Gravitation

INTRODUCTION

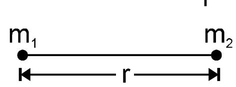
The motion of celestial bodies such as the sun, the moon, the earth and the planets etc. has been a subject of fascination since time immemorial. Indian astronomers of the ancient times have done brilliant work in this field, the most notable among them being Arya Bhatt the first person to assert that all planets including the earth revolve round the sun.

A millennium later the Danish astronomer Tycobrahe (1546-1601) conducted a detailed study of planetary motion which was interpreted by his pupil Johnaase Kepler (1571-1630), ironically after the master himself had passed away. Kepler formulated his important findings in three laws of planetary motion

UNIVERSAL LAW OF GRAVITATION

According to this law "Each particle attracts every other particle. The force of attraction between them is directly proportional to the product of their masses and inversely proportional to square of the distance between them".

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{or} \quad F = G\frac{m_1 m_2}{r^2}$$



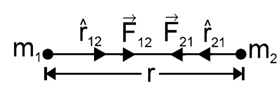
where G = 6.67×10^{-11} $Nm^2\ kg^{-2}$ is the universal gravitational constant. This law holds good irrespective of the nature of two objects (size, shape, mass etc.) at all places and all times. That is why it is known as universal law of gravitation.

Dimensional formula of G :

$$F = \frac{Fr^2}{m_1 m_2} = \frac{[MLT^{-2}]\,[L^2]}{[M^2]} = [M^{-1} L^3 T^{-2}]$$

Newton's Law of gravitation in vector form :

$$\vec{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{r}_{12} \quad \& \quad \vec{F}_{21} = \frac{Gm_1m_2}{r^2}\hat{r}_{21}$$



Where $\vec{F}_{12}$ is the force on mass m_1 exerted by mass m_2 and vice-versa.

Now $\hat{r}_{12} = -\hat{r}_{21}$,

Thus $\vec{F}_{21} = \frac{-G\,m_1\,m_2}{r^2}\hat{r}_{12}$.

Comparing above, we get $\vec{F}_{12} = -\vec{F}_{21}$

Important characteristics of gravitational force

(i) Gravitational force between two bodies form an action and reaction pair i.e. the forces are equal in magnitude but opposite in direction.

(ii) Gravitational force is a central force i.e. it acts along the line joining the centres of the two interacting bodies.

(iii) Gravitational force between two bodies is independent of the nature of the medium, in which they lie.

(iv) Gravitational force between two bodies does not depend upon the presence of other bodies.

(v) Gravitational force is negligible in case of light bodies but becomes appreciable in case of massive bodies like stars and planets.

(vi) Gravitational force is long range-force i.e., gravitational force between two bodies is effective even if their separation is very large. For example, gravitational force between the sun and the earth is of the order of 10^{27} N although distance between them is 1.5×10^7 km

Solved Examples

Ex.1 The centres of two identical spheres are at a distance 1.0 m apart. If the gravitational force between them is 1.0 N, then find the mass of each sphere.
($G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-1}$)

Sol. Gravitational force $F = \frac{Gm.m}{r^2}$

on substituting F = 1.0 N , r = 1.0 m and

$G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-1}$

we get $m = 1.225 \times 10^5$ kg

Ex-2 Two particles of masses m_1 and m_2, initially at rest at infinite distance from each other, move under the action of mutual gravitational pull. Show that at any instant their relative velocity of approach is $\sqrt{2G(m_1+m_2)/R}$, where R is their separation at that instant.

Sol. The gravitational force of attraction on m_1 due to m_2 at a separation r is

$$F_1 = \frac{Gm_1m_2}{r^2}$$

Therefore, the acceleration of m_1 is

$$a_1 = \frac{F_1}{m_1} = \frac{Gm_2}{r^2}$$

Similarly, the acceleration of m_2 due to m_1 is

$$a_2 = -\frac{Gm_1}{r^2}$$

the negative sign being put as a_2 is directed opposite to a_1. The relative acceleration of approach is

$$a = a_1 - a_2 = \frac{G(m_1+m_2)}{r^2} \qquad \text{.... (1)}$$

If v is the relative velocity, then

$$a = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt}.$$

But $-\frac{dr}{dt} = v$ (negative sign shows that r decreases with increaing t).

$$\therefore\ a = -\frac{dv}{dr}v. \qquad \text{.... (2)}$$

From (1) and (2), we have

$$v\,dv = -\frac{G(m_1+m_2)}{r^2}dr$$

Integrating, we get

$$\frac{v^2}{2} = \frac{G(m_1+m_2)}{r} + C$$

At $r = \infty$, $v = 0$ (given), and so $C = 0$.

$$\therefore\ v^2 = \frac{2G(m_1+m_2)}{r}$$

Let $v = v_R$ when $r = R$. Then

$$v_R = \sqrt{\left(\frac{2G(m_1+m_2)}{R}\right)}$$

Ex-3 Three identical bodies of mass M are located at the vertices of an equilateral triangle with side L. At what speed must they move if they all revolve under the influence of one another's gravity in a circular orbit circumscribing the triangle while still preserving the equilateral triangle ?

Sol. Let A, B and C be the three masses and O the centre of the circumscribing circle. The radius of this circle is

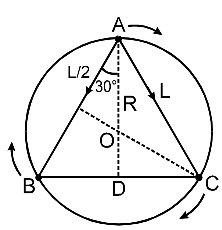
$$R = \frac{L}{2}\sec 30° = \frac{L}{2}\times\frac{2}{\sqrt{3}} = \frac{L}{\sqrt{3}}.$$

Let v be the speed of each mass M along the circle. Let us consider the motion of the mass at A. The force of gravitational attraction on it due to the masses at B and C are

$$\frac{GM^2}{L^2} \text{ along AB} \quad \text{and} \quad \frac{GM^2}{L^2} \text{ along AC}$$

The resultant force is therefore

$$2\frac{GM^2}{L^2}\cos 30° = \frac{\sqrt{3}\,GM^2}{L^2} \text{ along AD.}$$



This, for preserving the triangle, must be equal to the necessary centripetal force.

That is ,

$$\frac{\sqrt{3}\,GM^2}{L^2} = \frac{Mv^2}{R} = \frac{\sqrt{3}\,Mv^2}{L}$$

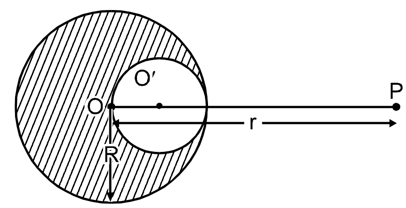
$$[\because R = L/\sqrt{3}\,] \quad \text{or} \quad v = \sqrt{\frac{GM}{L}}$$

Ex.4 Find out the time period of circular motion in above example

Ans. $\frac{(2\pi L^{3/2})}{\sqrt{3GM}}$

Ex.- 5 A solid sphere of lead has mass M and radius R.A spherical hollow is dug out from it (see figure). Its boundary passing through the centre and also touching the boundary of the solid sphere. Deduce the gravitational force on a mass m placed at P, which is distant r from O along the line of centres.

