Circle

DEFINITION

Circle is locus of a point which moves at a constant distance from a fixed point . This constant distance is called radius of the circle and fixed point is called centre of the circle.

STANDARD FORMS OF EQUATION OF A CIRCLE

General Form

The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$,

where g,f,c are constants.

For this circle, Centre = $(- g , -f)$

$= \left(-\frac{1}{2}\text{coef. of } x, -\frac{1}{2}\text{coef. of } y\right)$,

Radius = $\sqrt{g^2 + f^2 - c}$

Note :

(i) The above equation represents

* a real circle if $g^2 + f^2 > c$
* a point circle if $g^2 + f^2 = c$
* an imaginary circle if $g^2 + f^2 < c$

(ii) In the above equation

* If $c = 0 \Rightarrow$ the circles passes through the origin
* If $f = 0 \Rightarrow$ the centre is on x-axis
* If $g = 0 \Rightarrow$ the centre is on y-axis

(iii) The general eqaution of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if $a = b \neq 0$ and $h = 0$.

Note :

That every second degree equation in x & y, in which coefficient of x^2 is equal to coefficient of y^2 & the coefficient of xy is zero, always represents a circle.

Solved Examples

Ex.1 Find the radius of the circle

$x^2 + y^2 + 4x - 6y + 1 = 0$.

Sol. Here $x^2 + y^2 + 4x - 6y + 1 = 0$

on comparing from general eq^n.

$2g = 4 \Rightarrow g = 2$,

$2f = -6 \Rightarrow f = -3, \quad c = 1$

$\therefore \quad r = \sqrt{g^2 + f^2 - c} = \sqrt{4+9-1} = \sqrt{12} = 2\sqrt{3}$

Ex.2 If (4, –2) is the one extremity of diameter to the circle $x^2 + y^2 - 4x + 8y - 4 = 0$ then find its other extremity.

Sol. Centre of circle is (2, –4). Let the other extremity is (h, k)

$$\therefore \quad \left(\frac{4+h}{2}\right)=2, \left(\frac{-2+k}{2}\right)=-4 \;\Rightarrow\; (h, k) = (0, -6)$$

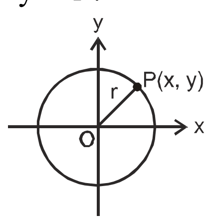
Ex.3 If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Sol. Centre of circle = (–3/2 , –2). This lies on diameter $y = 2x + m$

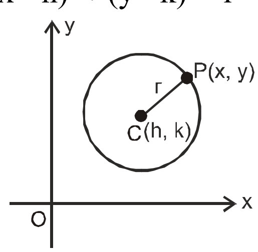
$\Rightarrow \quad -2 = -3/2 \times 2 + m \qquad \Rightarrow \quad m = 1$

EQUATION OF A CIRCLE IN SOME SPECIAL CASES

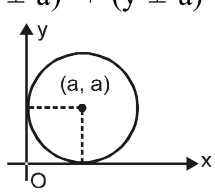
(a) The circle with centre as origin & radius ‘r’ has the equation; $x^2 + y^2 = r^2$.



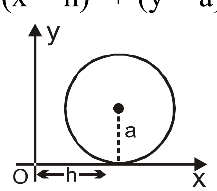
(b) The circle with centre (h, k) & radius ‘r’ has the equation; $(x - h)^2 + (y - k)^2 = r^2$.



(c) **Which touches both axes :** The equation of a circle with radius 'a' touching both coordinate axes is given by $(x \pm a)^2 + (y \pm a)^2 = a^2$



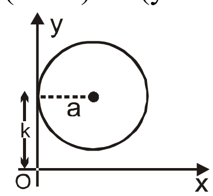
(d) **Which touches x-axis :** The equation of a circle with radius 'a' touching x-axis at a distance h from the origin is $(x - h)^2 + (y - a)^2 = a^2$



Note : The equation of a circle with radius 'a' touching x-axis at the origin is

$x^2 + (y \pm a)^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 \pm 2ay = 0$

(e) **Which touches y-axis :** The equation of a circle with radius 'a' touching y-axis at a distance k from the origin is $(x - a)^2 + (y - k)^2 = a^2$



Note : The equation of a circle with radius 'a' touching y-axis at the origin is

$(x \pm a)^2 + y^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 \pm 2ax = 0$

Solved Examples

Ex.4 If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches x-axis, then find the value of c.

Sol. Touches x-axis, hence radius = ordinate of centre.

Hence $\sqrt{g^2 + f^2 - c} = (-f)$ or $g^2 = c$.

Ex.5 Find the equation of a circle whose centre is (3, –1) and radius is 3

Sol. $(x - 3)^2 + (y + 1)^2 = 3^2$

$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = 9$

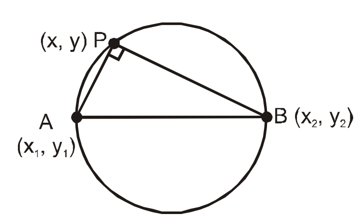
$\Rightarrow x^2 + y^2 - 6x + 2y + 1 = 0$

Ex.6 Find the equation of a circle with centre at the origin and which passes through the point (α, β).

Sol. Here radius = $\sqrt{\alpha^2 + \beta^2}$; so the required equation is $x^2 + y^2 = \alpha^2 + \beta^2$

The equation of circle with (x_1, y_1) & (x_2, y_2) as extremeties of its diameter is:

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.



This is obtained by the fact that angle in a semicircle is a right angle.

$\therefore$ (Slope of PA) (Slope of PB) = – 1

$\Rightarrow \dfrac{y-y_1}{x-x_1} \cdot \dfrac{y-y_2}{x-x_2} = -1$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Note that this will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).

Solved Examples

Ex.7 Find the equation of the circle whose centre is (1, –2) and radius is 4.

Sol. The equation of the circle is

$(x - 1)^2 + (y - (-2))^2 = 4^2$

$\Rightarrow (x - 1)^2 + (y + 2)^2 = 16$

$\Rightarrow x^2 + y^2 - 2x + 4y - 11 = 0$

Ex.8 Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is (2, – 3).

