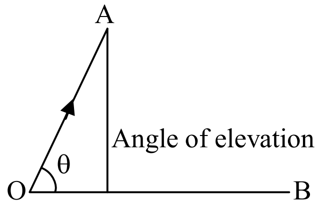
APPLICATION OF TRIGONOMETRY

CONTENTS

- Angle of Elevation
- Angle of Depression

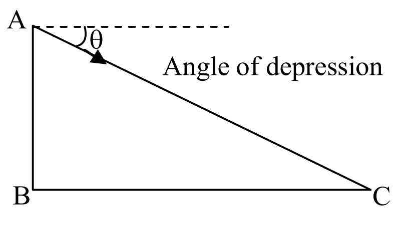
ANGLE OF ELEVATION

The angle of elevation of the point viewed is the angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level, i.e. the case when we raise our head to look at the object. (see fig.)



ANGLE OF DEPRESSION

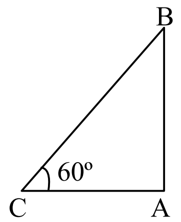
The angle of depression of a point on the object being viewed is the angle formed by the line of sight with the horizontal when the point is below the horizontal level, i.e. the case when we lower our head to look at the point being viewed. (See fig.)



❖ EXAMPLES ❖

Ex.1 The shadow of a building is 20 m long when the angle of elevation of the sun is 60º. Find the height of the building.

Sol. Let AB be the building and AC be its shadow.



Then, AC = 20 m and $\angle ACB = 60^\circ$.

Let AB = h.

Then, $\frac{AB}{AC} = \tan 60^\circ = \sqrt{3}$

$\Rightarrow \frac{h}{20} = \sqrt{3}$

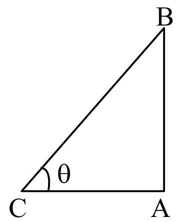
$\therefore$ $h = (20 \times \sqrt{3})m = (20 \times 1.732)$ m

= 34.64 m.

Ex.2 If a vertical pole 6m high has a shadow of length $2\sqrt{3}$ metres, find the angle of elevation of the sun.

Sol. Let AB be the vertical pole and AC be its shadow.

Let the angle of elevation be θ. Then,



AB = 6 m, AC = $2\sqrt{3}$ m

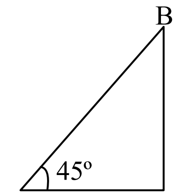
and $\angle ACB = \theta$.

Now, $\tan\theta = \dfrac{AB}{AC} = \dfrac{6}{2\sqrt{3}} = \sqrt{3} = \tan 60^\circ$.

$\therefore\ \theta = 60^\circ$.

Ex.3 A ladder against a vertical wall makes an angle of 45º with the ground. The foot of the ladder is 3m from the wall. Find the length of the ladder.

Sol. Let AB be the wall and CB, the ladder.



Then, AC = 3m and $\angle ACB = 45^\circ$

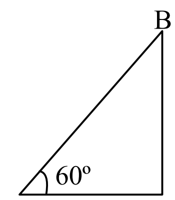
Now, $\dfrac{CB}{AC} = \sec 45^\circ = \sqrt{2} \Rightarrow \dfrac{CB}{3} = \sqrt{2}$

$\therefore$ Length of the ladder = CB = $3\sqrt{2}$

= (3 × 1.41) m = 4.23 m

Ex.4 A balloon is connected to a meteorological station by a cable of length 200 m, inclined at 60º to the horizontal. Find the height of the balloon from the ground. Assume that there is no slack in the cable.

Sol. Let B be the balloon and AB be the vertical height. Let C be the meteorological station and CB be the cable.



Then, BC = 200 m and $\angle ACB = 60^\circ$

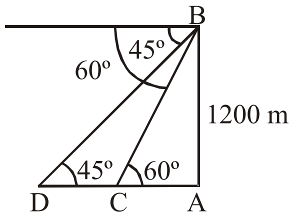
Then, $\dfrac{AB}{BC} = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \dfrac{AB}{200} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow AB = \left(\dfrac{200 \times \sqrt{3}}{2}\right) m = 173.2$ m.

Ex.5 The pilot of a helicopter, at an altitude of 1200m finds that the two ships are sailing towards it in the same direction. The angle of depression of the ships as observed from the helicopter are 60º and 45º respectively. Find the distance between the two ships.

Sol. Let B the position of the helicopter and let C, D be the ships. Let AB be the vertical height.



Then, AB = 1200 m,

$\angle ACB = 60^\circ$ and $\angle ADB = 45^\circ$.