Sol. Let O be the centre of the sphere and O' that of the hollow (figure). For an external point the sphere behaves as if its entire mass is concentrated at its centre. Therefore, the gravitatinal force on a mass \`m\` at P due to the original sphere (of mass M) is



$$F = G\frac{Mm}{r^2}, \text{ along PO.}$$

The diameter of the smaller sphere (which would be cut off) is R, so that its radius OO' is R/2. The force on m at P due to this sphere of mass M' (say) would be

$$F' = G\frac{M'm}{(r-\frac{R}{2})^2} \text{ along PO}'.$$

$$[\because \text{distance PO}' = r - \frac{R}{2}\,]$$

As the radius of this sphere is half of that of the original sphere, we have

$$M' = \frac{M}{8}.$$

$$\therefore \qquad F' = G\frac{Mm}{8(r-\frac{R}{2})^2} \text{ along PO}'.$$

As both F and F′ point along the same direction, the force due to the hollowed sphere is

$$F - F' = \frac{GMm}{r^2} - \frac{GMm}{8r^2(1-\frac{R}{2r})^2}$$

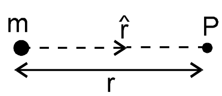
$$= \frac{GMm}{r^2}\left\{1-\frac{1}{8(1-\frac{R}{2r})^2}\right\}.$$

GRAVITATIONAL FIELD

The space surrounding the body within which its gravitational force of attraction is experienced by other bodies is called gravitational field. Gravitational field is very similar to electric field in electrostatics where charge 'q' is replaced by mass 'm' and electric constant 'K' is replaced by gravitational constant 'G'. The intensity of gravitational field at a points is defined as the force experienced by a unit mass placed at that point.

$$E = \frac{F}{m} = \frac{GM}{r^2}$$

The unit of the intensity of gravitational field is N kg^{-1}. In vector form $\vec{E} = -\frac{GM}{r^2}\hat{r}$



Dimensional formula of intensity of gravitational field

$$= \frac{F}{m} = \frac{[MLT^{-2}]}{[M]} = [M^0 LT^{-2}]$$

Solved Examples

Ex-6 Find the relation between the gravitational field on the surface of two planets A & B of masses m_A, m_B & radius R_A & R_B respectively if

(i) they have equal mass

(ii) they have equal (uniform) density

Let E_A & E_B be the gravitational field intensities on the surface of planets A & B.

then, $$E_A = \frac{Gm_A}{R_A^2} = \frac{G\frac{4}{3}\pi R_A^3 \rho_A}{R_A^2} = \frac{4G\pi}{3}\rho_A R_A$$

Similarly, $$E_B = \frac{Gm_B}{R_{B^2}} = \frac{4G}{3}\pi\ \rho_B R_B$$

(i) For $m_A = m_B$ $$\frac{E_A}{E_B} = \frac{R_B^2}{R_A^2}$$

(ii) For & $\rho_A = \rho_B$ $$\frac{E_A}{E_B} = \frac{R_A}{R_B}$$

GRAVITATIONAL POTENTIAL

The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy). Gravitational potential is very similar to electric potential in electrostatics.

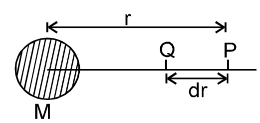
Let the unit mass be displaced through a distance dr towards mass M, then work done is given by

$$dW = F\,dr = \frac{GM}{r^2}dr \Rightarrow \int dW = \int_{\infty}^{r}\frac{GM}{r^2}dr = \frac{-GM}{r}.$$

Thus gravitational potential, $\boxed{V = -\frac{GM}{r}}$.

The unit of gravitational potential is J kg^{-1}. Dimensional Formula of gravitational potential

$$= \frac{\text{Work}}{\text{mass}} = \frac{[ML^2T^{-2}]}{[M]} = [M^\circ L^2 T^{-2}].$$



RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

The work done by an external agent to move unit mass from a point to another point in the direction of the field E, slowly through an infinitesimal distance dr = Force by external agent × distance moved = – Edr.

Thus $dV = -Edr \Rightarrow E = -\frac{dV}{dr}$.

Therefore, gravitational field at any point is equal to the negative gradient at that point.

Solved Examples

Ex.7 The gravitational field in a region is given by $\vec{E} = -(20\text{N/kg})\ (\hat{i} + \hat{j})$. Find the gravitational potential at the origin (0, 0) – (in J/kg)

(A*) zero (B) $20\sqrt{2}$

(C) $-20\sqrt{2}$ (D) can not be defined

Sol. $V = -\int E.dr = \left|\int Ex.dx + \int Ey.dy\right|$

$= 20x + 20y$

at origin V = 0

Ex.8 In above problem, find the gravitational potential at a point whose co-ordinates are (5, 4) – (in J/kg)

(A) – 180 (B*) 180
(C) – 90 (D) zero

Sol. $V = 20 \times 5 + 20 \times 4 = 180$ J/kg

Ex.9 In the above problem, find the work done in shifting a particle of mass 1 kg from origin (0, 0) to a point (5, 4) – (In J)

(A) – 180 (B*) 180
(C) – 90 (D) zero

Sol. $W = m(V_f - V_i) = 1(180 - 0) = 180$ J

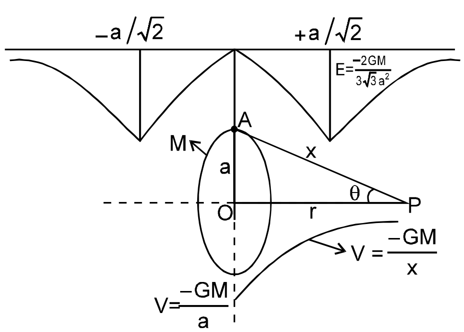
GRAVITATIONAL POTENTIAL & FIELD FOR DIFFERENT OBJECTS

I. Ring. $V = \frac{-GM}{x \text{ or } (a^2 + r^2)^{1/2}}$

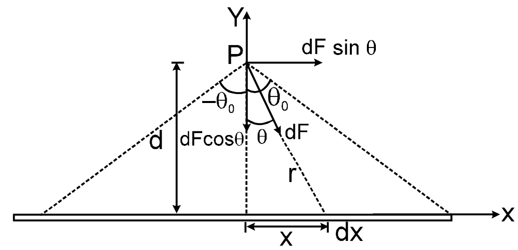
& $E = \frac{-GMr}{(a^2 + r^2)^{3/2}}\hat{r}$

or $E = -\frac{GM\cos\theta}{x^2}$

Gravitational field is maximum at a distance, $r = \pm a/\sqrt{2}$ and it is $-2GM/3\sqrt{3}a^2$