Sol. Let P be the point of intersection of the lines AB and LM whose equations are respectively

$3x - 2y - 1 = 0$ ....(i)

and $4x + y - 27 = 0$ ....(ii)

Solving (i) and (ii), we get x = 5, y = 7. So, coordinates of P are (5, 7). Let C(2, –3) be the centre of the circle. Since the circle passes through P, therefore

CP = radius = $\sqrt{(5-2)^2 + (7+3)^2}$

$\Rightarrow$ radius = $\sqrt{109}$.

Hence the equation of the required circle is

$(x - 2)^2 + (y + 3)^2 = \left(\sqrt{109}\right)^2$

Ex.9 Find the centre & radius of the circle whose equation is $x^2 + y^2 - 4x + 6y + 12 = 0$

Sol. Comparing it with the general equation

$x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2g = -4 \quad \Rightarrow \quad g = -2$

$2f = 6 \quad \Rightarrow \quad f = 3$

& $\quad c = 12$

$\therefore$ centre is (–g, –f) i.e. (2, –3)

and radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 - 12} = 1$

Ex.9 Find the equation of the circle, the coordinates of the end points of whose diameter are (–1, 2) and (4, –3)

Sol. We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1,\ x_2 = 4,\ y_1 = 2$ and $y_2 = -3$.

So, the equation of the required circle is

$(x + 1)(x - 4) + (y - 2)(y + 3) = 0$

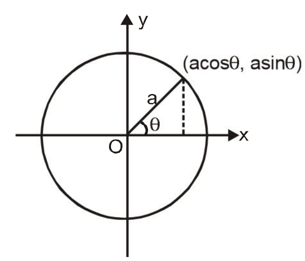
$\Rightarrow x^2 + y^2 - 3x + y - 10 = 0$.

Parametric Equation of a Circle

(a) The parametric equations of a circle

$x^2 + y^2 = a^2$ are $x = a\cos\theta$, $y = a\sin\theta$.

Hence parametric coordinates of any point lying on the circle $x^2 + y^2 = a^2$ are $(a\cos\theta, a\sin\theta)$



(b) The parametric equations of the circle

$(x-h)^2 + (y - k)^2 = a^2$ are

$x = h + a\cos\theta$, $y = k + a\sin\theta$.

Hence parametric coordinates of any point lying on the circle are $(h + a\cos\theta, k + a\sin\theta)$

(c) Parametric equations of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x = -g + \sqrt{g^2 + f^2 - c}\,\cos\theta$,

$y = -f + \sqrt{g^2 + f^2 - c}\,\sin\theta$

Solved Examples

Ex.10 Find the parametric coordinates of any point of the circle $x^2 + y^2 + 2x - 3y - 4 = 0$

Sol. Centre = $\left(-1, \dfrac{3}{2}\right)$ radius = $\sqrt{1 + \dfrac{4}{9} + 4} = \dfrac{7}{3}$

$\therefore$ Parametric coordinates of any point are

$\left(-1 + \dfrac{7}{3}\cos\theta,\ \dfrac{3}{2} + \dfrac{7}{3}\sin\theta\right)$

Ex.11 Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$

Sol. We have : $x^2 + y^2 - 4x - 2y + 1 = 0$

$\Rightarrow \quad (x^2 - 4x) + (y^2 - 2y) = -1$

$\Rightarrow \quad (x - 2)^2 + (y - 1)^2 = 2^2$

So, the parametric equations of this circle are

$x = 2 + 2\cos\theta$, $y = 1 + 2\sin\theta$.

Ex.12 Find the equations of the following curves in cartesian form. Also, find the centre and radius of the circle $x = a + c\cos\theta$, $y = b + c\sin\theta$

Sol. We have : $x = a + c\cos\theta$, $y = b + c\sin\theta$

$\Rightarrow \quad \cos\theta = \frac{x-a}{c}, \sin\theta = \frac{y-b}{c}$

$\Rightarrow \quad \left(\frac{x-a}{c}\right)^2 + \left(\frac{y-b}{c}\right)^2 = \cos^2\theta + \sin^2\theta$

$\Rightarrow \quad (x - a)^2 + (y - b)^2 = c^2$

Clearly, it is a circle with centre at (a, b) and radius c.

POSITION OF A POINT AND LINE WITH RESPECT TO A CIRCLE

1. Position of a point

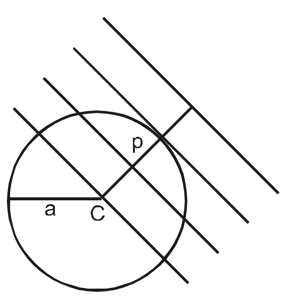
A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative. So

* $S_1 > 0 \Rightarrow (x_1, y_1)$ is outside the circle
* $S_1 = 0 \Rightarrow (x_1, y_1)$ is on the circle
* $S_1 < 0 \Rightarrow (x_1, y_1)$ is inside the circle

2. Position of a line

Let L = 0 be a line and S = 0 be a circle. If 'a' be the radius of the circle and 'p' be the length of the perpendicular from its centre on the line, then

* $p > a \Rightarrow$ line is outside the circle
* $p = a \Rightarrow$ line touches the circle
* $p < a \Rightarrow$ line is a chord of the circle
* $p = 0 \Rightarrow$ line is a diameter of the circle



CONDITION OF TANGENCY

A line L = 0 touches the circle S = 0, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle i.e., p = r. This is the condition of tangency for the line L = 0.

The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1+m^2}$

Thus, for every value of m,

the line $y = mx \pm a\sqrt{1+m^2}$

is a tangent of the circle $x^2 + y^2 = a^2$ and its point of contact is $\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\mp a}{\sqrt{1+m^2}}\right)$

Note :

* If $a^2(1 + m^2) - c^2 > 0$ line will meet the circle at real and different points.
* If $c^2 = a^2(1 + m^2)$ line will touch the circle.
* If $a^2(1 + m^2) - c^2 > 0$ line will meet circle at two imaginary points.

Solved Examples

Ex13 For what value of c will the line $y = 2x + c$ be a tangent to the circle $x^2 + y^2 = 5$?

