QUADRILATERALS

IMPORTANT POINTS

- A quadrilateral is a figure bounded by four line segments such that no three of them are parallel.
- Two sides of quadrilateral are consecutive or adjacent sides, if they have a common point (vertex).
- Two sides of a quadrilateral are opposite sides, if they have no common end-point (vertex).
- The consecutive angles of a quadrilateral are two angles which include a side in their intersection. In other words, two angles are consecutive, if they have a common arm.
- Two angles of a quadrilateral are said to be opposite angles if they do not have a common arm.
- The sum of the four angles of a quadrilateral is 360^o.

❖ EXAMPLES ❖

Ex.1 In a quadrilateral ABCD, the angles A, B, C and D are in the ratio 2 : 4 : 5 : 7. Find the measure of each angles of the quadrilateral.

Sol. We have $\angle A : \angle B : \angle C : \angle D = 2 : 4 : 5 : 7$. So, let $\angle A = 2x^o$, $\angle B = 4x^o$, $\angle C = 5x^o$, $\angle D = 7x^o$.

$\therefore \quad \angle A + \angle B + \angle C + \angle D = 360^o$

$\Rightarrow \quad 2x + 4x + 5x + 7x = 360^o$

$\Rightarrow \quad 18x = 360^o$

$\Rightarrow x = 20^o$

Thus, the angles are :

$\angle A = 40^o$, $\angle B = (4 \times 20)^o = 80^o$,

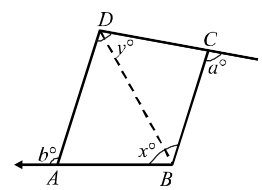
$\angle C = (5 \times 20)^o = 100^o$

and, $\angle D = (7x)^o = (7 \times 20)^o = 140^o$

Ex.2 The sides BA and DC of a quadrilateral ABCD are produced as shown in fig.

Prove that a + b = x + y.

Sol. Join BD. In ΔABD, we have



$\angle ABD + \angle ADB = b^o$ ....(i)

In ΔCBD, we have

$\angle CBD + \angle CDB = a^o$ ....(ii)

Adding (i) and (ii), we get

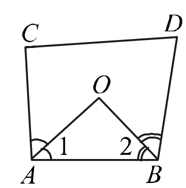
$(\angle ABD + \angle CBD) + (\angle ADB + \angle CDB) = a^o + b^o$

$\Rightarrow \quad x^o + y^o = a^o + b^o$

Hence, $\quad x + y = a + b$

Ex.3 In a quadrilateral ABCD, AO and BO are the bisectors of $\angle A$ and $\angle B$ respectively. Prove that $\angle AOB = \frac{1}{2}(\angle C + \angle D)$.

Sol. In ΔAOB, we have



$\angle AOB + \angle 1 + \angle 2 = 180^o$

$\Rightarrow \quad \angle AOB = 180^o - (\angle 1 + \angle 2)$

$\Rightarrow \quad \angle AOB = 180^o - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)$

$$\left[\because \angle 1 = \frac{1}{2}\angle A \text{ and } \angle 2 = \frac{1}{2}\angle B\right]$$

$\Rightarrow \angle AOB = 180^o - \frac{1}{2}(\angle A + \angle B)$

$\Rightarrow \angle AOB = 180^\circ - \frac{1}{2}[360^o - (\angle C + \angle D)]$

$[\because \angle A + \angle B + \angle C + \angle D = 360^o$

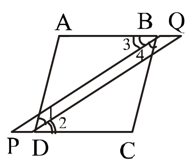
$\therefore \angle A + \angle B = 360^o - (\angle C + \angle D)]$

$\Rightarrow \angle AOB = 180^o - 180^o + \frac{1}{2}(\angle C + \angle D)$

$\Rightarrow \angle AOB = \frac{1}{2}(\angle C + \angle D)$

Ex.4 In figure bisectors of $\angle B$ and $\angle D$ of quadrilateral ABCD meet CD and AB produced at P and Q respectively. Prove that

$$\angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$$



Sol. In ΔPBC, we have

$\therefore \angle P + \angle 4 + \angle C = 180^o$

$\Rightarrow \angle P + \frac{1}{2}\angle B + \angle C = 180^o$ ...(i)

In ΔQAD, we have $\angle Q + \angle A + \angle 1 = 180^o$

$\Rightarrow \angle Q + \angle A + \frac{1}{2}\angle D = 180^o$ ...(ii)

Adding (i) and (ii), we get

$\angle P + \angle Q + \angle A + \angle C + \frac{1}{2}\angle B + \frac{1}{2}\angle D = 180^o + 180^o$

$\Rightarrow \angle P + \angle Q + \angle A + \angle C + \frac{1}{2}\angle B + \frac{1}{2}\angle D = 360^o$

$\Rightarrow \angle P + \angle Q + \angle A + \angle C + \frac{1}{2}(\angle B + \angle D) = \angle A + \angle B + \angle C + \angle D$

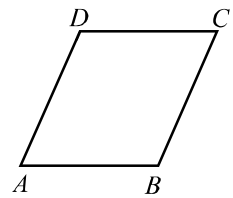
$[\therefore$ In a quadrilateral ABCD $\angle A + \angle B + \angle C + \angle D = 360^o]$

$\Rightarrow \angle P + \angle Q = \frac{1}{2}(\angle B + \angle D)$

$\Rightarrow \angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$

Ex.5 In a parallelogram ABCD, prove that sum of any two consecutive angles is 180^o.

Sol. Since ABCD is a parallelogram. Therefore, AD || BC.



Now, AD || BC and transversal AB intersects them at A and B respectively.

$\therefore \angle A + \angle B = 180^o$

[$\because$ Sum of the interior angles on the same side of the transversal is 180^o]

Similarly, we can prove that

$\angle B + \angle C = 180^o$, $\angle C + \angle D = 180^o$ and $\angle D + \angle A = 180^o$.