II. A linear mass of finite length on its axis :



(a) Potential : ⇒ $V = -\frac{GM}{L}\ell n(\sec\theta_0 + \tan\theta_0)$

$= -\frac{GM}{L}\ln\left\{\frac{L + \sqrt{L^2 + d^2}}{d}\right\}$

(b) Field intensity : ⇒ $E = -\frac{GM}{Ld}\sin\theta_0$

$= \frac{GM}{d\sqrt{L^2 + d^2}}$

III. An infinite uniform linear mass distribution of linear mass density λ, Here $\theta_0 = \frac{\pi}{2}$.

And noting that $\lambda = \frac{M}{2L}$ in case of a finite rod

we get, for field intensity $E = \frac{2G\lambda}{d}$

Potential for a mass-distribution extending to infinity is not defined. However even for such mass distributions potential-difference is defined. Here potential difference between points P_1 and P_2 respectively at distances d_1 and d_2 from the infinite rod, $v_{12} = 2G\lambda\, \ell n\frac{d_2}{d_1}$

IV. Uniform Solid Sphere

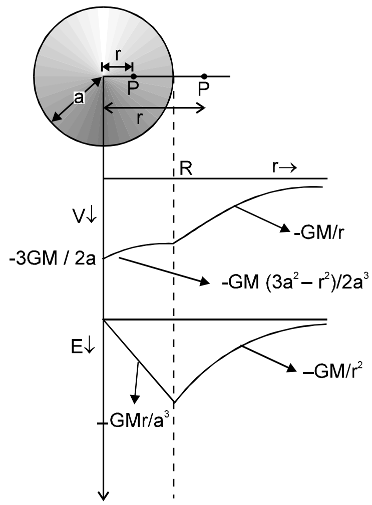
(a) Point P inside the shell. $r \le a$, then

$V = -\frac{GM}{2a^3}(3a^2 - r^2)$ & $E = -\frac{GMr}{a^3}$,

and at the centre $V = -\frac{3GM}{2a}$ and E = 0

(b) Point P outside the shell. $r \geq a$, then $V = -\dfrac{GM}{r}$

& $E = -\dfrac{GM}{r^2}$



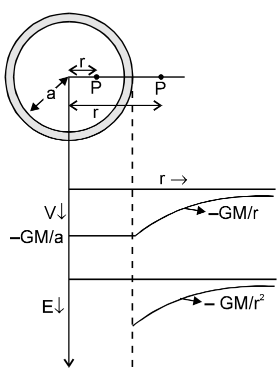
V. Uniform Thin Spherical Shell

(a) Point P Inside the shell.

$r \leq a$, then $V = \dfrac{-GM}{a}$ & $E = 0$

(b) Point P outside shell.

$r \geq a$, then $V = \dfrac{-GM}{r}$ & $E = -\dfrac{GM}{r^2}$



VI. Uniform Thick Spherical Shell

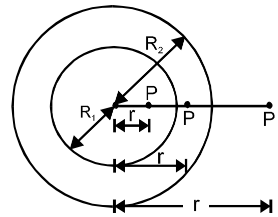
(a) Point outside the shell

$$V = -G\frac{M}{r}\ ;\ E = -G\frac{M}{r^2}$$

(b) Point inside the Shell

$$V = -\frac{3}{2}GM\left(\frac{R_2 + R_1}{R_2^2 + R_1R_2 + R_1^2}\right)$$

$E = 0$



(c) Point between the two surface

$$V = -\frac{GM}{2r}\left(\frac{3rR_2^2 - r^3 - 2R_1^3}{R_2^3 - R_1^3}\right)\ ;$$

$$E = -\frac{GM}{r^2}\ \frac{r^3 - R_1^3}{R_2^3 - R_1^3}$$

Solved Examples

Ex.10 Calculate the gravitational field intensity at the center of the base of a hollow hemisphere of mass M and radius R. (Assume the base of hemisphere to be open)

Sol. We consider the shaded elemental ring of mass, dm

$$= \frac{M}{(2\pi R^2)}\ 2\pi R \sin\theta\,(R d\theta)$$

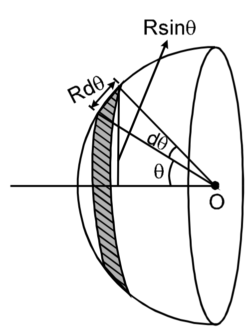
Field due to this ring at 0,

$$dE = \frac{Gdm R\cos\theta}{R^3}$$

(see formulae for field due to a ring)

or, $dE = \dfrac{GM}{R^2}\ \sin\theta \cos\theta\, d\theta$

Hence, $E = \displaystyle\int_0^{\pi/2} dE = \int_0^{\pi/2} \frac{GM}{R^2}\ \sin\theta\cos\theta\, d\theta$



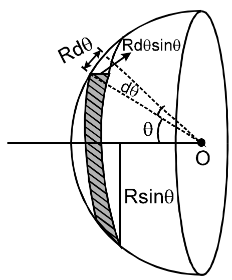
or, $E = \dfrac{GM}{2R^2}$

Ex.11 Calculate the gravitational field intensity and potential at the centre of the base of a solid hemisphere of mass m, radius R.

Sol. We consider the shaded elemental disc of radius R $\sin\theta$ and thickness $Rd\theta$

Its mass, $dM = \frac{M}{\frac{2}{3}\pi R^3}\pi(R\sin\theta)^2(Rd\theta\sin\theta)$

or $dM = \frac{3M}{2}\sin^3\theta\, d\theta$



Field due to this plate at O,

$$dE = \frac{2GdM(1-\cos\theta)}{(R\sin\theta)^2}$$

(see field due to a uniform disc)

$$\text{or } dE = \frac{3GM\sin\theta(1-\cos\theta)d\theta}{R^2}$$

$$\therefore\ E = \int_0^{\pi/2} dE = \int_0^{\pi/2}\frac{3GM\sin\theta(1-\cos\theta)}{R^2}$$

$$d\theta = \frac{3GM}{R^2}\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\pi/2}$$

$$\text{or } E = \frac{3GM}{2R^2}$$

Now potential due to the element under consideration at the centre of the base of the hemisphere,

$$dV = \frac{-2GdM}{r}(\text{cosec}\,\theta - \cot\theta)$$

(see potential due to a circular plate)

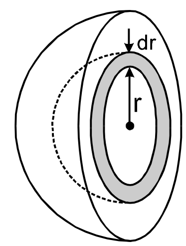
$$\text{or, } dV = \frac{-3GM\sin^3\theta(\text{cosec}\,\theta - \cot\theta)d\theta}{(R\sin\theta)}$$

$$\therefore\ V = -\frac{-3GM}{R}\int_0^{\pi/2}(\sin\theta - \cos\theta\sin\theta)d\theta$$

$$= -\frac{3GM}{R}\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\pi/2}$$

$$\text{or, } v = -\frac{3GM}{2R}$$

Aliter : Consider a hemispherical shell of radius r and thickness dr



Its mass, $dm = \frac{M}{\frac{2}{3}\pi R^3}(2\pi r^2 dr)$ or, $dm = \frac{3Mr^2\,dr}{R^3}$

Since all points of this hemispherical shell are at the same distance r from O. Hence potential at O due to it is ,

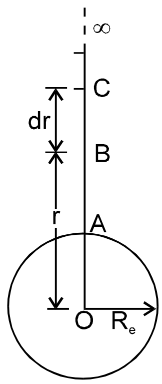
$$dV = \frac{-Gdm}{r} = \frac{-3GMrdr}{R^3}$$

$$\therefore\ V = \int_0^R dv = \frac{-3GM}{2R}$$

GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy of two mass system is equal to the work done by an external agent in assembling them, while their initial separation was infinity. Consider a body of mass m placed at a distance x from another body of mass M. The gravitational force of attraction between them is given by, $F = \frac{GMm}{r^2}$.

