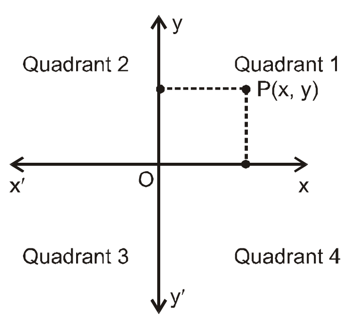
Point & Straight Line

POINT

RECTANGULAR CARTESIAN CO-ORDINATE SYSTEMS

We shall right now focus on two-dimensional co-ordinate geometry in which two perpendicular lines called co-ordinate axes (x-axis and y-axis) are used to locate a point in the plane.



O is called origin. Any point P in this plane can be represented by a unique ordered pair (x, y), which are called co-ordinates of that point. x is called x co-ordinate or abscissa and y is called y co-ordinate or ordinate. The two perpendicular lines xox′ and yoy′ divide the plane in four regions which are called quadrants, numbered as shown in the figure.

Let us look at some of the formulae linked with points now.

DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Note :

(i) In particular the Distance of a point $P(x, y)$ from the origin $= \sqrt{x^2 + y^2}$

(ii) Distance between two polar co-ordinates $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ is given by $AB = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$

Solved Examples

Ex.1 The distance between $P(3, -2)$ and $Q(-7, -5)$ is

[1] $\sqrt{115}$ [2] $\sqrt{109}$ [3] $\sqrt{91}$ [4] 11

Sol. $PQ = \sqrt{(3+7)^2 + (-2+5)^2} = \sqrt{100+9} = \sqrt{109}$

Ans. [2]

Ex.2 The distance between $P\left(2, -\frac{\pi}{6}\right)$ and $Q\left(3, \frac{\pi}{6}\right)$ is

[1] 3 [2] 1/2 [3] $\sqrt{7}$ [4] $\sqrt{5}$

Sol. $PQ = \sqrt{2^2 + 3^2 - 2.2.3\cos\left(-\frac{\pi}{6} - \frac{\pi}{6}\right)}$

$= \sqrt{13 - 12 \times \cos\left(-\frac{\pi}{3}\right)} = \sqrt{13 - 12 \times \frac{1}{2}} = \sqrt{7}$ **Ans. [3]**

Ex.3 Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5

Sol. Let P(x ,–1) and Q(3, 2) be the given points. Then PQ = 5 (given)

$$\sqrt{(x-3)^2+(-1-2)^2} = 5$$

$\Rightarrow (x-3)^2 + 9 = 25 \quad \Rightarrow x = 7$ or $x = -1$

APPLICATION OF DISTANCE FORMULAE

(i) For given three points A, B, C to decide whether they are collinear or vertices of a particular triangle. After finding AB, BC and CA we shall find that the point are:

* Collinear: If the sum of any two distances is equal to the third.
* Vertices of an equilateral triangle if AB = BC = CA
* Vertices of an isosceles triangle if AB = BC or BC = CA or CA = AB
* Vertices of a right angled triangle $AB^2 + BC^2 = CA^2$ etc.,

Solved Examples

Ex.4 If $A(2,-2); B(-2,1)$ and $C(5,2)$ are three points then A, B, C are

[1] collinear

[2] vertices of an equilateral triangle

[3] vertices of right angled triangle

[4] none of these

Sol.

$$AB = \sqrt{(-2-2)^2+(1+2)^2} = 5$$

$$BC = \sqrt{(5+2)^2+(2-1)^2} = 5\sqrt{2}$$

$$CA = \sqrt{(2-5)^2+(-2-2)^2} = 5$$

Since the sum of any two distances is not equal to the third so A, B, C are not collinear. They are vertices of a triangle. Also $AB^2 + CA^2 = BC^2$

$\Rightarrow A, B, C$ are vertices of a right angled triangle.

Ans. [3]

(ii) For given four points:

* $AB = BC = CD = DA;\ AC = BD \Rightarrow ABCD$ square
* $AB = BC = CD = DA;\ AC \neq BD \Rightarrow ABCD$ rhombus
* $AB = CD,\ BC = DA,\ AC = BD \Rightarrow ABCD$ is a rectangle
* $AB = CD,\ BC = DA,\ AC \neq BD$

  $\Rightarrow ABCD$ is a parallelogram

QUADRILATERAL DIAGONALS ANGLE BETWEEN DIAGONALS

(i)	Parallelogram	Not equal	$\theta \neq \frac{\pi}{2}$
(ii)	Rectangle	Equal	$\theta \neq \frac{\pi}{2}$
(iii)	Rhombus	Not equal	$\theta = \frac{\pi}{2}$
(iv)	Square	Equal	$\theta = \frac{\pi}{2}$

Note :

(i) Diagonal of square, rhombus, rectangle and parallelogram always bisect each other.

(ii) Diagonal of rhombus and square bisect each other at right angle.

(iii) Four given points are collinear, if area of quadrilateral is zero.

Solved Examples

Ex.5 Points $A(1,1), B(-2,7)$ and $C(3,-3)$ are

[1] collinear

[2] vertices of equilateral triangle

[3] vertices of isoscele triangle

[4] none of these

Sol. $AB = \sqrt{(1+2)^2+(1-7)^2} = \sqrt{9+36} = 3\sqrt{5}$

$BC = \sqrt{(-2-3)^2+(7+3)^2} = \sqrt{25+100} = 5\sqrt{5}$

$CA = \sqrt{(3-1)^2+(-3-1)^2} = \sqrt{4+16} = 2\sqrt{5}$

Clearly $BC = AB + AC$. Hence A, B, C are collinear.

Ans. [1]

SECTION FORMULA

Co-ordinates of a point which divides the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ are

(1) For internal division

$$= \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

(2) For external division $= \left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$

(3) Co-ordinates of mid point of PQ $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Put $m_1 = m_2$

(4) Co-ordinates of any point on the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are

$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right), \lambda \neq -1$$

1. DIVISION BY AXES: PQ is divided by

(i) x-axis in the ratio $= \frac{-y_1}{y_2}$

(ii) y-axis in the ratio $= -\frac{x_1}{x_2}$

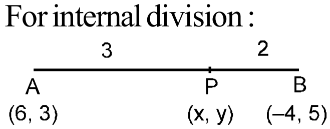
2. DIVISION BY A LINE: A line $ax + by + c = 0$ divides PQ in the ratio $-\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$

Solved Examples

Ex.6 Find the co-ordinates of the point which divides the line segment joining the points (6, 3) and (– 4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.

Sol. Let P (x, y) be the required point.

(i) For internal division :

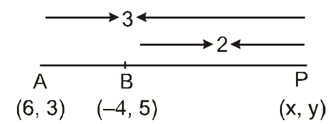


$x = \frac{3 \times -4 + 2 \times 6}{3 + 2}$ and $y = \frac{3 \times 5 + 2 \times 3}{3 + 2}$ or

$x = 0$ and $y = \frac{21}{5}$

So the co-ordinates of P are $\left(0, \frac{21}{5}\right)$

(ii) For external division



$x = \frac{3 \times -4 - 2 \times 6}{3 - 2}$ and $y = \frac{3 \times 5 - 2 \times 3}{3 - 2}$

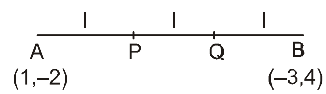
or $x = -24$ and $y = 9$

So the co-ordinates of P are (–24, 9)

Ex.7 Find the co-ordinates of points which trisect the line segment joining (1, – 2) and (– 3, 4).

Sol. Let A (1, –2) and B(–3, 4) be the given points. Let the points of trisection be P and Q.

Then AP = PQ = QB = λ (say)



$\therefore$ PB = PQ + QB = 2λ and AQ = AP + PQ = 2λ

$\Rightarrow$ AP : PB = $\lambda : 2\lambda$ = 1 : 2 and AQ : QB = $2\lambda : \lambda$ = 2 : 1

So P divides AB internally in the ratio 1 : 2 while Q divides internally in the ratio 2 : 1

$\therefore$ the co-ordinates of P are

$$\left(\frac{1 \times -3 + 2 \times 1}{1 + 2}, \frac{1 \times 4 + 2 \times -2}{1 + 2}\right) \text{ or } \left(-\frac{1}{3}, 0\right)$$

and the co-ordinates of Q are

$$\left(\frac{2 \times -3 + 1 \times 1}{2 + 1}, \frac{2 \times 4 + 1 \times (-2)}{2 + 1}\right) \text{ or } \left(-\frac{5}{3}, 2\right)$$

Hence, the points of trisection are $\left(-\frac{1}{3}, 0\right)$ and $\left(-\frac{5}{3}, 2\right)$.