Then, $\dfrac{AD}{AB} = \cot 45^\circ = 1$

$\Rightarrow \dfrac{AD}{1200} = 1 \Rightarrow AD = 1200$ m

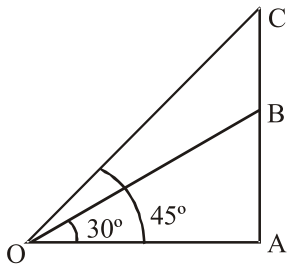
And, $\dfrac{AC}{AB} = \cot 60^\circ = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{AC}{1200} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow AC = \dfrac{1200}{\sqrt{3}} = 400\sqrt{3}$ m.

Ex.6 A vertical tower stands on a horizontal plane and is surmounted by a flagstaff of height 7m. At a point on the plane, the angle of elevation of the bottom of the flagstaff is 30º and that of the top of the flagstaff is 45º. Find the height of the tower.

Sol. Let AB be the tower and BC be the flagstaff.



Then, BC = 7 m. Let AB = h.

Let O be the point of observation.

Then, $\angle AOB = 30^\circ$ and $\angle AOC = 45^\circ$.

Now, $\frac{OA}{AC} = \cot 45^\circ = 1$

$\Rightarrow$ OA = AC = h + 7.

And, $\frac{OA}{AB} = \cot 30^\circ = \sqrt{3}$

$\Rightarrow \frac{OA}{h} = \sqrt{3} \Rightarrow OA = h\sqrt{3}$

$\therefore$ $h + 7 = h\sqrt{3}$

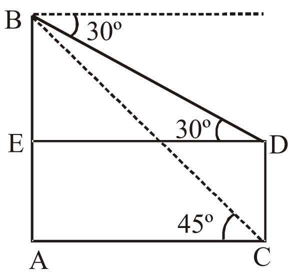
$$\Rightarrow h = \frac{7}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{7(\sqrt{3}+1)}{2} = 9.562 \text{ m}$$

Ex.7 From the top of a building 30 m high, the top and bottom of a tower are observed to have angles of depression 30° and 45° respectively. The height of the tower is :

(a) $15(1+\sqrt{3})$ m (b) $30(\sqrt{3}-1)$ m

(c) $30\left(1+\frac{1}{\sqrt{3}}\right)$ m (d) $30\left(1-\frac{1}{\sqrt{3}}\right)$ m

Sol. Let AB be the building and CD be the tower.



Then, AB = 30 m. Let DC = x.

Draw DE $\perp$ AB. Then AE = CD = x.

$\therefore$ BE = (30 – x) m.

Now, $\frac{AC}{AB} = \cot 45^\circ = 1$

$\Rightarrow \frac{AC}{30} = 1 \Rightarrow AC = 30$ m.

$\therefore$ DE = AC = 30 m.

$\frac{BE}{DE} = \tan 30^\circ = \frac{1}{\sqrt{3}} \Rightarrow \frac{BE}{30} = \frac{1}{\sqrt{3}}$

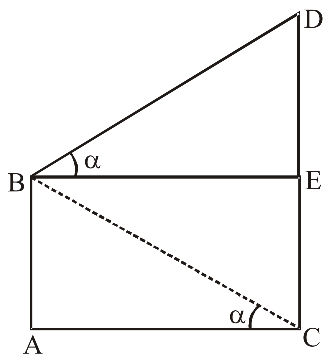
$\Rightarrow BE = \frac{30}{\sqrt{3}}$.

$\therefore$ $CD = AE = AB - BE = \left(30 - \frac{30}{\sqrt{3}}\right)$

$= 30\left(1-\frac{1}{\sqrt{3}}\right)$m

Ex.8 From the top of a cliff 25 m high the angle of elevation of a tower is found to be equal to the angle of depression of the foot of the tower. Find the height of the tower.

Sol. Let AB be the cliff and CD be the tower.



Then, AB = 25 m. From B draw BE $\perp$ CD.

Let $\angle EBD = \angle ACB = \alpha$.

Now, $\frac{DE}{BE} = \tan\alpha$ and $\frac{AB}{AC} = \tan\alpha$

$\therefore \frac{DE}{BE} = \frac{AB}{AC}$. So, DE = AB

[$\because$ BE = AC]

$\therefore$ CD = CE + DE = AB + AB = 2AB = 50m

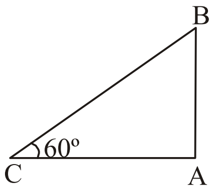
Ex.9 The altitude of the sun at any instant is 60º. The height of the vertical pole that will cast a shadow of 30 m is

(A) $30\sqrt{3}$ m (B) 15 m

(C) $\frac{30}{\sqrt{3}}$ m (D) $15\sqrt{2}$ m

Sol. Let AB be the pole and AC be its shadow.

Then, θ = 60º and AC = 30 m.