Sol. We have : $y = 2x + c$ or $2x - y + c = 0$ ......(i)

and $x^2 + y^2 = 5$ ....(ii)

If the line (i) touches the circle (ii), then

length of the $\perp$ from the centre (0, 0) = radius of circle (ii)

$\Rightarrow \quad \left|\frac{2\times 0 - 0 + c}{\sqrt{2^2 + (-1)^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5}$

$\Rightarrow \quad \frac{c}{\sqrt{5}} = \pm\sqrt{5} \quad \Rightarrow \quad c = \pm 5$

Hence, the line (i) touches the circle (ii) for $c = \pm 5$

Ex.14 If the line $y = mx + c$ touches the circle $x^2 + y^2 = 4y$ then find c.

Sol. Centre of the circle = (0, 2), radius = 2. So condition of tangency

$p = a \quad \Rightarrow \quad \frac{c-2}{\sqrt{m^2+1}} = 2 \quad \Rightarrow \quad c = 2\,(1 + \sqrt{m^2+1})$

Ex.15 Find the point at line $y = x + \sqrt{2a}$ touches the circle $x^2 + y^2 = a^2$.

Sol. The line $y = mx + a\sqrt{1+m^2}$ touches the circle $x^2 + y^2 = a^2$ at the point

$$\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$$

Here $m = 1$, $a = a$, so the required point is $\left(\frac{-a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

Point form of tangent :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is,
$x x_1 + y y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is: $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$.

Note : In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{x_1y + xy_1}{2}$ and c remains as c.

Parametric form of tangent :

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is
$x\cos\alpha + y\sin\alpha = a$.

NOTE : The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Ex.16 Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at $(4, -1)$.

Sol. Equation of tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$

or $4x - y - 15x - 60 + 3y - 3 + 109 = 0$ or $-11x + 2y + 46 = 0$

or $11x - 2y - 46 = 0$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$

Ex.17 Find the equation of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x + 3y + 5 = 0$

Sol. Given circle is $x^2 + y^2 - 6x + 4y - 12 = 0$ ....(i)

and given line is $4x + 3y + 5 = 0$ ....(ii)

Centre of circle (i) is $(3, -2)$ and its radius is 5.

Equation of any line

$4x + 3y + k = 0$ parallel to the line (ii) ......(iii)

If line (iii) is tangent to circle, (i) then

$$\frac{|4.3 + 3(-2) + k|}{\sqrt{4^2 + 3^2}} = 5 \text{ or } |6 + k| = 25$$

or $6 + k = \pm 25$ $\therefore$ $k = 19, -31$

Hence equation of required tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$

Equation of the Normal

The equation of the normal at the point (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $\Rightarrow \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$

Note : For the circle $x^2 + y^2 = a^2$ it becomes $\frac{x}{x_1} = \frac{y}{y_1}$

Solved Examples

Ex.18 Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point $(5, 6)$.

Sol. Since normal is line joining centre $\left(\frac{5}{2}, -1\right)$ and $(5, 6)$

Slope $= \frac{14}{5}$

Hence, the equation of the normal at $(5, 6)$ is

$y - 6 = (14/5)(x - 5) \Rightarrow 14x - 5y - 40 = 0$

Length of the tangent

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$PQ = PR = \sqrt{S_1}$

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Also area of the quadrilateral PQCR $= r\sqrt{S_1}$ and angle between tangents PQ and PR i.e.

$\angle QPR = 2\tan^{-1}\frac{r}{\sqrt{S_1}}$

Pair of tangents from a point :

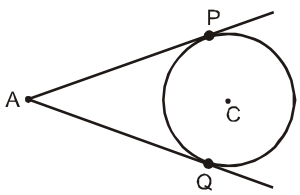
The equation of a pair of tangents drawn from the point A (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$;

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.



Solved Examples

Ex.19 Write the equation of the tangent to the circle $(x - 1)^2 + (y + 2)^2 = 10$ at the point (2, 1)

Sol. The eqaution of the given circle can be written as $x^2 + y^2 - 2x + 4y - 5 = 0$.

So the equation of the tangent at (2, 1) will be

$x(2) + y(1) - (x + 2) + 2(y + 1) - 5 = 0$

$\Rightarrow \quad x + 3y - 5 = 0$

Ex.20 Find the equation of the normal to the circle $x^2 + y^2 + 6x + 8y + 1 = 0$ passign through (0, 0)

Sol. Centre of the circle = (–3, –4). So the normal is a line passing through (0, 0) and (–3, –4). Consequently its equation is

$$y - 0 = \frac{-4}{-3}(x - 0) \qquad \Rightarrow \qquad 4x - 3y = 0$$

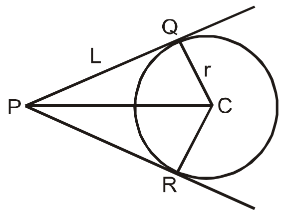
Ex.21 Two tangents PQ and PR drawn to the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ from point P(16, 7). If the centre of the cirlce is C then find the area of quadrilateral PQCR.

Sol. Area PQCR = $2\Delta PQC = 2 \times \frac{1}{2} L \times r$

Where L = length of tangent and r = radius of circle.

$L = \sqrt{S_1}$ and $r = \sqrt{1 + 4 + 20} = 5$

Hence the required area = 75 sq. units.



Ex.22 A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. Then find equation of the pair of tangent.

Sol. Equation of pair of tangents is given by $SS_1 = T^2$.

or $S = x^2 + y^2 + 20(x + y) + 20$, $S_1 = 20$,

$T = 10(x + y) + 20 = 0$

$\therefore \; SS_1 = T^2$

$\Rightarrow \; 20\{x^2 + y^2 + 20(x + y) + 20\} = 10^2 (x + y + 2)^2$

$\Rightarrow \; 4x^2 + 4y^2 + 10xy = 0 \quad \Rightarrow \quad 2x^2 + 2y^2 + 5xy = 0$

Ex.23 Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Sol. Given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ....(i)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5$

Clearly P lies outside the circle

and $T \equiv x \,.\, 0 + y \,.\, 1 - (x + 0) + 2(y + 1)$

i.e. $T \equiv -x + 3y + 2$.