- A quadrilateral having exactly one pair of parallel sides, is called a trapezium.
- A trapezium is said to be an isoscels trapezium, if its non-parallel sides are equal.
- A quadrilateral is a parallelogram if its both pairs of opposite sides are parallel.
- A parallelogram having all sides equal is called a rhombus.
- A parallelogram whose each angle is a right angle, is called a rectangle.
- A square is a rectangle with a pair of adjacent sides equal.
- A quadrilateral is a kite if it has two pairs of equal adjacent sides and unequal opposite sides.
- A diagonal of a parallelogram divides it into two congruent triangles.
- In a parallelogram, opposite sides are equal.
- The opposite angles of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.
- In a parallelogram, the bisectors of any two consecutive angles intersect at right angle.
- If diagonal of a parallelogram bisects one of the angles of the parallelogram, it also bisects the second angle.
- The angle bisectors of a parallegram form a rectangle.

Ex.6 In a parallelogram ABCD, $\angle D = 115^\circ$, determine the measure of $\angle A$ and $\angle B$.

Sol. Since the sum of any two consecutive angles of a parallelogram is 180º. Therefore,

$$\angle A + \angle D = 180^\circ \text{ and } \angle A + \angle B = 180^\circ$$

Now, $\angle A + \angle D = 180^\circ$

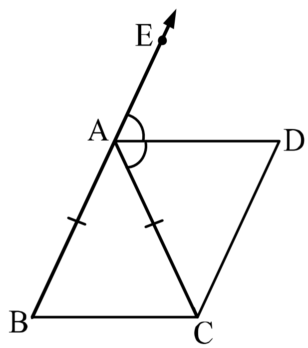
$\Rightarrow \angle A + 115^\circ = 180^\circ$ [$\because \angle D = 115^\circ$ (given)]

$\Rightarrow \angle A = 65^\circ$ and $\angle A + \angle B = 180^\circ$

$\Rightarrow 65^\circ + \angle B = 180^\circ \Rightarrow \angle B = 115^\circ$

Thus, $\angle A = 65^\circ$ and $\angle B = 115^\circ$

Ex.7 In figure, AB = AC, $\angle EAD = \angle CAD$ and CD || AB. Show that ABCD is a parallelogram.



Sol. In ΔABC, AB = AC [Given]

$\Rightarrow \angle ABC = \angle ACB$ ....(1)

(Angles opposite the equal sides are equal)

$\angle EAD = \angle CAD$ [Given] ....(2)

Now, $\angle EAC = \angle ABC + \angle ACB$

$\left(\begin{array}{l}\text{An exterior angle is equal to sum of}\\ \text{two interior opposite angles of a triangles}\end{array}\right)$

$\Rightarrow \angle EAD + \angle CAD = \angle ABC + \angle ACB$

$\Rightarrow \angle CAD + \angle CAD = \angle ACB + \angle ACB$

By (1) and (2)

$\Rightarrow 2\angle CAD = 2\angle ACB$

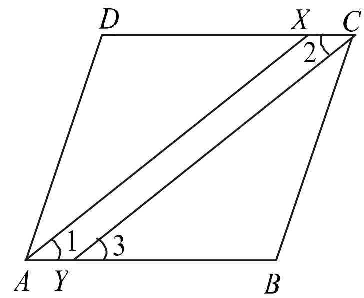
$\Rightarrow \angle CAD = \angle ACB$

$\Rightarrow$ BC | | AD

Also, CD | | AB [Given]

Thus, we have both pairs of opposite sides of quadrilateral ABCD parallel. Therefore, ABCD is a parallelogram.

Ex.8 ABCD is a parallelogram and line segments AX,CY are angle bisector of $\angle A$ and $\angle C$ respectively then show AX || CY.



Sol. Since opposite angles are equal in a parallelogram. Therefore, in parallelogram ABCD, we have $\angle A = \angle C$

$\Rightarrow \frac{1}{2}\angle A = \frac{1}{2}\angle C$

$\Rightarrow \angle 1 = \angle 2$ ....(i)

[$\because$ AX and CY are bisectors of $\angle A$ and $\angle C$ respectively]

Now, AB || DC and the transversal CY intersects them.

$\therefore \angle 2 = \angle 3$ ....(ii)

[$\because$ Alternate interior angles are equal]

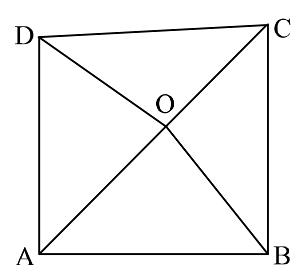
From (i) and (ii), we get

$\angle 1 = \angle 3$

Thus, transversal AB intersects AX and YC at A and Y such that $\angle 1 = \angle 3$ i.e. corresponding angles are equal.

$\therefore$ AX || CY

Ex.9 In the adjoining figure, a point O is taken inside an equilateral quad. ABCD such that OB = OD. Show that A, O and C are in the same straight line.



Sol. Given a quad. ABCD in which AB = BC = CD = DA and O is a point within it such that OB = OD.

To prove $\angle AOB + \angle COB = 180^o$

Proof In ΔOAB and OAD, we have

AB = AD (given), OA = OA

(common) and OB = OD (given)

$\therefore$ $\Delta OAB \cong \Delta OAD$

$\therefore$ $\angle AOB = \angle AOD$ ….(i) (c.p.c.t.)

Similarly, $\Delta OBC \cong \Delta ODC$

$\therefore$ $\angle COB = \angle COD$ ….(ii)

Now, $\angle AOB + \angle COB + \angle COD + \angle AOD$

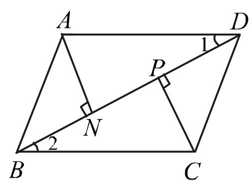
$= 360^o$ [$\angle$ at a point]

$\Rightarrow 2(\angle AOB + \angle COB) = 360^o$

$\Rightarrow \angle AOB + \angle COB = 180^o$

Ex.10 In figure AN and CP are perpendiculars to the diagonal BD of a parallelogram ABCD. Prove that :

(i) $\Delta ADN \cong \Delta CBP$ (ii) AN = CP



Sol. Since ABCD is a parallelogram.

