AREA OF PARALLELOGRAM AND TRIANGLES

➢	**IMPORTANT POINTS**
❖	Parallelograms on the same base and between the same parallels are equal in area.
❖	Area of a parallelogram is the product of its any side and the corresponding altitude.
❖	Parallelogram on the same base and having equal areas lie between the same parallels.
❖	If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle, is half the area of the parallelogram
❖	Two congruent figures having same area.

❖ EXAMPLES ❖

Ex.1 ABCD is a quadrilateral and BD is one of its diagonals as shown in fig. Show that ABCD is a parallelogram and find its area.

Sol. Since diagonal BD intersects transversals AB and DC at B and D respectively such that

$\angle ABD = \angle CDB$ [Each equal to 90°]

i.e., alternate interior angles are equal.

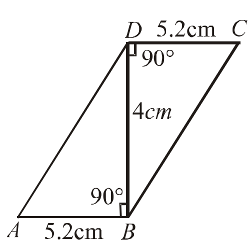
$\therefore$ AB || DC

Also, AB = DC [Each equal to 5.2 cms (Given)]

Thus, one pair of opposite sides AB and DC of quadrilateral ABCD are equal and parallel.

Hence, ABCD is a parallelogram.

Now



ar ($||^{gm}$ ABCD) = Base × Corresponding altitude

= AB × BD = 5.2 × 4 sq.cm

Ex.2 In parallelogram ABCD, AB = 10 cm. The altitudes corresponding to the sides AB and AD are respectively 7 cm and 8 cm. Find AD.

Sol. We have,

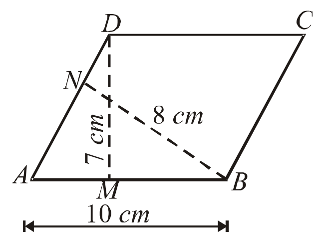
Area of a $||^{gm}$ = Base × Height.

$\therefore$ ar ($||^{gm}$ ABCD) = AB × DM

$= (10 \times 7)\ cm^2$ .... (i)

Also, ar ($||^{gm}$ABCD) = AD × BN

$= (AD \times 8)\ cm^2$ ....(ii)

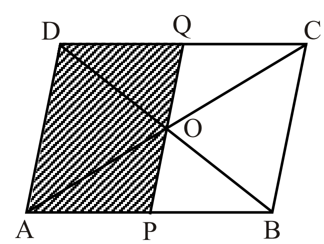


From (i) and (ii), we get

$10 \times 7 = AD \times 8$

$\Rightarrow \quad AD = \frac{10 \times 7}{8}$ cm = 8.75 cm.

Ex.3 In the adjoining figure, ABCD is a ||gm whose diagonals AC and BD intersect at O. A line segment through O meets AB at P and DC at Q. Prove that are (`APQD) = $\frac{1}{2}$ ar (||gm ABCD).



Sol. Diagonal AC of ||gm ABCD divides it into two triangles of equal area.

$\therefore$ ar(ΔACD) = $\frac{1}{2}$ ar(||gm ABCD) ….(i)

In ΔOAP and OCQ, we have

OA = OC

[diagonals of a ||gm bisect each other]

∠AOP = ∠COQ [vert. opp. ∠]

∠PAO = ∠QCO [alt. int. ∠]

$\therefore$ ΔOAP ≅ ΔOCQ

$\therefore$ ar(ΔOAP) = ar(ΔOCQ)

⇒ ar(ΔOAP) + ar(quad. AOQD)

= ar(ΔOCQ) + ar(quad. AOQD)

⇒ ar(quad. APQD) = ar(ΔACD)

= $\frac{1}{2}$ ar(||gm ABCD) [using (i)]

$\therefore$ ar(ΔAPQD) = $\frac{1}{2}$ ar(||gm ABCD)

- Triangles on the same base and between the same parallels are equal in area.
- The area of a triangle is half the product of any of its sides and the corresponding altitude.
- If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half of the parallelogram.
- The area of a trapezium is half the product of its height and the sum of parallel sides.
- Triangles having equal areas and having one side of one of the triangles equal to one side of the other, have their corresponding altitudes equal.

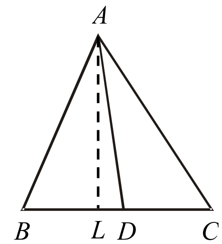
Ex.4 Show that a median of a triangle divides it into two triangles of equal area.

Sol. Given : A Δ ABC in which AD is the median.

To Prove ar (Δ ABD) = ar (Δ ADC)

Construction : Draw AL ⊥ BC.

Proof Since AD is the median of Δ ABC. Therefore,



D is the mid-point of BC.

⇒ BD = DC

⇒ BD × AL = DC × AL

[Multiplying both sides by AL]

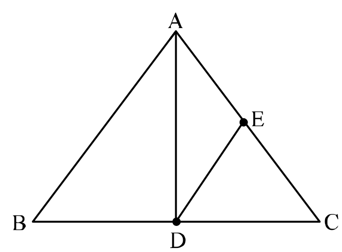
⇒ $\frac{1}{2}$ (BD × AL) = $\frac{1}{2}$ (DC × AL)

⇒ ar (Δ ABD) = ar (Δ ADC)

<u>ALITER</u> Since Δs ABD and ADC have equal bases and the same altitude AL. Therefore, ar (Δ ABC) = ar (Δ ADC).

Ex.5 In figure, AD is a median of ΔABC and DE is a median of ΔDAC. Show that

ar(ΔAED) = $\frac{1}{4}$ ar(ΔABC)



Sol. AD is a median of ΔABC

⇒ ar (ΔABD) = ar (ΔACD)

⇒ ar (ΔACD) = $\frac{1}{2}$ ar (ΔABC) .... (1)

DE is a median of ΔDAC

⇒ ar (ΔAED) = $\frac{1}{2}$ ar (ΔACD) .... (2)

From (1) and (2),

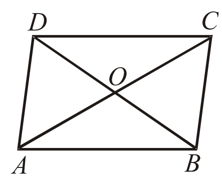
ar (ΔAED) = $\frac{1}{2}\left\{\frac{1}{2}\text{ar}(\Delta ABC)\right\} = \frac{1}{4}$ ar (ΔABC)

Ex.6 The diagonals of ABCD, AC and BD intersect in O. Prove that if BO = OD, the triangles ABC and ADC are equal in area.

Sol. Given : A quadrilateral ABCD in which its diagonals AC and BD intersect at O such that BO = OD.

