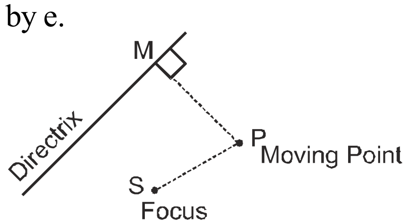
Parabola

CONIC SECTIONS

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

* The fixed point is called the **Focus.**
* The fixed straight line is called the **Directrix.**
* The constant ratio is called the **Eccentricity** denoted by e.



$$\frac{PS}{PM} = e$$

* The line passing through the focus & perpendicular to the directrix is called the **Axis**.
* A point of intersection of a conic with its axis is called a **Vertex**.

If S is (p, q) & directrix is $\ell x + my + n = 0$

then $\quad PS = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$ &

$$PM = \frac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$$

$$\frac{PS}{PM} = e \quad \Rightarrow (\ell^2 + m^2)\,[(x-p)^2 + (y-q)^2]$$
$$= e^2 (\ell x + my + n)^2$$

Which is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Section of right circular cone by different planes

A right circular cone is as shown in the figure – 1

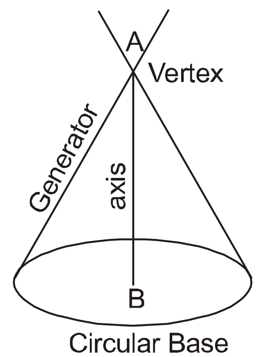


Figure 1

(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure - 2.

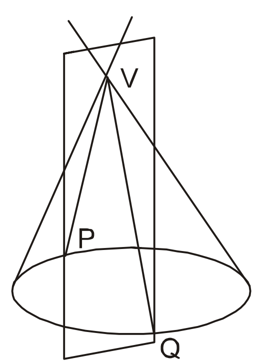


Figure –2

(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the **figure – 3**.

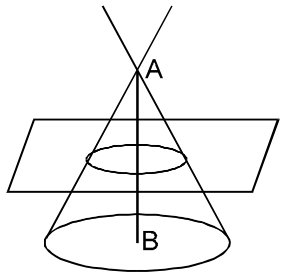


Figure- 3

(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the figure–4.

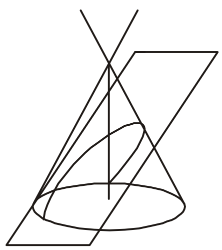


Figure-4

(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the **figure – 5 & 6**.

Figure -5

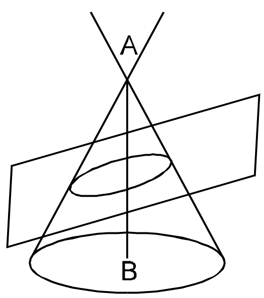
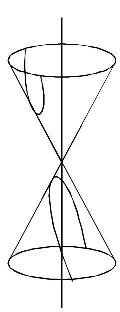


Figure -6



Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) When The Focus Lies On The Directrix.

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if:

$e > 1 \equiv h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1 \equiv h^2 \geq ab$ the lines will coincident.

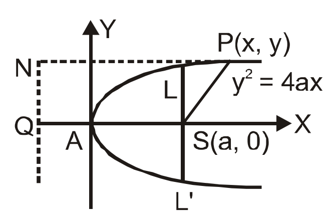
$e < 1 \equiv h^2 < ab$ the lines will be imaginary.

Case (II) When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1; \Delta \neq 0,$	$0 < e < 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab; a+b=0$

DEFINITION

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed straight line (called the directrix).



Let S be the focus. QN be the directrix and P be any point on the parabola. Then by definition. PS = PN where PN is the length of the perpendicular from P on the directrix QN.

TERMS RELATED TO PARABOLA

Axis : A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.

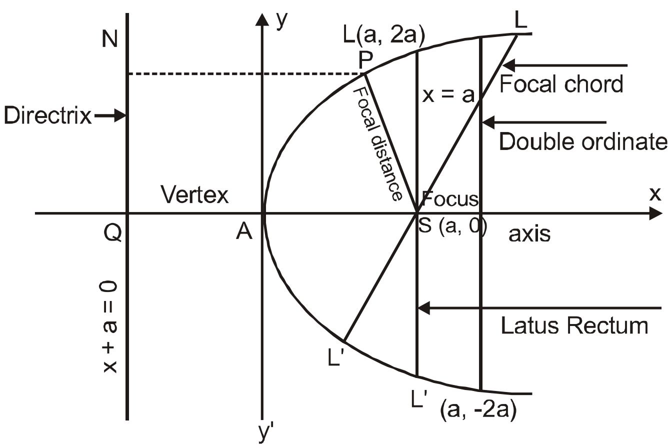
Vertex : The point of intersection of a parabola and its axis is called the vertex of the parabola.