$\therefore$ AD || BC

Now, AD || BC and transversal BD intersects them at B and D.

$\therefore$ $\angle 1 = \angle 2$

[$\because$ Alternate interior angles are equal]

Now, in Δs ADN and CBP, we have

$\angle 1 = \angle 2$

$\angle AND = \angle CPD$ and, AD = BC

[$\because$ Opposite sides of a $||^{gm}$ are equal]

So, by AAS criterion of congruence

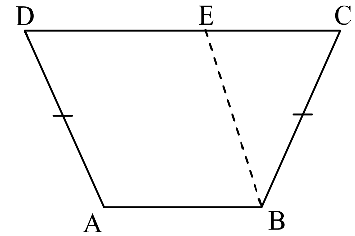
$\Delta ADN \cong \Delta CBP$

AN = CP

[$\because$Corresponding parts of congruent triangles are equal]

Ex.11 In figure, ABCD is a trapezium such that

AB || CD and AD = BC.



BE || AD and BE meets BC at E.

Show that (i) ABED is a parallelogram.

(ii) $\angle A + \angle C = \angle B + \angle D = 180^o$.

Sol. Here, AB || CD (Given)

$\Rightarrow$ AB || DE ….(1)

Also, BE || AD (Given) ….(2)

From (1) and (2),

ABED is a parallelogram

$\Rightarrow$ AD = BE ….(3)

Also, AD = BC (Given) ….(4)

From (3) and (4),

BE = BC

$\Rightarrow$ $\angle BEC = \angle BCE$ ….(5)

Also, $\angle BAD = \angle BED$

(opposite angles of parallelogram ABED)

i.e., $\angle BED = \angle BAD$ ….(6)

Now, $\angle BED + \angle BEC = 180^o$ (Linear pair of angles)

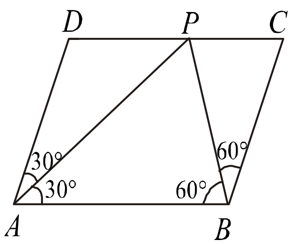
$\Rightarrow$ $\angle BAD + \angle BCE = 180^o$

By (5) and (6)

$\Rightarrow$ $\angle A + \angle C = 180^o$

Similarly, $\angle B + \angle D = 180^o$

Ex.12 In figure ABCD is a parallelogram and $\angle DAB = 60^\circ$. If the bisectors AP and BP of angles A and B respectively, meet at P on CD, prove that P is the mid-point of CD.



Sol. We have, $\angle DAB = 60^\circ$

$\angle A + \angle B = 180^\circ$

$\therefore \quad 60^\circ + \angle B = 180^\circ \Rightarrow \angle B = 120^\circ$

Now, AB || DC and transversal AP intersects them.

$\therefore \quad \angle PAB = \angle APD$

$\Rightarrow \quad \angle APD = 30^\circ \qquad [\because \angle PAB = 30^\circ]$

Thus, in ΔAPD, we have

$\angle PAD = \angle APD$ [Each equal to 30°]

$\Rightarrow \quad AD = PD \qquad \text{.... (i)}$

[$\because$ Angles opposite to equal sides are equal]

Since BP is the bisector of $\angle B$. Therefore,

$\angle ABP = \angle PBC = 60^\circ$

Now, AB || DC and transversal BP intersects them.

$\therefore \quad \angle CPB = \angle ABP$

$\Rightarrow \quad \angle CPB = 60^\circ \qquad [\because \angle ABP = 60^\circ]$

Thus, in ΔCBP, we have

$\angle CBP = \angle CPB$ [Each equal to 60°]

$\Rightarrow \quad CP = BC$

$\because$ [Sides opp, to equal angles are equal]

$\Rightarrow \quad CP = AD \qquad \text{.... (ii)}$

[$\because$ ABCD is a $\|^{gm}$ $\therefore$ AD = BC]

From (i) and (ii), we get

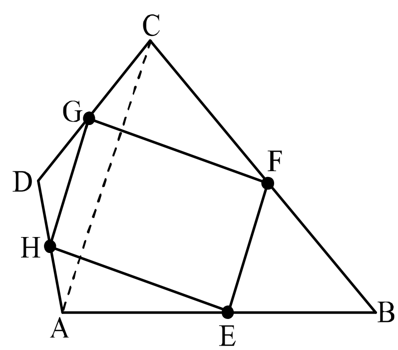
PD = CP

$\Rightarrow$ P is the mid point of CD.

- A quadrilateral is a parallelogam if its opposite sides are equal.
- A quadrilateral is a parallelogram if its opposite angles are equal.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
- A quadrilateral is a parallelogram, if its one pair of opposite sides are equal and parallel.

Ex.13 Prove that the line segments joining the mid-point of the sides of a quadrilateral forms a parallelogram.

Sol. Points E, F, G and H are the mid-points of the sides AB, BC, CD and DA respectively, of the quadrilateral ABCD. We have to prove that EFGH is a parallelogram.



Join the diagonal AC of the quadrilateral ABCD.

Now, in ΔABC, we have E and F mid-points of the sides BA and BC.

$\Rightarrow \quad EF \parallel AC$

and $\quad EF = \frac{1}{2} AC \qquad \text{.... (1)}$

Similarly, from ΔADC, we have

$GH \parallel AC$

and $\quad GH = \frac{1}{2} AC \qquad \text{....(2)}$

Then from (1) and (2), we have

$EF \parallel GH$

and $\quad EF = GH$

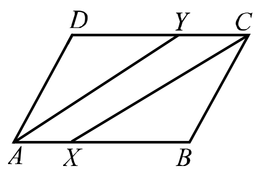
This proves that EFGH is a parallelogram.

Ex.14 In figure ABCD is a parallelogram and X, Y are the mid-points of sides AB and DC respectively. Show that AXCY is a parallelogram.