To Prove : ar (Δ ABC) = ar (Δ ADC)

Proof : In Δ ABD, we have BO = OD. [Given]



$\Rightarrow$ O is the mid-point of BD

$\Rightarrow$ AO is the median

$\Rightarrow$ ar (Δ AOB) = ar (Δ AOD) ....(i)

$\left[\because \text{Median divides a } \Delta \text{ into two } \Delta\text{s of equal area}\right]$

In Δ CBD, O is the mid-point of BD.

$\therefore$ CO is a median

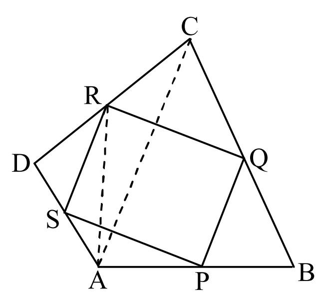
$\Rightarrow$ ar (Δ COB) = ar (Δ COD) ....(ii)

Adding (i) and (ii), we get

ar (Δ AOB) + ar (Δ COB) = ar(Δ AOD) + ar(Δ COD)

$\Rightarrow$ ar (Δ ABC) = ar (Δ ADC).

Ex.7 Let P, Q, R, S be respectively the midpoints of the sides AB, BC, CD and DA of quad. ABCD. Show that PQRS is a parallelogram such that ar(||gm PQRS) = $\frac{1}{2}$ ar(quad. ABCD).



Sol. Join AC and AR.

In ΔABC, P and Q are midpoints of AB and BC respectively.

$\therefore$ PQ || AC and PQ = $\frac{1}{2}$ AC

In ΔDAC, S and R are midpoints of AD and DC respectively.

$\therefore$ SR || AC and SR = $\frac{1}{2}$ AC

Thus, PQ || SR and PQ = SR.

$\therefore$ PQRS is a ||gm.

Now, median AR divides ΔACD into two Δ of equal area.

$\therefore$ ar (ΔARD) = $\frac{1}{2}$ ar (ΔACD) ….(1)

Median RS divides ΔARD into two Δ of equal area.

$\therefore$ ar (ΔDSR) = $\frac{1}{2}$ ar (ΔARD) ….(2)

From (1) and (2), we get

ar (ΔDSR) = $\frac{1}{4}$ ar (ΔACD)

Similarly, ar (ΔBQP) = $\frac{1}{4}$ ar (ΔABC)

$\Rightarrow$ ar (ΔDSR) + ar (ΔBQP)

= $\frac{1}{4}$ [ar (ΔACD) + ar (ΔABC)]

$\Rightarrow$ ar (ΔDSR) + ar (ΔBQP)

= $\frac{1}{4}$ ar [quad. ABCD) …..(3)

Similarly, ar (ΔCRQ) + ar (ΔASP)

= $\frac{1}{4}$ ar (quad. ABCD) ….(4)

Adding (3) and (4), we get

ar (ΔDSR) + ar (ΔBQP) + ar (ΔCRQ) + ar (ΔASP)

= $\frac{1}{2}$ ar (quad. ABCD) ….(5)

But, ar (ΔDSR) + ar (ΔBQP) + ar (ΔCRQ) + ar (ΔASP) + ar (||gm PQRS)

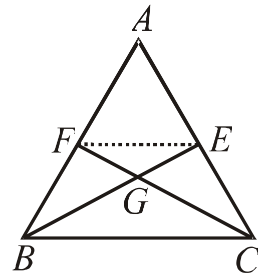
= ar (quad. ABCD) ….(6)

Subtracting (5) from (6), we get

ar (||gm PQRS) = $\frac{1}{2}$ ar (quad. ABCD)

Ex.8 The medians BE and CF of a triangle ABC intersect at G. Prove that area of ΔGBC = area of quadrilateral AFGE.

Sol. Join EF. Since the line segment Joining the mid-points of two sides of a triangle is parallel to the third side. So, EF || BC.



Clearly, ΔsBEF and CEF are on the same base EF and between the same parallel lines. So,

ar (ΔBEF) = ar (ΔCEF)

⇒ ar(ΔBEF) − ar(ΔGEF)

= ar(ΔCEF) − ar (ΔGEF)

⇒ ar (ΔBFG) = ar (ΔCEG) .... (i)

We know that a median of a triangle divides it into two triangels of equal area.

Therefore,

ar (ΔBEC) = ar (ΔABE)

⇒ ar(ΔBGC) + ar (ΔCEG) = ar(quad. AFGE) + ar (ΔBFG)

⇒ ar(ΔBGC) + ar(ΔBFG) = ar (quad. AFGE) + ar (ΔBFG) [Using (i)]

⇒ ar (ΔBGC) = ar (quad. AFGE)

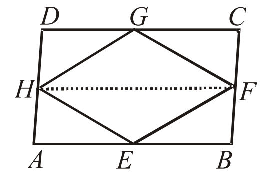
Ex.9 E, F, G, H are respectively, the mid-points of the sides AB, BC, CD and DA of parallelogram ABCD. Show that the area of quadrilateral EFGH is half the area of the parallelogram ABCD.

Sol. Given: A quadrilateral ABCD in which E, F, G, H are respectively the mid-points of the sides AB, BC, CD and DA.

To Prove :

(i) ar ($||^{gm}$ EFGH) = $\frac{1}{2}$ ar ($||^{gm}$ ABCD)

Construction : Join AC and HF



Since ΔHGF and $||^{gm}$ HDCF are on the same base HF and between the same parallel lines.

∴ ar(ΔHGF) = $\frac{1}{2}$ ar ($||^{gm}$ HDCF) ....(i)

Similarly, ΔHEF and $||^{gm}$ HABF are on the same base HF and between the same parallels.

∴ ar (ΔHEF) = $\frac{1}{2}$ ar ($||^{gm}$ HABF) ....(ii)

Adding (iii) and (iv),we get

ar (ΔHGF) + ar (ΔHEF) = $\frac{1}{2}$ [ar($||^{gm}$ HDCF)

+ ar($||^{gm}$ HABF)]

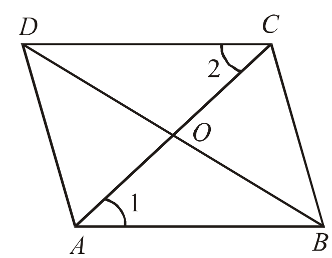
⇒ ar ($||^{gm}$ EFGH) = $\frac{1}{2}$ ar ($||^{gm}$ ABCD).