The vertex is the middle point of the focus and the point of intersection of axis and directrix.

Eccentricity : If P be a point on the parabola and PN and PS are the distance from the directrix and focus S respectively then the ratio PS/PN is called the eccentricity of the parabola which is denoted by **e**.

By the definition for the parabola e = 1.

If $e > 1 \Rightarrow$ hyperbola, $e = 0 \Rightarrow$ circle, $e < 1 \Rightarrow$ ellipse



Latus Rectum

Let the given parabola be $y^2 = 4ax$. In the figure LSL' (a line through focus $\perp$ to axis) is the latus rectum.

Also by definition,

$$LSL' = 2\,(\sqrt{4a.a}) = 4a$$

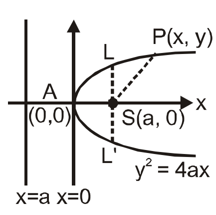
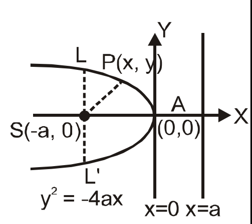
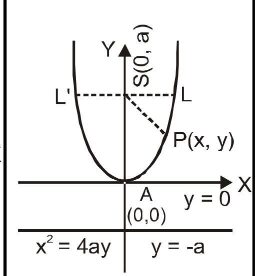
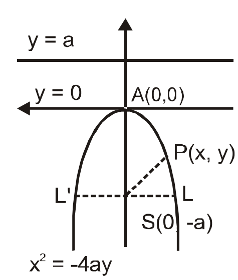
= double ordinate (Any chord of the parabola $y^2 = 4ax$ which is $\perp$ to its axis is called the double ordinate) through the focus S.

Note : Two parabolas are said to be equal when their latus recta are equal.

Focal Chord

Any chord to the parabola which passes through the focus is called a focal chord of the parabola.

FOUR STANDARD FORMS OF THE PARABOLA

Standard Equation	$y^2 = 4ax\ (a>0)$	$y^2 = -4ax(a>0)$	$x^2 = 4ay(a>0)$	$x^2 = -4ay(a > 0)$
Shape of Parabola				
Vertex	A(0, 0)	A(0, 0)	A(0, 0)	A(0, 0)
Focus		S(a, 0)	S(–a, 0)	S(0, a) S(0, –a)
Equation of directrix	x = –a	x = a	y = –a	y = a

Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Extermities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Equation of tangents at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Focal distance of a point P(x, y)	$x + a$	$x - a$	$y + a$	$y - a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

Solved Examples

Ex.1 Find the equation of the parabola whose vertex is $(-3, 0)$ and directrix is $x + 5 = 0$.

Sol. A line passing through the vertex $(-3, 0)$ and perpendicular to directrix $x + 5 = 0$ is the axis of the parabola. Let focus of the parabola is $(a, 0)$. Since vertex, is the middle point of $(-5, 0)$ and focus S, therefore

$$-3 = \frac{(a-5)}{2} \Rightarrow a = -1$$

$\therefore$ Focus $= (-1, 0)$

Thus the equation to the parabola is

$$(x + 1)^2 + y^2 = (x + 5)^2 \quad \Rightarrow \quad y^2 = 8(x + 3)$$

REDUCTION OF STANDARD EQUATION

If the equation of a parabola contains second degree term either in y or in x(but not in both) then it can be reduced into standard form. For this we change the given equation into the following forms-

$(y - k)^2 = 4a\ (x - h)$ or $(x - p)^2 = 4b\ (y - q)$

Then we compare from the following table for the results related to parabola.

EquationofParabola	Vertex	Axis	Focus	Directrix	Equation of L.R.	Length of L.R.
$(y-k)^2 = 4a(x - h)$	(h, k)	$y = k$	$(h + a, k)$	$x + a - h = 0$	$x = a + h$	$4a$
$(x-p)^2 = 4b(y - q)$	(p, q)	$x = p$	$(p, b + q)$	$y + b - q = 0$	$y = b + q$	$4b$

Solved Examples

Ex.2 Find the vertex of the parabola

$x^2 - 8y - x + 19 = 0.$

Sol. Given equation can be written as

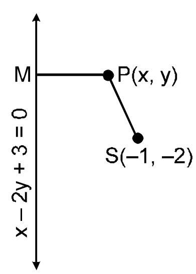
$$\left(x-\frac{1}{2}\right)^2 - 8y + 19 - \frac{1}{4} = 0$$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2 = 8y - \frac{76-1}{4} \Rightarrow \left(x-\frac{1}{2}\right)^2 = 8\left(y-\frac{75}{32}\right)$$

$$\therefore \text{ vertex} = \left(\frac{1}{2}, \frac{75}{32}\right)$$

Ex.3 Find the equation of the parabola whose focus is at $(-1, -2)$ and the directrix is $x - 2y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,