Ex.8 The co-ordinates of point of internal and external division of the line segment joining two points $(3, -1)$ and $(3, 4)$ in the ratio 2 : 3 are respectively

[1] (2, 3) (11, 3) [2] (3, 1) (3, –11)
[3] (1, 3) (–11, 3) [4] (1, –3) (11, –3)

Sol. Internal division

$$x = \frac{2(3)+3(3)}{2+3} = 3 \qquad y = \frac{2(4)+3(-1)}{2+3} = 1$$

Hence point $(3,1)$

External division

$$x = \frac{2(3)-3(3)}{2-3} = 3 \qquad y = \frac{2(4)-3(-1)}{2-3} = -11$$

Hence point $(3,-11)$ **Ans.[2]**

Ex.9 The ratio in which the line $3x+4y=7$ divides the line segment joining the points $(1,2)$ and $(-2,1)$ is

[1] $\frac{4}{9}$ [2] $\frac{9}{4}$ [3] $\frac{1}{3}$ [4] $\frac{3}{4}$

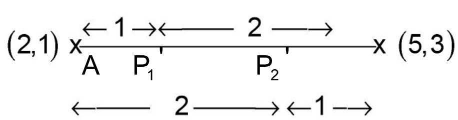
Sol. Required ratio $= -\dfrac{ax_1+by_1+c}{ax_2+by_2+c}$

$$= -\frac{3(1)+4(2)-7}{3(-2)+4(1)-7} = -\frac{4}{-9} = \frac{4}{9}$$ **Ans. [1]**

Ex.10 The points of trisection of line joining the points $A(2,1)$ and $B(5,3)$ are

[1] $\left(3,\frac{5}{3}\right),\left(4,\frac{7}{3}\right)$ [2] $\left(3,\frac{3}{5}\right)\left(4,\frac{3}{7}\right)$

[3] $\left(-3,\frac{5}{3}\right)\left(4,-\frac{7}{3}\right)$ [4] $\left(3,-\frac{5}{3}\right)\left(4,\frac{3}{7}\right)$

Sol.


$$P_1(x,y) = \left(\frac{1\times5+2\times2}{1+2}, \frac{1\times3+2\times1}{1+2}\right) = \left(3,\frac{5}{3}\right)$$

$$P_2(x,y) = \left(\frac{2\times5+1\times2}{2+1}, \frac{2\times3+1\times1}{2+1}\right) = \left(4,\frac{7}{3}\right)$$

Ans.[1]

Ex.11 The ratio in which the lines joining the $(3,-4)$ and $(-5,6)$ divided by x-axis is

[1] 3 : 2 [2] 2 : 3 [3] 4 : 3 [3] 3 : 4

Sol. Required ratio $= -\dfrac{y_1}{y_2} = -\left(\dfrac{-4}{6}\right) = 2:3$ **Ans.[2]**

Centroid, Incentre & Excentre :

If A (x_1, y_1), B(x_2, y_2), C(x_3, y_3) are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the co-ordinates of the special points of triangle ABC are as follows :

Centroid G $\equiv \left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

Incentre I $\equiv \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$,and

Excentre (to A) I_1

$\equiv \left(\dfrac{-ax_1+bx_2+cx_3}{-a+b+c}, \dfrac{-ay_1+by_2+cy_3}{-a+b+c}\right)$ and so on.

Notes :

(i) Incentre divides the angle bisectors in the ratio, $(b+c) : a; (c+a) : b \ \& \ (a+b) : c$.

(ii) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.

(iii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2 : 1.

(iv) In an isosceles triangle G, O, I & C lie on the same line and in an equilateral triangle, all these four points coincide.

(v) In a right angled triangle orthocentre is at right angled vertex and circumcentre is mid point of hypotenuse

(vi) In case of an obtuse angled triangle circumcentre and orthocentre both are out side the triangle.

Solved Examples

Ex.12 Find the co-ordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Sol. (i) We know that the co-ordinates of the centroid of a triangle whose angular points are $(x_1, y_1), (x_2, y_2)\ (x_3, y_3)$ are $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

So the co-ordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are $\left(\dfrac{0+8+8}{3}, \dfrac{6+12+0}{3}\right)$ or $\left(\dfrac{16}{3}, 6\right)$.

(ii) Let A (0, 6), B (8, 12) and C(8, 0) be the vertices of triangle ABC.

Then c = AB = $\sqrt{(0-8)^2+(6-12)^2}$ = 10,

b = CA = $\sqrt{(0-8)^2+(6-0)^2}$ = 10 and

a = BC = $\sqrt{(8-8)^2+(12-0)^2}$ = 12.

The co-ordinates of the in-centre are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \text{or}$$

$$\left(\frac{12\times 0+10\times 8+10\times 8}{12+10+10}, \frac{12\times 6+10\times 12+10\times 0}{12+10+10}\right) \quad \text{or}$$

$$\left(\frac{160}{32}, \frac{192}{32}\right) \text{ or } (5, 6)$$

Ex.13 Centroid of the triangle whose vertices are $(0,0),(2,5)$ and $(7,4)$ is

[1] (4, 3) [2] (3, 4) [3] (3, 3) [4] (3, 5)

Sol. $\left(\frac{0+2+7}{3}, \frac{0+5+4}{3}\right) = (3,3)$ **Ans.[3]**

Ex.14 Incentre of triangle whose vertices are $A(-36,7), B(20,7), C(0,-8)$ is

[1] (1, 1) [2] (0, –1) [3] (–1, 0) [4] (1, 0)

Sol. Using distance formula

$$a = BC = \sqrt{20^2+(7+8)^2} = 25,$$

$$b = CA = \sqrt{36^2+(7+8)^2} = 39$$

$$c = AB = \sqrt{(36+20)^2+(7-7)^2} = 56,$$

$$I = \left(\frac{25(-36)+39(20)+56(0)}{25+39+56}, \frac{25(7)+39(7)+56(-8)}{25+39+56}\right)$$

$I = (-1,0)$ **Ans.[3]**

Ex.15 If $(1,4)$ is the centroid of a triangle and its two vertices are $(4,-3)$ and $(-9,7)$ then third vertices is

[1] (7, 8) [2] (8, 8) [3] (8, 7) [4] (6, 8)

Sol. Let the third vertices of triangle be (x,y) then

$$1 = \frac{x+4-9}{3} \Rightarrow x = 8$$

$$4 = \frac{y-3+7}{3} \Rightarrow y = 8$$

So third vertex is (8, 8).

Ans.[2]

Ex.16 If $(0,1),(1,1)$ and $(1,0)$ are middle points of the sides of a triangle, then find its incentre is

[1] $\left(2-\sqrt{2},\ 2-\sqrt{2}\right)$ [2] $\left(2-\sqrt{2},\ -2+\sqrt{2}\right)$

[3] $\left(2+\sqrt{2},\ 2+\sqrt{2}\right)$ [4] $\left(2+\sqrt{2},\ -2-\sqrt{2}\right)$

Sol. Let $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$x_1+x_2=0,\ x_2+x_3=2,\ x_3+x_1=2$$

$$y_1+y_2=2,\ y_2+y_3=2,\ y_3+y_1=0$$

Solving these equations, we get $A(0,0), B(0,2)$ and $C(2,0)$ Now $a = BC = 2\sqrt{2}, b = CA = 2, c = AB = 2$

Thus incentre of ΔABC is $\left(2-\sqrt{2}, 2-\sqrt{2}\right)$ **Ans.[1]**

Ex. 17 Two vertices of a triagle are (5, –1) and (–2, 3). If origin is the orthocentre, then the third vertex of the triangle is

(1) (4, –7) (2) (–4, 7)

(3) (–4, –7) (4) (4 , 7)

Sol. Let C (α , β) be the third vertex

$$\overline{AO} \perp \overline{BC} = \left(\frac{\beta-3}{\alpha+2}\right)\left(\frac{-1}{5}\right) = -1$$

$\Rightarrow 5\alpha - \beta = -13$ .... (1)

$$\overline{BO} \perp \overline{AC} \Rightarrow \left(\frac{\beta+1}{\alpha-5}\right)\left(\frac{3}{-2}\right) = -1$$

$\Rightarrow 2\alpha - 3\beta = 13$ ..... (2)

Solving (1) and (2) , $(\alpha , \beta) = (-4 , -7)$ **Ans.[3]**

Ex.18 If $O(0,0)$; $A(3,0)$ and $B(0,4)$ are vertices of a triangle, then its circumcentre is

[1] (1, 1) [2] $\left(\frac{3}{2}, 2\right)$ [3] $\left(1, \frac{4}{3}\right)$ [4] $\left(2, \frac{3}{2}\right)$

Sol. Let it be $P(x,y)$. Then $PO^2 = PA^2 = PB^2$

$$\Rightarrow x^2 + y^2 = (x-3)^2 + y^2 = x^2 + (y-4)^2$$

$$\Rightarrow 0 = -6x + 9 = -8y + 16 \qquad \Rightarrow x = \frac{3}{2},\ y = 2$$

$\therefore$ circumcentre $\equiv \left(\frac{3}{2}, 2\right)$ **Ans.[2]**

AREA OF TRIANGLE AND QUADRILATERAL

1. AREA OF TRIANGLE (Cartesian Coordinates)

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

Area of triangle $ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Condition of collinearity:

Three points $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ are collinear if the area of $\Delta ABC = 0$ i.e.,

if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Particular cases:

(i) When one vertex is origin i.e., if the vertices are $(0,0)$; (x_1,y_1) and (x_2,y_2) then its area $\Delta = \frac{1}{2}|x_1y_2 - x_2y_1|$

(ii) When abscissae or ordinates of all vertices are equal then its area is zero.

(iii) When two vertices be on x-axis say $(a,0)$, $(b,0)$ and third vertex be (h,k), then its area $= \frac{1}{2}|a-b|k$

(iv) When two vertices be on y-axis say $(0,c)$, $(0,d)$ and third vertex be (h,k), then its area $= \frac{1}{2}|c-d|h$

(v) Area of the triangle formed by coordinate axes and the line $ax + by + c = 0$ is $\frac{c^2}{2ab}$

(vi) When ABC is right angled triangle and $\angle B = 90^0$, then $\Delta = \frac{1}{2}(AB \times BC)$

(vii) When ABC is equilateral triangle, then $\Delta = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{1}{\sqrt{3}}(\text{height})^2$

(viii) When D, E, F are the mid points of the sides AB, BC, CA of the triangle ABC, then its area $\Delta = 4(\Delta DEF)$

Note:

Area of a triangle is always taken to be non-negative. So always use mod sign while using area formula.