Now equation of pair of tangents from P(0, 1) to circle (1) is $SS_1 = T^2$

or $5(x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $\quad 4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $\quad 2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ....(ii)

Separate equation of pair of tangents :

From (ii), $2x^2 + 3(y - 1)x - 2(2y^2 - 4y + 2) = 0$

$$\therefore \quad x = \frac{-3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$$

or $\quad 4x + 3y - 3 = \pm\sqrt{25y^2 - 50y + 25}$

$= \pm 5(y - 1)$

$\therefore$ Separate equations of tangents are

$2x - y + 1 = 0$ and $x + 2y - 2 = 0$

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents of a circle is called the director circle of that circle.

The equation of the director circle of $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$

It may be easily seen that

* Centre of the director circle = centre of the given circle.
* Radius of the director circle = $\sqrt{2}$ (radius of the given circle)

CHORD OF CONTACT :

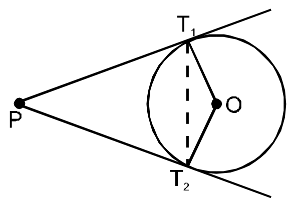
If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2+y^2+2gx+2fy+c=0$, then the equation of the chord of contact T_1T_2 is:

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

Note : Here R = radius; L = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.

(b) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2+L^2}}$.



(c) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2+L^2}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\dfrac{2RL}{L^2-R^2}\right)$

(e) Equation of the circle circumscribing the triangle PT_1T_2 is:

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0.$

Solved Examples

Ex.24 Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$

Sol. Given circle is $x^2 + y^2 - 2x + 4y + 7 = 0$ ....(i)

Let P = (1, 2)

For point P (1, 2), $x^2 + y^2 - 2x + 4y + 7$

$= 1 + 4 - 2 + 8 + 7 = 18 > 0$

Hence point P lies outside the circle

For point P (1, 2), $T = x \cdot 1 + y \cdot 2 - (x + 1) + 2(y + 2) + 7$

i.e. $T = 4y + 10$

Now equation of the chord of contact of point P(1, 2) w.r.t. circle (i) will be

$4y + 10 = 0$ or $2y + 5 = 0$

Ex. 25 Find the distance between the chord of contact with respect to point (0, 0) and (g, f) of circle $x^2 + y^2 + 2gx + 2fy + c = 0.$

Sol. Chord of contact with respect to (0, 0)

$gx + fy + c = 0$ ....(i)

Chord of contact with respect to (g, f)

$gx + fy + g(x + g) + f(y + f) + c = 0$

$\Rightarrow 2gx + 2fy + g^2 + f^2 + c = 0$

$\Rightarrow gx + fy + \dfrac{1}{2}(g^2 + f^2 + c) = 0$ ....(ii)

Distance betwen (i) and (ii) is

$$= \frac{\frac{1}{2}(g^2+f^2+c)-c}{\sqrt{g^2+f^2}} = \frac{g^2+f^2-c}{2\sqrt{g^2+f^2}}$$

Ex.26 Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents.

Sol. Given circles are $S_1 \equiv x^2 + y^2 - 12 = 0$ ..... (i)

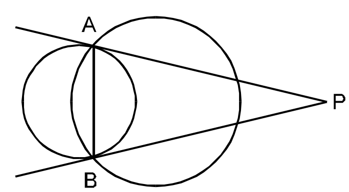
and $S_2 = x^2 + y^2 - 5x + 3y - 2 = 0$ ...... (ii)

Now equation of common chord of circle (i) and (ii) is

$S_1 - S_2 = 0$ i.e. $5x - 3y - 10 = 0$ .... (iii)

Let this line meet circle (i) [or (ii)] at A and B

Let the tangents to circle (i) at A and B meet at $P(\alpha, \beta)$, then AB will be the chord of contact of the tangents to the circle (i) from P, therefore equation of AB will be



$x\alpha + y\beta - 12 = 0$ .... (iv)

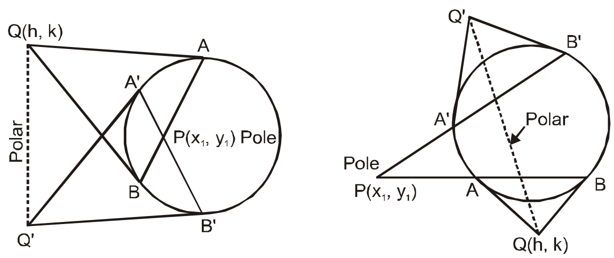
Now lines (iii) and (iv) are same, therefore, equations (iii) and (iv) are identical

$\therefore \dfrac{\alpha}{5} = \dfrac{\beta}{-3} = \dfrac{-12}{-10}$ $\therefore \alpha = 6, \beta = -\dfrac{18}{5}$

Hence $P = \left(6, -\dfrac{18}{5}\right)$

POLE & POLAR

Let $P(x_1, y_1)$ be any point inside or outside the circle. Draw chords AB and A' B' pasing through P. If tangent to the circle at A and B meet at Q (h, k), then locus of Q is called polar of P.w.r.t. circle and P is called the pole and if tangent to the circle at A' and B' meet at Q', then the straigt line QQ' is polar with P' as its pole.



1. Equation of polar

* Equation of polar of the pole $P(x_1, y_1)$ w.r.t. $x^2 + y^2 = a^2$ is

$xx_1 + yy_1 = a^2$

* Equation of polar of the pole (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

2. Coordinates of pole

* Pole of polar $Ax + By + C = 0$ w.r.t. circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa^2}{C}, -\frac{Ba^2}{C}\right)$

* Pole of polar $Ax + By + C = 0$ with respect to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by the equation

$$\frac{x_1 + g}{A} = \frac{y_1 + f}{B} = \frac{gx_1 + fy_1 + c}{C}$$

3. Conjugate points and Conjugate lines

(i) Conjugate points :- Two points are called conjugate points with respect to a circle if each point lies on the polar of the other point with respect to the same circle.

(ii) Conjugate lines :- Two lines are called conjugate lines with respect to a circle if the pole of each line lies on the other line.