$SP = PM$

$\Rightarrow SP^2 = PM^2$

$\Rightarrow (x+1)^2 + (y+2)^2$

$$= \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$\Rightarrow 5\,[(x+1)^2 + (y+2)^2] = (x - 2y + 3)^2$

$\Rightarrow 5(x^2 + y^2 + 2x + 4y + 5)$

$= (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

Ex.4 Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches. $4y^2 + 12x - 20y + 67 = 0$

Sol. The given equation is

$4y^2 + 12x - 20y + 67 = 0$

$$\Rightarrow y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow y^2 - 5y = -3x - \frac{67}{4}$$

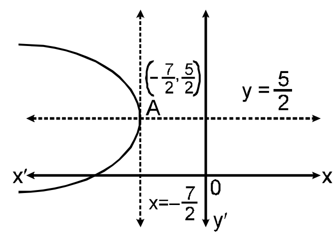
$$\Rightarrow y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2 = -3x - \frac{42}{4}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2 = -3\left(x+\frac{7}{2}\right) \qquad \text{....(i)}$$

Let $\quad x = X - \dfrac{7}{2},\ y = Y + \dfrac{5}{2}$ ....(ii)

Using these relations, equation (i) reduces to



$Y^2 = -3X$ ....(iii)

This is of the form $Y^2 = -4aX$. On comparing, we get $4a = 3 \Rightarrow a = 3/4$.

Vertex - The coordinates of the vertex are $(X = 0, Y = 0)$

So, the coordinates of the vertex are

$\left(-\dfrac{7}{2}, \dfrac{5}{2}\right)$ [Putting X = 0, Y = 0 in (ii)]

Axis

The equation of the axis of the parabola is $Y = 0$.

So, the equation of the axis is

$y = \dfrac{5}{2}$ [Putting Y = 0 in (ii)]

Focus-

The coordinates of the focus are $(X = -a, Y = 0)$ i.e. $(X = -3/4, Y = 0)$.

So, the coordinates of the focus are

$(-17/4, 5/2)$ [Putting X = 3/4 in (ii)]

Directrix -

The equation of the directrix is $X = a$ i.e. $X = \dfrac{3}{4}$.

So, the equation of the directrix is

$x = -\dfrac{11}{4}$ [Putting X = 3/4 in (ii)]

Latusrectum -

The length of the latusrectum of the given parabola is $4a = 3$.

GENERAL EQUATION OF A PARABOLA

If (h, k) be the locus of a parabola and the euqation of directrix is ax + by + c = 0, then its equation is given by

$\sqrt{(x-h)^2+(y-k)^2} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$ which gives

$(bx - ay)^2 + 2gx + 2fy + d = 0$

where g, f, d are the constant.

Note

- The general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a parabola, if

  (a) $h^2 = ab$

  (b) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

Parametric representation:

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Parametric form for :	$y^2 = -4ax$	$(-at^2, 2at)$
	$x^2 = 4ay$	$(2at, at^2)$
	$x^2 = -4ay$	$(2at, -at^2)$

Solved Examples

Ex.5 Find the parametric equation of the parabola $(x - 1)^2 = -12 (y - 2)$

Sol. $\because$ $4a = -12 \Rightarrow a = -3,\quad y - 2 = at^2$

$x - 1 = 2at \Rightarrow x = 1 - 6t,\quad y = 2 - 3t^2$

Ex.6 Find the parameter t of a point (4, –6) of the parabola $y^2 = 9x$.

Sol. Parametric coordinates of any point on parabola $y^2 = 4ax$ are

$(at^2, 2at)$

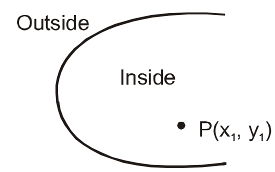
Here $4a = 9 \Rightarrow a = 9/4$

since y coordinates is = –6

$\therefore$ $2(9/4)\,t = -6 \Rightarrow t = -4/3$

Position of a point relative to a parabola:

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



$S_1 : y_1^2 - 4ax_1$

$S_1 < 0 \rightarrow$ Inside

$S_1 > 0 \rightarrow$ Outside

Solved Examples

Ex.7 Check whether the point (3, 4) lies inside or outside the paabola $y^2 = 4x$.