Ex.10 Two segments AC and BD bisect each other at O. Prove that ABCD is a parallelogram.

Sol. Given: AC and BD are two segments bisecting each other at O.

To Prove: ABCD is a parallelogram.

Construction : Join AB, BC, CD and DA.



Proof : In Δs AOB and COD, we have

AO = CO [Given]

BO = DO [Given]

and, ∠AOB = ∠COD [Vertically opp. ∠s]

So, by SAS criterion of congruence

ΔAOB ≅ ΔCOD

⇒ AB = CD [∵ Corresponding parts of congruent triangles are equal]

and, ∠1 = ∠2.

Thus, AB and DC intersect AC at A and C respectively such that ∠1 = ∠2 i.e. alternate interior angles are equal.

∴ AB || DC

Thus, in quadrilateral ABCD, we have

AB = DC and AB || DC

i.e. a pair of opposite sides are equal and parallel. Hence, ABCD is a parallelogram.

Hence, ABCD is a parallelogram.

Ex.11 ABCD is a parallelogram. L and M are points on AB and DC respectively and AL = CM. Prove that LM and BD bisect each other.

Sol. We have, AL = CM

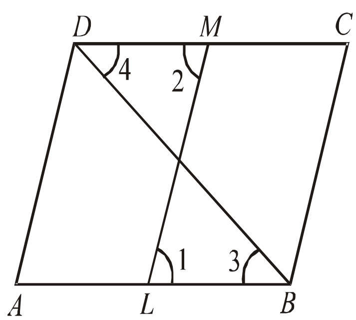
⇒ AB – BL = CD – DM

⇒ – BL = – DM [∵ ABCD is a ||gm

∴ AB = DC]

⇒ BL = DM .... (i)

Now, AB || DC and transversals BD and LM intersect them.



∴ ∠3 = ∠4 and ∠1 = ∠2 ....(ii)

Thus, in Δs OBL and ODM, we have

∠1 = ∠2 [From (ii)]

BL = MD [From (i)]

∠3 = ∠4 [From (ii)]

So, by ASA criterion of congruence

ΔOBL ≅ ΔODM

⇒ OB = OD and OL = OM

[∵ Corresponding parts of congruent triangles are equal]

⇒ O is the mid-point of BD and LM both.

⇒ BD and LM bisect each other.

Ex.12 A point O inside a rectangle ABCD is joined to the vertices. Prove that the sum of the areas of a pair of opposite triangles so formed is equal to the sum of the other pair of triangles.

Sol. Given : A rectangle ABCD and O is a point inside it, OA, OB, OC and OD have been joined.

To Prove : ar (Δ AOD) + ar (Δ BOC)

= ar (Δ AOB) + ar (Δ COD).

Construction : Draw EOF|| AB and LOM || AD.

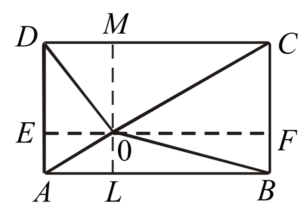
Proof : We have,

ar (ΔAOD) + ar (ΔBOC)

$= \frac{1}{2}(AD \times OE) + \frac{1}{2}(BC \times OF)$

$= \frac{1}{2}(AD \times OE) + \frac{1}{2}(AD \times OF)$ [∴ AD = BC]

$= \frac{1}{2}(AD \times (OE + OF))$



$= \frac{1}{2}(AD \times EF)$

$= \frac{1}{2}(AD \times AB)$ [∴ EF = AB]

$= \frac{1}{2}$ ar (rect ABCD) and,

ar(Δ AOB)+ ar (ΔCOD)

$= \frac{1}{2}(AB \times OL) + \frac{1}{2}(CD \times OM)\frac{1}{2}$

$= (AB \times OL) + \frac{1}{2}(AB \times OM)$ [∵ AB = CD]

$= \frac{1}{2} AB \times (OL + OM)$

$= \frac{1}{2}(AB \times LM)$ [∵ LM = AD]

$\frac{1}{2} = (AB \times AD) = \frac{1}{2}$ ar (rect ABCD)

∴ ar (Δ AOD) + ar (Δ BOC)

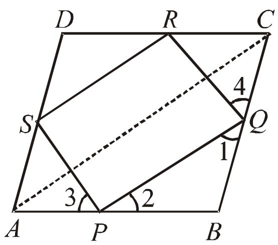
= ar(Δ AOB)+ ar (Δ COD)

Ex.13 `ABCD is a rhombus and P, Q, R, S are the mid-points of AB, BC, CD, DA respectively. Prove that `PQRS is a rectangle.

Sol. Given: A rhombus `ABCD in which P, Q, R, S are the mid-points of sides AB, BC, CD and DA respectively. PQ, QR, RS and SP are joined.

To Prove: `PQRS is rectangle.

Construction : Join AC.



Proof : In order to prove that PQRS is a rectangle, it is sufficient to show that it is a parallelogram whose one angle is a right angle. First we shall prove that PQRS is parallelogram.

In ΔABC, P and Q are the mid-points of AB and BC respectively.

$\therefore$ PQ || AC and PQ = $\frac{1}{2}$ AC ....(i)

In ΔADC, R and S are the mid-points of CD and AD respectively.

$\therefore$ RS || AC and RS = $\frac{1}{2}$ AC ....(ii)

From (i) and (ii), we have

PQ || RS and PQ = RS

Thus, PQRS is a quadrilateral such that one pair of opposite sides PQ and SR is equal and parallel.

So, PQRS is a parallelogram.

Now, we shall prove that one angle of parallelogram PQRS is a right angle.