Ex.19 If the vertices of a triangle are $(1,2)(4,-6)$ and $(3,5)$ then its area is

[1] $\frac{25}{2}$ sq. unit [2] 12 sq. unit

[3] 5 sq. unit [4] 25 sq. unit

Sol. $\Delta = \frac{1}{2}\left[1(-6-5) + 4(5-2) + 3(2+6)\right]$

$= \frac{1}{2}[-11 + 12 + 24] = \frac{25}{2}$ square unit **Ans.[1]**

2. AREA OF QUADRILATERAL

If $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ and (x_4,y_4) are vertices of a quadrilateral then its area

$$= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\right]$$

Note:

(i) If the area of quadrilateral joining four points is zero then those four points are collinear.

(ii) If two opposite vertex of rectangle are (x_1, y_1) and (x_2, y_2) then its area may be $= |(y_2 - y_1)(x_2 - x_1)|$

(iii) If two opposite vertex of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$ then its area is

$$= \frac{1}{2}AC^2 = \frac{1}{2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]$$

Solved Examples

Ex.20 If $(1,1)(3,4)(5,-2)$ and $(4,-7)$ are vertices of a quadrilateral then its area is

[1] $\frac{41}{2}$ sq. units [2] 41 sq. units

[3] 20 sq. units [4] 22 sq. units

Sol. $= \frac{1}{2}[1(4) - 3(1) + 3(-2) - 5(4) + 5(-7) - 4(-2) + 4(1) - 1(-7)]$

$= \frac{1}{2}[4 - 3 - 6 - 20 - 35 + 8 + 4 + 7] = \frac{41}{2}$ sq. units

Ans.[1]

Ex.21 If the coordinates of two opposite vertex of a square are (a,b) and (b,a) then area of square is

[1] $(a + b)^2$ [2] $2(a + b)^2$

[3] $(a - b)^2$ [4] $2(a - b)^2$

Sol. We know that area of square $= \frac{1}{2}d^2$

$= \frac{1}{2}\left[(a-b)^2 + (b-a)^2\right] = (a-b)^2$ **Ans.[3]**

Ex.22 If the co-ordinates of two points A and B are (3, 4) and (5, –2) respectively. Find the co-ordinates of any point P if PA = PB and Area of ΔPAB = 10.

Sol. Let the co-ordinates of P be (x, y). Then

PA = PB $\Rightarrow$ $PA^2 = PB^2$

$\Rightarrow (x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2$

$\Rightarrow x - 3y - 1 = 0$

Now, Area of ΔPAB = 10

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$\Rightarrow 6x + 2y - 26 = \pm 20$

$\Rightarrow 6x + 2y - 46 = 0$ or $6x + 2y - 6 = 0$

$\Rightarrow 3x + y - 23 = 0$ or $3x + y - 3 = 0$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$ we get x = 7, y = 2. Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get x = 1, y = 0. Thus, the co-ordinates of P are (7, 2) or (1, 0)

3. AREA OF A TRIANGLE (Polar Coordinates)

If (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) are vertices of a triangle then its area Δ.

$$\Delta = \frac{1}{2}\left[r_1r_2\sin(\theta_2 - \theta_1) + r_2r_3\sin(\theta_3 - \theta_2) + r_3r_1(\theta_1 - \theta_3)\right]$$

Solved Examples

Ex.23 The area of a triangle with vertices (a,θ); $\left(2a, \theta + \frac{\pi}{3}\right)$; $\left(3a, \theta + \frac{2\pi}{3}\right)$

[1] $\frac{5\sqrt{2}}{4}a^2$ [2] $\frac{2\sqrt{5}}{4}a^2$

[3] $\frac{5\sqrt{3}}{4}a^2$ [4] $\frac{2\sqrt{3}}{4}a^2$

Sol. $= \frac{1}{2}\left[2a^2\sin\frac{\pi}{3} + 6a^2\sin\frac{\pi}{3} + 3a^2\sin\left(\frac{-2\pi}{3}\right)\right] = \frac{5\sqrt{3}}{4}a^2$

Ans.[3]

4. AREA OF A TRIANGLE WHEN EQUATIONS OF ITS SIDES ARE GIVEN:

If $a_rx + b_ry + c_r = 0 (r = 1,2,3)$ are sides of a triangle then its area is given by

$$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

when C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 in the determinant.

Ex.24 If $x - y = 1$, $x + 2y = 0$ and $2x + y = 3$ are sides of a triangle, then its area is

[1] $\frac{2}{3}$ [2] $\frac{3}{2}$ [3] 2 [4] $\frac{1}{2}$

Sol. Area = $\frac{1}{2(-3)(-3)3}\begin{vmatrix} 1 & -1 & -1 \\ 1 & 2 & 0 \\ 2 & 1 & -3 \end{vmatrix}^2 = \frac{1}{54}\times 36 = \frac{2}{3}$.

Ans.[1]

LOCUS OF A POINT

The locus of a moving point is the path traced out by that point under one or more given conditions.

How to find the locus of a point : Let (x_1, y_1) be the co-ordinate of the moving points say P. Now apply the geometrical conditions on x_1, y_1. This gives a relation between x_1, and y_1. Now replace x_1 by x and y_1 by y in the eleminant and resulting equation would be the equation of the locus.

Note :

(i) Locus of of a point P which is equidistant from the two point A and B is straight line and is a perpendicular bisector of line AB.

(ii) In above case if $PA = KPB$ where $K \neq 1$ then the locus of P is a circle.

(iii) Locus of P if A and B are fixed.

(a) Circle if $\angle APB = \text{constant}$

(b) Circle with diameter AB if $\angle APB = \frac{\pi}{2}$

(c) Ellipse if $PA + PB = \text{constant}$

(d) Hyperbola if $PA - PB = \text{constant}$

Solved Examples

Ex.25 The locus of a point such that the sum of its distances from the points $(0,2)$ and $(0,-2)$ is 6 is

[1] $9x^2 + 5y^2 = 45$

[2] $5x^2 + 9y^2 = 45$

[3] $4x^2 + 7y^2 = 35$

[4] $9x^2 + 5y^2 = 50$

Sol. Let $P(h,k)$ be any point on the locus and let $A(0,2)$ and $B(0,-2)$ be the given points.

By the given condition $PA + PB = 6$

$$\Rightarrow \sqrt{(h-0)^2 + (k-2)^2} + \sqrt{(h-0)^2 + (k+2)^2} = 6$$

$$\Rightarrow \sqrt{h^2 + (k-2)^2} = 6 - \sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow h^2 + (k-2)^2 = 36 - 12\sqrt{h^2 + (k+2)^2} + h^2 + (k+2)^2$$

$$\Rightarrow -8k - 36 = -12\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow (2k+9)^2 = 9\left(h^2 + (k+2)^2\right)$$

$$\Rightarrow 4k^2 + 36k + 81 = 9h^2 + 9k^2 + 36k + 36$$

$$\Rightarrow 9h^2 + 5k^2 = 45$$

Hence, locus of (h,k) is $9x^2 + 5y^2 = 45$

Ans.[1]

STRAIGHT LINE

EQUATION OF STRAIGHT LINE

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called the equation of straight line. Every linear equation in two variable x and y always represents a straight line

eg. $3x + 4y = 5$

General form of straight line is given by

$ax + by + c = 0$

EQUATION OF THE AXES

(1) Equation of x-axis : $y = 0$

(2) Equation a line parallel to x axis at a distance 'a' from it $\Rightarrow$ $y = a$

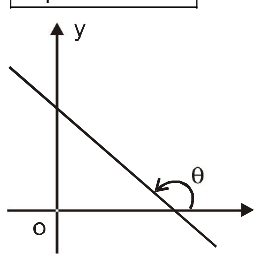
(3) Equation of y-axis : $x = 0$

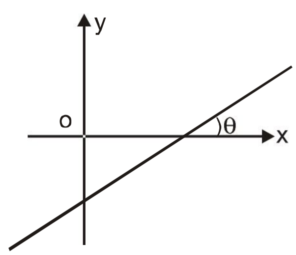
(4) Equation of a line parallel to y axis at a distance 'a' from it $\Rightarrow$ $x = a$

SLOPE OF A LINE

The slope of a line is equal to the tangent of the angle which it makes with the positive side of x-axis and it is generally denoted by m.

Thus if a line makes an angle θ with x-axis then its $\boxed{\text{slope} = m = \tan\theta}$





The slope of a line joining two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

Note :

(i) $-\infty < m \le \infty$

(ii) Slope of x axis or a line parallel to x axis is $\tan 0° = 0$

(iii) Slope of y axis or a line parallel to y axis is $\tan 90° = \infty$

DIFFERENT FORMS OF THE EQUATION OF STRAIGHT LINE

1. General form = $ax + by + c = 0$

where a , b, c are any real numbers and a, b are not zero both at a time.

Particular case ; In $ax + by + c = 0$

$a = 0 \Rightarrow by + c = 0$

which is a line parallel to x axis

$b = 0 \Rightarrow ax + c = 0$

which is a line parallel to y axis

$c = 0 \Rightarrow ax + by = 0$

which is a line passing through origin.

Solved Examples

Ex.1 What is the slope of a line whose inclination with the positive direction of x-axis is :

(i) 0º (ii) 90º (iii) 120º (iv) 150º

Sol. (i) Here $\theta = 0º$

Slope = $\tan\theta = \tan 0º = 0$.

(ii) Here $\theta = 90º$

$\therefore$ The slope of line is not defined.

(iii) Here $\theta = 120º$

$\therefore$ Slope = $\tan\theta = \tan 120º = \tan(180º - 60º)$

$= -\tan 60º = -\sqrt{3}$.