Sol. $y^2 - 4x = 0$

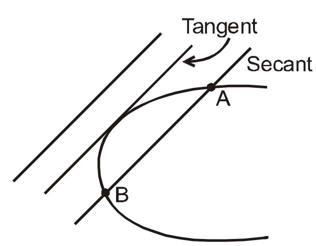
$\because$ $S_1 \equiv y_1^2 - 4x_1 = 16 - 12 = 4 > 0$

$\therefore$ (3, 4) lies outside the parabola.

Line & a parabola:

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm \Rightarrow$ condition of tangency is, $c = a/m$.

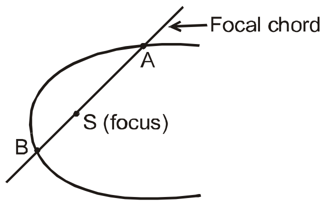
Length of the chord intercepted by the parabola on the line $y = mx + c$ is :



$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

NOTE :

1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\, y + 2\, at_1 t_2 = 0$.
2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



3. Length of the focal chord making an angle α with the x– axis is $4a\operatorname{cosec}^2 \alpha$.

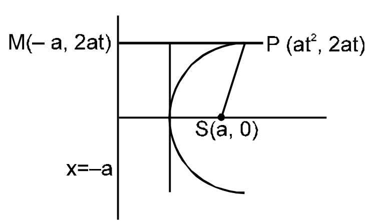
Solved Examples

Ex.8 Discuss the position of line $y = x + 1$ with respect to parabola $y^2 = 4x$.

Sol. Solving we get $(x + 1)^2 = 4x \Rightarrow (x - 1)^2 = 0$ so $y = x + 1$ is tangent to the parabola.

Ex.9 Prove that focal distance of a point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ $(a > 0)$ is $a(1 + t^2)$.

Sol.

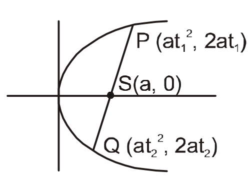


$\because$ PS = PM = $a + at^2$

PS = $a(1 + t^2)$.

Ex.10 If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Sol. Let parabola is $y^2 = 4ax$



since P, S & Q are collinear

$\therefore\ m_{PQ} = m_{PS}$

$\Rightarrow \dfrac{2}{t_1 + t_2} = \dfrac{2t_1}{t_1^2 - 1}$

$\Rightarrow t_1^2 - 1 = t_1^2 + t_1t_2$

$\Rightarrow t_1t_2 = -1$

Ex.11 If the endpoint t_1, t_2 of a chord satisfy the relation $t_1\, t_2 = k$ (const.) then prove that the chord always passes through a fixed point. Find the point?

Sol. Equation of chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$y - 2at_1 = \dfrac{2}{t_1 + t_2}(x - at_1^2)$

$(t_1 + t_2)\, y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$y = \dfrac{2}{t_1 + t_2}(x + ak) \qquad (\because \quad t_1t_2 = k)$

$\therefore$ This line passes through a fixed point $(-ak, 0)$.

Ex.12 If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, find the value of k.

Sol. Given $\quad x = \dfrac{3y + k}{2} \quad$ .....(i)

and $\quad y^2 = 6x \quad$ .....(ii)

$\Rightarrow \quad y^2 = 6\left(\dfrac{3y + k}{2}\right)$

$\Rightarrow \quad y^2 = 3(3y + k) \Rightarrow y^2 - 9y - 3k = 0$ .....(iii)

If line (i) touches parabola (ii) then roots of quadratic equation (iii) is equal

so $(-9)^2 = 4 \times 1 \times (-3k) \Rightarrow k = -\dfrac{27}{4}$

EQUATION OF TANGENT IN DIFFERENT FORMS

(i) Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$yy_1 = 2a\,(x + x_1)$

Note :

The equation of the tangent at (x_1, y_1) can also obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$,

y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of parabola is polynomial of second degree in x and y.

(ii) Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$ty = x + at^2$.

(iii) Slope Form

The equation of tangent to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$y = mx + \frac{a}{m}$.

The coordinate of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Solved Examples

Ex.13 Find the slope of tangent lines drawn from (3, 8) to the parabola $y^2 = -12x$.

Sol. Since $8^2 + 12 \times 3 \neq 0$, therefore the point (3, 8) is not on the parabola

Now equation of any tangent to the parabola $y^2 = -12x$ is written as

$y = mx - \left(\frac{3}{m}\right)$

Since this line passes through (3, 8)

so $8 = 3m - \left(\frac{3}{m}\right) \Rightarrow 3m^2 - 8m - 3 = 0$

Solving this equation, we get $m = 3, -\frac{1}{3}$

Ex.14 Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \frac{a}{m}$

Sol. Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \frac{a}{m} \Rightarrow y = mx + a\left(m + \frac{1}{m}\right)$

but the given tangent is $y = mx + c$

$\therefore\ c = am + \frac{a}{m}$

Ex.15 A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Sol. Slope of required tangent's are $m = \frac{3 \pm 1}{1 \mp 3}$

$m_1 = -2,\ m_2 = \frac{1}{2}$

$\because$ Equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

$\therefore$ tangent's $y = -2x - 1$ at $\left(\frac{1}{2}, -2\right)$

$y = \frac{1}{2}x + 4$ at (8, 8)

Ex.16 Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Sol. Equation of tangent to parabola

$y^2 = 9x$ is $y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\therefore\ 10 = 4m + \frac{9}{4m} \Rightarrow 16m^2 - 40m + 9 = 0$

$m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ equation of tangent's are

$y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$.