Sol. Since X and Y are the mid-points of AB and DC respectively. Therefore,

$AX = \frac{1}{2}AB$ and $CY = \frac{1}{2}DC$ ... (i)

But, AB = DC [$\because$ ABCD is a $\|^{gm}$]



$\Rightarrow \frac{1}{2}AB = \frac{1}{2}DC$

$\Rightarrow$ AX = CY .... (ii)

Also, AB || DC

$\Rightarrow$ AX || YC .... (iii)

Thus, in quadrilateral AXCY, we have

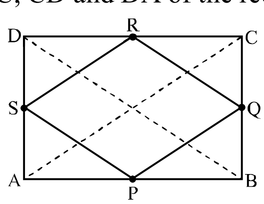
AX || YC and AX = YC

[From (ii) and (iii)]

Hence, quadrilateral AXCY is a parallelogram.

Ex.15 Prove that the line segments joining the mid-points of the sides of a rectangle forms a rhombus.

Sol. P, Q, R and S are the mid-points of the sides AB, BC, CD and DA of the rectangle ABCD.



Here, AC = BD ($\because \Delta ABC \cong \Delta BAD$)

Now, SR || AC and $SR = \frac{1}{2}AC$

and PQ || AC and $PQ = \frac{1}{2}AC$

$\Rightarrow$ SR || PQ and $SR = PQ = \frac{1}{2}AC$

Similarly, PS || QR and $PS = QR = \frac{1}{2}BD$

$\Rightarrow$ SR || PQ, PS || QR

and SR = PQ = PS = QR ($\because$ AC = BD)

PQRS is a rhombus.

Ex.16 In figure ABCD is a parallelogram and X and Y are points on the diagonal BD such that DX = BY. Prove that

(i) AXCY is a parallelogram

(ii) AX = CY, AY = CX

(iii) $\Delta AYB \cong \Delta CXD$

Sol. Given : ABCD is a parallelogram. X and Y are points on the diagonal BD such that DX = BY

To Prove :

(i) AXCY is a parallelogram

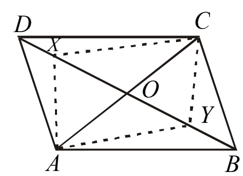
(ii) AX = CY, AY = CX

(iii) $\Delta AYB \cong \Delta CXD$

Construction : join AC to meet BD at O.

Proof :

(i) We know that the diagonals of a parallelogram bisect each other. Therefore, AC and BD bisect each other at O.



$\therefore$ OB = OD

But, BY = DX

$\therefore$ OB – BY = OD – DX

$\Rightarrow$ OY = OX

Thus, in quadrilateral AXCY diagonals AC and XY are such that OX = OY and OA = OC i.e. the diagonals AC and XY bisect each other.

Hence, AXCY is a parallelogram.

(ii) Since AXCY is a parallelogram

$\therefore$ AX = CY and AY = CX

(iii) In triangles AYB and CXD, we have

AY = CX [From (ii)]

AB = CD

[$\because$ ABCD is a parallelogram]

BY = DX [Given]

So, by SSS-criterion of congruence, we have

$\Delta AYB \cong \Delta CXD$

Ex.17 In fig. ABC is an isosceles triangle in which AB = AC. CP || AB and AP is the bisector of exterior ∠CAD of ΔABC. Prove that ∠PAC = ∠BCA and ABCP is a parallelogram.

Sol. Given : An isosceles ΔABC having AB = AC.AP is the bisector of ext ∠ CAD and CP || AB.

To Prove : ∠PAC = ∠BCA and ABCP

Proof : In ΔABC, we have

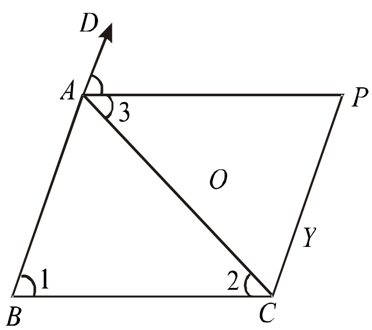
AB = AC [Given]

⇒ ∠1 = ∠2 .... (i)

[∵ Angles opposite to equal sides in a Δ are equal]

Now, in Δ ABC, we have

ext ∠CAD = ∠1 + ∠2



[∵ An exterior angles is equal to the sum of two opposite interior angles]

⇒ ext ∠CAD = 2∠2 [∵ ∠1 = ∠2 (from (i))]

⇒ 2∠3 = 2∠2

[∵ AP is the bisector of ext.∠CAD ∴∠CAD = 2∠3]

⇒ ∠3 = ∠2

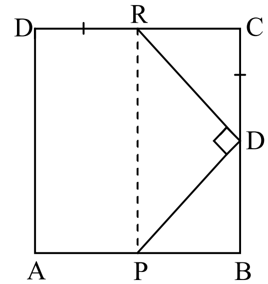
Thus, AC intersects lines AP and BC at A and C respectively such that ∠3 = ∠2 i.e., alternate interior angles are equal. Therefore,

AP || BC.

But,CP || AB [Gvien]

Thus, ABCP is a quadrilateral such that AP || BC and CP || AB. Hence, ABCP is a parallelogram.

Ex.18 In the given figure, ABCD is a square and ∠PQR = 90º. If PB = QC = DR, prove that



(i) QB = RC, (ii) PQ = QR, (iii) ∠QPR = 45º.

Sol. BC = DC, CQ = DR ⇒ BC – CQ = ΔCDR

⇒ QB = RC

From ΔCQR, ∠RQB = ∠QCR + ∠QRC

⇒ ∠RQP + ∠PQB = 90º + ∠QRC

⇒ 90º + ∠PQB = 90º + ∠QRC

Now, ΔRCQ ≅ ΔQBP and therefore,

QR = PQ

PQ = QR ⇒ ∠QPR = ∠PRQ

Bur, ∠QPR + ∠PRQ = 90º.

So, ∠QPR = 45º

- Each of the four angles of a rectangel is a right angle.
- Each of the four sides of a rhombus is of the same length.
- Each of the angles of a square is a right angle and each of the four sides is of the same length.
- The diagonals of a rectangle are of equal length.
- If the two diagonals of parallelogram are equal, it is a rectangle.
- The diagonals of a rhombus are perpendicular to each other.
- If the diagonals of a parallelogram are perpendicular, then it is a rhombus.
- The diagonals of a square are equal and perpendicular to each other.
- If the diagonals of a parallelogram are equal and intersect at right angles then the parallelogram is a square.