(iv) Here $\theta = 150º$

$\therefore$ Slope = $\tan\theta = \tan 150º = \tan(180º - 30º)$

$= -\tan 30º = -\frac{1}{\sqrt{3}}$

Ex.2 Find the slope of the line passing through the points :

(i) (1, 6) and (– 4, 2) (ii) (5, 9) and (2, 9)

Sol. (i) Let A = (1, 6) and B = (– 4, 2)

$\therefore$ Slope of AB = $\frac{2-6}{-4-1} = \frac{-4}{-5} = \frac{4}{5}$

$\left(\text{Using slope} = \frac{y_2 - y_1}{x_2 - x_1}\right)$

(ii) Let A = (5, 9), B = (2, 9)

$\therefore$ Slope of AB = $\frac{9-9}{2-5} = \frac{0}{-3} = 0$

2. Point -Slope form :

$y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

Solved Examples

Ex.3 Find the equation of a line passing through (2, –3) and inclined at an angle of 135º with the positive direction of x-axis.

Sol. Here, m = slope of the line = tan 135º = tan (90º + 45º) = – cot 45º = –1, $x_1 = 2$, $y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or

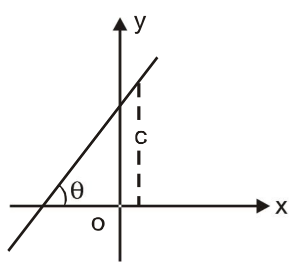
$y + 3 = -x + 2$ or $x + y + 1 = 0$

3. Slope form :- $\boxed{y = mx + c}$

Where 'm' is the slope of the line and 'c' is the length of the intercept made by it on y axis

for general form $ax + by + c = 0$

$$\text{slope} = \frac{-a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$$



Solved Examples

Ex.4 Find the equation of a line with slope –1 and cutting off an intercept of 4 units on negative direction of y-axis.

Sol. Here m = –1 and c = – 4. So, the equation of the line is $y = mx + c$

i.e. $y = -x - 4$ or $x + y + 4 = 0$

Ex.5 Equation of a line which is passing through (3 , –4) and making an angle of 45° with x axis is

[1] $x - y - 7 = 0$ [2] $x + y + 7 = 0$
[3] $x - y + 7 = 0$ [4] $x + y - 7 = 0$

Sol. $y - (-4) = \tan 45°(x - 3) \Rightarrow y + 4 = x - 3$

$\Rightarrow x - y - 7 = 0$ **Ans.[1]**

4. Two point form : The equation of a line passing through two given points (x_1, y_1) and $(x_2 - y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Solved Examples

Ex.6 Equation of a line passing through (3 , – 4) and (4, 3) is

[1] $y = 7x + 25$ [2] $y = 7x - 25$
[3] $y = -7x - 25$ [4] $y = -7x + 25$

Sol. $y + 4 = \frac{3+4}{4-3}(x - 3)$ or $y = 7x - 25$ **Ans.[2]**

Ex.7 Find the equation of the line joining the points (– 1, 3) and (4, – 2)

Sol. Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{3-(-2)}{-1-4}(x + 1) \Rightarrow y - 3 = -x - 1$$

$\Rightarrow x + y - 2 = 0$

5. Determinant form : Equation of line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Solved Examples

Ex.8 Find the equation of line passing through (2, 4) & (– 1, 3).

Sol. $\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \Rightarrow x - 3y + 10 = 0$

6. Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$

It is the equation of straight line which cuts off intercepts a and b on the axes of x and y respectively.

(a) length of the intercept of the line between coordinate axes $AB = \sqrt{a^2 + b^2}$

(b) Area of Δ OAB = $\frac{ab}{2}$

(c) Coordinates of the points P which divides AB in the ratio $m_1 : m_2$ are $\left(\frac{m_2 a}{m_1 + m_2}, \frac{m_1 b}{m_1 + m_2}\right)$

Solved Examples

Ex.9 Equation of a line which makes intercepts 3 and 4 on x axis and y axis respectively is

[1] $4x + 3y = 12$ [2] $3x + 4y = 12$
[3] $6x + y = 12$ [4] $4x - 3y = 12$

Sol. $\frac{x}{3}+\frac{y}{4}=1 \Rightarrow 4x + 3y = 12$ **Ans.[1]**

Ex.10 Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

Sol. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ....(i)

This passes through (3, 4), therefore $\frac{3}{a} + \frac{4}{b} = 1$ ..(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$.

Putting $b = 14 - a$ in (ii), we get $\frac{3}{a} + \frac{4}{14-a} = 1$

$\Rightarrow a^2 - 13a + 42 = 0$

$\Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7, 6$

For $a = 7$, $b = 14 - 7 = 7$ and

for $a = 6$, $b = 14 - 6 = 8$.

Putting the values of a and b in (i), we get the equations of the lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or

$x + y = 7$ and $4x + 3y = 24$

7. Perpendicular/Normal form :

$x\cos\alpha + y\sin\alpha = p$ (where $p > 0, 0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x–axis.

Solved Examples

Ex.11 Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30º with the positive direction of the x-axis.

Sol. Here $p = 3$, $\alpha = 30º$

$\therefore$ Equation of the line in the normal form is

$x \cos 30º + y \sin 30º = 3$ or $x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3$ or

$\sqrt{3}x + y = 6$

Ex.12 Equation of a line on which length of perpendicular from origin is 4 and inclination of this perpendicular is 60° with the positive direction of x axis is

[1] $x + y\sqrt{3} = 8$ [2] $x - y\sqrt{3} = 8$
[3] $y - x\sqrt{3} = 8$ [4] $y + x\sqrt{3} = 8$

Sol. $x \cos 60° + y \sin 60° = 4$

$\frac{x}{2}+\frac{y\sqrt{3}}{2}=4 \Rightarrow x+y\sqrt{3}=8$ **Ans.[1]**

8. Parametric form

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

It is the equation of a straight line passes through a given point A (x_1, y_1) and makes an angle θ with x axis. The coordinates (x , y) of any point P on this line are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. The distance of this point P from the given point A is $\sqrt{(x_1+r\cos\theta-x_1)^2+(y_1+r\sin\theta-y_1)^2} = r$

Solved Examples

Ex.13 Equation of a line which passes through point A(2,3) and makes an angle of 45° with x axis. If this line meet the line $x + y + 1 = 0$ at point P then distance AP is

[1] $2\sqrt{3}$ [2] $3\sqrt{2}$ [3] $5\sqrt{2}$ [4] $2\sqrt{5}$

Sol. Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$

hence $\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow x - 2 = y - 3$

$\Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$

if this point is on line $x + y + 1 = 0$ then

$\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right) + 1 = 0$

$\Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$ **Ans.[2]**

Ex.14 Find the equation of the line through the point A(2, 3) and making an angle of 45º with the x-axis. Also determine the length of intercept on it between A and the line $x + y + 1 = 0$

Sol. The equation of a line through A and making an angle of 45º with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \text{ or}$$

$x - y + 1 = 0$

Suppose this line meets the line $x + y + 1 = 0$ at P such that AP = r. Then the co-ordinates of P are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$\Rightarrow x = 2 + r\cos 45^\circ,\ y = 3 + r\sin 45^\circ$

$\Rightarrow x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$

Thus, the co-ordinates of P are $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$

Since P lies on $x + y + 1 = 0$,

so $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2}$

$\Rightarrow$ length $AP = |r| = 3\sqrt{2}$

Thus, the length of the intercept $= 3\sqrt{2}$.

REDUCTION OF GENERAL FORM OF EQUATION INTO STANDARD FORM

General form of equation $ax + by + c = 0$ then its

1. **Slope intercept form is**

   $y = -\frac{ax}{b} - \frac{c}{b}$, here slope $m = -\frac{a}{b}$,

   intercept $c = \frac{-c}{b}$

2. **Intercept form is**

   $\frac{x}{-c/a} + \frac{y}{-c/b} = 1$, here x intercept is $= -\frac{c}{a}$,

   y intercept is $= -\frac{c}{b}$

3. **Normal form is :-** To change the general form of a line into normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$ like

   $$-\frac{ax}{\sqrt{a^2+b^2}} - \frac{by}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}},$$

   here $\cos\alpha = -\frac{a}{\sqrt{a^2+b^2}}$

   $\sin\alpha = -\frac{b}{\sqrt{a^2+b^2}}$ and $p = \frac{c}{\sqrt{a^2+b^2}}$

Solved Examples

Ex.15 Find slope, x-intercept & y-intercept of the line $2x - 3y + 5 = 0$.

Sol. Here, $a = 2,\ b = -3,\ c = 5$

$\therefore$ slope $= -\frac{a}{b} = \frac{2}{3}$

x-intercept $= -\frac{c}{a} = -\frac{5}{2}$ y-intercept $= \frac{5}{3}$

ANGLE BETWEEN TWO LINES

1. The angle θ between two lines whose slopes are m_1 and m_2 is given by

   $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

   (i) If the lines are parallel, then $m_1 = m_2$

   (ii) If the lines are perpendicular, then $m_1 m_2 = -1$

2. If the given equation of straight line are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the angle between the lines 'θ' is given by

   $\tan\theta = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$

   (i) If the lines are parallel, then

   $a_2b_1 - a_1b_2 = 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

   (ii) If the lines are perpendicular, then $a_1a_2 + b_1b_2 = 0$

   (iii) Coincident :- $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

   (iv) In particulars, angle between lines $a_1x + b_1y + c_1 = 0$ and $b_2y + c_2 = 0$ is $\tan^{-1}\left|\frac{a_1}{b_1}\right|$

   and angle between lines $a_1x + b_1y + c_1 = 0$ and $a_2x + c_2 = 0$ is $\tan^{-1}\left|\frac{b_1}{a_1}\right|$

   (v) Angle between lines

   $x\cos\alpha + y\sin\alpha = p_1$ and $x\cos\beta + y\sin\beta = p_2$

   $= |\alpha - \beta|$

Solved Examples

Ex.16 The acute angle between two lines is $\pi/4$ and slope of one of them is 1/2. Find the slope of the other line.

