x)}>0,\ \frac{P(x)}{Q(x)}<0,\ \frac{P(x)}{Q(x)}\ge 0 \text{ and } \frac{P(x)}{Q(x)}\le 0$$ are known as rational algebraic inequations.

To solve these inequations we use the sign method as

(a) Obtain P(x) and Q(x).

(b) Factorize P(x) and Q(x) into linear factors.

(c) Make the coefficient of x positive in all factors.

(d) Obtain critical points by equating all factors to zero.

(e) Plot the critical points on the number line. If these are n critical points, then they divide the number into (n + 1) regions.

(f) In the right most region the expression $\frac{P(x)}{Q(x)}$ bears positive sign and in other region the expression bears positive and negative signs depending on the exponents of the factors.