Ex.17 Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Sol. Equation of tangent to $y^2 = 4ax$ is

$y = mx + \frac{a}{m}$ ....(i)

Equation of tangent to $x^2 = 4by$ is

$x = m_1y + \frac{b}{m_1} \Rightarrow y = \frac{1}{m_1}x - \frac{b}{(m_1)^2}$ ....(ii)

for common tangent, (i) & (ii) must represent same line.

$\therefore\ \frac{1}{m_1} = m$ & $\frac{a}{m} = -\frac{b}{m_1^2}$

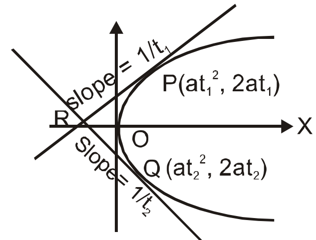
$\Rightarrow \frac{a}{m} = -bm^2 \Rightarrow m = \left(-\frac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is

$y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}$.

POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



$R = (at_1t_2, a(t_1 + t_2))$.

Note :

- Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y_2 = 4ax$ is
$$\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$$
- The G.M. of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The A.M. of the y-coodinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

EQUATIONS OF NORMAL IN DIFFERENT

FORMS

(i) Point form

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

(ii) Parametric form

The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$y + tx = 2at + at^3$.

(iii) Slope form

The eqaution of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$y = mx - 2am - am^3$

Note :

The coordinates of the point of contact are $(am^2 - 2am)$.

Solved Examples

Ex.18 Find the equation of a normal at the parabola $y^2 = 4x$ which passes through (3, 0).

Sol. Equation of normal $y = mx - 2am - am^3$

Here a = 1 and it passes through (3, 0)

$0 = 3m - 2m - m^3$

$\Rightarrow m^3 - m = 0$

$\Rightarrow m = 0, \pm 1$

for $m = 0 \Rightarrow y = 0$

$m = 1 \Rightarrow y = x - 3$

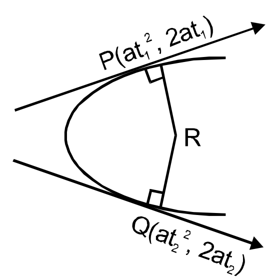
$m = -1 \Rightarrow y = -x + 3$

Condition for Normality

The line $y = mx + c$ is normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$ and $x^2 = 4ay$ if $c = 2a + \frac{a}{m^2}$

Point of Intersection of Normals

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

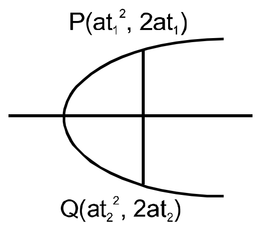


$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2) - at_1t_2(t_1 + t_2)]$

Note :

- If the normal at the point $P(at_1^{\,2}, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^{\,2}, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$



It is clear that PQ is normal to the parabola at P and not at Q.

- If the normals at the points $(at_1^{\,2}, 2at_1)$ and $(at_2^{\,2}, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$
- Length of normal chord is given by $a(t_1 - t_2)\sqrt{(t_1+t_2)^2+4} = \frac{4a(t_1^2+1)^{3/2}}{t_1^2}$
- The normal at the extremities of the latus rectum of a parabola intersect at right angle on the axis of the parabola.

Co-normal Points

Any three points on a parabola normals at which pass through a common point are called co-normal points

Note :

This implies that if three normals are drawn through a point (x_1, y_1) then their slopes are the roots of the cubic:

$y_1 = mx_1 - 2am - am^3$ which gives three values of m. Let these values are m_1, m_2, m_3 then from the eq^n.