❖ EXA MPLES ❖

Ex.19 Prove that in a parallelogram

(i) opposite sides are equal

(ii) opposite angles are equal

(iii) each diagonal bisects the parallelogram

Sol. Given : A ||gm ABCD in which AB || DC and AD || BC.

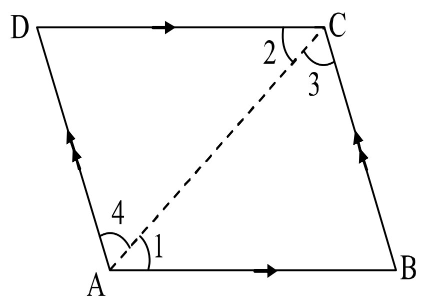
To prove (i) AB = CD and BC = AD;

(ii) $\angle B = \angle D$ and $\angle A = \angle C$,

(iii) $\Delta ABC = \Delta CDA$ and $\Delta ABD = \Delta CDB$

Construction join A and C.

In ΔABC and CDA, we have,



$\angle 1 = \angle 2$

[Alt. int. $\angle$, as AB || DC and CA cuts them]

$\angle 3 = \angle 4$

[Alt. int. $\angle$, as BC || AD and CA cuts them]

AC = CA (common)

$\therefore$ $\Delta ABC \cong \Delta CDA$ [AAS-criterial]

(i) $\Delta ABC \cong \Delta CDA$ (proved)

$\therefore$ AB = CD and BC = AD (c.p.c.t.)

(ii) $\Delta ABC \cong \Delta CDA$ (proved)

$\therefore$ $\angle B = \angle D$ (c.p.c.t.)

Also, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$

$\angle 1 + \angle 4 = \angle 2 + \angle 3 \Rightarrow \angle A = \angle C$

Hence, $\angle B = \angle D$ and $\angle A = \angle C$

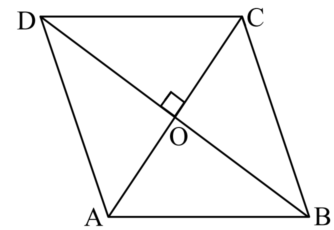
(iii) Since $\Delta ABC \cong \Delta CDA$ and congruent triangles are equal in area,

So we have $\Delta ABC = \Delta CDA$

Similarly, $\Delta ABD = \Delta CDB$

Ex.20 If the diagonals of a parallelogram are perpendicular to each other, prove that it is a rhombus.

Sol. Since the diagonals of a ||gm bisect each other,



we have, OA = OC and OB = OD.

Now, in ΔAOD and COD, we have

OA = OC, $\angle AOD = \angle COD = 90°$

and OD is common

$\therefore$ $\Delta AOD \cong \Delta COD$

$\therefore$ AD = CD (c.p.c.t.)

Now, AB = CD and AD = BC

(opp. sides of a ||gm)

and AD = CD (proved)

$\therefore$ AB = CD = AD = BC

Hence, ABCD is a rhombus.

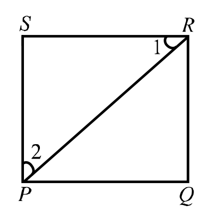
Ex.21 PQRS is a square. Determine $\angle SRP$.

Sol. PQRS is a square.

$\therefore$ PS = SR and $\angle PSR = 90°$

Now, in Δ PSR, we have

PS = SR



$\Rightarrow$ $\angle 1 = \angle 2$ $\left[\begin{array}{c}\because \text{Angles opp. to} \\ \text{equal sides are equal}\end{array}\right]$

But, $\angle 1 + \angle 2 + \angle PSR = 180°$

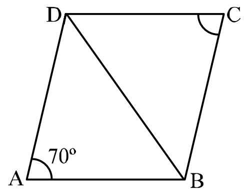
$\therefore$ $2\angle 1 + 90° = 180°$ [$\because \angle PSR = 90°$]

$\Rightarrow$ $2\angle 1 = 90°$

$\Rightarrow$ $\angle 1 = 45°$

Ex.22 In the adjoining figure, ABCD is a rhombus. If $\angle A = 70^o$, find $\angle CDB$

Sol.



We have $\angle C = \angle A = 70^o$

(opposite $\angle$ of a $\|$gm)

Let $\angle CDB = x^o$

In ΔCDB, we have

$CD = CB \Rightarrow \angle CBD = \angle CDB = x^o$

$\therefore$ $\angle CDB + \angle CBD + \angle DCB = 180^o$

(angles of a triangle)

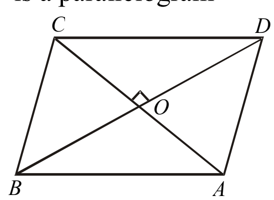
$\Rightarrow x^o + x^o + 70^o = 180^o$

$\Rightarrow 2x = 110$, i.e., $x = 55$

Hence, $\angle CDB = 55^o$

Ex.23 ABCD is a rhombus with $\angle ABC = 56°$. Determine $\angle ACD$.

Sol. ABCD is a parallelogram



$\Rightarrow$ $\angle ABC = \angle ADC$

$\Rightarrow$ $\angle ADC = 56°$ [$\because$ $\angle ABC = 56°$ (Given)]

$\Rightarrow$ $\angle ODC = 28°$ [$\because$ $\angle ODC = \frac{1}{2}\angle ADC$]

Now, ΔOCD we have,

$\angle OCD + \angle ODC + \angle COD = 180°$

$\Rightarrow$ $\angle ODC + 28° + 90° = 180°$

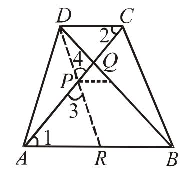
$\Rightarrow$ $\angle OCD = 62° \Rightarrow \angle ACD = 62°$.

◈ The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half of it.