Now, Let the body of mass m is displaced from point. C to B through a distance 'dr' towards the mass M, then work done by internal conservative force (gravitational) is given by,



$$dW = F\,dr = \frac{GMm}{r^2}\,dr \Rightarrow \int dW = \int_{\infty}^{r} \frac{GMm}{r^2}\,dr$$

$\therefore$ Gravitational potential energy, $\boxed{U = -\frac{GMm}{r}}$

Special Cases:

(i) From above equation, it is clear that gravitational potential energy of two mass system increases with increase in separation (r) (i.e. it becomes less negative).

(ii) Gravitational P.E. becomes maximum (or zero) at r $= \infty$.

(iii) If the body of mass m moves from a distance r_1 to r_2 ($r_1 > r_2$), then work done or change in gravitational P.E. is given by

$$dU = \int_{r_1}^{r_2} \frac{GM_e m}{r^2}\,dr = GM_e m$$

$$\int_{r_1}^{r_2} r^{-2}dr = -GM_e m\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

Since $r_1 > r_2$, so change in gravitational P.E. of the body is negative. It means, when the body is brought near to the earth, P.E. of the earth-mass system decrease.

(iv) When the body of mass m is moved from the surface of earth (i.e., $r_1 = R_e$) to a height h (i.e., $r_2 = R_e + h$), then change in P.E. of the earth-mass system s given by

$$dU = -GM_e m\left[\frac{1}{R_e + h} - \frac{1}{R_e}\right]$$

$$= \frac{GM_e m}{R_e}\left[1 - \left(\frac{1}{1 + h/R_e}\right)\right] = \frac{GM_e m}{R_e}\left[1 - \left(1 + \frac{h}{R_e}\right)^{-1}\right]$$

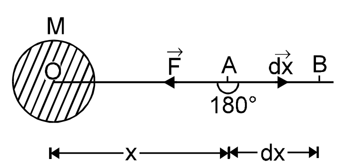
Using binomial expansion $\Rightarrow$

$$dU = \frac{GM_e m}{R_e}\left[1 - \left(1 - \frac{h}{R_e}\right)\right] \approx \frac{GM_e mh}{R_e^2}$$

Since $g = \frac{GM_e}{R_e^2}$ then $dU = mgh$

Gravitational potential difference is defined as the work done by an external agent to move a unit mass from one point to the other point in the gravitational field. According to the definition, E is the force experienced by a unit mass at A. The direction of this force is towards the body of mass M. Now the work done to move the unit mass from A to B is given by

$$dW = \vec{F} \cdot \vec{dx} = Edx \cos 180^\circ = -Edx$$



This work done is equal to the gravitational potential difference (dV).

Where $\frac{dV}{dx}$ is called potential gradient.

Solved Examples

Ex.12 Calculate the velocity with which a body must be thrown vertically upward from the surface of the earth so that it may reach a height of 10 R, where R is the radius of the earth and is equal to 6.4×10^8 m. (Earth's mass = 6×10^{24} kg, Gravitational constant $G = 6.7 \times 10^{-11}$ nt-m^2/kg^2)

Sol. The gravitational potential energy of a body of mass m on earth's surface is

$$U(R) = -\frac{GMm}{R}$$

where M is the mass of the earth (supposed to be concentrated at its centre) and R is the radius of the earth (distance of the particle from the centre of the earth). The gravitational energy of the same body at a height 10 R from earth's surface, i.e. at a distance 11R from earth's centre is

$$U(11R) = -\frac{GMm}{R}$$

$\therefore$ change in potential energy $U(11R) - U(R)$

$$= -\frac{GMm}{11R} - \left(-\frac{GMm}{R}\right) = \frac{10}{11}\frac{GMm}{R}$$

This difference must come from the initial kinetic energy given to the body in sending it to that height. Now, supose the body is thrown up with a vertical speed v, so that its initial kinetic energy is $\frac{1}{2}mv^2$.

Then $\frac{1}{2}mv^2 = \frac{10}{11}\frac{GMm}{R}$ or $v = \sqrt{\left(\frac{20}{11}\frac{GMm}{R}\right)}$.

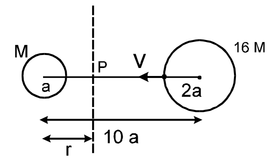
Putting the given values :

$$v = \sqrt{\left(\frac{20 \times (6.7 \times 10^{-11} nt - m^2/kg^2) \times (6 \times 10^{24} kg)}{11(6.4 \times 10^6 m)}\right)}$$

$= 1.07 \times 10^4$ m/s.

Ex.13 Distance between centres of two stars is 10 a. The masses of these stars are M and 16 M and their radii are a & 2a resp. A body is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star?

Sol. Let P be the point on the line joining the centres of the two planets s.t. the net field at it is zero



Then, $\frac{GM}{r^2} - \frac{G.16M}{(10a - r)^2} = 0 \Rightarrow (10a - r)^2 = 16r^2$

$\Rightarrow 10a - r = 4r \Rightarrow r = 2a$

Potential at point P, $v_P = \frac{-GM}{r} - \frac{G.16M}{(10a - r)}$

$$= \frac{-GM}{2a} - \frac{2GM}{a} = \frac{-5GM}{2a}.$$

Now if the particle projected from the larger planet has enough energy to cross this point, it will reach the smaller planet.

For this, the K.E. imparted to the body must be just enough to raise its total mechanical energy to a value which is equal to P.E. at point P.

i.e. $\frac{1}{2}mv^2 - \frac{G(16M)m}{2a}\frac{-GMm}{8a} = mv_P$

or, $\frac{v^2}{2}\frac{-8GM}{a}\frac{-GM}{8a} = \frac{-5GMm}{2a}$ or,

$v^2 = \frac{45GM}{4a}$ or, $v_{min} = \frac{3}{2}\sqrt{\frac{5GM}{a}}$

GRAVITATIONAL SELF-ENERGY

The gravitational self-energy of a body (or a system of particles) is defined as the workdone by an external agent in assembling the body (or system of particles) from infinitesimal elements (or particles) that are initially an infinite distance apart.