$\Rightarrow \quad am^3 + (2a - x_1)\, m + y_1 = 0$

- The sum of the slopes of the normals at co-normal points is zero, i.e., $m_1 + m_2 + m_3 = 0$. and $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x_1}{a}$ and $m_1m_2m_3 = -\frac{y_1}{a}$
- The sum of the ordinates of the co-normal points is zero (i.e., $-2am_1 - 2am_2 - 2am_3 = -2a(m_1+m_2+m_3) = 0$.
- The centroid of the triangle formed by the co-normal points lies on the axis of the parabola
- The vertices of the triangle formed by the co-normal points are $(am_1^{\,2} - 2am_1)$, $(am_2^{\,2}, -2am_2)$ and $(am_3^{\,2}, -2am_3)$. Thus, y-coordinate of the centroid becomes

$$\frac{-2a(m_1+m_2+m_3)}{3} = \frac{-2a}{3} \times 0 = 0.$$

i.e., centroid of triangle

$$\left(\frac{am_1^2+am_2^2+am_3^2}{3}, \frac{2am_1+2am_2+2am_3}{3}\right)$$

$$= \left(\frac{am_1^2+am_2^2+am_3^2}{3}, 0\right)$$

Hence, the centroid lies on the x-axis i.e., axis of the parabola.]

- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then h > 2a.

Solved Examples

Ex.19 If two of the normal of the parabola $y^2 = 4x$, that pass through (15, 12) are $4x + y = 72$, and $3x - y = 33$, then prove that its third normal is $y = x - 3$.

Sol. Here, If m_1, m_2, m_3 are slopes of normal then $m_1 + m_2 + m_3 = 0$

and $\quad m_1 m_2 m_3 = -\frac{y_1}{a}$

here $\quad a = 1, \; m_1 = -4, \; m_2 = 3$

$\therefore \; -4 + 3 + m_3 = 0 \;\Rightarrow\; m_3 = 1$

Also $(-4)(3)(1) = -\frac{12}{1}$ is satisfied.

But (15, 12) satisfies $y = x - 3$.

Ex.20 If two normal drawn from any point to the parabola $y^2 = 4ax$ makes angle α and β with the axis such that $\tan\alpha \,.\, \tan\beta = 2$, then find the locus of this point,

Sol. Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

passes through (h, k)

$k = mk - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation

then $m_1 . m_2 . m_3 = -\frac{k}{a}$

but $m_1 m_2 = 2,\ m_3 = -\frac{k}{2a}$

m_3 is root of (i)

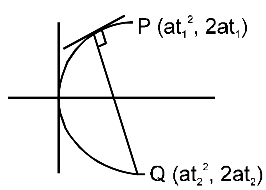
$$a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a-h) + k = 0 \Rightarrow k^2 = 4ah$$

Thus locus is $y^2 = 4ax$.

Ex.21 If the normal at point 't_1' intersects the parabola again at 't_2' then show that $t_2 = -t_1 - \frac{2}{t_1}$

Sol. Slope of normal at P = $-t_1$ and slope of chord

$PQ = \frac{2}{t_1 + t_2}$



$\therefore\ -t_1 = \frac{2}{t_1 + t_2}$

$t_1 + t_2 = -\frac{2}{t_1} \Rightarrow t_2 = -t_1 - \frac{2}{t_1}$.

Ex.22 If the normals at points t_1, t_2 meet at the point t_3 on the parabola then prove that

(i) $t_1 t_2 = 2$ (ii) $t_1 + t_2 + t_3 = 0$

Sol. Since normal at t_1 & t_2 meet the curve at t_3

$\therefore\ t_3 = -t_1 - \frac{2}{t_1}$ .....(i)

$t_3 = -t_2 - \frac{2}{t_2}$ .....(ii)

$\Rightarrow (t_1^2 + 2)\, t_2 = t_1 (t_2^2 + 2)$

$t_1 t_2 (t_1 - t_2) + 2 (t_2 - t_1) = 0$

$\because\ t_1 \neq t_2,\ t_1 t_2 = 2$ ......(iii)

Hence (i) $t_1 t_2 = 2$

from equation (i) & (iii), we get $t_3 = -t_1 - t_2$

Hence (ii) $t_1 + t_2 + t_3 = 0$

Ex.23 Find the locus of the point N from which 3 normals are drawn to the parabola $y^2 = 4ax$ are such that

(i) Two of them are equally inclined to x-axis

(ii) Two of them are perpendicular to each other

Sol. Equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Let the normal passes through N(h, k)

$\therefore\ k = mh - 2am - am^3$

$\Rightarrow am^3 + (2a - h)\, m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's of above equation

$\therefore\ m_1 + m_2 + m_3 = 0$ ......(i)

$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a-h}{a}$ ......(ii)

$m_1 m_2 m_3 = -\frac{k}{a}$ ......(iii)

(i) If two normal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore\ m_3 = 0 \Rightarrow y = 0$

(ii) If two normal's are perpendicular

$\therefore\ m_1 m_2 = -1$

from (3) $m_3 = \frac{k}{a}$ .....(iv)

from (2) $-1 + \frac{k}{a}(m_1 + m_2) = \frac{2a-h}{a}$ .....(v)