$\therefore \ \frac{AB}{AC} = \tan 60^\circ$

$\Rightarrow \ \frac{AB}{30} = \sqrt{3} \Rightarrow AB = 30\sqrt{3}$ m

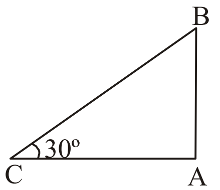
Ex.10 When the sun is 30º above the horizontal, the length of shadow cast by a building 50m high is-

(A) $\frac{50}{\sqrt{3}}$ m (B) $50\sqrt{3}$ m

(C) 25 m (D) $25\sqrt{3}$ m

Sol. Let AB be the building and AC be its shadow.

Then, AB = 50 m and θ = 30º.



$\therefore \ \frac{AC}{AB} = \cot 30^\circ = \sqrt{3}$

$\Rightarrow \frac{AC}{50} = \sqrt{3}$

$\Rightarrow$ AC = $50\sqrt{3}$ cm.

Ex.11 If the elevation of the sun changed from 30º to 60º, then the difference between the lengths of shadows of a pole 15 m high, made at these two positions, is–

(A) 7.5 m (B) 15 m

(C) $10\sqrt{3}$ m (D) $\frac{15}{\sqrt{3}}$ m

Sol. When AB = 15m, θ = 30º, then $\frac{AC}{AB} = \tan 30^\circ$

$\Rightarrow$ AC = $\frac{15}{\sqrt{3}}$ m.

When AB = 15m, θ = 60º, then $\frac{AC}{AB} = \tan 60^\circ$

$\Rightarrow$ AC = $15\sqrt{3}$ m

$\therefore$ Diff. in lengths of shadows

$= \left(15\sqrt{3} - \frac{15}{\sqrt{3}}\right)$

$= \frac{30}{\sqrt{3}} = 10\sqrt{3}$ m.

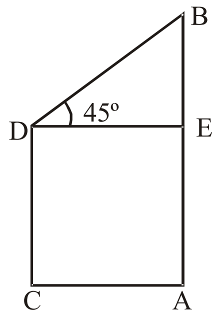
Ex.12 The heights of two poles are 80 m and 62.5 m. If the line joining their tops makes an angle of 45º with the horizontal, then the distance between the poles, is -

(A) 17.5 m (B) 56.4 m

(C) 12.33 m (D) 44 m

Sol. Let AB and CD be the poles such that

AB = 80 m and CD = 62.5 m.



Draw DE ⊥ AB. Then,

$\angle EDB = 45^\circ$

Now, BE = AB – AE = AB – CD = 17.5

$$\frac{DE}{BE} = \cot 45^\circ = 1$$

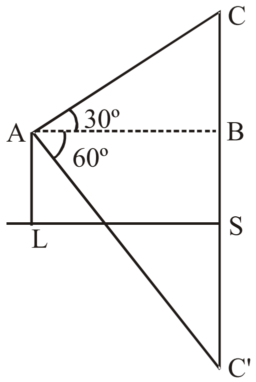
$\Rightarrow$ DE = BE = 17.5 m.

Ex.13 If the angle of elevation of cloud from a point 200 m above a lake is 30º and the angle of depression of its reflection in the lake is 60º, then the height of the cloud above the lake, is

(A) 200 m (B) 500 m

(C) 30 m (D) None of these

Sol. Let C be the cloud and C′ be its reflection in the lake.



Let CS = C′S = x.

Now, $\frac{BC}{AB} = \tan 30^\circ = \frac{1}{\sqrt{3}}$

$\Rightarrow$ $x - 200 = \frac{AB}{\sqrt{3}}$

Also, $\frac{BC'}{AB} = \tan 60^\circ = \sqrt{3}$

$\Rightarrow$ $x + 200 = (AB)\sqrt{3}$.

$\therefore$ $\sqrt{3}(x - 200) = \frac{x + 200}{\sqrt{3}}$ or x = 400.

$\therefore$ CS = 400 m.

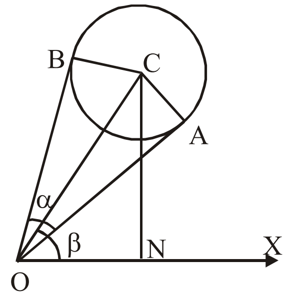
Ex.14 A balloon of radius γ makes an angle α at the eye of an observer and the angle of elevation of its centre is β. The height of its centre from the ground level is given by :

(A) $\gamma \cos\frac{\beta}{2}\sec\alpha$ (B) $\gamma \cos\beta\sec\frac{\alpha}{2}$

(C) $\gamma \sin\frac{\beta}{2}\operatorname{cosec}\alpha$ (D) $\gamma \sin\beta\operatorname{cosec}\frac{\alpha}{2}$

Sol. Let C be the centre of the balloon and O be the position of the observer at the horizontal line OX. Let OA and OB be the tangents to the balloon so that $\angle AOB = \alpha$, $\angle XOC = \beta$ and CA = CB = γ.

Clearly, right angled triangles OAC and OBC are congruent.