Solved Examples

Ex.27 Find the equation of polar of point (4, 4) with respect to circle $(x - 1)^2 + (y - 2)^2 = 1$.

Sol. $(x - 1)^2 + (y - 2)^2 = 1$

$x^2 + y^2 - 2x - 4y + 4 = 0$

equation of polar of point (4 , 4) is

$4x + 4y - (x + 4) - 2(y + 4) + 4 = 0$

$\Rightarrow \quad 3x - 2y - 8 = 0$

Ex.28 Find the pole of the line $\frac{x}{a} + \frac{y}{b} = 1$ with respect to circle $x^2 + y^2 = c^2$.

Sol. Let the pole is (h , k)

Hence polar of this pole is $xh + yk - c^2 = 0$ .....(1)

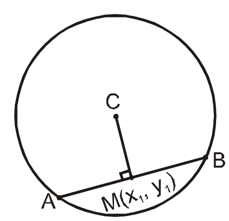
but polar is $\frac{x}{a} + \frac{y}{b} = 0$ .....(2)

comparing the coefficient of x and y

$$\frac{h}{(1/a)} = \frac{k}{(1/b)} = \frac{-c^2}{-1} \Rightarrow h = \frac{c^2}{a}, \; k = \frac{c^2}{b}$$

Equation of the chord with a given middle point:

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.



Notes :

(i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.

(ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

Solved Examples

Ex.29 Find the equation of chord of the circle $x^2 + y^2 = 8x$ bisected at the point (4 , 3).

Sol. $T = S_1 \Rightarrow x(4) + y(3) - 4(x + 4) = 16 + 9 - 32$

$\Rightarrow 3y - 9 = 0 \Rightarrow y = 3$

Ex.30 Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is $(1, -1)$

Sol. Equation of given circle is

$S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$

Let $L \equiv (1, -1)$

For point $L(1, -1)$, $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$ and

$T \equiv x.1 + y(-1) + 3(x + 1) + 4(y - 1) - 11$

i.e. $T \equiv 4x + 3y - 12$

Now equation of the chord of circle (i) whose middle point is $L(1, -1)$ is

$T = S_1$ or $4x + 3y - 12 = -11$ or $4x + 3y - 1 = 0$

Second Method : Let C be the centre of the given circle, then $C \equiv (-3, -4)$. $L \equiv (1, -1)$ slope of CL

$= \dfrac{-4+1}{-3-1} = \dfrac{3}{4}$

$\therefore$ Equation of chord of circle whose middle point is L, is $y + 1 = -\dfrac{4}{3}(x - 1)$

($\because$ chord is perpendicular to CL) or $4x + 3y - 1 = 0$

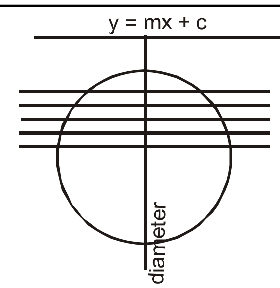
Equation of the chord joining two points of circle :

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points $P(\alpha)$ and $Q(\beta)$ on it is given by the equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}.$$

DIAMETER OF A CIRCLE

The locus of middle points of a system of parallel chords of a circle is called the diameter of that circle. The diameter of the circle $x^2 + y^2 = r^2$ corresponding to the system of parallel chords $y = mx + c$ is $\mathbf{x + my = 0}$



Note :

* Every Diameter passes through the centre of the circle.
* A diameter is perpendicular to the system of parallel chords.

Solved Examples

Ex.31 Find the diameter of the circle $x^2 + y^2 - 4x + 2y - 11 = 0$ corresponding to a system of chords parallel to the line $x - 2y + 1 = 0$.

Sol. Centre of the circle = (2, –1).

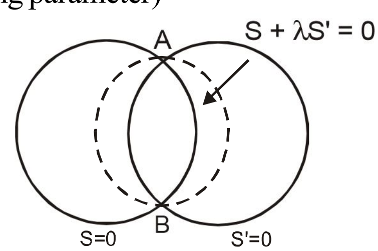
The equation of the line perpendicular to chord $x - 2y + 1 = 0$ is $2x + y + k = 0$

Since the line passes through the point (2, 1). So $k = -3$

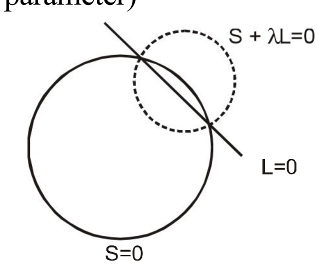
The equation of diameter is $2x + y - 3 = 0$

FAMILY OF CIRCLES

1. If S = 0 and S' = 0 are two intersecting circles, the $S + \lambda S' = 0$ $(\lambda \neq -1)$ represents family of circles passing throguh intersection points of S = 0 and S' = 0 (λ being parameter)



2. If S = 0 and L = 0 represent a circle and a line, then $S + \lambda L = 0$ represent family of circles passing through intersection points of circle S= 0 and line L = 0 (λ being parameter)



3. Family of circles touching a line $ax + by + c = 0$ at (x_1, y_1) on it, is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(ax + by + c) = 0$

(a) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

$= 0$, where K is a parameter.

(b) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided co–efficient of $xy = 0$ and co–efficient of x^2 = co–efficient of y^2.

(c) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $u L_1L_3 + \lambda L_2L_4 = 0$ where values of u & λ can be found out by using condition that co–efficient of x^2 = co–efficient of y^2 and co–efficient of $xy = 0$.