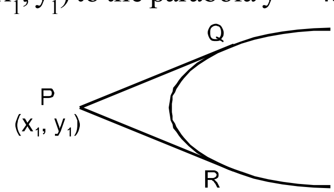
from (1) $m_1 + m_2 = -\frac{k}{a}$ .....(vi)

from (5) & (6), we get

$-1 - \frac{k^2}{a} = 2 - \frac{h}{a}$ $y^2 = a(x - 3a)$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$.



where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$

Locus of point of intersection

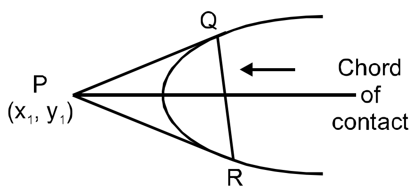
The locus of point of intersection of tangent to the parabola $y^2 = 4ax$ which are having an angle θ between them given by $y^2 - 4ax = (a + x)^2 \tan^2 \theta$

Note :

- If $\theta = 0°$ or π then locus is $(y^2 - 4ax) = 0$ which is the given parabola.
- If $\theta = 90°$, then locus is $x + a = 0$ which is the directrix of the parabola.

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.



Note :

- The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- Lengths of the chord of contact is $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$
- Area of triangle formed by tangents drawn from (x_1, y_1) and their chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$.

Solved Examples

Ex.24 Find the area of triangle made by the chord of contact and tangent drawn from point (4, 6) to the parabola $y^2 = 8x$.

Sol. Area $= \frac{1}{2a}(y_1^{\,2} - 4ax_1)^{3/2}$

Here $a = 2$, $(x_1, y_1) = (4, 6)$

$\therefore$ Area of triangle $= \frac{1}{2.2}(36 - 32)^{3/2} = \frac{(4)^{3/2}}{4} = 2$

CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$T = S_1$

where $T \equiv yy_1 - 2a(x + x_1)$ and $S_1 \equiv y_1^{\,2} - 4ax$.

Solved Examples

Ex25 Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point $P(-1, 2)$

Sol. We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore (y^2 - 4x)(4 + 4) = (2y - 2(x - 1))^2$

$\Rightarrow 8y^2 - 32x = 4y^2 + 4x^2 + 4 - 8xy + 8y - 8x$

$\Rightarrow y^2 - x^2 + 2xy - 6x - 2y = 1$

Ex.26 Find the focus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(i) $m_1 + m_2 = m_0$ (const) (ii) $\theta_1 + \theta_2 = \theta_0$ (const)

Sol. Equation of tangent to $y^2 = 4ax$, is

$y = mx + \frac{a}{m}$

Let it passes through P(h, k)

$\therefore m^2h - mk + a = 0$

(i) $m_1 + m_2 = m_0 = \frac{k}{h} \Rightarrow y = m_0x$

(ii) $\tan\theta_0 = \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{k/h}{1 - a/h}$

$\Rightarrow y = (x - a)\tan\theta_0$

Director circle:

Locus of the point of intersection of the perpendicular tangents to a curve is called the Director Circle. For parabola $y^2 = 4ax$ it's equation is $x + a = 0$ which is parabola's own directrix.

Solved Examples

Ex.27 Find the length of chord of contact of the tangents drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$.

Sol. Let tangent at $P(t_1)$ & $Q(t_2)$ meet at (x_1, y_1)

$\therefore\ at_1t_2 = x_1 \qquad \& \qquad a(t_1 + t_2) = y_1$

$\because\ PQ = \sqrt{(at_1^2 - at_2^2)^2 + (2a(t_1 - t_2))^2}$

$= a\sqrt{((t_1 + t_2)^2 - 4t_1t_2)((t_1 + t_2)^2 + 4)}$

$= \sqrt{\dfrac{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}{a^2}}$

Ex.28 If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Sol. Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0$ .....(i)

But given is $x - y - 1 = 0$

$\therefore \quad \dfrac{4}{1} = \dfrac{-k}{-1} = \dfrac{4h}{-1}$

$\Rightarrow \quad h = -1,\ k = 4$

$\therefore \quad$ point $\equiv (-1, 4)$

Ex.29 Find the locus of point whose chord of contact w.r.t to the parabola $y^2 = 4bx$ is the tangents of the parabola $y^2 = 4ax$.

Sol. Equation of tangent to $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$ ......(i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$y = \dfrac{2b}{k}x + \dfrac{2bh}{k}$ .....(ii)

From (i) & (ii)

$m = \dfrac{2b}{k},\ \dfrac{a}{m} = \dfrac{2bh}{k} \quad \Rightarrow \quad a = \dfrac{4b^2h}{k^2}$

locus of P is $y^2 = \dfrac{4b^2}{a}x$.