Gravitational self energy of a system of n particles

Potential energy of n particles at an average distance 'r' due to their mutual gravitational attraction is equal to the sum of the potential energy of all pairs of particle, i.e.,

$$U_s = -G \sum_{\substack{\text{all pairs} \\ j \neq i}} \frac{m_i m_j}{r_{ij}}$$

This expression can be written as

$$U_s = -\frac{1}{2} G \sum_{i=1}^{i=n} \sum_{\substack{j=1 \\ j\neq i}}^{j=n} \frac{m_i m_j}{r_{ij}}$$

If consider a system of 'n' particles, each of same mass 'm' and seperated from each other by the same average distance 'r', then self energy

or $$U_s = -\frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{m^2}{r}\right)_{ij}$$

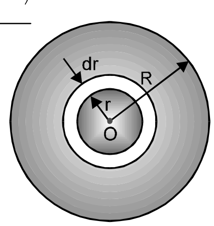
Thus on the right handside 'i' comes 'n' times while 'j' comes (n – 1) times. Thus

$$U_s = -\frac{1}{2} Gn(n-1) \frac{m^2}{r}$$

Gravitational Self energy of a Uniform Sphere (star)

$$U_{shell} = -G \frac{\left(\frac{4}{3}\pi r^3 \rho\right)\left(4\pi r^2 dr\, \rho\right)}{r}$$

where $\rho = \dfrac{M}{\left(\frac{4}{3}\right)\pi R^3}$



$$= -\frac{1}{3} G (4\pi\rho)^2 r^4 dr,$$

$$U_{star} = -\frac{1}{3} G (4\pi\rho)^2 \int_0^R r^4 dr$$

$$= -\frac{1}{3} G (4\pi\rho)^2 \left[\frac{r^5}{5}\right]_0^R$$

$$= -\frac{3}{5} G \left(\frac{4\pi}{3} R^3 \rho\right)^2 \frac{1}{R}.$$

$$\therefore\ U_{star} = -\frac{3}{5} \frac{GM^2}{R}$$

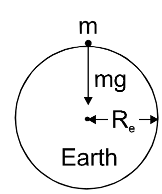
ACCELERATION DUE TO GRAVITY

It is the acceleration, a freely falling body near the earth's surface acquires due to the earth's gravitational pull. The property by virtue of which a body experiences or exerts a gravitational pull on another body is called **gravitational mass m_G**, and the property by virtue of which a body opposes any change in its state of rest or uniform motion is called its **inertial mass m_I** thus if $\vec{E}$ is the gravitational field intensity due to the earth at a point P, and $\vec{g}$ is acceleration due to gravity at the same point, then $m_I \vec{g} = m_G \vec{E}$.

Now the value of inertial & gravitational mass happen to be exactly same to a great degree of accuracy for all bodies. Hence, $\vec{g} = \vec{E}$

The gravitational field intensity on the surface of earth is therefore numerically equal to the acceleration due to gravity (g), there. Thus we get,

$$g = \frac{GM_e}{R_e^2}$$



where , M_e = Mass of earth

R_e = Radius of earth

Note : Here the distribution of mass in the earth is taken to be spherical symmetrical so that its entire mass can be assumed to be concentrated at its center for the purpose of calculation of g.

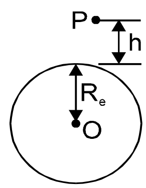
VARIATION OF ACCELERATION DUE TO GRAVITY

(a) Effect of Altitude

Acceleration due to gravity on the surface of the earth is given by, $g = \frac{GM_e}{R_e^2}$

Now, consider the body at a height 'h' above the surface of the earth, then the acceleration due to gravity at height 'h' given by

$$g_h = \frac{GM_e}{(R_e + h)^2} = g\left(1 + \frac{h}{R_e}\right)^{-2}$$



$$\simeq g\left(1 - \frac{2h}{R_e}\right) \text{ when } h << R.$$

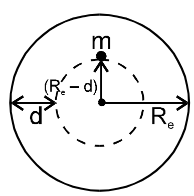
The decrease in the value of 'g' with height $h = g - g_h = \frac{2gh}{R_e}$. Then percentage decrease in the value of

$$\text{'g'} = \frac{g - g_h}{g} \times 100 = \frac{2h}{R_e} \times 100\%$$

(b) Effect of depth

The gravitational pull on the surface is equal to its weight i.e. $mg = \frac{GM_e m}{R_e^2}$

$$\therefore\ mg = \frac{G \times \frac{4}{3}\pi R_e^3 \rho m}{R_e^2} \text{ or}$$



$$g = \frac{4}{3}\pi G R_e \rho \text{....(1)}$$

When the body is taken to a depth d, the mass of the sphere of radius $(R_e - d)$ will only be effective for the gravitational pull and the outward shall will have no resultant effect on the mass. If the acceleration due to gravity on the surface of the solid sphere is g_d, then

$$g_d = \frac{4}{3}\pi G (R_e - d)\rho$$

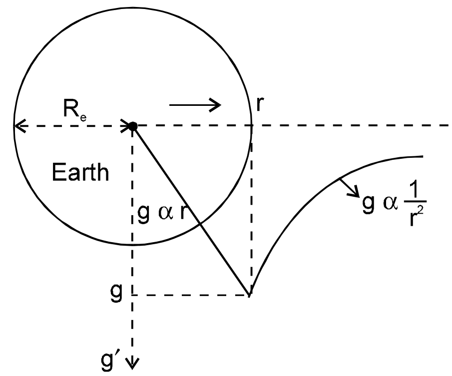
...............(2)

By dividing equation (2) by equation (1)

$$\Rightarrow\quad g_d = g\left(1 - \frac{d}{R_e}\right)$$

IMPORTANT POINTS

(i) At the center of the earth, $d = R_e$, so $g_{centre} = g\left(1 - \frac{R_e}{R_e}\right) = 0$. Thus weight (mg) of the body at the centre of the earth is zero.



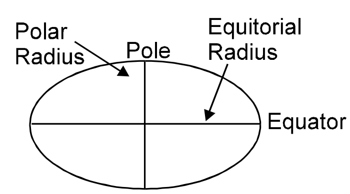
(ii) Percentage decrease in the value of 'g' with the depth

$$= \left(\frac{g - g_d}{g}\right) \times 100 = \frac{d}{R_e} \times 100.$$

(c) Effect of the surface of Earth

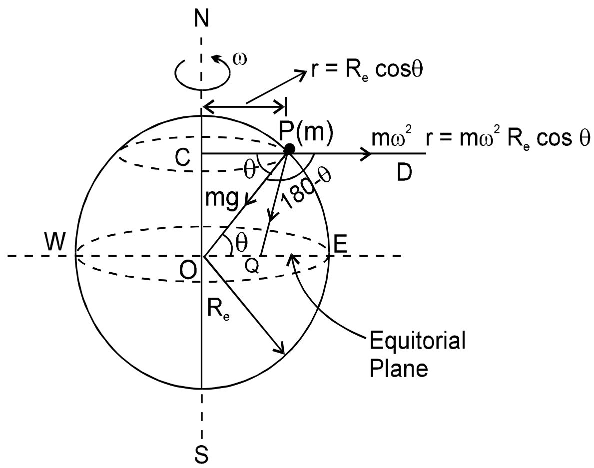
The equatorial radius is about 21 km longer than its polar radius.

We know, $g = \frac{GM_e}{R_e^2}$ Hence $g_{pole} > g_{equator}$. The weight of the body increase as the body taken from the equator to the pole.



(d) Effect of rotation of the Earth

The earth rotates around its axis with angular velocity ω. Consider a particle of mass m at latitude θ. The angular velocity of the particle is also ω.