ABCD is a rhombus

$\Rightarrow$ AB = BC

[$\because$ All sides of a rhombus are equal]

$\Rightarrow \frac{1}{2}AB = \frac{1}{2}BC$

$\Rightarrow$ PB = BQ ....(i)

$\left[\because \text{ P and Q are the mid-points of AB and BC respectively}\right]$

Now, in ΔPBQ we have,

PB = BQ

$\Rightarrow \angle 1 = \angle 2$ ....(ii)

$\left[\text{Angles opposite to equal sides are equal}\right]$

Again, ABCD is a rhombus

$\Rightarrow$ AB = BC = CD = AD

$\Rightarrow$ AB = BC, CD = AD

$\Rightarrow \frac{1}{2}AB = \frac{1}{2}BC, \frac{1}{2}CD = \frac{1}{2}AD$

$\Rightarrow$ AP = CQ, CR = AS ....(iii)

Now, in Δs APS and CQR, we have

AP = CQ [From (iii)]

AS = CR [From (iii)]

and PS = QR [$\because$ PQRS is ||gm $\therefore$ PS = QR]

So, by SSS criterion of congruence

ΔAPS $\cong$ ΔCQR

$\Rightarrow \angle 3 = \angle 4$ ....(iv)

$\left[\because \text{ Corresponding parts of congruent triangles are equal}\right]$

Now, $\angle 3 + \angle SPQ + \angle 2 = 180^\circ$

and, $\angle 1 + \angle PQR + \angle 4 = 180^\circ$

$\therefore \angle 3 + \angle SPQ + \angle 2 = \angle 1 + \angle PQR + \angle 4$

$\Rightarrow \angle SPQ = \angle PQR$ ...(v)

$\left[\because \angle 1 = \angle 2, \text{ from (ii) and } \angle 3 = \angle 4, \text{ from (iv)}\right]$

Now, transversal PQ cuts parallel lines SP and RQ at P and Q respectively.

$\therefore \angle SPQ + \angle PQR = 180^\circ$

$\Rightarrow \angle SPQ + \angle SPQ = 180^\circ$ [Using (v)]

$\Rightarrow \angle SPQ = 90^\circ$

Thus, PQRS is a parallelogram such that $\angle SPQ = 90^\circ$.

Hence, PQRS is a rectangle.

Ex.14 Triangles ABC and DBC are on the same base BC with A, D on opposite sides of line BC, such that ar (Δ ABC) = ar (Δ DBC). Show that BC bisects AD.

Sol. Since Δs ABC and DBC are equal in area and have a common side BC. Therefore the altitudes corresponding to BC are equal i.e.

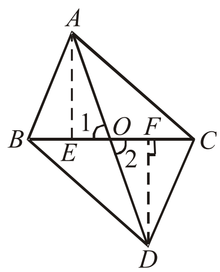
AE = DF.

Now, in Δs AEO and DFO, we have

$\angle 1 = \angle 2$ [Vertically opp. angles]

$\angle AEO = \angle DFO$ [Each equal to 90°]

and, AE = DF



So, by AAS criterion of congruence,

Δ AEO $\cong$ Δ DFO

$\Rightarrow$ AO = DO

$\Rightarrow$ BC bisects AD.

Ex.15 ABCD is a parallelogram and O is any point in its interior. Prove that :

(i) ar(ΔAOB) + ar(ΔCOD) = $\frac{1}{2}$ ar ($\|^{gm}$ ABCD)

(ii) ar (ΔAOB) + ar Δ(COD)

= ar (ΔBOC) + ar (ΔAOD)

Sol. Given: A parallelogram ABCD and O is a point in its interior.

(i) Since ΔAOB and parallelogram ABFE are on the same base AB and between the same parallel lines AB and EF.

$\therefore$ ar (ΔAOB) = $\frac{1}{2}$ ar ($\|^{gm}$ ABFE) ....(i)

Similarly,

ar (ΔCOD) = $\frac{1}{2}$ ar ($\|^{gm}$ DEFC) ....(ii)

Adding (i) and (ii), we get

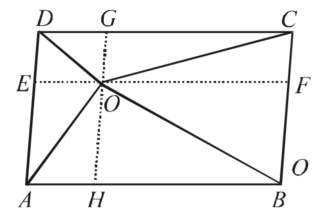
ar (ΔAOB) + ar (ΔCOD)

= $\frac{1}{2}$ ar ($\|^{gm}$ ABCD)

(ii) To Prove: ar(ΔAOB) + ar(ΔCOD)

= ar (ΔBOC) + ar (ΔAOD).

Construction: Draw EOF $\|$AB and GOH $\|$ AD.



Proof: Since GH $\|$ DE and EF $\|$ DC

$\therefore$ OG $\|$ DE and OE $\|$ GD

$\Rightarrow$ EOGD is a parallelogram

Similarly, EAHO, HBFO and FOGC are parallelograms.

Now, OD is a diagonal of parallelogram EOGD

$\Rightarrow$ ar (ΔEOD) = ar (ΔDOG) .... (iii)

OA is a diagonal of parallelogram EAHO

$\Rightarrow$ ar (ΔEOA) = ar (ΔAOH) ....(iv)

OB is a diagonal of parallelogram HBFO

$\Rightarrow$ ar (ΔBOF) = ar (ΔBOH) ....(v)

OC is a diagonal of parallelogram FOGC

$\Rightarrow$ ar (ΔFOC) = ar (ΔCOG)

Adding (iii), (iv) and (v), we get

ar (ΔEOD) + ar (ΔEOA) + ar (ΔBOF) + ar(ΔFOC)

= ar (ΔDOG) + ar (ΔAOH) + ar (ΔBOH) + ar (ΔCOG)

$\Rightarrow$ ar (ΔAOD) + ar (ΔBOC)

= ar (ΔAOB) + ar (ΔCOD)

Ex.16 A quadrilateral ABCD is such that diagonal BD divides its area in two equal parts. Prove that BD bisects AC.

Sol. Given: A quadrilateral ABCD in which diagonal BD bisects it, i.e.

ar (Δ ABD) = ar (Δ BDC)

Construction: Join AC.

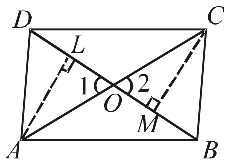
Suppose AC and BD intersect at O. Draw AL ⊥ BD and CM ⊥ BD.

To Prove : AO = OC.

Proof: We have, ar (Δ ABD) = ar(Δ BDC)

Thus, Δs ABD and ABC are on the same base AB and have equal area. Therefore, their corresponding altitudes are equal.

i.e., AL = CM



Now, in Δs ALO and CMO, we have

∠1 = ∠2 [Vertically opposite angles]

∠ALO = ∠CMO [Each equal to 90°]

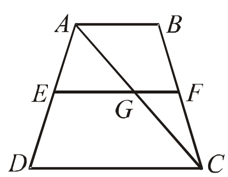
and, AL = CM [Proved above]

So, by AAS criterion of congruence

Δ ALO ≅ Δ CMO

⇒ AO = OC ⇒ BD bisects AC.

Ex.17 In Fig. ABCD is a trapezium in which side AB is parallel to side DC and E is the mid-point of side AD. If F is a point on the side BC such that the segment EF is parallel to side DC.