Sol. If θ be the acute angle between the lines with slopes m_1 and m_2, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $\theta = \dfrac{\pi}{4}$ and $m_1 = \dfrac{1}{2}$

$\therefore \tan\dfrac{\pi}{4} = \left|\dfrac{\frac{1}{2} - m_2}{1 + \frac{1}{2}m_2}\right|$

$\Rightarrow 1 = \left|\dfrac{1 - 2m_2}{2 + m_2}\right|$

$\Rightarrow \dfrac{1 - 2m_2}{2 + m_2} = +1$ or -1

Ex.17 Find the equation of the straight line which passes through the origin and making angle 60º with the line $x + \sqrt{3}y + 3\sqrt{3} = 0$.

Sol. Given line is $x + \sqrt{3}y + 3\sqrt{3} = 0$.

$\Rightarrow y = \left(-\dfrac{1}{\sqrt{3}}\right)x - 3$

$\therefore$ Slope of (1) $= -\dfrac{1}{\sqrt{3}}$.

Let slope of the required line be m. Also between these lines is given to be 60º.

$\Rightarrow \tan 60^\circ = \left|\dfrac{m - (-1/\sqrt{3})}{1 + m(-1/\sqrt{3})}\right|$

$\Rightarrow \sqrt{3} = \left|\dfrac{\sqrt{3}m + 1}{\sqrt{3} - m}\right| \quad \Rightarrow \dfrac{\sqrt{3}m + 1}{\sqrt{3} - m} = \pm\sqrt{3}$

$\dfrac{\sqrt{3}m + 1}{\sqrt{3} - m} = \sqrt{3} \quad \Rightarrow \sqrt{3}m + 1 = 3 - \sqrt{3}m$

$\Rightarrow m = \dfrac{1}{\sqrt{3}}$

Using $y = mx + c$, the equation of the required line is

$y = \dfrac{1}{\sqrt{3}}x + 0$

i.e. $x - \sqrt{3}y = 0$.

($\because$ This passes through origin, so $c = 0$)

$\dfrac{\sqrt{3}m + 1}{\sqrt{3} - m} = -\sqrt{3} \quad \Rightarrow \sqrt{3}m + 1 = -3 + \sqrt{3}m$

$\Rightarrow$ m is not defined

$\therefore$ The slope of the required line is not defined. Thus, the required line is a vertical line. This line is to pass through the origin.

$\therefore$ The equation of the required line is $x = 0$

Ex.18 The angle between the lines $2x - y + 1 = 0$ and $x + 5 = 0$ is given by

[1] $\tan^{-1}\dfrac{1}{2}$ [2] $\tan^{-1} 1$

[3] $\tan^{-1} 2$ [4] none of these

Sol. $\tan^{-1}\left|\dfrac{-1}{2}\right| = \tan^{-1}(1/2)$ (use form (iv))

Ans.[1]

Ex.19 The angle between the lines $2x - y + 1 = 0$ and $2y + 1 = 0$ is given by

[1] $\tan^{-1}\dfrac{1}{2}$ [2] $\tan^{-1} 1$

[3] $\tan^{-1} 2$ [4] none of these

Sol. $\tan^{-1}\left|\dfrac{2}{-1}\right| = \tan^{-1} 2$ (use form (iv)) **Ans.[3]**

Ex.20 Angle between $y = x + 6$ and $y = \sqrt{3}x + 7$ is here $m_1 = 1$ $m_2 = \sqrt{3}$

[1] 75 [2] 45 [3] 15 [4] 30

Sol. $\tan\theta = \left|\dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}\right| \Rightarrow \theta = \tan^{-1}\dfrac{1 - \sqrt{3}}{1 + \sqrt{3}} = 15^\circ$ **Ans.[3]**

Ex.21 If A(1,2), B(–1, 3) and C (3, –5) be vertices of a triangle then find $\angle B$

[1] $\tan^{-1}\dfrac{1}{2}$ [2] $\tan^{-1} 1$

[3] $\tan^{-1} 2$ [4] $\tan^{-1}\dfrac{3}{4}$

Sol. Slope of AB $= m_1 = -1/2$

Slope of BC $= m_2 = -2$

$\therefore \angle B = \tan^{-1}\left|\dfrac{-1/2 + 2}{1 + (-1/2)(-2)}\right| = \tan^{-1}(3/4)$ **Ans.[4]**

Ex.22 If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is

(1) 3 (2) 4 (3) –1 (4) –4

Sol. $m_1 = -\frac{1}{4}$ $\quad m_2 = -\frac{4}{k}$

Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right)\times\left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1$ **Ans.[3]**

Ex.23 If $7x + 3y + 9 = 0$ and $y = kx + 7$ are two parallel lines then k is

(1) 3/7 (2) –7/3 (3) 3 (4) 7

Sol. $m_1 = -\frac{7}{3}$ $\quad m_2 = k$

Two lines are parallel if $m_1 = m_2$; $k = -7/3$

Ans.[2]

Parallel Lines :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where 'd' is a parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

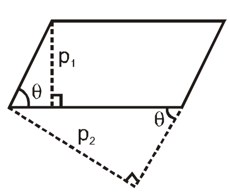
Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is $= \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

Note that coefficients of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given

by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$



Solved Examples

Ex.24 Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line $2x - 3y = 7$.

Sol. Given line is $2x - 3y = 7$

(1) $\Rightarrow$ $3y = 2x - 7$ $\Rightarrow$ $y = \frac{2}{3}x - \frac{7}{3}$

$\therefore$ Slope of (1) is 2/3

The required line is parallel to (1), so its slope is also 2/3, y-intercept of required line = 4

$\therefore$ By using $y = mx + c$ form, the equation of the required line is

$y = \frac{2}{3}x + 4$ or $2x - 3y + 12 = 0$

Ex.25 Two sides of a square lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area ?

Sol. Clearly the length of the side of the square is equal to the distance between the parallel lines

$x + y - 1 = 0$ ....(i) and

$x + y + 2 = 0$ ....(ii)

Putting $x = 0$ in (i), we get $y = 1$. So (0, 1) is a point on line (i).

Now, Distance between the parallel lines

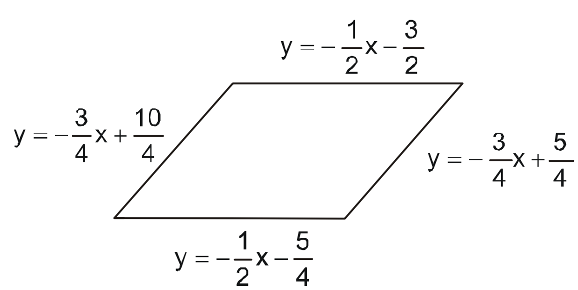
= length of the $\perp$ from (0, 1) to $x + y + 2 = 0$

$= \frac{|0+1+2|}{\sqrt{1^2+1^2}} = \frac{3}{\sqrt{2}} \; \frac{3}{\sqrt{2}}$

Thus, the length of the side of the square is $\frac{3}{\sqrt{2}}$ and hence its area $= \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$

Ex.26 Find the area of the parallelogram whose sides are $x + 2y + 3 = 0$, $3x + 4y - 5 = 0$, $2x + 4y + 5 = 0$ and $3x + 4y - 10 = 0$

Sol.



Here, $c_1 = -\frac{3}{2}$, $c_2 = -\frac{5}{4}$,

$d_1 = \frac{10}{4}$, $d_2 = \frac{5}{4}$, $m_1 = -\frac{1}{2}$, $m_2 = -\frac{3}{4}$

$$\therefore \text{ Area} = \left| \frac{\left(-\frac{3}{2}+\frac{5}{4}\right)\left(\frac{10}{4}-\frac{5}{4}\right)}{\left(-\frac{1}{2}+\frac{3}{4}\right)} \right| = \frac{5}{4} \text{ sq. units}$$

Ex.27 The distance between the lines $3x + 4y - 5 = 0$ and $3x + 4y + 7 = 0$

[1] 12 [2] 2 [3] 12/5 [4] 5/12

Sol. $= \frac{|-5-7|}{\sqrt{9+16}} = \frac{12}{5}$

Ex.28 Equation of a line which passes through (4,6) and parallel to $3x - 7y + 2 = 0$ is

(1) $3x - 7y + 30 = 0$ (2) $3x + 7y + 30 = 0$
(3) $3x - 7y - 30 = 0$ (4) $3x + 7y - 30 = 0$

Sol. Let the equation is $3x - 7y + k = 0$ this line passes through (4, 6)

Hence $3(4) - 7(6) + k = 0 \Rightarrow k = 30$

the requiired equation is $3x - 7y + 30 = 0$

Ans.[1]

Perpendicular Lines:

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form

$y = -\frac{1}{m} x + d$, where 'd' is any parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where 'k' is any parameter.

Solved Examples

Ex.29 Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$

Sol. The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$2x - 3y + \lambda = 0$ ....(i)

This passes through the point (3, 4)

$\therefore 3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting $\lambda = 6$ in (i), we get $2x - 3y + 6 = 0$, which is the required equation.

Aliter

The slope of the given line is –3/2. Since the required line is perpendicular to the given line. So, the slope of the required line is 2/3. As it passes through (3, 4). So, its equation is $y - 4 = \frac{2}{3}(x - 3)$ or $2x - 3y + 6 = 0$

POSITION OF A POINT RELATIVE TO A LINE

(i) The point (x_1, y_1) lies on the line $ax + by + c = 0$ if, $ax_1 + by_1 + c = 0$

(ii) If $P(x_1, y_1)$ and $Q(x_2, y_2)$ do not lie on the line $ax + by + c = 0$ then they are on the same side of the line, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign and they lie on the opposite side of line if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the opposite sign.

(iii) (x_1, y_1) is on origin or non origin sides of the line $ax + by + c = 0$ if $ax_1 + by_1 + c$ and c are of the same or opposite signs.