◈ The line drawn through the mid-point of one side of a triangle, parallel to another side, intersects the third side at its mid-point.

Ex.24 Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to each of the parallel sides and is equal to half the difference of these sides.

Sol. Given : A trapezium ABCD in which AB $\|$ DC and P and Q are the mid-points of its diagonals AC and BD respectively.



To Prove : (i) PQ $\|$ AB or DC

(ii) $PQ = \frac{1}{2}(AB - DC)$

Construction : Join DP and produce DP to meet AB in R.

Proof : Since AB $\|$ DC and transversal AC cuts them at A and C respectively.

$\angle 1 = \angle 2$ .... (i)

[$\therefore$ Alternate angles are equal]

Now, in Δs APR and DPC, we have

$\angle 1 = \angle 2$ [From (i)]

AP = CP [$\because$ P is the mid-point of AC]

and, $\angle 3 = \angle 4$ [Vertically opposite angles]

So, by ASA criterion of congruence

$\Delta APR \cong \Delta DPC$

$\Rightarrow$ AR = DC and PR = DP ....(ii)

[$\because$ Corresponding parts of congruent triangles are equal]

In ΔDRB, P and Q are the mid-points of sides DR and DB respectively.

$\therefore$ PQ $\|$ RB

$\Rightarrow$ PQ $\|$ AB [$\because$ RB is a part of AB]

$\Rightarrow$ PQ $\|$ AB and DC [$\because$ AB $\|$ DC (Given)]

This proves (i).

Again, P and Q are the mid-points of sides DR and DB respectively in ΔDRB.

$\therefore$ $PQ = \frac{1}{2}RB \Rightarrow PQ = \frac{1}{2}(AB - AR)$

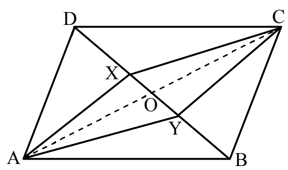
$\Rightarrow$ $PQ = \frac{1}{2}(AB - DC)$ [From (ii), AR = DC]

This proves (ii).

- ◈ A diagonal of a parallelogram divides it into two triangles of equal area.
- ◈ For each base of a parallelogram, the corresponding altitude is the line segment from a point on the base, perpendicular to the line containing the opposite side.
- ◈ Parallelograms on the same base and between the same parallels are equal in area.
- ◈ A parallelogram and a rectangle on the same base and between the same parallels are equal in area.
- ◈ The area of a parallelogram is the product of its base and the corresponding altitude.
- ◈ Parallelograms on equal bases and between the same parallels are equal in area.

❖ EXAMPLES ❖

Ex.25 In the adjoining figure, ABCD is parallelogram and X, Y are the points on diagonal BD such that DX = BY. Prove that CXAY is a parallelogram.



Sol. Join AC, meeting BD at O.

Since the diagonals of a parallelogram bisect each other, we have OA = OC and OD = OB.

Now, OD = OB and DX = BY

⇒ OD – DX = OB – BY ⇒ OX = OY

Now, OA = OC and OX = OY

∴ CXAY is a quadrilateral whose diagonals bisect each other.

∴ CXAY is a ||gm

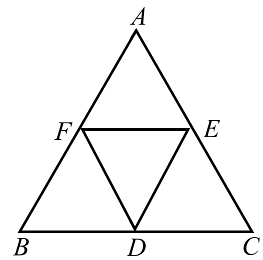
Ex.26 Prove that the four triangles formed by joining in pairs, the mid-points of three sides of a triangle, are concurrent to each other.

Sol. Given : A triangle ABC and D,E,F are the mid-points of sides BC, CA and AB respectively.

To Prove :

Δ AFE ≅ ΔFBD ≅ ΔEDC ≅ ΔDEF.

Proof : Since the segment joining the mid-points of the sides of a triangle is half of the third side. Therefore,



$DE = \frac{1}{2}AB \Rightarrow DE = AF = BF$ .... (i)

$EF = \frac{1}{2}BC \Rightarrow EF = BD = CD$ .... (ii)

$DF = \frac{1}{2}AC \Rightarrow DF = AE = EC$ ....(iii)

Now, in Δs DEF and AFE, we have

DE = AF [From (i)]

DF = AE [From (ii)]

and,EF = FE [Common]

So, by SSS criterion of congruence,

Δ DEF ≅ Δ AFE

Similarly, Δ DEF ≅ Δ FBD and Δ DEF ≅ Δ EDC

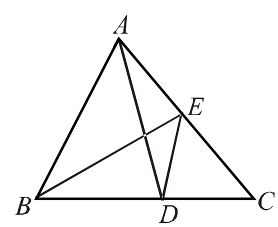
Hence, Δ AFE ≅ Δ FBD ≅ Δ EDC ≅ Δ DEF

Ex.27 In fig, AD is the median and DE || AB. Prove that BE is the median.

Sol. In order to prove that BE is the median, it is sufficient to show that E is the mid-point of AC.

Now, AD is the median in ΔABC

⇒ D is the mid-point of BC.



Since DE is a line drawn through the mid-point of side BC of ΔABC and is parallel to AB (given). Therefore, E is the mid-point of AC. Hence, BE is the median of ΔABC.

Ex.28 Let ABC be an isosceles triangle with AB = AC and let D,E,F be the mid-points of BC, CA and AB respectively. Show that AD ⊥ FE and AD is bisected by FE.

Sol. Given : An isosceles triangle ABC with D, E and F as the mid-points of sides BC, CA and AB respectively such that AB = AC. AD intersects FE at O.

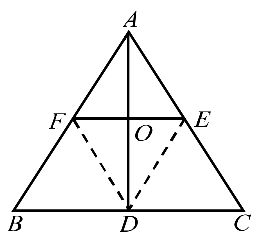
To Prove : AD ⊥ FE and AD is bisected by FE.

Constructon : Join DE and DF.