According to parallelogram law of vector addition, the resultant force acting on mass m along PQ is

$F = [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 + \{2mg \times m\omega^2 R_e \cos\theta\} \cos(180-\theta)]^{1/2} = [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 - (2m^2 g\omega^2 R_e \cos\theta)\cos\theta]^{1/2}$

$$= mg\left[1+\left(\frac{R_e\,\omega^2}{g}\right)^2\cos^2\theta - 2\frac{R_e\,\omega^2}{g}\cos^2\theta\right]^{1/2}$$

At pole $\theta = 90° \Rightarrow g_{pole} = g$, At equator $\theta = 0 \Rightarrow$

$g_{equator} = g\left[1-\frac{R_e\,\omega^2}{g}\right]$. Hence $g_{pole} > g_{equator}$

If the body is taken from pole to the equator, then

$g' = g\left(1-\frac{R_e\omega^2}{g}\right)$. Hence % change in weight =

$$\frac{mg-mg\left(1-\frac{R_e\omega^2}{g}\right)}{mg}\times 100 = \frac{mR_e\,\omega^2}{mg}\times 100 = \frac{R_e\,\omega^2}{g}\times 100$$

ESCAPE SPEED

The minimum speed required to project a body from the surface of the earth so that it never returns to the surface of the earth is called escape speed.

A body thrown with escape speed goes out of the gravitational pull of the earth.

Work done to displace the body from the surface of the earth

($r = R_e$) to infinity ($r = \infty$) is given by

$$\int dW = \int_{R_e}^{\infty}\frac{GM_e m}{r^2}dr \quad \text{or} \quad W = GM_e\, m\int_{R_e}^{\infty}\frac{1}{r^2}dr$$

$$= -GM_e\, m\left[\frac{1}{r}\right]_{R_e}^{\infty} = -GM_e\, m\left[\frac{1}{\infty}-\frac{1}{R_e}\right] \Rightarrow$$

$$W = \frac{GM_e m}{R_e}$$

Let v_e be the escape speed of the body of mass m, then kinetic energy of the body is given by

$$\frac{1}{2}mv_e^2 = \frac{GM_e m}{R_e} \Rightarrow v_e = \sqrt{2gR_e} = 11.2\ \text{km s}^{-1}.$$

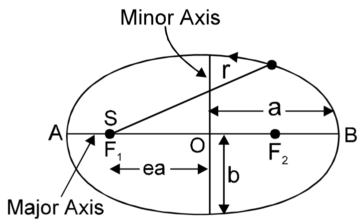
Important Points

1. Escape speed depends on the mass and size of the planet. That is why escape velocity on the Jupiter is more than on the earth.
2. Escape speed is independent of the mass of the body.
3. Any body thrown upward with escape speed start moving around the sun.

MOTION OF SATELLITES AND KEPLER LAWS

A heavenly body revolving around a planet in an orbit is called natural satellite. For example, moon revolves around the planet the earth, so moon is the satellite of the earth. Their motions can be sttudied with the help of kepler's laws, as stated :

I . **Law of orbit :** Each Planet moves arround the sun in an elliptical orbit with the sun at one of the foci as shown in figure. The eccentricity of an ellipse is defined as the ratio of the distance SO and AO i.e. e $= \frac{SO}{AO}$

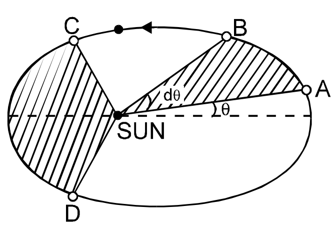


$\therefore\ e = \frac{SO}{a} \qquad SO = ea$

The distance of closest approach with the sun at F_1 is AS. This distance is called perigee. The greatest distance (BS) of the planet from the sun is called apogee.

Perigee (AS) = AO – OS = a – ea = a (1 – e)

Apogee (BS) = OB + OS = a + ea = a (1 + e)



II . **Law of Areas :** The line joining the sun and a planet sweeps out equal areas in equal intervals of time. A planet takes the same time to travel from A to B as from C to D as shown in figure. (The shaded areas are equal). Naturally the planet has to move faster from C to D. The law of areas is identical with the law of conservation of angular momentum.

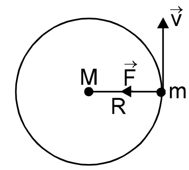
Areal velocity $= \frac{\text{area swept}}{\text{time}} = \frac{\frac{1}{2}r\,(rd\theta)}{dt}$

$= \frac{1}{2} r^2 \frac{d\theta}{dt}$ = constant   Hence $\frac{1}{2} r^2 \omega$ = constant.

III. **Law of periods :** The square of the time for the planet to complete a revolution about the sun is proportional to the cube of semimajor axis of the elliptical orbit.

i.e. Centripetal force = Gravitational force

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow \frac{GM}{R} = v^2$$



Now, speed of the planet is

$$v = \frac{\text{Circumference of the circular orbit}}{\text{Time period}} = \frac{2\pi R}{T}$$

Substituting value in above equation ⇒

$$\frac{GM}{R} = \frac{4\pi^2 R^2}{T^2} \text{ or } T^2 = \frac{4\pi^2 R^3}{GM}$$

Since $\left(\frac{4\pi^2}{GM}\right)$ is constant,

$\therefore\ T^2 \propto R^3$ or $\frac{T^2}{R^3}$ = constant

Solved Examples

Ex.14 A satellite is launched into a circular orbit 1600 km above the surface of the earth. Find the period of revolution if the radius of the earth is R = 6400 km and the acceleration due to gravity is 9.8 m/sec^2. At what height from the ground should it be launched so that it may appear stationary over a point on the earth's equator ?

Sol. The orbiting period of a satellite at a height h from earth's surface is $T = \frac{2\pi r^{3/2}}{gR^2}$ where r = R + h

$$\text{then, } T = \frac{2\pi(R+h)}{R}\sqrt{\left(\frac{R+h}{g}\right)}$$

Here, R = 6400 km, h = 1600 km = R/4. Then

$$T = \frac{2\pi\left(R+\frac{R}{4}\right)}{R}\sqrt{\left(\frac{R+\frac{R}{4}}{g}\right)} = 2\pi(1-2J)^{3/2}\sqrt{\frac{R}{g}}$$

Putting the given values : T = 2 × 3.14 × $\sqrt{\left(\frac{6.4\times10^6 m}{9.8\ m/s^2}\right)}$ $(1.25)^{3/2}$ = 7092 sec = 1.97 hours

Now, a satellite will appear stationary in the sky over a point on the earth's equator if its period of revolution round the earth is equal to the period of revolution of the earth round its own axis which is 24 hours. Let us find the height h of such a satellite above the earth's surface in terms of the earth's radius. Let it be nR. then

$$T = \frac{2\pi(R+nR)}{R}\sqrt{\left(\frac{R+nR}{g}\right)} = 2\pi\sqrt{\left(\frac{R}{g}\right)}\ (1+n)^{3./2}$$

$$= 2 \times 3.14\sqrt{\left(\frac{6.4\times10^{6}\ \text{meter/sec}}{9.8\ \text{meter/sec}^{2}}\right)}\ (1+n)^{3/2}$$

$$= (5075\ \text{sec})\ (1+n)^{3/2} = (1.41\text{hours})\ (1+n)^{3/2}$$

For T = 24 hours, we have

$(24\ \text{hours}) = (1.41)\ \text{hours})\ (1+n)^{3/2}$

or $(1+n)^{3/2} = \dfrac{24}{1.41} = 17$

or $1 + n = (17)^{2/3} = 6.61$ or $n = 5.61$

The height of the geo-stationary satellite above the earth's surface is $nR = 5.61 \times 6400$ km

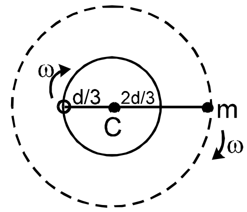
$= 3.59 \times 10^4$ km.