Solved Examples

Ex.30 Points (3,4) and (–9,6) lie the line $7x + 5y - 9 = 0$

[1] on same side

[2] on opposite side

[3] on the line

[4] nothing can be predicted

Sol. Putting both the points in the given equation

$\Rightarrow 7 \times 3 + 5 \times 4 - 9 = 32$ and $7 \times (-9) + 5(6) - 9 = -42$

as both are of opposite sign so they lie opposite side of the given line

Ans.[2]

Ex.31 Show that (1, 4) and (0, –3) lie on the opposite sides of the line $x + 3y + 7 = 0$.

Sol. At (1, 4), the value of $x + 3y + 7 = 1 + 3(4) + 7 = 20 > 0$.

At (0, – 3), the value of $x + 3y + 7 = 0 + 3(-3) + 7 = -2 < 0$

$\therefore$ The points (1, 4) and (0, – 3) are on the opposite sides of the given line.

The ratio in which a given line divides the line segment joining two points :

Let the given line $ax + by + c = 0$ divide the line segment joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}$. If A & B are on the same side of the given line then m/n is negative but if A & B are on opposite sides of the given line, then m/n is positive

Solved Examples

Ex.32 Find the ratio in which the line joining the points A (1, 2) and B(– 3, 4) is divided by the line $x + y - 5 = 0$.

Sol. Let the line $x + y = 5$ divides AB in the ratio k : 1 at P

$\therefore$ co-ordinate of P are $\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$

Since P lies on $x + y - 5 = 0$

$\therefore \frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0 \Rightarrow k = -\frac{1}{2}$

$\therefore$ Required ratio is 1 : 2 extrenally.

Aliter Let the ratio is m : n

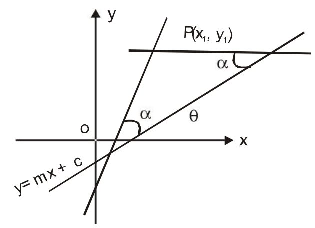
$\therefore \frac{m}{n} = -\frac{(1\times1+1\times2-5)}{1\times(-3)+1\times4-5} = -\frac{1}{2}$

$\therefore$ ratio is 1 : 2 externally.

EQUATION OF A LINE MAKING A GIVEN ANGLE WITH ANOTHER LINE

The equation of a line passing through (x_1, y_1) and making an angle α with the line $y = mx + c$ is given by

$$\boxed{y - y_1 = \frac{m \mp \tan\alpha}{1 \pm m\tan\alpha}(x - x_1)}$$



Solved Examples

Ex.33 One vertex of an equilateral triangle is (2,3) and the equation of line opposite to the vertex is $x + y = 2$, then equation of remaining two sides are

(1) $y - 3 = \pm 2(x - 2)$

(2) $y - 3 = (\sqrt{3} \pm 1)(x - 2)$

(3) $y - 3 = (2 \pm \sqrt{3})(x - 2)$

(4) None of these

Sol. Since the two sides make an angle of 60° each with side $x + y = 2$. Therefore equation of these sides will be

$$y - 3 = \frac{-1 \pm \tan 60^\circ}{1 \mp (-1)\tan 60^\circ}(x - 2) = \frac{-1 \pm \sqrt{3}}{1 \pm \sqrt{3}}(x - 2)$$

$\Rightarrow y - 3 = (2 \pm \sqrt{3})(x - 2)$ **Ans.[3]**

CONCURRENCY OF THREE LINES

Three lines $a_1x + b_1y + c_1 = 0$,
$a_2x + b_2y + c_2 = 0$
and $a_3x + b_3y + c_3 = 0$

are concurrent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note :

If P = 0, Q = 0, R = 0 are equations of three lines, then $P + Q + R = 0 \Rightarrow$ the lines are concurrent. But its converse may not be true i.e., if the three lines are concurrent then it is not necessary that $P + Q + R = 0$.

Solved Examples

Ex.34 If the lines $x + y = 1$, $2x - y = 0$ and $x + 2y + \lambda = 0$ are concurrent then find λ.

[1] –5/3 [2] 5/3 [3] 3/5 [4] –3/5

Sol. Using condition of concurrency, we have

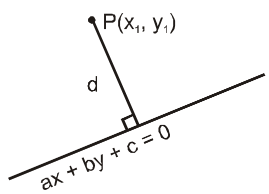
$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 1(-\lambda) - 1(2\lambda) - (4 + 1) = 0$

$\Rightarrow -3\lambda - 5 = 0 \quad \Rightarrow \lambda = -5/3.$ **Ans.[1]**

Length of perpendicular from a point on a line :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.



Solved Examples

Ex.35 Find the distance between the line $12x - 5y + 9 = 0$ and the point (2, 1)

Sol. The required distance $= \left|\dfrac{12 \times 2 - 5 \times 1 + 9}{\sqrt{12^2 + (-5)^2}}\right|$

$= \dfrac{|24 - 5 + 9|}{13} = \dfrac{28}{13}$

Ex.36 Find all points on $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.

Sol. Note that the co-ordinates of an arbitrary point on $x + y = 4$ can be obtained by putting $x = t$ (or $y = t$) and then obtaining y (or x) from the equation of the line, where t is a parameter.

Putting $x = t$ in the equation $x + y = 4$ of the given line, we obtain $y = 4 - t$. So, co-ordinates of an arbitrary point on the given line are $P(t, 4 - t)$. Let $P(t, 4 - t)$ be the required point. Then, distance of P from the line $4x + 3y - 10 = 0$ is unity i.e.

$\Rightarrow \left|\dfrac{4t + 3(4 - t) - 10}{\sqrt{4^2 + 3^2}}\right| = 1 \Rightarrow \quad |t + 2| = 5$

$\Rightarrow t + 2 = \pm 5 \quad \Rightarrow \quad t = -7$ or $t = 3$

Hence, required points are (–7, 11) and (3, 1)

Self practice problem :

Find the length of the altitudes from the vertices of the triangle with vertices :(–1, 1), (5, 2) and (3, –1).

Answer : $\dfrac{16}{\sqrt{13}}, \dfrac{8}{\sqrt{5}}, \dfrac{16}{\sqrt{37}}$

Reflection of a point about a line :

(i) Foot of the perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

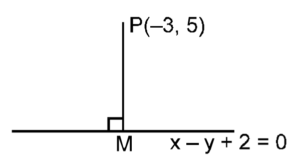
(ii) The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right).$$

Solved Examples

Ex.37 Find the foot of perpendicular of the line drawn from P (– 3, 5) on the line $x - y + 2 = 0$.

Sol. Slope of PM = – 1



$\therefore$ Equation of PM is

$x + y - 2 = 0$ ....(i)

solving equation (i) with $x - y + 2 = 0$,

we get co-ordinates of M (0, 2)

<u>Aliter</u>

Here, $\dfrac{x + 3}{1} = \dfrac{y - 5}{-1} = -\dfrac{(1 \times (-3) + (-1) \times 5 + 2)}{(1)^2 + (-1)^2}$

$\Rightarrow \dfrac{x + 3}{1} = \dfrac{y - 5}{-1} = 3$

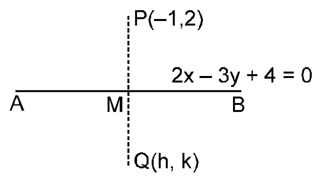
$\Rightarrow x + 3 = 3 \quad \Rightarrow \quad x = 0$ and

$y - 5 = -3 \quad \Rightarrow \quad y = 2$

$\therefore$ M is (0, 2)

Ex.38 Find the image of the point P(–1, 2) in the line mirror $2x - 3y + 4 = 0$.

Sol. Let image of P is Q.



$\therefore$ PM = MQ & PQ $\perp$ AB

Let Q is (h, k)

$\therefore$ M is $\left(\frac{h-1}{2}, \frac{k+2}{2}\right)$

It lies on $2x - 3y + 4 = 0$.

$\therefore\ 2\left(\frac{h-1}{2}\right) - 3\left(\frac{k+2}{2}\right) + 4 = 0.$

or $2h - 3k = 0$ ....(i)

slope of PQ = $\frac{k-2}{h+1}$

PQ $\perp$ AB

$\therefore\ \frac{k-2}{h+1} \times \frac{2}{3} = -1.$

$\Rightarrow 3h + 2k - 1 = 0.$ ....(ii)

soving (i) & (ii), we get $h = \frac{3}{13}, k = \frac{2}{13}$

$\therefore$ Image of P(– 1, 2) is $Q\left(\frac{3}{13}, \frac{2}{13}\right)$

Aliter

The image of P (– 1, 2) about the line $2x - 3y + 4 = 0$

is $\frac{x+1}{2} = \frac{y-2}{-3} = -2\,\frac{[2(-1) - 3(2) + 4]}{2^2 + (-3)^2}$

$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13}$

$\Rightarrow 13x + 13 = 16 \quad \Rightarrow \quad x = \frac{3}{13}$ &

$13y - 26 = -24 \quad \Rightarrow \quad y = \frac{2}{13}$

$\therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

Self practice problems :

1. Find the foot of perpendicular of the line drawn from (– 2, – 3) on the line $3x - 2y - 1 = 0$.

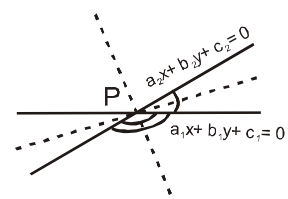
2. Find the image of the point (1, 2) in y-axis.

Answers (27) $\left(\frac{-23}{13}, \frac{-41}{13}\right)$ (28) (– 1, 2)

BISECTORS OF THE ANGLES BETWEEN TWO LINES

Bisectors of the two angles between two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$



First write the equation of the lines so that the constant terms are positive. Then

(a) If $a_1a_2 + b_1b_2 > 0$ then on taking positive sign in the above bisectors equation we shall get the obtuse angle bisector and on taking negative sign we shall get the acute angle bisector.