$\therefore$ $\angle AOC = \angle BOC = \frac{\alpha}{2}$.

Let CN $\perp$ OX.

Now, $\frac{OC}{CA} = \operatorname{cosec}\frac{\alpha}{2}$

$\Rightarrow$ $OC = \gamma \operatorname{cosec}\frac{\alpha}{2}$ ....(i)

Also, $\frac{CN}{OC} = \sin\beta$

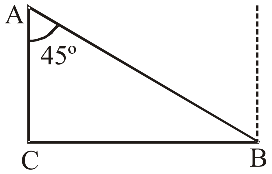
$\Rightarrow$ $CN = OC\sin\beta = \gamma\operatorname{cosec}\frac{\alpha}{2}\sin\beta$ [Using(i)]

Ex.15 The banks of a river are parallel. A swimmer starts from a point on one of the banks and swims in a straight line inclined to the bank at 45º and reaches the opposite bank at a point 20 m from the point opposite to the starting point. The breadth of the river is -

(A) 20 m (B) 28.28 m

(C) 14.14 m (D) 40 m

Sol. Let A be the starting point and B, the end point of the swimmer. Then AB = 20 m and $\angle BAC = 45^\circ$.



Now, $\frac{BC}{AB} = \sin 45° = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{BC}{20} = \frac{1}{\sqrt{2}}$

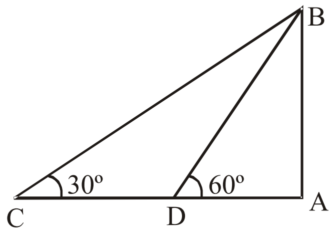
$\Rightarrow BC = \frac{20 \times \sqrt{2}}{2} = 14.14$ m.

Ex.16 A man on a cliff observes a fishing trawler at an angle of depression of 30º which is approaching the shore to the point immediately beneath the observer with a uniform speed. 6 minutes later, the angle of depression of the trawler is found to be 60º. The time taken by the trawler to reach the shore is -

(A) $3\sqrt{3}$ min (B) $\sqrt{3}$ min

(C) 1.5 min (D) 3 min

Sol. Let AB be the cliff and C and D be the two positions of the fishing trawler.



Then, ∠ACB = 30º and ∠ADB = 60º

Let AB = h.

Now, $\frac{AD}{AB} = \cot 60° = \frac{1}{\sqrt{3}}$

$\Rightarrow AD = \frac{h}{\sqrt{3}}$.

And, $\frac{AC}{AB} = \cot 30° = \sqrt{3}$

$\Rightarrow AC = \sqrt{3}\,h$

$CD = AC - AD = \left(\sqrt{3}\,h - \frac{h}{\sqrt{3}}\right) = \frac{2h}{\sqrt{3}}$

Let u m/min be the uniform speed of the trawler.

Distance covered in 6 min = 6u metres.

$\therefore CD = 6u \Rightarrow \frac{2h}{\sqrt{3}} = 6u \Rightarrow h = 3\sqrt{3}\,u$

Now, $AD = \frac{h}{\sqrt{3}} = \frac{3\sqrt{3}\,u}{\sqrt{3}} = 3u$.

Time taken by trawler to reach A

$= \frac{\text{distance AD}}{\text{speed}} \Rightarrow A = \frac{3u}{u} = 3$ min

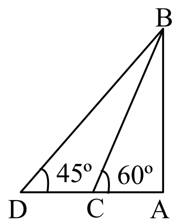
Ex.17 A boat is being rowed away from a cliff 150m high. At the top of the cliff the angle of depression of the boat changes from 60º to 45º in 2 minutes. The speed of the boat is –

(A) 2 km/hr (B) 1.9 km/hr

(C) 2.4 km/hr (D) 3 km/hr

Sol. Let AB be the cliff and C and D be the two positions of the ship. Then, AB = 150 m,

∠ACB = 60º and ∠ADB = 45º.



Now, $\frac{AD}{AB} = \cot 45° = 1$

$\Rightarrow \frac{AD}{150} = 1 \Rightarrow AD = 150$ m.

$\frac{AC}{AB} = \cot 60° = \frac{1}{\sqrt{3}} \Rightarrow \frac{AC}{150} = \frac{1}{\sqrt{3}}$

$\Rightarrow AC = \frac{150}{\sqrt{3}} = 50\sqrt{3} = 86.6$ m.

$\therefore$ CD = AD – AC = (150 – 86.6) m = 63.4 m

Thus, distance covered in 2 min. = 63.4 m

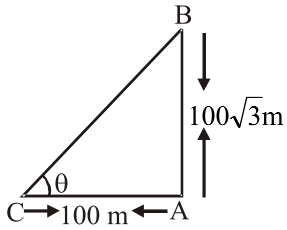
$\therefore$ Speed of the boat

$= \left(\frac{63.4}{2} \times \frac{60}{1000}\right)$ km/hr. = 1.9 km/hr.