Prove that EF = $\frac{1}{2}$ (AB + DC).

Sol. Given : A trapezium ABCD in which AB || DC, E is the mid-point of AD and F is a point on BC such that EF || DC.

To Prove: EF = $\frac{1}{2}$ (AB + DC)

Proof: In ΔADC, E is the mid-point of AD and EG || DC (Given)

∴ G is the mid-point of AC

Since segment joining the mid-points of two sides of a triangle is half of the third side.

∴ EG = $\frac{1}{2}$ DC ....(i)

Now, ABCD is a trapezium in which AB || DC.

But, EF || DC

∴ EF || AB

⇒ GF || AB

In ΔABC, G is the mid-point of AC (proved above) and EF || AB.

∴ F is the mid-point of BC

⇒ GF = $\frac{1}{2}$ AB ....(ii)

$\left[\begin{array}{l}\therefore \text{Segment joining the mid-points of} \\ \text{two sides of a } \Delta \text{ is half of the third sides}\end{array}\right]$

From (i) and (ii), we have

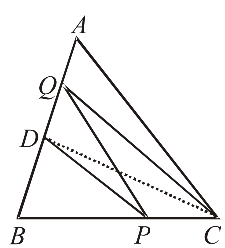
$$EG + GF = \frac{1}{2}(DC) + \frac{1}{2}(AB)$$

$$\Rightarrow EF = \frac{1}{2}(AB + DC)$$

Ex.18 In ΔABC, D is the mid-point of AB. P is any point of BC. CQ || PD meets AB in Q. Show that ar (ΔBPQ) = $\frac{1}{2}$ ar (ΔABC).

Sol. To Prove: ar (ΔBPQ) = $\frac{1}{2}$ ar (ΔABC)

Construction: Join CD.



Proof: Since D is the mid-point of AB. So, in ΔABC, CD is the median.

ar (ΔBCD) = $\frac{1}{2}$ ar (ΔABC) .... (i)

Since ΔsPDQ and PDC are on the same base PD and between the same parallel lines PD and QC.

∴ ar(ΔPDQ) = ar (ΔPDC) ....(ii)

Now, from (i)

$$\text{ar}(\Delta BCD) = \frac{1}{2}\text{ar}(\Delta ABC)$$

$$\Rightarrow \text{ar}(\Delta BPD) + \text{ar}(\Delta PDC) = \frac{1}{2}\text{ar}(\Delta ABC)$$

$$\Rightarrow \text{ar}(\Delta BPD) + \text{ar}(\Delta PDQ) = \frac{1}{2}\text{ar}(\Delta ABC)$$

[Using (ii)]

$$\Rightarrow \text{ar}(\Delta BPQ) = \frac{1}{2}\text{ar}(\Delta ABC)$$

Ex.19 If the medians of a ΔABC intersect at G, show that ar(Δ AGB) = ar(Δ AGC) = ar(Δ BGC) = $\frac{1}{3}$ ar(Δ ABC).

Sol. Given: A Δ ABC such that its medians AD, BE and CF intersect at G.

To Prove : ar (Δ AGB) = ar (Δ BGC) = ar (CGA) = $\frac{1}{3}$ ar (Δ ABC)

Proof : We know that the median of a triangle divides it into two triangles of equal area.

In Δ ABC, AD is the median

$\Rightarrow$ ar (Δ ABD) = ar (Δ ACD) .... (i)

In Δ GBC, GD is the median

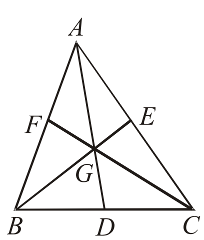
$\Rightarrow$ ar (Δ GBD) = ar (Δ GCD) ....(ii)

Subtracting (ii) from (i), we get

ar (Δ ABD) – ar (Δ GBD) = ar (Δ ACD) – ar (Δ GCD)

$\Rightarrow$ ar (Δ AGB) = ar (Δ AGC) ....(iii)

Similarly,



ar (Δ AGB) = ar (Δ BGC) ....(iv)

From (iii) and (iv), we get

ar(Δ AGB) = ar(Δ BGC) = ar(Δ AGC) ....(v)

But, ar (Δ AGB) + ar (Δ BGC) + ar (Δ AGC) = ar (Δ ABC)

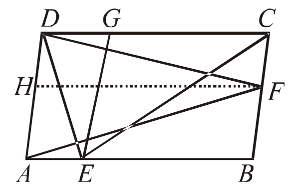
$\therefore$ 3 ar (Δ AGB) = ar (Δ ABC)

$$\Rightarrow \text{ar}(\Delta AGB) = \frac{1}{3}\text{ar}(\Delta ABC)$$

Hence, ar (Δ AGB) = ar (Δ AGC) = ar (Δ BGC) = $\frac{1}{3}$ ar (Δ ABC).

Ex.20 In a parallelogram ABCD, E, F are any two point on the sides AB and BC respectively. Show that ar (ΔADF) = ar (ΔDCE).

Sol. Construction: Draw EG || AD and FH || AB.



Proof: Since FH || AB (by construction). Therefore, ABFH is a parallelogram.

Now, AF is a diagonal of $||^{gm}$ ABFH

$$\therefore \text{ar}(\Delta AFH) = \frac{1}{2}\text{ar}(||^{gm} ABFH) \quad .... (i)$$

In $||^{gm}$ DCFH, DF is a diagonal.

$$\therefore \text{ar}(\Delta DFH) = \frac{1}{2}\text{ar}(||^{gm} DCFH) \quad ....(ii)$$

From (i) and (ii), we have

$$\text{ar}(\Delta AFH) + \text{ar}(\Delta DFH) = \frac{1}{2}\text{ar}(||^{gm} ABFH) + \frac{1}{2}\text{ar}(||^{gm} DCFH)$$

$$\Rightarrow \text{ar}(\Delta AFH) + \text{ar}(\Delta DFH) = \frac{1}{2}[\text{ar}(||^{gm} ABFH) + (\text{ar}(||^{gm} DCFH)]$$

$$\Rightarrow \text{ar}(\Delta AFH) + \text{ar}(\Delta DFH) = \frac{1}{2}\text{ar}(||^{gm} ABCD)$$

$$\Rightarrow \text{ar}(\Delta ADF) = \frac{1}{2}\text{ar}(||^{gm} ABCD) \quad ....(iii)$$

In $||^{gm}$ AEGD, DE is a diagonal.