Proof : Since the segment joining the mid-points of two sides of a triangle is parallel to third side and is half of it. Therefore,

$$DE \parallel AB \text{ and } DE = \frac{1}{2}AB$$

Also, $DF \parallel AC$ and $DF = \frac{1}{2}AC$



But,AB = AC [Given]

$\Rightarrow \frac{1}{2}AB = \frac{1}{2}AC \Rightarrow DE = DF$ .... (i)

Now, $DE = \frac{1}{2}AB \Rightarrow DE = AF$ .... (ii)

and,$DF = \frac{1}{2}AC \Rightarrow DF = AE$ ....(iii)

From (i), (ii) and (iii) we have

DE = AE = AF = DF

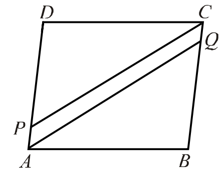
⇒ DEAF is a rhombus.

⇒ Diagonals AD and FE bisect each other at right angle.

AD ⊥ FE and AD is bisected by FE.

Ex.29 ABCD is a parallelogram. P is a point on AD such that $AP = \frac{1}{3}AD$ and Q is a point on BC such that $CQ = \frac{1}{3}BP$. Prove that AQCP is a parallelogram.

Sol. ABCD is a parallelogram.



⇒ AD = BC and AD || BC

$\Rightarrow \frac{1}{3}AD = \frac{1}{3}BC$ and AD || BC

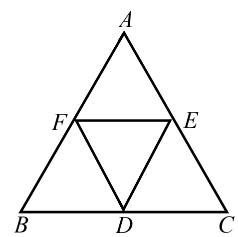
⇒ AP = CQ and AP || CQ

Thus, APCQ is a quadrilateral such that one pair of opposite side AP and CQ are parallel and equal.

Hence, APCQ is a parallelogram.

Ex.30 In fig. D,E and F are, respectively the mid-points of sides BC, CA and AB of an equilateral triangle ABC. Prove that DEF is also an equilateral triangle.

Sol. Since the segment joining the mid-points of two sides of a triangle is half of the third side. Therefore, D and E are mid-points of BC and AC respectively.



$\Rightarrow DE = \frac{1}{2}AB$ .... (i)

E and F are the mid-points of AC and AB respectively.

$\therefore EF = \frac{1}{2}BC$ .... (ii)

F and D are the mid-points AB and BC respectively.

$\Rightarrow FD = \frac{1}{2}AC$

Now, ΔABC is an equilateral triangle

$\Rightarrow AB = BC = CA \Rightarrow \frac{1}{2}AB = \frac{1}{2}BC = \frac{1}{2}CA$

⇒ DE = EF = FD

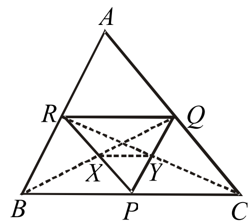
[Using (i), (ii) and (iii)]

Hence, ΔDEF is an equilateral triangle.

Ex.31 P,Q and R are, respectively, the mid-points of sides BC, CA and AB of a triangle ABC. PR and BQ meet at X. CR and PQ meet at Y. Prove that XY = $\frac{1}{4}BC$

Sol. Given : A ΔABC with P,Q and R as the mid-points of BC, CA and AB respectively. PR and BQ meet at X and CR and PQ meet at Y.

Construction : Join "X and Y.



Proof: Since the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half of it. Therefore, Q and R are mid-points of AC and AB respectively.

$\therefore$ RQ || BC and RQ = $\frac{1}{2}$ BC .... (i)

$$\left[\begin{array}{c}\because \text{P is the mid-point} \\ \text{of BC} \therefore \frac{1}{2}\text{BC} = \text{BP}\end{array}\right]$$

$\Rightarrow$ RQ || BP and RQ = BP

$\Rightarrow$ BPQR is a parallelogram.

Since the diagonals of a parallelogram bisect each other.

$\therefore$ X is the mid-point of PQ.

$$\left[\begin{array}{c}\because \text{X is the point of intersection of} \\ \text{diagonals BQ and PR of } ||^{gm} \text{ BPQR}\end{array}\right]$$

Similarly, Y is the mid-point of PQ.

Now, consider ΔPQR. XY is the line segment joining the mid-points of sides PR and PQ.

$\therefore$ XY = $\frac{1}{2}$RQ .... (i)

But RQ = $\frac{1}{2}$BC [From (i)]

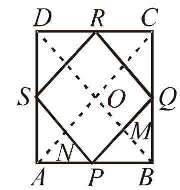
Hence, XY = $\frac{1}{4}BC$.

Ex.32 Show that the quadrilateral, formed by joining the mid-points of the sides of a square, is also a square.

Sol. Given : A square ABCD in which P, Q, R, S are the mid-points of sides AB, BC, CD, DA respectively. PQ, QR, RS and SP are joined.

To Prove : PQRS is a square.

Construction : Join AC and BD.



Proof : In ΔABC, P and Q are the mid-points of sides AB and BC respectively.

$\therefore$ PQ || AC and PQ = $\frac{1}{2}$AC .... (i)

In ΔADC, R and S are the mid-points of CD and AD respectively.

$\therefore$ RS || AC and RS = $\frac{1}{2}$AC ....(ii)

From (i) and (ii), we have

PQ || RS and PQ = RS ....(iii)

Thus, in quadrilateral PQRS one pair of opposite sides are equal and parallel.

Hence, PQRS is a parallelogram.

Now, in Δs PBQ and RCQ, we have

PB = RC

$$\left[\begin{array}{l}\because \text{ABCD, is a square} \therefore \text{AB} = \text{BC} = \text{CD} = \text{DA} \\ \Rightarrow \frac{1}{2}\text{AB} = \frac{1}{2}\text{CD and } \frac{1}{2}\text{AB} = \frac{1}{2}\text{BC}\end{array}\right]$$

BQ = CQ [$\Rightarrow$ PB = CR and BQ = CQ]

and ∠PBQ = ∠RCQ [Each equal to 90°]

So, by SAS criterion of congruence

ΔPBQ ≅ ΔRCQ

$\Rightarrow$ PQ = QR ....(iv)

[$\because$ Corresponding parts of congruent Δs are equal]

From (iii) and (iv), we have

PQ = QR = RS

But,PQRS is a||^{gm}.