Ex.18 A tower is $100\sqrt{3}$ metres high. Find the angle of elevation of its top from a point 100 metres away from its foot.

Sol. Let AB be the tower of height $100\sqrt{3}$ metres, and let C be a point at a distance of 100 metres from the foot of the tower.

Let θ be the angle of elevation of the top of the tower from point C.



In ΔCAB, we have

$$\tan\theta = \frac{AB}{AC}$$

$$\Rightarrow \tan\theta = \frac{100\sqrt{3}}{100} = \sqrt{3}$$

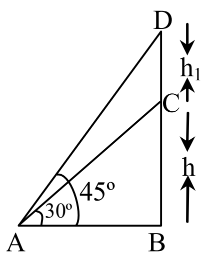
$$\Rightarrow \theta = 60^\circ$$

Hence, the angle of elevation of the top of the tower from a point 100 metres away from its foot is 60º.

Ex.19 From a point on the ground 40 m away from the foot of a tower, the angle of elevation of the top of the tower is 30º. The angle of elevation of the top of a water tank (on the top of the tower) is 45º. Find the (i) height of the tower (ii) the depth of the tank.

Sol. Let BC be the tower of height h metre and CD be the water tank of height h_1 metre.

Let A be a point on the ground at a distance of 40 m away from the foot B of the tower.



In ΔABD, we have $\tan 45^\circ = \frac{BD}{AB}$

$$\Rightarrow 1 = \frac{h + h_1}{40} \Rightarrow h + h_1 = 40 \text{ m} \quad ....(i)$$

In ΔABC, we have

$$\tan 30^\circ = \frac{BC}{AB} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{40}$$

$$\Rightarrow h = \frac{40}{\sqrt{3}} \text{ m} = \frac{40\sqrt{3}}{3} \text{ m} = 23.1 \text{ m}$$

Substituting the value of h in (i), we get

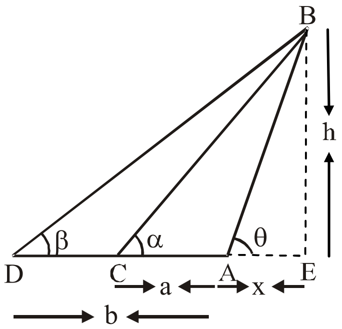
$$23.1 + h_1 = 40$$

$$\Rightarrow h_1 = (40 - 23.1)\text{m} = 16.9 \text{ m}$$

Ex.20 Two stations due south of a leaning tower which leans towards the north are at distance a and b from its foot. If α, β be the elevations of the top of the tower from these stations, prove that its inclination θ to the horizontal is given by

$$\cot\theta = \frac{b\cot\alpha - a\cot\beta}{b - a}$$

Sol. Let AB be the leaning tower and let C and D be two given stations at distances a and b respectively from the foot A of the tower.



Let AE = x and BE = h

In ΔABE, we have

$$\tan\theta = \frac{BE}{AE} \quad \Rightarrow \tan\theta = \frac{h}{x}$$

$$\Rightarrow x = h\cot\theta \quad ....(i)$$

In ΔCBE, we have

$$\tan\alpha = \frac{BE}{CE}$$

$\Rightarrow$ $\tan\alpha = \dfrac{h}{a+x}$

$\Rightarrow$ $a + x = h\cot\alpha$

$\Rightarrow$ $x = h\cot\alpha - a$ ....(ii)

In Δ DBE, we have

$\tan\beta = \dfrac{BE}{DE}$

$\Rightarrow$ $\tan\beta = \dfrac{h}{b+x}$

$\Rightarrow$ $b + x = h\cot\beta$

$\Rightarrow$ $x = h\cot\beta - b$ ....(iii)

From equations (i) and (ii), we have

$h\cot\theta = h\cot\alpha - a$

$\Rightarrow$ $h(\cot\alpha - \cot\theta) = a$

$\Rightarrow$ $h = \dfrac{a}{\cot\alpha - \cot\theta}$ ....(iv)

From equation (i) and (iii), we get

$h\cot\theta = h\cot\beta - b$

$\Rightarrow$ $h(\cot\beta - \cot\theta) = b$

$\Rightarrow$ $h = \dfrac{b}{\cot\beta - \cot\theta}$

Equating the values of h from equations (iv) and (v), we get

$$\frac{a}{\cot\alpha - \cot\theta} = \frac{b}{\cot\beta - \cot\theta}$$

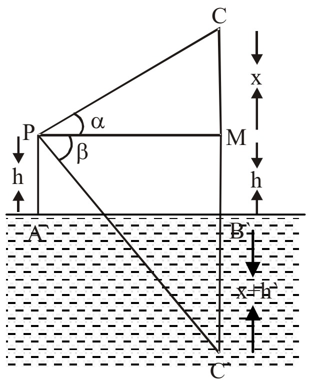
$\Rightarrow$ $a(\cot\beta - \cot\theta) = b(\cot\alpha - \cot\theta)$

$\Rightarrow$ $(b - a)\cot\theta = b\cot\alpha - a\cot\beta$

$\Rightarrow$ $\cot\theta = \dfrac{b\cot\alpha - a\cot\beta}{b-a}$

Ex.21 If the angle of elevation of a cloud from a point h metres above a lake is α and the angle of depression of its reflection in the lake is β, prove that the height of the cloud is $\dfrac{h(\tan\beta + \tan\alpha)}{\tan\beta - \tan\alpha}$.