$$\therefore \text{ar}(\Delta DEG) = \frac{1}{2}\text{ar}(||^{gm} AEGD) \quad ....(iv)$$

In $||^{gm}$ CBEG, CE is a diagonal.

$$\therefore \text{ar}(\Delta CEG) = \frac{1}{2}\text{ar}(||^{gm} CBEG) \quad ....(v)$$

From (iv) and (v),we have

ar (ΔDEG) + ar (ΔCEG)

$= \frac{1}{2}$ ar ((||^{gm} AEGD) + $\frac{1}{2}$ ar ((||^{gm} CBEG)

⇒ ar (ΔDEG) + ar (ΔCEG)

$= \frac{1}{2}$ [ar (||^{gm} AEGD + $\frac{1}{2}$ ar (||^{gm} CBEG)]

⇒ ar(ΔDEG)+ ar(ΔCEG) = $\frac{1}{2}$ [ar(||^{gm} ABCD)

⇒ ar (ΔDCE) = $\frac{1}{2}$ ar (||^{gm} ABCD) ....(vi)

From (iii) and (vi), we get

ar (ΔADF) = ar (ΔDCE).

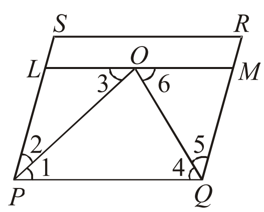
Ex.21 In Fig. PQRS is a parallelogram, PQ and QO are respectively, the angle bisectors of ∠P and ∠Q. Line LOM is drawn parallel to PQ. Prove that :

(i) PL = QM (ii) LO = OM.

Sol. Since PQRS is a parallelogram.

∴ PS || QR ⇒ PL || QM

Thus, we have



PL || QM and LM || PQ [Given]

⇒ PQML is parallelogram.

⇒ PL = QM [∵ Opp. sides of a || gm are equal]

This proves (i).

Now, OP is the bisector of ∠P

∴ ∠1 = ∠2 ....(i)

Now, PQ || LM and transversal OP intersects them

∴ ∠1 = ∠3 ....(ii)

From (i) and (ii), we get

∠2 = ∠3

Thus, in ΔOPL, we have

∠2 = ∠3

⇒ OL = PL ...(iii)

[∵ Opposite sides of equal angles in triangle are equal]

Since OQ is the bisector of ∠Q

∴ ∠4 = ∠5 ....(iv)

Also, PQ || LM and transversal OQ intersects them

∠4 = ∠6 ....(v)

From (iv) and (v), we get

∠5 = ∠6

Thus, in ΔOQM, we have

∠5 = ∠6

⇒ OM = QM ....(vi)

[∵ Opp. sides of equal angles are equal]

But,PL = QM ....(vii) [As proved above]

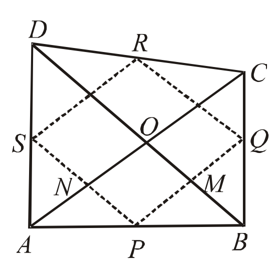
So, from (iii), (vi) and (vii), we get

OL = OM.

Ex.22 The diagonals of a quadrilateral ABCD are perpendicular. Show that the quadrilateral, formed by joining the mid-points of its sides, is a rectangle.

Sol. Given: A quadrilateral whose diagonals AC and BD are perpendicular to each other, P, Q, R, S are the mid-points of sides AB, BC, CD and DA respectively. PQ, QR, RS and SP are joined.

To Prove: PQRS is a rectangle.



Proof: In ΔABC, P and Q are the mid-points of AB and BC respectively.

∴ PQ || AC and PQ = $\frac{1}{2}$ AC .... (i)

In ΔADC, R and S are the mid-points of CD and AD respectively.

$\therefore$ RS || AC and RS = $\frac{1}{2}$ AC ....(ii)

From (i) and (ii), we have

PQ || RS and PQ = RS

Thus, in quadrilateral PQRS, a pair of opposite sides are equal and parallel. So, PQRS is a parallelogram.

Suppose the diagonals AC and BD of quadrilateral ABCD intersect at O.

Now in Δ ABD, P is the mid-point of AB and S is the mid-point of AD.

$\therefore$ PS || BD $\Rightarrow$ PN || MO

Also, from (i) , PQ || AC

$\Rightarrow$ PM || NO

Thus, in quadrilateral PMON, we have

PN || MO and PM || NO

$\Rightarrow$ PMON is a parallelogram.

$\Rightarrow$ $\angle MPN = \angle MON$

[$\because$ Opposite angles of a $||^{gm}$ are equal]

$\Rightarrow$ $\angle MPN = \angle BOA$ [$\because \angle BOA = \angle MON$]

$\Rightarrow$ $\angle MPN = 90^\circ$ [$\because AC \perp BD \therefore \angle BOA = 90^\circ$]

$\Rightarrow$ $\angle QPS = 90^\circ$ [$\because \angle MPN = \angle QPS$]

Thus, PQRS is a parallelogram whose one angle $\angle QPS = 90^\circ$ Hence PQRS is a rectangle.

Ex.23 In a parallelogram ABCD diagonals AC and BD intersect at O and AC = 6.8cm and BD = 13.6 cm. Find the measures of OC and CD.

Sol. Since the diagonals of a parallelogram bisect each other. Therefore, O is the mid-point of AC and BD.

$\therefore$ $OC = \frac{1}{2} AC = \frac{1}{2} \times 6.8 \text{ cm} = 3.4 \text{ cm}$

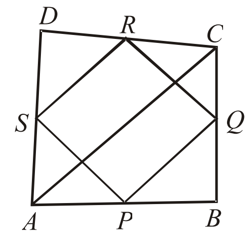
and, $OD = \frac{1}{2} BD = \frac{1}{2} \times 5.6 \text{ cm} = 2.8 \text{ cm}$

Ex.24 Prove that the figure formed by joining the mid-points of the pairs of consecutive sides of a quadrilateral is a parallelogram.

Sol. Given: ABCD is a quadrilateral in which P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively.

To Prove: PQRS is a parallelogram.

Construction: Join A and C.



Proof: In ΔABC, P and Q are the mid-points of sides AB and BC respectively.

$\therefore$ PQ || AC and PQ = $\frac{1}{2}$ AC .... (i)

In ΔADC, R and S are the mid-points of CD and AD respectively.