Solved Examples

Ex.32 Find the equation of the circle pasing through the origin and through the points of intersection of two circles $x^2 + y^2 - 10x + 9 = 0$ and $x^2 + y^2 = 4$

Sol. Let the circle be $(x^2 + y^2 - 10x + 9) + \lambda(x^2 + y^2 - 4) = 0$

Since it passes through (0,0), so we have

$9 - 4\lambda = 0 \Rightarrow \lambda = 9/4$

So the required equation is

$4(x^2 + y^2 - 10x + 9) + 9(x^2 + y^2 - 4) = 0$

$\Rightarrow \quad 13(x^2 + y^2) - 40x = 0$

Ex.33 Find the equation of the circle passing through the origin and through the points of intersection of the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ and the line $x + y - 1 = 0$

Sol. Let the required equation be

$(x^2 + y^2 - 2x + 4y - 20) + \lambda(x + y - 1) = 0$

Since it passes through (0,0), so we have $-20 - \lambda = 0$

$\Rightarrow \quad \lambda = -20$

Hence the required equation is

$(x^2 + y^2 - 2x + 4y - 20) - 20(x + y - 1) = 0$

$\Rightarrow \quad x^2 + y^2 - 22x - 16y = 0$

Ex.34 Find the equations of the circles passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

Sol. Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$ or $(x^2 + y^2 - 2x - 4y - 4) + \lambda(x^2 + y^2 - 10x - 12y + 40) = 0$

or $(x^2 + y^2) - 2\dfrac{(1+5\lambda)}{1+\lambda}x - 2\dfrac{(2+6\lambda)}{1+\lambda}y + \dfrac{40\lambda - 4}{1+\lambda} = 0$ ....(i)

$r = \sqrt{g^2 + f^2 - c} = 4$, given

$\therefore \quad 16 = \dfrac{(1+5\lambda)^2}{(1+\lambda)^2} + \dfrac{(2+6\lambda)^2}{(1+\lambda)^2} - \dfrac{40\lambda - 4}{1+\lambda}$

$16(1 + 2\lambda + \lambda^2) = 1 + 10\lambda + 25\lambda^2 + 4 + 24\lambda + 36\lambda^2 - 40\lambda^2 - 40\lambda + 4 + 4\lambda$

or $\quad 16 + 32\lambda + 16\lambda^2 = 21\lambda^2 - 2\lambda + 9$

or $\quad 5\lambda^2 - 34\lambda - 7 = 0$

$\therefore \quad (\lambda - 7)(5\lambda + 1) = 0$

$\therefore \quad \lambda = 7, -1/5$

Putting the values of λ in (i) the required circles are

$2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and

$x^2 + y^2 - 2y - 15 = 0$

Ex.35 Find the equations of circles which touches $2x - y + 3 = 0$ and pass through the points of intersection of the line $x + 2y - 1 = 0$ and the circle $x^2 + y^2 - 2x + 1 = 0$.

Sol. The required circle by $S + \lambda P = 0$ is

$x^2 + y^2 - 2x + 1 + \lambda(x + 2y - 1) = 0$

or $\quad x^2 + y^2 - x(2 - \lambda) + 2\lambda y + (1 - \lambda) = 0$

centre $(-g, -f)$ is $\left(\dfrac{2-\lambda}{2}, -\lambda\right)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{\dfrac{(2-\lambda)^2}{4} + \lambda^2 - (1-\lambda)} = \dfrac{1}{2}\sqrt{5\lambda^2} = \dfrac{\sqrt{5}}{2}|\lambda|$.

Since the circle touches the line $2x - y + 3 = 0$ therefore perpendicular from centre is equal to radius

$\left|\dfrac{2.((2-\lambda)/2) - (-\lambda) + 3}{\sqrt{5}}\right| = \dfrac{|\lambda|}{2}\sqrt{5}$.

$\therefore \quad \lambda = \pm 2$

Putting the values of λ in (i) the required circles are

$x^2 + y^2 + 4y - 1 = 0$

$x^2 + y^2 - 4x - 4y + 3 = 0$.

Ex.36 Find the equation of circle passing through the points A(1, 1) & B(2, 2) and whose radius is 1.

Sol. Equation of AB is $x - y = 0$

$\therefore$ equation of circle is

$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$

or $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

radius $= \sqrt{\frac{(\lambda-3)^2}{4} + \frac{(\lambda+3)^2}{4} - 4}$

But radius = 1 (given)

$\therefore \sqrt{\frac{(\lambda-3)^2}{4} + \frac{(\lambda+3)^2}{4} - 4} = 1$

or $(\lambda - 3)^2 + (\lambda + 3)^2 - 16 = 4 \Rightarrow \lambda = \pm 1$

$\therefore$ equation of circles are $x^2 + y^2 - 2x - 4y + 4 = 0$ & $x^2 + y^2 - 4x - 2y + 4 = 0$

Ex.37 Find the equation of the circle passing through the point (2, 1) and touching the line $x + 2y - 1 = 0$ at the point (3, – 1).

Sol. Equation of circle is

$(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$

Since it passes through the point (2, 1),

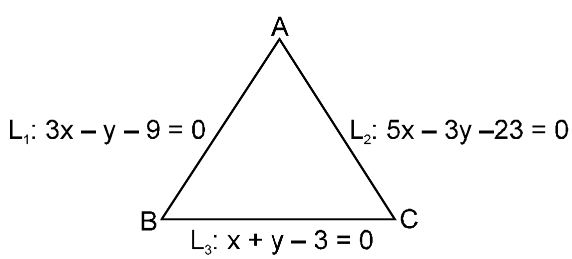
$1 + 4 + \lambda(2 + 2 - 1) = 0 \Rightarrow \lambda = -\frac{5}{3}$

$\therefore$ circle is $(x - 3)^2 + (y + 1)^2 - \frac{5}{3}(x + 2y - 1) = 0$

$\Rightarrow 3x^2 + 3y^2 - 23x - 4y + 35 = 0$

Ex.38 Find the equation of circle circumcscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$.

Sol.



$L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$(3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$(15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69) + \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$(5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu) + y(50 - 14\lambda - 6\mu) + (207 + 69\lambda + 27\mu) = 0$ ....(i)

coefficient of x^2 = coefficient of y^2

$\Rightarrow 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$

$2\lambda + \mu + 3 = 0$ ....(ii)

coefficient of xy = 0

$\Rightarrow \lambda + \mu - 7 = 0$ ....(iii)

Solving (ii) and (iii), we have

$\lambda = -10, \mu = 17$

Puting these values of λ & μ in equation (i), we get $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

COMMON CHORD OF TWO CIRCLES

The line joining the points of intersection of two circles is called the common chord. If the equation of two circle.