QR = PS

So, PQ = QR = RS = PS ....(v)

Now, PQ || AC [From (i)]

$\Rightarrow$ PM || NO ....(vi)

Since P and S are the mid-points of AB and AD respectively.

PS || BD

$\Rightarrow$ PM || MO ....(vii)

Thus, in quadrilateral PMON, we have

PM || NO [From (vi)]

PN || MO [From (vii)]

So, PMON is a parallelogram.

$\Rightarrow \angle MPN = \angle MON$

$\Rightarrow \angle MPN = \angle BOA$ [$\because \angle MON = \angle BOA$]

$\Rightarrow \angle MPN = 90^\circ$

$$z\begin{bmatrix} \because & \text{Diagonals of square are } \perp \\ \therefore & AC \perp BD \Rightarrow \angle BOA = 90^\circ \end{bmatrix}$$

$\Rightarrow \angle QPS = 90^\circ$

Thus, PQRS is a quadrilateral such that PQ = QR = RS = SP and $\angle$ QPS = 90°.

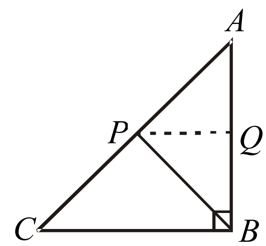
Hence, PQRS is a square.

Ex.33 ΔABC is a triangle right angled at B ; and P is the mid-point of AC. Prove that PB = PA = $\frac{1}{2}$ AC.

Sol. Given : ΔABC right angled at B, P is the mid-point of AC.

To Prove : PB = PA = $\frac{1}{2}$ AC.

Construction : Through P draw PQ || BC meeting AB at Q.



Proof : Since PQ || BC. Therefore,

$\angle$ AQP = $\angle$ ABC [Corresponding angles]

$\Rightarrow \angle AQP = 90^\circ$
[$\because \angle ABC = 90^\circ$]

But,$\angle$ AQP + $\angle$ BQP = 180°

[$\because \angle AQP$ & $\angle BQP$ are angles of a linear pair]

$\therefore$ 90° + $\angle$ BQP = 180°

$\Rightarrow \angle BQP = 90^\circ$

Thus, $\angle$ AQP = $\angle$ BQP = 90°

Now, in Δ ABC, P is the mid-point of AC and PQ || BC. Therefore, Q is the mid-point of AB i.e, AQ = BQ.

Consider now Δs APQ and BPQ.

we have, AQ = BC [Proved above]

$\angle$ AQP = $\angle$ BQP [From (i)]

and, PQ = PQ

So, by SAS cirterion of congruence

Δ APQ $\cong \angle$ BPQ

$\Rightarrow$ PA = PB

Also,

PS = $\frac{1}{2}$ AC, since P is the mid-point of AC

Hence, PA = PB = $\frac{1}{2}$ AC.

Ex.34 Show that the quadrilateral formed by joining the mid-points of the consecutive sides of a rectangle is a rhombus.

Sol. Given : A rectangle ABCD in which P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively. PQ, QR, RS and SP are joined.

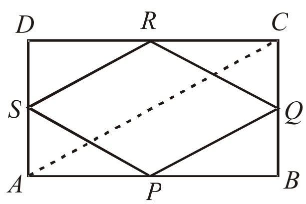
To Prove : PQRS is rhombus.

Construction : Join AC.

Proof : In ΔABC, P and Q are the mid-points of sides AB and BC respectively.

$\therefore$ PQ || AC and PQ = $\frac{1}{2}$ AC .... (i)

In Δ ADC, R and S are the mid-points of CD and AD respectively.



$\therefore$ SR || AC and SR = $\frac{1}{2}$ AC .... (ii)

From (i) and (ii), we get

PQ || SR and PQ = SR ....(iii)

$\Rightarrow$ PQRS is a parallelogram.

Now, ABCD is a rectangle.

$\Rightarrow$ AD = BC $\Rightarrow$ $\frac{1}{2}$ AD = $\frac{1}{2}$ BC

$\Rightarrow$ AS = BQ ....(iv)

In Δs APS and BPQ , we have

AP = BP [$\therefore$ P is the mid-point of AB]

$\angle$ PAS = $\angle$ PBQ [Each equal to 90°]

and, AS = BQ [From (iv)]

So, by SAS criterion of congruence

ΔAPS ≅ Δ BPQ

PS = PQ ....(v)

[$\because$ Corresponding parts of congruent triangles are equal]

From (iii) and (v), we obtain that PQRS is a parallelogram such that PS = PQ i.e., two adjacent sides are equal.

Hence, PQRS is a rhombus.

IMPORTANT POINTS TO BE REMEMBERED

1. Sum of the angles of a quadrilateral is 360º.

2. A diagonal of a parallelogram divides it into two congruent triangles.

3. Two opposite angles of a parallelogram are equal.

4. The diagonals of a parallelogram bisect each other.

5. In a parallelogram, the bisectors of any two consecutive angles intersect at right angle.

6. If a diagonal of a parallelogram bisects one of the angles of the parallelogram it also bisects the second angle.

7. The angles bisectors of a parallelogram form a rectangle.

8. A quadrilateral is a parallelogram if its opposite sides are equal.

9. A quadrilateral is a parallelogram iff its opposite angles are equal.

10. The diagonals of a quadrilateral bisect each other, iff it is a parallelogram.

11. A quadrilateral is a parallelogram if its one pair of opposite sides are equal and parallel.

12. Each of the four angles of a rectangle is a right angle.

13. Each of the four sides of a rhombus of the same length.

14. The diagonals of a rectangle are of equal length.

15. Diagonals of a parallelogram are equal if and only if it is a rectangle.

16. The diagonals of a rhombus are perpendicular to each other.

17. Diagonals of a parallelogram are perpendicular if and only if it is a rhombus.

18. The diagonals of a square are equal and perpendicular to each other.

19. If the diagonals of a parallelogram are equal and intersect at right angle, then it is a square.

20. The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half of it.

21. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

22. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.