$\therefore$ RS || AC and RS = $\frac{1}{2}$ AC ....(ii)

From (i) and (ii), we have

PQ = RS and PQ || RS

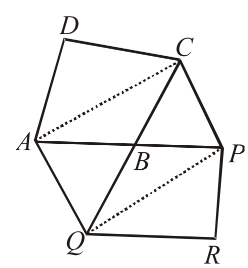
Thus, in quadrilateral PQRS one pair of opposite sides are equal and parallel.

Hence, PQRS is a parallelogram.

Ex.25 The side AB of a parallelogram ABCD is produced to any point P. A line through A parallel to CF meets CB produced in Q and the parallelogram PBQR completed.

Show that ar ($||^{gm}$ ABCD) = ar ($||^{gm}$ BPRQ).

Sol. Construction: Join AC and PQ.



To prove: ar ($||^{gm}$ ABCD) = ar($||^{gm}$ BPRQ)

Proof: Since AC and PQ are diagonals of parallelograms ABCD and BPQR respectively.

$\therefore$ ar (ΔABC) = $\frac{1}{2}$ ar ($\|^{gm}$ ABCD) ....(i)

and, ar (ΔPBQ) = $\frac{1}{2}$ ar ($\|^{gm}$ BPRQ) ....(ii)

Now, Δs ACQ and AQP are on the same base AQ and between the same parallels AQ and CP

$\therefore$ ar(ΔACQ) = ar (ΔAQP)

$\Rightarrow$ ar(ΔACQ) – ar (ΔABQ)

= ar (ΔAQP)–ar(ΔABQ)

[Subtracting ar (ΔABQ) from both sides]

$\Rightarrow$ ar (ΔABC) = ar (ΔBPQ)

$\Rightarrow$ $\frac{1}{2}$ ar ($\|^{gm}$ ABCD) = $\frac{1}{2}$ ar ($\|^{gm}$ BPRQ)

[Using (i) and (ii)]

$\Rightarrow$ ar($\|^{gm}$ ABCD) = ar($\|^{gm}$ BPRQ).

Ex.26 In a parallelogram ABCD, the bisector of $\angle$A also bisects BC at X. Prove that AD = 2AB.

Sol. Since AX is the bisector of $\angle$A.

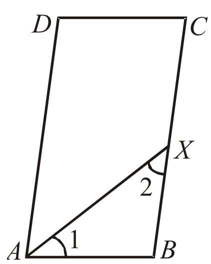
$\therefore$ $\angle 1 = \frac{1}{2}\angle A$ .... (i)

Since ABCD is a parallelogram.

Therefore, AD $\|$ BC and AB intersects them.

$\Rightarrow$ $\angle A + \angle B = 180^\circ$

[$\because$ Sum of interior angles is 180º]



$\Rightarrow$ $\angle B = 180^\circ - \angle A$

In ΔABX, we have

$\angle 1 + \angle 2 + \angle B = 180^\circ$

$\Rightarrow$ $\frac{1}{2}\angle A + \angle 2 + 180^\circ - \angle A = 180^\circ$

$\Rightarrow$ $\angle 2 - \frac{1}{2}\angle A = 0$

$\Rightarrow$ $\angle 2 = \frac{1}{2}\angle A$ ....(ii)

From (i) and (ii), we have
$\angle 1 = \angle 2$.

Thus, in ΔABX, we have

$\angle 1 = \angle 2$.

$\Rightarrow$ BX = AB [$\because$ Sides opposite to equal angles in a Δ are equal]

$\Rightarrow$ 2BX = 2AB [Multiplying both sides by 2]

$\Rightarrow$ BC = 2AB [$\because$ X is the mid-point of BC

$\therefore$ AD = BC]

$\Rightarrow$ AD = 2AB

[$\because$ ABCD is a $\|^{gm}$ $\therefore$ AD = BC]

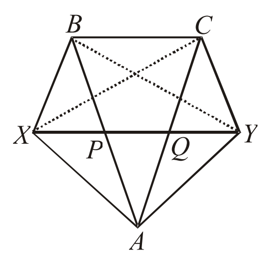
Ex.27 In Fig. BC $\|$ XY, BX $\|$ CA and AB $\|$ YC. Prove that:

ar (ΔABX) = ar (ΔACY).

Sol. Join XC and BY.

Since Δs BXC and BCY are on the same base BC and between the sum parallels BC and XY

$\therefore$ ar(ΔBXC) = ar(ΔBCY) .... (i)



Also, Δs BXC and ABX are on the same base BX and between the same parallels BX and AC.

$\therefore$ ar (ΔBXC) = ar (ΔABX) ....(ii)

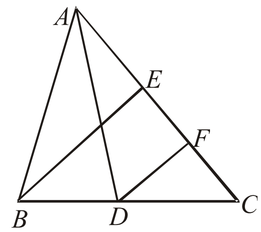
Clearly, ΔsBCY and ACY are on the same base CY and between the same parallels AB and CY.

$\therefore$ ar(ΔBCY) = ar (ΔACY) ....(iii)

From (i), (ii) and (iii), we get

ar(ΔABX) = ar (ΔACY).

Ex.28 In Fig. AD and BE are medians of ΔABC and BE $\|$ DF.



Prove that CF = $\frac{1}{4}$ AC

Sol. In ΔBEC, DF is a line through the mid-point D of BC and parallel to BE intersecting CE at F. Therefore, F is the mid-point of CE. Because the line drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.

Now, F is the mid-point of CE

$\Rightarrow$ CF = $\frac{1}{2}$ CE

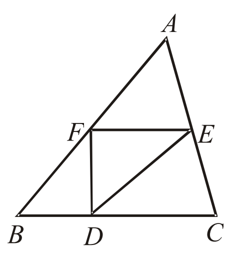
$\Rightarrow$ CF = $\frac{1}{2}\left(\frac{1}{2}AC\right)\left[\begin{array}{l}\because \text{ E is the mid-point} \\ \text{of AC} \therefore \text{EC} = \frac{1}{2}\text{AC}\end{array}\right]$

$\Rightarrow$ CF = $\frac{1}{4}$ AC

Ex.29 D, E, F are the mid-points of the sides BC, CA and AB respectively of ΔABC, prove that BDEF is a parallelogram whose area is half that of ΔABC. Also, show that

ar (ΔDEF) = $\frac{1}{4}$ are (ΔABC).

Sol. Since D and E are the mid-points of sides BC and AC respectively.



Therefore, DE || BA $\Rightarrow$ DE || BF

Similarly, FE || BD. So, BDEF is a parallelogram. Similarly, DCEF and AFDE are parallelograms.