Ex.15 In a double star, two stars (one of mass m and the other of 2m) distant d apart rotate about their common centre of mass. Deduce an expression ofr the period of revolution. Show that the ratio of their angular momenta about the centre of mass is the same as the ratio of their kinetic energies.

Sol. The centre of mass C will be at distances d/3 and 2d/3 from the masses 2m and m respectively. Both the stars rotate round C in their respective orbits with the same angular velocity ω. The gravitational force acting on each star due to the other supplies the necessary centripetal force.

The gravitational force on either star is $\frac{G(2m)m}{d^2}$. If we consider the rotation of the smaller star, the centripetal force ($m r \omega^2$) is $\left[m\left(\frac{2d}{3}\right)\omega^2\right]$ and for bigger star $\left[\frac{2md\omega^2}{3}\right]$ i.e.same

$$\therefore \quad \frac{G(2m)m}{d^2} = m\left(\frac{2d}{3}\right)\omega^2$$



or $\quad \omega = \sqrt{\left(\frac{3Gm}{d^3}\right)}$

Therefore, the period of revolution is given by T =

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{d^3}{3Gm}\right)}$$

The ratio of the angular momenta is

$$\frac{(I\omega)_{big}}{(I\omega)_{small}} = \frac{I_{big}}{I_{small}} = \frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2} = \frac{1}{2},$$

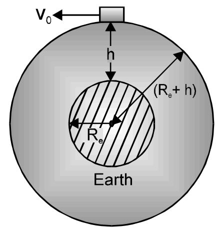
since ω is same for both. The ratio of their kinetic energies is $\frac{(\frac{1}{2}I\omega^2)_{big}}{(\frac{1}{2}I\omega^2)_{small}} = \frac{I_{big}}{I_{small}} = \frac{1}{2},$

which is the same as the ratio of their angular momenta.

SATELLITE SPEED (OR ORBITAL SPEED)

The speed required to put the satellite into its orbit around the earth is called orbital speed.

The gravitational attraction between satellite and the earth provides the necessary centripetal force.



$$\frac{GM_e m}{(R_e+h)^2}=\frac{mv_0^2}{(R_e+h)} \text{ or } v_0^2=\frac{GM_e}{(R_e+h)}$$

$$\text{or, } v_0=\left[\frac{GM_e}{(R_e+h)}\right]^{\frac{1}{2}}=\left[\frac{gR_e^2}{(R_e+h)}\right]^{\frac{1}{2}}$$

When $h << R_e$ then $v_0=\sqrt{gR_e}$

$\therefore\ v_0=\sqrt{9.8\times6.4\times10^6}=7.92\times10^3$ $ms^{-1}=7.92$ km s^{-1}

Time period of Satellite

Time period,

$$T=\frac{\text{Circumference of the orbit}}{\text{orbital speed}}=\frac{2\pi(R_e+h)}{v_0}$$

$$\text{But } v_0=\left[\frac{gR_e^2}{(R_e+h)}\right]^{\frac{1}{2}} \quad \therefore\ T=\frac{2\pi(R_e+h)}{\left[\frac{gR_e^2}{(R_e+h)}\right]^{\frac{1}{2}}}$$

$$=\frac{2\pi}{R_e}\left[\frac{(R_e+h)^3}{g}\right]^{\frac{1}{2}}$$

Height of the satellite above the earth's surface

Time period of satellite is given by,

$$T=\frac{2\pi}{R_e}\left[\frac{(R_e+h)^3}{g}\right]^{\frac{1}{2}}$$

$$T^2=\frac{4\pi^2}{R_e^2}\frac{(R_e+h)^3}{g} \quad \text{or} \quad (R_e+h)^3=\frac{T^2R_e^2g}{4\pi^2}$$

$$(R_e+h)=\left(\frac{T^2R_e^2g}{4\pi^2}\right)^{\frac{1}{3}} \qquad h=\left(\frac{T^2R_e^2g}{4\pi^2}\right)^{\frac{1}{3}}-R_e$$

Energy of a Satellite

P.E. of a satellite of mass m revolving around the earth in a circular orbit of the earth is given by

$$U=\frac{-GM_em}{r} \text{ and K.E. } =\frac{1}{2}mv_0^2$$

$$\frac{mv_0^2}{r}=\frac{GM_em}{r^2} \text{ or } mv_0^2=\frac{GM_em}{r}.$$

$$\text{Hence K.E. }=\frac{GM_em}{2r}$$

Total Energy E = U + K.E.

$$=\frac{-GM_em}{r}+\frac{-GM_em}{2r} \quad \text{or} \quad E=-\frac{-GM_em}{2r}$$

Since total energy is negative, so it implies that satellite is bound to the earth. If satellite is close to the surface of the earth then total energy →

$$E=-\frac{GM_em}{2R_e}$$

GEO-STATIONARY SATELLITES OR GEO-SYNCHRONOUS SATELLITES

(i) The time period of the satellite around the earth must be equal to the rotational period of the earth (i.e. 24 hours.)

(ii) The direction of motion of the satellite must be same as that of the earth. i.e. from west to east.

The height of the geio-stationary satellite from the surface of the earth can be calculated from the

$$\text{equation } h=\left(\frac{T^2R_e^2g}{4\pi^2}\right)^{\frac{1}{2}}-R_e$$

Now $T=$ 24 hours $=24\times3600$ s,

$R_e=6.4\times10^6$ m, $g=9.8$ ms^{-2}

$$\therefore\ h=\left[\frac{(24\times3600)^2\times(6.4\times10^6)^2\times9.8}{4\pi^2}\right]^{\frac{1}{2}}-6.4\times10^6$$

or $h=35930\times10^3$ m $=35930$ km.

Uses of Artificial Satellites

Some important uses of artificial satellites are :

(i) They are used as communication satellites to send messages to distant places.

(ii) They are used as weather satellites to forecast weather.

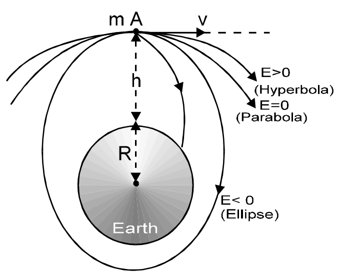
(iii) They are used to explore the upper region of the atmosphere.