Sol. Let AB be the surface of the lake and let P be a point of observation such that AP = h metres. Let C be the position of the cloud and C′ be its reflection in the lake. Then, CB = C′B. Let PM be perpendicular from P on CB. Then, $\angle CPM = \alpha$ and $\angle MPC' = \beta$. Let CM = x.



Then, CB = CM + MB = CM + PA = x + h.

In ΔCPM, we have

$\tan\alpha = \dfrac{CM}{PM}$

$\Rightarrow$ $\tan\alpha = \dfrac{x}{AB}$ [$\because$ PM = AB]

$\Rightarrow$ $AB = x\cot\alpha$ ....(i)

In Δ PMC′, we have

$\tan\beta = \dfrac{C'M}{PM}$

$\Rightarrow$ $\tan\beta = \dfrac{x+2h}{AB}$

[$\because$ C′M = C′B + BM = x + h + h]

$\Rightarrow$ $AB = (x + 2h)\cot\beta$ ....(ii)

From (i) and (ii), we have

$x\cot\alpha = (x + 2h)\cot\beta$

$\Rightarrow$ $x(\cot\alpha - \cot\beta) = 2h\cot\beta$

$\Rightarrow$ $x\left(\dfrac{1}{\tan\alpha} - \dfrac{1}{\tan\beta}\right) = \dfrac{2h}{\tan\beta}$

$\Rightarrow$ $x\left(\dfrac{\tan\beta - \tan\alpha}{\tan\alpha\tan\beta}\right) = \dfrac{2h}{\tan\beta}$

$\Rightarrow$ $x = \dfrac{2h\tan\alpha}{\tan\beta - \tan\alpha}$

Hence,

Height of the cloud = x + h

$$= \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} + h$$

$$= \frac{2h\tan\alpha + h\tan\beta - h\tan\alpha}{\tan\beta - \tan\alpha}$$

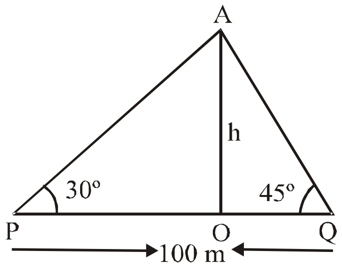
$$= \frac{h(\tan\alpha + \tan\beta)}{\tan\beta - \tan\alpha}$$

Ex.22 There is a small island in the middle of a 100 m wide river and a tall tree stands on the island. P and Q are points directly opposite to each other on two banks, and in line with the tree. If the angles of elevation of the top of the tree from P and Q are respectively 30º and 45º, find the height of the tree.

Sol. Let OA be the tree of height h metre.

In triangles POA and QOA, we have

$$\tan 30^\circ = \frac{OA}{OP} \text{ and } \tan 45^\circ = \frac{OA}{OQ}$$



$\Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{h}{OP}$ and $1 = \dfrac{h}{OQ}$

$\Rightarrow$ OP = $\sqrt{3}$ h and OQ = h

$\Rightarrow$ OP + OQ = $\sqrt{3}$ h + h $\Rightarrow$ PQ = ($\sqrt{3}$ + 1)h

$\Rightarrow$ 100 = ($\sqrt{3}$ + 1)h [$\because$ PQ = 100 m]

$\Rightarrow h = \dfrac{100}{\sqrt{3}+1}$ m $\Rightarrow h = \dfrac{100(\sqrt{3}-1)}{2}$ m

$\Rightarrow$ h = 50(1.732 – 1) m = 36.6 m

Hence, the height of the tree is 36.6 m

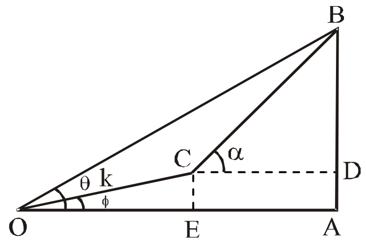
Ex.23 The angle of elevation of a cliff from a fixed point is θ. After going up a distance of k metres towards the top of cliff at an angle of φ, it is found that the angle of elevation is α. Show that the height of the cliff is

$\dfrac{k(\cos\phi - \sin\phi\cot\alpha)}{\cot\theta - \cot\alpha}$ metres

Sol. Let AB be the cliff and O be the fixed point such that the angle of elevation of the cliff from O is θ i.e. ∠AOB = θ. Let ∠AOC = φ and OC = k metres. From C draw CD and CE perpendiculars on AB and OA respectively.