Now, DF is a diagonal of $||^{gm}$ BDEF.

$\therefore$ ar (ΔBDF) = ar (ΔDEF) .... (i)

DE is a diagonal of $||^{gm}$ DCEF

$\therefore$ ar (ΔDCE) = ar (ΔDEF) ....(ii)

FE is a diagonal of $||^{gm}$ AFDE

$\therefore$ ar (ΔAFE) = ar (ΔDEF) ....(iii)

From (i), (ii) and (iii), we have

ar (ΔBDF) = ar (ΔDCE) = ar (ΔAFE) = ar (ΔDEF)

But, ar (ΔBDF) + ar (ΔDCE) + ar (ΔAFE) + ar (ΔDEF) = ar (ΔABC)

$\therefore$ 4 ar (ΔDEF) = ar (ΔABC)

$\Rightarrow$ ar (ΔDEF) = $\frac{1}{4}$ ar (ΔABC).

Now, ar ($||^{gm}$ BDEF) = 2 ar (ΔDEF)

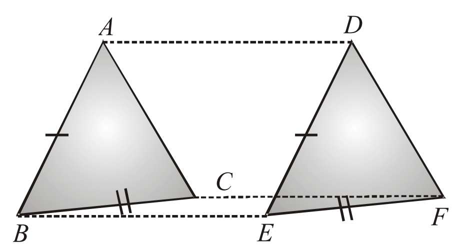
$\Rightarrow$ ar ($||^{gm}$ BDEF) = 2 × $\frac{1}{4}$ ar(ΔABC)

= $\frac{1}{2}$ ar (ΔABC)

Ex.30 ΔABC and ΔDEF are two triangles such that AB, BC are respectively equal and parallel to DE, EF; show that AC is equal and parallel to DF.

Sol. Given:Two triangles ABC and DEF such that AB = DE and AB || DE. Also BC = EF and BC || EF

To Prove: AC = DF and AC || DF



Proof : Consider the quadrilateral ABED.

We have, AB = DE and AB || DE

$\Rightarrow$ One pair of opposite sides are equal and parallel

$\Rightarrow$ ABED is a parallelogram.

$\Rightarrow$ AD = BE and AD || BE ....(i)

Now, consider quadrilateral BCFE.

We have, BC = EF and BC || EF

$\Rightarrow$ One pair of opposite sides are equal and parallel

$\Rightarrow$ BCFE is a parallelogram.

$\Rightarrow$ CF = BE and CF || BE ....(ii)

From (i) and (ii), we have

AD = CF and AD || CF

$\Rightarrow$ ACFD is a parallelogram

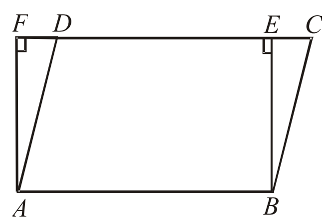
AC = DF and AC || DF

Ex.31 Parallelogram ABCD & rectangle ABEF have the same base AB and also have equal areas. Show that perimeter of the parallelogram is greater than that of the rectangle.

Sol. Given: A $||^{gm}$ ABCD and a rectangle ABEF with the same base AB and equal areas.

To Prove: Perimeter of $||^{gm}$ ABCD > Perimeter of rectangle ABEF

i.e. AB + BC + CD + AD > AB + BE + EF + AF.



Proof: Since opposite sides of a parallelogram and a rectangle are equal.

$\therefore$ AB = DC [$\because$ ABCD is a $||^{gm}$]

and, AB = EF [$\because$ ABEF is a rectangle]

$\therefore$ DC = EF .... (i)

$\Rightarrow$ AB + DC = AB + EF ....(ii)

Since, of all the segments that can be drawn to a given line from a point not lying on it, the perpendicular segment is the shortest.

$\therefore$ BE < BC and AF < AD $\Rightarrow$ BC > BE and AD > AF

$\Rightarrow$ BC + AD > BE + AF ....(iii)

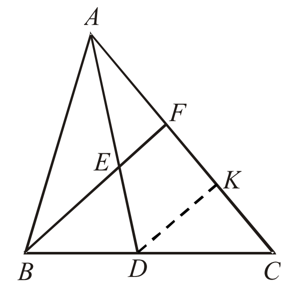
Adding (ii) and (iii), we get

AB + DC + BC + AD > AB + EF + BE + AF

$\Rightarrow$ AB + BC + CD + DA > AB + BE + EF + FA.

Ex.32 In ΔABC, AD is the median through A and E is the mid-point of AD. BE produced meets AC in F. Prove that AF = $\frac{1}{3}$AC.

Sol. Through D, draw DK || BF. In ΔADK, E is the mid-point of AD and EF || DK.



$\therefore$ F is the mid-point of AK

$\Rightarrow$ AF = FK .... (i)

In ΔBCF, D is the mid-point of BC and DK || BF

$\therefore$ K is the mid-point of FC

$\therefore$ FK = KC ....(ii)

From (i) and (ii), we have

AF = FK = KC ....(iii)

Now, AC = AF + FK + KC

$\Rightarrow$ AC = AF + AF + AF [Using (iii)]

$\Rightarrow$ AC = 3 (AF)

$\Rightarrow$ AF = $\frac{1}{3}$AC

IMPORTANT POINTS TO BE REMEMBERED

1. Two figures are said to be on the same base and between the same parallels, if they have a common side (base) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.
2. Two congruent figures have equal areas but the converse need not be true.
3. A diagonal of a parallelogram divides it into two triangles of equal area.
4. Parallelograms on the same base and between the same parallels are equal in area.
5. The area of a parallelogram is the product of its base and the corresponding altitude.
6. Parallelograms on equal bases and between the same parallels are equal in area.
7. Triangles on the same bases and between the same parallels are equal in area.
8. The area of a triangle is half the product of any of its sides and the corresponding altitude.
9. If a triangle and a parallelogram are on the same base and between the same parallels, then the area of the triangle is equal to the half of the parallelogram.
10. The area of a trapezium is half the product of its height and the sum of parallel sides.
11. Triangles having equal areas and having one side of one of the triangles, equal to one side of the other, have their corresponding altitudes equal.
12. If each diagonal of a quadrilateral separates it into two triangles of equal area, then the quadrilateral is a parallelogram.
13. The area of a rhombus is half the product of the lengths of its diagonals.
14. Diagonals of a parallelogram divide it into four triangles of equal area.
15. A median of a triangle divides it into two triangles of equal area.