$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$, then equation of common chord is $S - S' = 0$

$\Rightarrow 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Also the length of the common chord AB is given by

$AB = 2\sqrt{a^2 - p^2}$

where 'a' is the radius of one of the given circles and 'p' is the distance of its centre from their common chord.

Solved Examples

Ex.39 Find the length and equation of the common chord of circles $x^2 + y^2 = 10x$ and $x^2 + y^2 = 4$

Sol. The equation of the common chord is

$(x^2 + y^2 - 10x) - (x^2 + y^2 - 4) = 0$

$\Rightarrow -10x + 4 = 0 \Rightarrow 5x - 2 = 0$

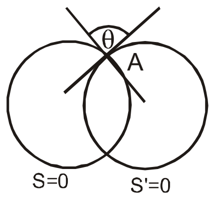
Also with respect to second circle a = 2 , p = 2/5

$\therefore$ length of common chord $= 2\sqrt{4 - 4/25} = \frac{8\sqrt{6}}{5}$

ANGLE OF INTERSECTION OF TWO CIRCLES

The angle of intersection between two circles S = 0 and S' = 0 is defined

as the angle between their tangent at their point of intersection.



If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2} \quad \text{or}$$

$$\cos\theta = \frac{2(g_1 g_2 + f_1 f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$$

Condition of Orthogonality

If the angle of intersecton of the two circle is a right angle then such circle are called Orthogonal circle.

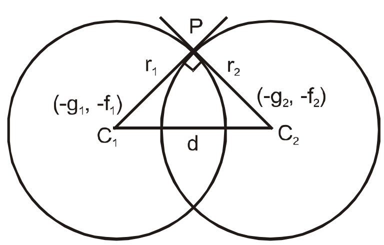
In ΔPC_1C_2

$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2 \quad \Rightarrow \quad d^2 = r_1^{\,2} + r_2^{\,2}$

$\Rightarrow \; (g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^{\,2} + f_1^{\,2} - c_1 + g_2^{\,2} + f_2^{\,2} - c_2$

$\Rightarrow \; 2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$

(Condition of Orthogonality)



Solved Examples

Ex.40 For what value of m the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + m = 0$ cuts orthogonally

Sol. Let the two circles be $x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$,

where $g_1 = 5/2$, $f_1 = 3/2$, $c_1 = 7$, $g_2 = -4$, $f_2 = 3$ and $c_2 = m$.

If the two circles intersects orthogonally, then

$2(g_1 g_2 + f_1 f_2) = c_1 + c_2 \Rightarrow 2\left(-10 + \frac{9}{2}\right) = 7 + m$

$\Rightarrow \quad 11 = 7 + m \Rightarrow m = -18$

Ex.41 Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Sol. Given circles are $x^2 + y^2 + 3x - 5y + 6 = 0$ .....(i)

and $4x^2 + 4y^2 - 28x + 29 = 0$

or $\quad x^2 + y^2 - 7x + \frac{29}{4} = 0.$ ....(ii)

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(iii)

Since circle (iii) cuts circles (i) and (ii) orthogonally

$\therefore \quad 2g\left(\frac{3}{2}\right) + 2f\left(-\frac{5}{2}\right) = c + 6$ or

$3g - 5f = c + 6$ ....(iv)

and $2g\left(-\frac{7}{2}\right) + 2f.0 = c + \frac{29}{4}$ or

$-7g = c + \frac{29}{4}$ ....(v)

From (iv) & (v), we get $10g - 5f = -\frac{5}{4}$

or $\quad 40g - 20f = -5.$ ....(vi)

Given line is $3x + 4y = -1$ ....(vii)

Since centre $(-g, -f)$ of circle (iii) lies on line (vii),

$\therefore \quad -3g - 4f = -1$ ....(viii)

Solving (vi) & (viii), we get $g = 0$, $f = \frac{1}{4}$

$\therefore \quad$ from (5), $c = -\frac{29}{4}$

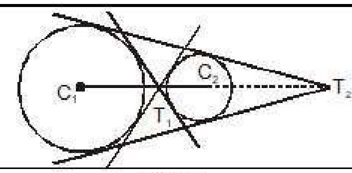
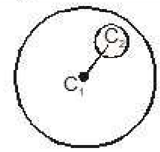
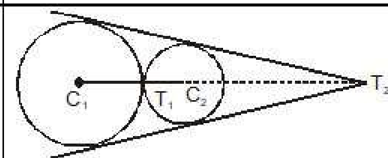
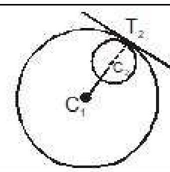
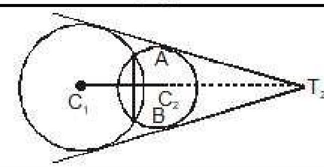
$\therefore \quad$ from (iii), required circle is

$x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$ or

$4(x^2 + y^2) + 2y - 29 = 0$

POSITION OF TWO CIRCLES

Let C_1 (h_1, k_1) and C_2 (h_2, k_2) be the centre of two circle and r_1, r_2 be their radius then

	CONDITION	POSITION	DIAGRAM	NO. OF COMMON TANGENTS
(i)	$C_1C_2 > r_1 + r_2$	do not intersect or one outside the other		4
(ii)	$C_1C_2 < \lvert r_1 - r_2 \rvert$	one inside the other		0
(iii)	$C_1C_2 = r_1 + r_2$	external touch		3
(iv)	$C_1C_2 = \lvert r_1 - r_2 \rvert$	internal touch		1
(v)	$\lvert r_1 - r_2 \rvert < C_1C_2 < r_1 + r_2$	intersection at two real points		2

Point of intersection of common tangents :

The points T_1 and T_2 (points of intersection of indirect and direct common tangents) divide C_1C_2 internally and externally in the ratio $r_1 : r_2$.

Equation of the common tangent at point of contact : $S_1 - S_2 = 0$

Point of contact : The point of contact C_1C_2 in the ratio $r_1 : r_2$ internally or externally as the case may be.