Ex.30 Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given point (p, q).

Sol. Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore\ qk - 2a(p + h) = k^2 - 4ah$

$\therefore$ Required locus is

$y^2 - 2ax - qy + 2ap = 0.$

Ex.31 Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ whose slope is 'm'.

Sol. Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

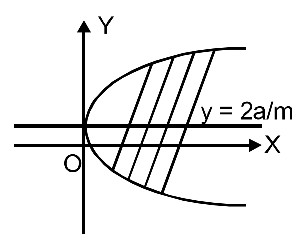
but slope $= \dfrac{2a}{k} = m$

$\therefore$ locus is $y = \dfrac{2a}{m}$

DIAMETER OF A PARABOLA

Diameter of a parabola is the locus of middle points of a series of its parallel chords.

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \dfrac{2a}{m}$.



Solved Examples

Ex.32 The equation of system of parallel chords of the parabola $y^2 = \dfrac{2}{3}x$ is $y + 2x + 1 = 0$ then find its diameter.

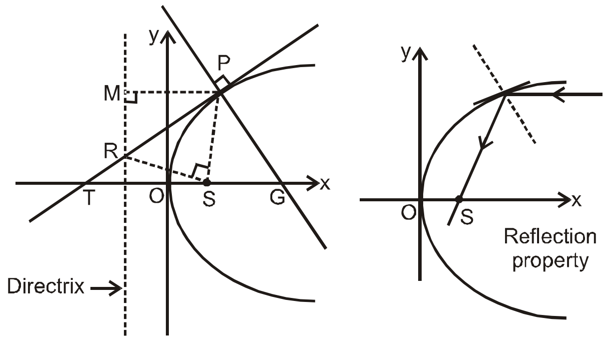
Sol. Here $4a = \dfrac{2}{3} \quad \Rightarrow \quad a = \dfrac{1}{6}$ and $m = -2$

Diameter is $y = \dfrac{2a}{m}$

$\Rightarrow \quad y = -1/6$. This is the required equation of diameter.

Important Highlights:

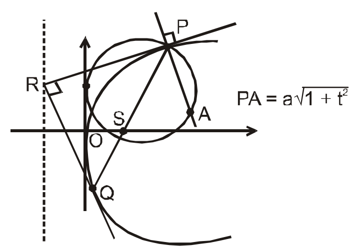
(i) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.



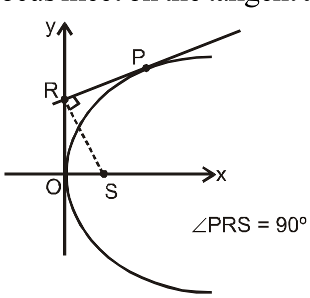
(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus.

See figure above.

(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.



(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

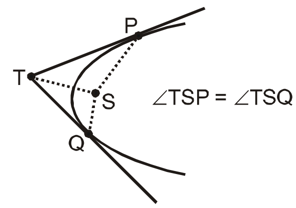


(v) If the tangents at P and Q meet in T, then:

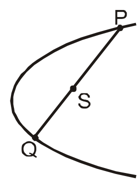
$\Rightarrow$ TP and TQ subtend equal angles at the focus S.

$\Rightarrow$ $ST^2 = SP.\ SQ$ &

$\Rightarrow$ The triangles SPT and STQ are similar.



(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



$$\frac{2(PS)(SQ)}{PS + SQ} = 2a$$

(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h, k) to the parabola $y^2 = 4ax$ then

$k = mh - 2am - am^3$

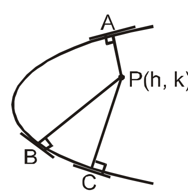
i.e. $am^3 + m(2a - h) + k = 0$.

$m_1 + m_2 + m_3 = 0$;

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$;

$m_1 m_2 m_3 = -\frac{k}{a}$.

Where m_1, m_2 & m_3 are the slopes of the three concurrent normals. Note that



A, B, C $\rightarrow$ Conormal points

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

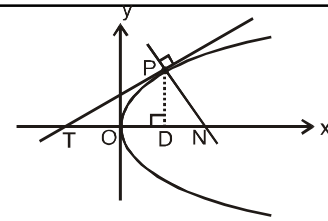
$\Rightarrow$ algebraic sum of the ordinates of the three conormal points on the parabola is zero

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn froma point P (h, k) is

$h > 2a$ & $k^2 < \frac{4}{27a}(h - 2a)^3$.

(ix) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.



TD = 2(OD) , DN = 2a

(x) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum.

See figure above.

Note : Students must try to proof all the above properties.