(b) If $a_1a_2 + b_1b_2 < 0$ then positive sign give the acute angle bisector and negative sign gives the obtuse angle bisector.

(c) On taking positive sign we shall get equation of the bisector of the angle which contains the origin and negative sign gives the equation of the bisector which does not contain origin.

Solved Examples

Ex.39 Find the equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$

[1] $11x - 3y + 9 = 0$ [2] $9x - 11y + 9 = 0$
[3] $11x - 3y - 9 = 0$ [4] $11x + 3y - 9 = 0$

Sol. The given equation can be written as

$3x - 4y + 7 = 0$ and $-12x - 5y + 2 = 0$

Here $a_1a_2 + b_1b_2 = 3(-12) - 4)(-5) < 0$, so positive sign gives the acute angle bisector which is

$$\frac{3x-4y+7}{5} = \frac{-12x-5y+2}{13}$$

$\Rightarrow 99x - 27y + 81 = 0 \quad \Rightarrow 11x - 3y + 9 = 0$

Ans.[1]

Note : This is also the bisector of the angle in which origin lies (since c_1, c_2 are positive and it has been obtained by taking positive sign)

Ex.40 Find the equations of the bisectors of the angle between the straight lines

$3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$.

Sol. The equations of the bisectors of the angles between $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ are

$$\frac{3x-4y+7}{\sqrt{3^2+(-4)^2}} = \pm \frac{12x-5y-8}{\sqrt{12^2+(-5)^2}}$$

or $$\frac{3x-4y+7}{5} = \pm \frac{12x-5y-8}{13}$$

or $39x - 52y + 91 = \pm (60x - 25y - 40)$

Taking the positive sign, we get $21x + 27y - 131 = 0$ as one bisector

Taking the negative sign, we get $99x - 77y + 51 = 0$ as the other bisector.

Ex.41 For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them;

(ii) bisector of the acute angle between them;

Sol. (i) The equations of the given straight lines are

$4x + 3y - 6 = 0$ ....(1)

$5x + 12y + 9 = 0$ ....(2)

The equation of the bisectors of the angles between lines (1) and (2) are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm \frac{5x+12y+9}{\sqrt{5^2+12^2}} \quad \text{or}$$

$$\frac{4x+3y-6}{5} = \pm \frac{5x+12y+9}{13}$$

Taking the positive sign, we have $\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13}$

or $52x + 39y - 78 = 25x + 60y + 45$

or $27x - 21y - 123 = 0$

or $9x - 7y - 41 = 0$

Taking the negative sign, we have $\frac{4x+3y-6}{5} = -\frac{5x+12y+9}{13}$

or $52x + 39y - 78 = -25x - 60y - 45$

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$

Hence the equation of the bisectors are

$9x - 7y - 41 = 0$ ....(3)

and $7x + 9y - 3 = 0$ ....(4)

Now slope of line (1) $= -\frac{4}{3}$ and slope of the bisector (3) $= \frac{9}{7}$.

If θ be the acute angle between the line (1) and the bisector (3), then

$$\tan\theta = \left|\frac{\frac{9}{7}+\frac{4}{3}}{1+\frac{9}{7}\left(-\frac{4}{3}\right)}\right| = \left|\frac{27+28}{21-36}\right| = \left|\frac{55}{-15}\right| = \frac{11}{3} > 1$$

$\therefore \theta > 45^\circ$

Hence $9x - 7y - 41 = 0$ is the bisector of the obtuse angle between the given lines (1) and (2)

(ii) Since $9x - 7y - 41$ is the bisector of the obtuse angle between the given lines, therefore the other bisector $7x + 9y - 3 = 0$ will be the bisector of the acute angle between the given lines.

__2nd Method__ :

Writing the equation of the lines so that constants become positive we have

$-4x - 3y + 6 = 0$ ....(1)

and $5x + 12y + 9 = 0$ ....(2)

Here $a_1 = -4$, $a_2 = 5$, $b_1 = -3$, $b_2 = 12$

Now $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$

$\therefore$ origin does not lie in the obtuse angle between lines (1) and (2) and hence equation of the bisector of the obtuse angle between lines (1) and (2) will be

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = -\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $13(-4x - 3y + 6) = -5(5x + 12y + 9)$

or $27x - 21y - 123 = 0$ or $9x - 7y - 41 = 0$

and the equation of the bisector of the acute angle will be (origin lies in the acute angle)

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$

To discriminate between the bisector of the angle containing a point :

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then ; $\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point (α, β) is,

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \text{ or } \frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

according as $a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

Solved Examples

Ex.42 For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the bisector of the angle which contains the origin.

Sol. For point O(0, 0), $4x + 3y - 6 = -6 < 0$ and $5x + 12y + 9 = 9 > 0$

Hence for point O(0, 0) $4x + 3y - 6$ and $5x + 12y + 9$ are of opposite signs.

Hence equation of the bisector of the angle between the given lines containing the origin will be

$$\frac{4x+3y-6}{\sqrt{(4)^2+(3)^2}} = -\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $\frac{4x+3y-6}{5} = -\frac{5x+12y+9}{13}$

or $52x + 39y - 78 = -25x - 60y - 45$.

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$

LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

If equation of two lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ then the equation of the lines passing through the point of intersection of these lines is $L_1 + \lambda L_2 = 0$ or $(a_1x + b_1y + c = 0) + \lambda (a_2x + b_2y + c_2 = 0) = 0$, value of λ is obtained with the help of the additional infomation given in the problem.

Solved Examples

Ex.43 Equation of a line passing through the point of intersection of $x + y - 3 = 0$ and $2x - y + 1 = 0$ and a point $(2, -3)$ is

[1] $4x + y - 5 = 0$ [2] $4x - y - 5 = 0$

[3] $4x + 5y + 5 = 0$ [4] $4x + 2y - 5 = 0$

Sol. $(x + y - 3) + \lambda (2x - y + 1) = 0$

as this line also passes through $(2, -3)$

Hence $(2 - 3 - 3) + \lambda (2 \times 2 - (-3) + 1) = 0$

$-4 + 8\lambda = 0$; $\lambda = 1/2$

equation is $(x + y - 3) + 1/2 (2x - y + 1) = 0$

$4x + y - 5 = 0$ **Ans.[1]**

Ex.44 Find the equation of the straight line which passes through the point (2, –3) and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.

Sol. Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$(x + y + 4) + \lambda (3x - y - 8) = 0$ ....(i)

This will pass through (2, –3) if

$(2 - 3 + 4) + \lambda (6 + 3 - 8) = 0$ or $3 + \lambda = 0$

$\Rightarrow \lambda = -3$.

Putting the value of λ in (i), the required line is

$(x + y + 4) + (-3)(3x - y - 8) = 0$

or $-8x + 4y + 28 = 0$ or $2x - y - 7 = 0$

Aliter

Solving the equations $x + y + 4 = 0$ and $3x - y - 8 = 0$ by cross-multiplication, we get $x = 1, y = -5$

So the two lines intersect at the point (1, –5). Hence the required line passes through (2, –3) and (1, –5) and so its equation is $y + 3 = \frac{-5+3}{1-2}(x - 2)$

or $2x - y - 7 = 0$

Ex.45 Obtain the equations of the lines passing through the intersection of lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and equally inclined to the axes.

Sol. The equation of any line through the intersection of the given lines is

$(4x - 3y - 1) + \lambda (2x - 5y + 3) = 0$

or $x(2\lambda + 4) - y(5\lambda + 3) + 3\lambda - 1 = 0$ ....(i)

Let m be the slope of this line. Then $m = \frac{2\lambda + 4}{5\lambda + 3}$

As the line is equally inclined with the axes, therefore $m = \tan 45^\circ$ or $m = \tan 135^\circ$

$\Rightarrow m = \pm 1, \frac{2\lambda + 4}{5\lambda + 3} = \pm 1$

$\Rightarrow \lambda = -1$ or $\frac{1}{3}$, putting the values of λ in (i), we get $2x + 2y - 4 = 0$ and $14x - 14y = 0$

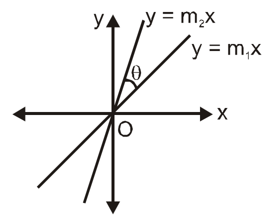
i.e. $x + y - 2 = 0$ and $x = y$ as the equations of the required lines.

A Pair of straight lines through origin:

(i) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab$ $\Rightarrow$ lines are real & distinct .

(b) $h^2 = ab$ $\Rightarrow$ lines are coincident .



(c) $h^2 < ab$ $\Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

This equation is obtained by multiplying the two equations of lines $(m_1x - y)(m_2x - y) = 0$

$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(iv) The condition that these lines are :

(a) at right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(b) coincident is $h^2 = ab$.

(c) equally inclined to the axis of x is $h = 0$. i.e. coeff. of $xy = 0$.

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angles between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Solved Examples

Ex.46 Show that the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.

Sol. The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain $a = 6$, $b = 1$ and $2h = -5$.

$\therefore\ h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0 \Rightarrow\ h^2 > ab$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now, $6x^2 - 5xy + y^2 = 0$

$\Rightarrow \left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0$

$\Rightarrow \left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0$

$\Rightarrow \left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$

$\Rightarrow \frac{y}{x} - 3 = 0$ or $\frac{y}{x} - 2 = 0 \Rightarrow y - 3x = 0$ or $y - 2x = 0$

So the given equation represents the straight lines $y - 3x = 0$ and $y - 2x = 0$.

Ex.47 Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 3y^2 = 0$.

Sol. We have $2x^2 - 7xy + 3y^2 = 0$.