Then, ∠DCB = α.

Let h be the height of the cliff AB.



In ΔOCE, we have

$$\sin\phi = \frac{CE}{OC}$$

$\Rightarrow \sin\phi = \dfrac{CE}{k}$

$\Rightarrow$ CE = k sin φ ...(i) [$\because$CE = AD]

$\Rightarrow$ AD = k sin φ

And, $\cos\phi = \dfrac{OE}{OC}$

$\Rightarrow \cos\phi = \dfrac{OE}{k}$

$\Rightarrow$ OE = k cos φ ....(ii)

In Δ OAB, we have

$$\tan\theta = \frac{AB}{OA}$$

$\Rightarrow \tan\theta = \dfrac{h}{OA}$

$\Rightarrow$ OA = h cot θ ....(iii)

$\therefore$ CD = EA = OA – OE

= h cot θ – k cos φ .....(iv)

[Using eqs.(ii) and (iii)]

and, BD = AB – AD = AB – CE

$= (h - k \sin \phi)$ ....(v)

[Using equation (i)]

In ΔBCD, we have

$$\tan \alpha = \frac{BD}{CD} \Rightarrow \tan \alpha = \frac{h - k\sin\phi}{h\cot\theta - k\cos\phi}$$

[Using equations (iv) and (v)]

$$\Rightarrow \frac{1}{\cot\alpha} = \frac{h - k\sin\phi}{h\cot\theta - k\cos\phi}$$

$$\Rightarrow h \cot \alpha - k \sin \phi \cot \alpha = h \cot \theta - k \cos \phi$$

$$\Rightarrow h(\cot \theta - \cot \alpha) = k(\cos \phi - \sin \phi \cot \alpha)$$

$$\Rightarrow h = \frac{k(\cos\phi - \sin\phi\cot\alpha)}{\cot\theta - \cot\alpha}$$

Ex.24 At the foot of a mountain the elevation of its summit is 45º; after ascending 1000 m towards the mountain up a slope of 30º inclination is found to be 60º. Find the height of the mountain.

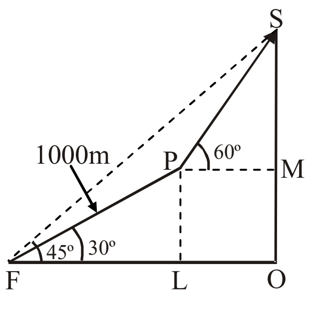
Sol. Let F be the foot and S be the summit of the mountain FOS. Then, $\angle OFS = 45^\circ$ and therefore $\angle OSF = 45^\circ$. Consequently,

OF = OS = h km(say).

Let FP = 1000 m = 1 km be the slope so that $\angle OFP = 30^\circ$. Draw $PM \perp OS$ and $PL \perp OF$.

Join PS. It is given that $\angle MPS = 60^\circ$.

In ΔFPL, We have



$$\sin 30^\circ = \frac{PL}{PF}$$

$$\Rightarrow PL = PF \sin 30^\circ = \left(1 \times \frac{1}{2}\right) \text{km.} = \frac{1}{2} \text{km.}$$

$$\therefore OM = PL = \frac{1}{2} \text{km}$$

$$\Rightarrow MS = OS - OM = \left(h - \frac{1}{2}\right) \text{km} \quad \text{….(i)}$$

Also, $\cos 30^\circ = \frac{FL}{PF}$

$$\Rightarrow FL = PF \cos 30^\circ = \left(1 \times \frac{\sqrt{3}}{2}\right) \text{km} = \frac{\sqrt{3}}{2} \text{km}$$

Now, h = OS = OF = OL + LF

$$\Rightarrow h = OL + \frac{\sqrt{3}}{2}$$

$$\Rightarrow OL = \left(h - \frac{\sqrt{3}}{2}\right) \text{km}$$

$$\Rightarrow PM = \left(h - \frac{\sqrt{3}}{2}\right) \text{km}$$

In ΔPSM, we have

$$\tan 60^\circ = \frac{SM}{PM}$$

$$\Rightarrow SM = PM. \tan 60^\circ \quad \text{…..(ii)}$$

$$\Rightarrow \left(h - \frac{1}{2}\right) = \left(h - \frac{\sqrt{3}}{2}\right)\sqrt{3}$$

[Using equations (i) and (ii)]

$$\Rightarrow h - \frac{1}{2} = h\sqrt{3} - \frac{3}{2}$$

$$\Rightarrow h(\sqrt{3} - 1) = 1$$

$$\Rightarrow h = \frac{1}{\sqrt{3} - 1}$$

$$\Rightarrow h = \frac{\sqrt{3} + 1}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{\sqrt{3} + 1}{2}$$

$$= \frac{2.732}{2} = 1.336 \text{ km}$$

Hence, the height of the mountain is 1.366 km.