Notes :

(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

(ii) Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by: $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$,

where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

Solved Examples

Ex.42 Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Sol. Given circles are $x^2 + y^2 - 2x - 4y = 0$ ....(i)

and $x^2 + y^2 - 8y - 4 = 0$ ....(ii)

Let A and B be the centres and r_1 and r_2 the radii of circles (i) and (ii) respectively, then

$A \equiv (1, 2)$, $B \equiv (0, 4)$, $r_1 = \sqrt{5}$, $r_2 = 2\sqrt{5}$

Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and

$r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = \sqrt{5}$

Thus $AB = |r_1 - r_2|$, hence the two circles touch each other internally.

Ex.43 A circle with radius 5 touches another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at point (5, 5). Find its eqaution.

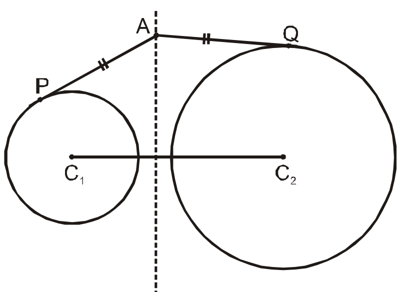
Sol. The centre of the given circle $C_1 \equiv (1, 2)$ and radius $= \sqrt{1+4=20} = 5$. Since the radii of two circles are equal so they touch externally. If $C_2(h, k)$ be the centre of the required circle then the point of contact (5, 5) is the mid point of C_1C_2. Hence $C_2 \equiv (9, 8)$ and the reqd. eq^n will be $(x - 9)^2 + (y - 8)^2 = 25$

$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0$ a

Radical axis and radical centre:

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given by

$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.



The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

Notes :

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other, then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

(d) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(e) Radical axis bisects a common tangent between the two circles.

(f) A system of circles, every two which have the same radical axis, is called a coaxial system.

(g) Pairs of circles which do not have radical axis are concentric.

Solved Examples

Ex.44 Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$$3x^2 + 3y^2 + 4x - 6y - 1 = 0$$
$$2x^2 + 2y^2 - 3x - 2y - 4 = 0$$
$$2x^2 + 2y^2 - x + y - 1 = 0$$

Sol. Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of x^2 and y^2 be each unity. Subtracting in pairs the three radical axes are

$$\frac{17}{6}x - y + \frac{5}{3} = 0 \ ; \qquad -x - \frac{3}{2}y - \frac{3}{2} = 0$$

$$-\frac{11}{6}x + \frac{5}{2}y - \frac{1}{6} = 0.$$

solving any two, we get the point $\left(-\frac{16}{21} - \frac{31}{63}\right)$ which satisfies the third also. This point is

Ex.45 Find the equation opf the radical axis of the circles $2x^2 + 2y^2 - 7x = 0$ and $x^2 + y^2 - 4y - 7 = 0$.

Sol. The equations of the given circles may be written as

$2x^2 + 2y^2 - 7x = 0$ and $2x^2 + 2y^2 - 8y - 4 = 0$

The equation of their radical axis is given by

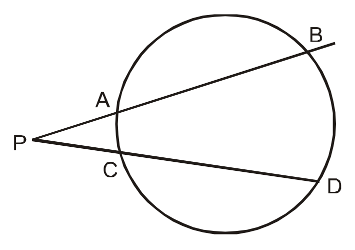
$S - S' = 0$

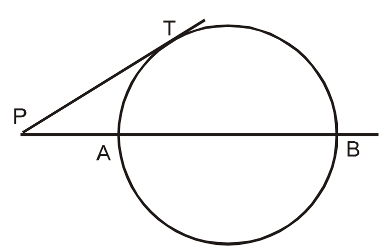
$\Rightarrow (2x^2 + 2y^2 - 7x) - (2x^2 + 2y^2 - 8y - 14) = 0$

$\Rightarrow 7x - 8y - 14 = 0$

SOME IMPORTANT RESULTS

* If the line $lx + my + n = 0$ is a tangent to the circle $x^2 + y^2 = a^2$, then $a^2 (l^2 + m^2) = n^2$.
* If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is a point circle then $g^2 + f^2 = c$.
* If the radius of the given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ be r and it touches both the axes then $g = f = \sqrt{c} = r$.
* The length of the tangent drawn from any point on the circle $x_2 + y_2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{c - c_1}$.
* If the circles $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{c^2}$.
* If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches x-axis and y-axis, then $g^2 = c$ and $f^2 = c$ respectively.
* The length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$
* The length of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\frac{2ab}{\sqrt{a^2 + b^2}}$.
* If two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other, then $g^2 + f^2 = 2c$.
* If the line $y = mx + c$ is a normal to the circle with radius r and centre at (a, b), then $b = ma + c$.
* If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinates axes at the points A and B and O is the origin, then the area of the triangle OAB is $\frac{r^4}{2ab}$.
* If θ is the angle subtended at $P(x_1, y_1)$ by the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, then
$$\cot\frac{\theta}{2} = \frac{\sqrt{S_1}}{\sqrt{g^2 + f^2 - c}}$$
* If the line $lx + my + n = 0$ is a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$, then $(hl + km + n)^2 = a^2 (l^2 + m^2)$.
* The length of the chord intercepted by the circle $x^2 + y^2 = r^2$ on the line $\frac{x}{a} + \frac{y}{b} = 1$ is $2\sqrt{\left(\frac{r^2(a^2 + b^2) - a^2b^2}{a^2 + b^2}\right)}$
* The distance between the chord of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\frac{1}{2}\frac{g^2 + f^2 - c}{\sqrt{g^2 + f^2}}$.
* The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2 + \beta^2 - a^2}}\right)$.
* If lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ cut the axes at concyclic points, then $l_1l_2 = m_1m_2$.
* The area of the triangle formed by the tangents from the points (h, k) to the circle $x^2 + y^2 = a^2$ and their chord of contact is $\frac{a}{h^2 + k^2}(h^2 + k^2 - a^2)^{3/2}$.
* If O is the origin and OP, OQ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the circumcentre of the triangle OPQ is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$.
* If chord of a circle AB and CD meet at some point P, then PA . PB = PC . PD and if AB chord and tangent at T meet at P, then PA . PB = PT^2.





* If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of the circle then the area of the quadrilateral OABC is $\sqrt{c(g^2 + f^2 - c)}$.