$\Rightarrow 2x^2 - 6xy - xy + 3y^2 = 0$

$\Rightarrow 2x(x - 3y) - y(x - 3y) = 0$

$\Rightarrow (x - 3y)(2x - y) = 0$

$\Rightarrow x - 3y = 0$ or $2x - y = 0$

Thus the given equation represents the lines $x - 3y = 0$ and $2x - y = 0$. The equations of the lines passing through the origin and perpendicular to the given lines are $y - 0 = -3(x - 0)$ and $y - 0 = -\frac{1}{2}(x - 0)$

[$\because$ (Slope of $x - 3y = 0$) is 1/3 and (Slope of $2x - y = 0$) is 2]

$\Rightarrow y + 3x = 0$ and $2y + x = 0$

Ex.48 Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$

Here a = coeff. of $x^2 = 4$, b = coeff. of $y^2 = 11$ and $2h$ = coeff. of $xy = 24$ $\therefore\ h = 12$

Now $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{144 - 44}}{4 + 11}\right| = \frac{4}{3}$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}\left(\frac{4}{3}\right)$ and obtuse angle between them is $\pi - \tan^{-1}\left(\frac{4}{3}\right)$

Ex.49 Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$

Sol. Given equation is $3x^2 - 5xy + 4y^2 = 0$ ....(1)

comparing it with the equation $ax^2 + 2hxy + by^2 = 0$ ....(2)

we have $a = 3$, $2h = -5$; and $b = 4$

Now the equation of the bisectors of the angle between the pair of lines (1) is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

or $\frac{x^2 - y^2}{3 - 4} = \frac{xy}{-\frac{5}{2}}$; or $\frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$

or $5x^2 - 2xy - 5y^2 = 0$

Ex.50 Lines represented by $2x^2 - 7xy + 3y^2 = 0$ are

[1] $x + 2y = 0$, $x - 3y = 0$

[2] $x - 2y = 0$, $x + 3y = 0$

[3] $2x - y = 0$, $3x - y = 0$

[4] $2x - y = 0$, $x - 3y = 0$

Sol. $2x^2 - 7xy + 3y^2 = 0 = (2x - y)(x - 3y)$

$\therefore\ 2x - y = 0$, $x - 3y = 0$ **Ans.[4]**

Ex.51 Angle between the lines $2x^2 - 7xy + 3y^2 = 0$ is

[1] 45 [2] 30 [3] 60 [4] 90

Sol. $\tan\theta = \frac{2\sqrt{(-7/2)^2 - 2\times 3}}{2 + 3} \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°$

Ans.[1]

General equation of second degree representing a pair of Straight lines :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Such an equation is obtained again by multiplying the two equation of lines

$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Solved Examples

Ex.52 Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection.

Sol. Given equation is $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$

Writing the equation (1) as a quadratic equation in x we have

$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$

$$\therefore\ x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - 4.2(3y^2 + 7y + 4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^2 + 4y + 4}}{4} = \frac{-(5y+6) \pm (y+2)}{4}$$

$$\therefore\ x = \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4}$$

or $4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$

or $x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence equation (1) represents a pair of straight lines whose equation are

$x + y + 1 = 0$ .....(1)

and $2x + 3y + 4 = 0$ .....(2)

Solving these two equations, the required point of intersection is $(1, -2)$.

EQUATION OF LINES JOINING THE INTERSECTION POINTS OF A LINE & A CURVE TO THE ORIGIN

Let $lx + my + n = 0$ .... (1) and the second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ....(2)

then their joint equation is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$$

i.e. making the equation (2) homogenous using equation (1)

Solved Examples

Ex.53 Equation of pair of straight lines which are formed by joining the origin and the points of intersection of a circle $x^2 + y^2 = a^2$ and a line $y = mx + c$ is

[1] $x^2(c^2 + a^2m^2) + y(c^2 + a^2) - 2ma^2xy = 0$
[2] $x^2(c^2 - a^2m^2) - y(c^2 - a^2) + 2ma^2xy = 0$
[3] $x^2(c^2 - a^2m^2) + y(c^2 - a^2) - 2ma^2xy = 0$
[4] $x^2(c^2 - a^2m^2) + y(c^2 + a^2) - 2ma^2xy = 0$

Sol. $x^2 + y^2 = a^2\left(\frac{y - mx}{c}\right)^2$

$\Rightarrow x^2(c^2 - a^2m^2) + y(c^2 - a^2) - 2ma^2xy = 0$

Ans.[3]

Ex.54 For what value of k, the equation $kx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines

[1] 2 [2] 3
[3] 8 [4] 4

Sol. Here $a = k$, $b = 12$, $c = -3$,

$h = -5$, $g = 5/2$, $f = -8$

$\therefore \Delta = k(12)(-3) + 2(-8)(5/2)(-5) - k(-8)^2 - 12(5/2)^2 - (-3)(-5)^2 = 0$

$\Rightarrow k = 2$ **Ans.[1]**

Ex.55 The angle and equation of bisectors of angle between the lines represented by the eqation $2x^2 - 7xy + 3y^2 = 0$ are

[1] $45°, 7x^2 - 2xy - 7y^2 = 0$

[2] $60°, 7x^2 - 2xy - 7y^2 = 0$

[3] $120°, 7x^2 - 2xy - 7y^2 = 0$

[4] $45°, 7x^2 + 2xy + 7y^2 = 0$

Sol. Since $2x^2 - 7xy + 3y^2 = (2x - y)(x - 3y)$

$\therefore$ given lines are $2x - y = 0$ and $x - 3y = 0$

since $a = 2$, $h = -7/2$, $b = 3$

Therefore angle between lines given by

$$\tan\theta = \frac{2\sqrt{(-7/2)^2 - 6}}{2+3} = 1 \Rightarrow \theta = 45°$$

Again equations of bisectors are given by

$$\frac{x^2 - y^2}{2-3} = \frac{xy}{-7/2} \Rightarrow 7x^2 - 2xy - 7y^2 = 0$$

Ans.[1]

Ex.56 Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$.

Sol. Equation of the given curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ and equation of the given straight line is

$y - 3x = 2$; $\therefore \frac{y - 3x}{2} = 1$

Making equation (1) homogeneous equation of the second degree in x any y with the help of (1), we have

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right) - 11\left(\frac{y-3x}{2}\right)^2 = 0$$

or $x^2 + 2xy + 3y^2 + \frac{1}{2}(4xy + 8y^2 - 12x^2 - 24xy) - \frac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$

or $-119x^2 + 34xy + 17y^2 = 0$

or $119x^2 - 34xy - 17y^2 = 0$

or $7x^2 - 2xy - y^2 = 0$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$

we have $a = 7$, $b = -1$ and $2h = -2$ i.e. $h = -1$

If θ be the acute angle between pair of lines (3), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

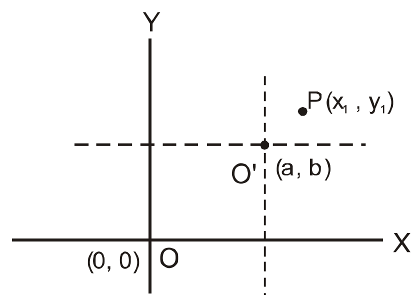
$\therefore \theta = \tan^{-1}\frac{2\sqrt{2}}{3}$.

TRANSFORMATION OF AXES

1. Parallel transformation

Let origin $O(0,0)$ be shifted to a point (a,b) by moving the x-axis and y-axis, parallel to themselves. If the co-ordinate of point P with reference to old axis are (x_1, y_1) then co-ordinate of this point with respect to new axis will be $(x_1 - a, y_1 - b)$

$P(x', y') = P(x_1 - a, y_1 - b)$



2. Rotational transformation:

Let OX and OY be the old axes and OX' and OY' be the new axes obtained by rotating the old axes OX and OY through an angle θ, again, if co-ordinates of any point $P(x,y)$ with reference to new axes will be (x', y'), then

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

The above relation between (x , y) and (x' , y') can be easily obtained with the help of following table

	x↓	y↓
x'→	$\cos\theta$	$\sin\theta$
y'→	$-\sin\theta$	$\cos\theta$

3 Reflection (image) of a point:

Let (x, y) be any point, then its image with respect to

(i) $x-\text{axis} \Rightarrow (x,-y)$

(ii) $y-\text{axis} \Rightarrow (-x,y)$

(iii) $\text{origin} \Rightarrow (-x,-y)$

(iv) line $y = x \Rightarrow (y,x)$

Solved Examples

Ex.57 If axes are transformed from origin to the point $(-2,1)$ then new co-ordinates of $(4,-5)$ is

[1] (6, –6) [2] (–6, 6)

[3] (4, –4) [4] (6, 6)

Sol. $\left[4-(-2),-5-1\right] = (6,-6)$ **Ans.[1]**

Ex.58 If the new coordinates of a point after the rotation of axes in the negative direction by an angle of $\frac{\pi}{3}$ are $(4,2)$ then coordinates with respect to old axes are

[1] $\left(2+\sqrt{3},-2\sqrt{3}+1\right)$

[2] $\left(2-\sqrt{3},-2\sqrt{3}-1\right)$

[3] $\left(2+\sqrt{3},\ 2\sqrt{3}+1\right)$

[4] $\left(2+\sqrt{3},-2\sqrt{3}-1\right)$

Sol.

	x	y
4	$\cos\left(-60^0\right)$	$\sin\left(-60^0\right)$
2	$-\sin\left(-60^0\right)$	$\cos\left(-60^0\right)$

$\therefore x = x'\cos\alpha - y'\sin\alpha$

$= 4\cos(-60) - 2\sin(-60) = 2+\sqrt{3}$

$y = x'\sin\alpha + y'\cos\alpha$

$= 4\sin(-60) + 2\cos(-60) = -2\sqrt{3}+1$

Hence the coordinates are $\left(2+\sqrt{3},-2\sqrt{3}+1\right)$

Ans.[1]