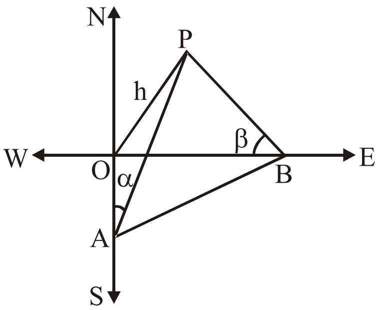
Ex.25 The angle of elevation of the top of a tower from a point A due south of the tower is α and from B due east of the tower is β. If AB = d, show that the height of the tower is $\frac{d}{\sqrt{\cot^2\alpha+\cot^2\beta}}$.

Sol. Let OP be the tower and let A and B be two points due south and east respectively of the tower such that ∠OAP = α and ∠OBP = β. Let OP = h. In ΔOAP, we have

$$\tan\alpha = \frac{h}{OA}$$

$$\Rightarrow\ OA = h\cot\alpha \qquad ....(i)$$

In Δ OBP, we have



$$\tan\beta = \frac{h}{OB}$$

$$\Rightarrow\ OB = h\cot\beta. \qquad ....(ii)$$

Since OAB is a right angled triangle. Therefore,

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow\ d^2 = h^2\cot^2\alpha + h^2\cot^2\beta$$

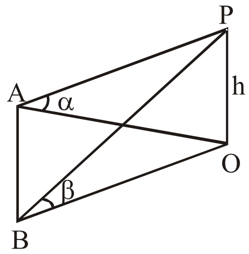
$$\Rightarrow\ h = \frac{d}{\sqrt{\cot^2\alpha+\cot^2\beta}}$$

[Using (i) and (ii)]

Ex.26 The elevation of a tower at a station A due north of it is α and at a station B due west of A is β. Prove that the height of the tower is $\frac{AB\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha-\sin^2\beta}}$.

Sol. Let OP be the tower and let A be a point due north of the tower OP and let B be the point due west of A. Such that ∠OAP = α and ∠OBP = β. Let h be the height of the tower.

In right angled triangles OAP and OBP, we have



$$\tan\alpha = \frac{h}{OA} \text{ and } \tan\beta = \frac{h}{OB}$$

$$\Rightarrow\ OA = h\cot\alpha \text{ and } OB = h\cot\beta.$$

In ΔOAB, we have

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow\ AB^2 = OB^2 - OA^2$$

$$\Rightarrow\ AB^2 = h^2\cot^2\beta - h^2\cot^2\alpha$$

$$\Rightarrow\ AB^2 = h^2[\cot^2\beta - \cot^2\alpha]$$

$$\Rightarrow\ AB^2 = h^2[(\text{cosec}^2\beta - 1) - (\text{cosec}^2\alpha - 1)]$$

$$\Rightarrow\ AB^2 = h^2(\text{cosec}^2\beta - \text{cosec}^2\alpha)$$

$$\Rightarrow\ AB^2 = h^2\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\sin^2\beta}\right)$$

$$\Rightarrow\ h = \frac{AB\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha-\sin^2\beta}}$$

Ex.27 An aeroplane when flying at a height of 4000m from the ground passes vertically above another aeroplane at an instant when the angles of the elevation of the two planes from the same point on the ground are 60º and 45º respectively. Find the vertical distance between the aeroplanes at that instant.

Sol. Let P and Q be the positions of two aeroplanes when Q is vertically below P and OP = 4000 m. Let the angles of elevation of P and Q at a point A on the ground be 60º and 45º respectively.

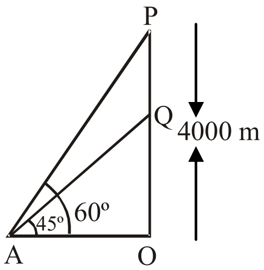
In triangles AOP and AOQ, we have

$$\tan 60^\circ = \frac{OP}{OA} \text{ and } \tan 45^\circ = \frac{OQ}{OA}$$

$$\Rightarrow\ \sqrt{3} = \frac{4000}{OA} \text{ and } 1 = \frac{OQ}{OA}$$

$\Rightarrow$ OA = $\dfrac{4000}{\sqrt{3}}$ and OQ = OA

$\Rightarrow$ OQ = $\dfrac{4000}{\sqrt{3}}$ m



$\therefore$ Vertical distance between the aeroplanes

= PQ = OP – OQ

$= \left(4000 - \dfrac{4000}{\sqrt{3}}\right) \text{m} = 4000\dfrac{(\sqrt{3}-1)}{\sqrt{3}}$ m

= 1690.53 m