(iv) They are used to telecast T.V. programs to distant places.

(v) They are used to know the exact shape of the earth.

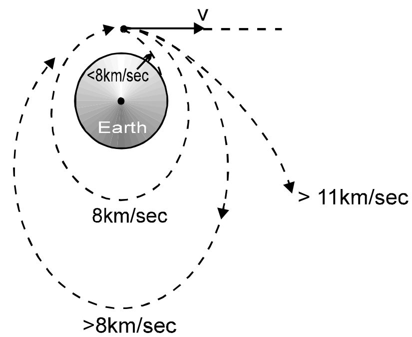
LAUNCHING OF AN ARTIFICIAL SATELLITE AROUND THE EARTH

The satellite is placed upon the rocket which is launched from the earth. After the rocket reaches its maximum vertical height h, a spherical mechanism gives a thrust to the satellite at point A (fig.) producing a horizontal speed v. The total energy of the satellite at A is thus.

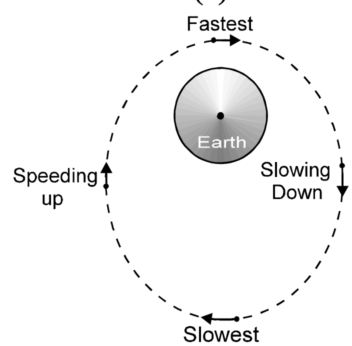


$$E = \frac{1}{2}mv^2 - \frac{GMm}{R+h}$$

The orbit will be an ellipse (closed path), a parabola, or an hyperbola depending on whether E is negative, zero, or positive. In all cases the centre of the earth is at one focus of the path. If the energy is too low, the elliptical orbit will intersect the earth and the satellite will fall back. Otherwise it will keep on moving in a closed orbit, or will escape from the Earth, depending on the values of v and R.



(a)



(b)

Hence a satellite carried to a height h (<< R) and given a horizontal speed of 8 km/sec will be placed almost in a circular orbit around the earth (fig.) If launched at less than 8 km/sec, it would get closer and closer to the earth until it hits the ground. Thus 8 km/sec is the critical (minimum) speed.

(a) Orbits and Speed:

For a body on the earth's surface, projected horizontally with a speed v, the trajectory depends on the value of its speed v.

Velocity (v)	Trajectory
(i) Less than the orbital speed $v < \sqrt{gR_e}$	(i) Body returns to the earth
(ii) Equal to orbital speed $v = \sqrt{gR_e}$	(ii) Body acquires a near the earth circular orbit
(iii) Between orbital and escape speed $\sqrt{gR_e} < v < \sqrt{2gR_e}$	(iii) Body acquires an eiliptical orbit with the earth as the near focus
(iv) Equal to escape speed $v = \sqrt{2gR_e}$	(iv) Body just escapes the earth's gravity along in a parabolic path.
(v) Greater then escape speed$v = \sqrt{2gR_e}$	(v) Body escape's the earth's gravity in ah hyperbolic path.

Ex.16 A rocket starts vertically upward with speed v_0. Shown that its speed v at height h is given by

$$v_0^2 - v^2 = \frac{2hg}{1+\frac{h}{R}},$$

where R is the radius of the earth and g is acceleration due to gravity at earth's surface. Hence deduce an expression for maximum height reached by a rocket fired with speed 0.9 times the escape velocity.

Sol. The gravitational potential energy of a mass m on earth's surface and that a height h is given by

$$U(R) = -\frac{GMm}{R} \text{ and } U(R+h) = -\frac{GMm}{R+h}$$

$$\therefore U(R+h) - U(R) = -GMm\left(\frac{1}{R+h} - \frac{1}{R}\right)$$

$$= \frac{GMmh}{(R+h)R} = \frac{mhg}{1+\frac{h}{R}} \quad [\because GM = gR^2]$$

This increase in potential energy occurs at the cost of kinetic energy which correspondingly decreases. If v is the velocity of the rocket at height h, then the decrease in kinetic energy is $\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$.

Thus, $\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2 = \frac{mhg}{1+\frac{h}{R}}$,

or $v_0^2 - v^2 = \frac{2gh}{1+\frac{h}{R}}$

Let h_{max} be the maximum height reached by the rocket, at which its velocity has been reduced to zero. Thus, substituting v = 0 and h = h_{max} in the last expression, we have

$$v_0^2 = \frac{2gh_{max}}{1+\frac{h_{max}}{R}} \quad \text{or} \quad v_0^2\left(1+\frac{h_{max}}{R}\right) = 2gh_{max}$$

$$\text{or } v_0^2 = h_{max}\left(2g - \frac{v_0^2}{R}\right) \quad \text{or } h_{max} = \frac{v_0^2}{2g - \frac{v_0^2}{R}}$$

Now, it is given that $v_0 = 0.9 \times$ escape velocity

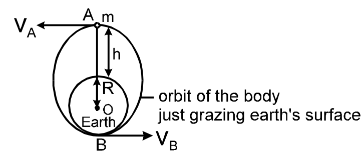
$= 0.9 \times \sqrt{(2gR)}$

$$\therefore h_{max} = \frac{(09 \times 0.9)2gR}{2g - \frac{(09\times 0.9)2gR}{R}}$$

$$= \frac{1.62gR}{2g - 1.62R} = \frac{1.62R}{0.38} = 4.26R$$

Ex- 17 For a particle projected in a transverse direction from a height h above Earth's surface, find the minimum initial velocity so that it just grazes the surface of earth path of this particle would be an ellipse with center of earth as the farther focus, point of projection as the apojee and a diametrically opposite point on earth's surface as perigee.

Sol. Suppose velocity of projection at point A is v_A & at point B, the velocity of the particle is v_B.



then applying Newton's 2nd law at point A & B, we get, $\frac{mv_A^2}{\rho_A} = \frac{GM_em}{(R+n)^2}$ & $\frac{mv^2}{\rho_B} = \frac{GM_em}{R^2}$

Where ρ_A & ρ_B are radius of curvature of the orbit at points A & B of the ellipse,

but $\rho_A = \rho_B = \rho$(say).

Now applying conservation of energy at points A & B

$$\frac{-GM_em}{R+h} + \frac{1}{2}mv_A^2 = \frac{-GM_em}{R} + \frac{1}{2}mv_B{}^2$$

$$\Rightarrow GM_em\left(\frac{1}{R} - \frac{1}{(R+h)}\right)$$

$$= \frac{1}{2}(mv_B{}^2 - mv_A{}^2) = \left(\frac{1}{2}\rho GM_em\left(\frac{1}{R^2} - \frac{1}{(R+h)^2}\right)\right)$$

$$\text{or, } \rho = \frac{2R(R+h)}{2R+h} = \frac{2Rr}{R+r}$$

$$\therefore V_A{}^2 = \frac{\rho GM_e}{(R+h)^2} = 2GM_e\frac{R}{r(r+R)}$$

where r = distance of point of projection from earth's centre = R + h.