System of Particles and Rotational Motion

CENTRE OF MASS

Every physical system has associated with it a certain point whose motion characterises the motion of the whole system. When the system moves under some external forces, then this point moves as if the entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is called the centre of mass of the system.

Centre of mass of system of N point masses is that point about which moment of mass of the system is zero. It means that if about a particular origin the moment of mass of system of N point masses is zero then that particular origin is the centre of mass of the system.

MOMENT OF POINT MASS 'M' ABOUT AN ORIGIN 'O'

Mass Moment :

It is defined as the product of mass of the particle and distance of the particle from the point about which mass moment is taken. It is a vector quantity and its direction is directed from the point about which it is taken to the particle

Let P be the point where mass 'm' is located. Take position vector of point P with respect to origin O. The moment of point mass m about origin O is defined as

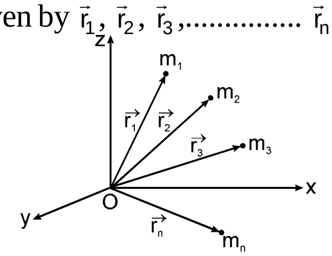
$$\vec{M} = m\vec{r}$$

The physical significance of moment of mass is that when differentiated with respect to time it gives momentum of the particle.

It is worth noting that moment of point mass depends on choice of origin.

MOMENT OF SYSTEM OF N POINT MASSES ABOUT AN ORIGIN 'O'

Consider a system of N point masses m_1, m_2, m_3 .... m_n whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, ... \vec{r}_n$ respectively.



The moment of system of point masses about origin O is the sum of individual moment of each point mass about origin O.

$$\vec{M} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n$$

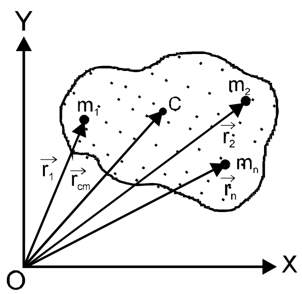
CENTRE OF MASS OF A SYSTEM OF 'N' DISCRETE PARTICLES

Consider a system of N point masses m_1, m_2, m_3, ... m_n whose position vectors from origin O are given by $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$,................ $\vec{r}_n$ respectively. Then the position vector of the centre of mass C of the system is given by.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$$

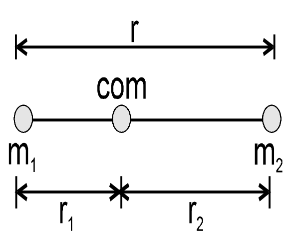
$$;\ \vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \qquad \vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

where $M\left(=\sum_{i=1}^{n} m_i\right)$ is the total mass of the system.



POSITION OF COM OF TWO PARTICLES

Centre of mass of two particles of mass m_1 and m_2 separated by a distance r lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m)



i.e. $\quad r \propto \frac{1}{m}$

or $\quad \frac{r_1}{r_2} = \frac{m_2}{m_1}$

or $\quad m_1r_1 = m_2r_2$

or $\quad r_1 = \left(\frac{m_2}{m_2 + m_1}\right)r$ and $r_2 = \left(\frac{m_1}{m_1 + m_2}\right)r$

Here, $\quad r_1$ = distance or COM from m_1

and $\quad r_2$ = distance or COM from m_2

From the above discussion, we see that

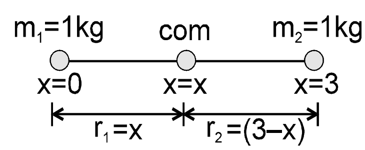
$r_1 = r_2 = \frac{1}{2}$ if $m_1 = m_2$, i.e., COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_2 < m_1$, i.e., COM is nearer to the particle having larger mass.

Solved Examples

Ex.1 Two particles of mass 1 kg and 2 kg are located at x = 0 and x = 3 m. Find the position of their centre of mass.

Sol.



Since, both the particles lies on x-axis, the COM will also lie on x-axis. Let the COM is located at x = x, then

r_1 = distance of COM from the particle of mass 1 kg = x and r_2 = distance of COM from the particle of mass 2 kg = (3 – x) $\quad$ Using $\frac{r_1}{r_2} = \frac{m_2}{m_1}$

or $\quad \frac{x}{3-x} = \frac{2}{1}$ or x = 2 m

Thus, the COM of the two particles is located at x = 2 m. **Ans.**

Ex.2 The position vector of three particles of mass m_1 = 1 kg, m_2 = 2 kg and m_3 = 3 kg are $\vec{r}_1 = (\hat{i} + 4\hat{j} + \hat{k})m$, $\vec{r}_2 = (\hat{i} + \hat{j} + \hat{k})m$

and $\vec{r}_3 = (2\hat{i} - \hat{j} - 2\hat{k})m$ respectively. Find the position vector of their centre of mass.

Sol. The position vector of COM of the three particles will be given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

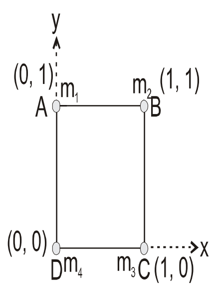
Substituting the values, we get

$$\vec{r}_{COM} = \frac{(1)(\hat{i} + 4\hat{j} + \hat{k}) + (2)(\hat{i} + \hat{j} + \hat{k}) + (3)(2\hat{i} - \hat{j} - 2\hat{k})}{1+2+3}$$

$$= \frac{9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{6} \qquad \vec{r}_{COM} = \frac{1}{2}(3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\text{m}$$ **Ans.**

Ex.3 Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find the position of centre of mass of the particles.

Sol. Assuming D as the origin, DC as x -axis and DA as y-axis, we have



m_1 = 1 kg, (x_1, y_1) = (0, 1m)

m_2 = 2 kg, (x_2, y_2) = (1m, 1m)

m_3 = 3 kg, (x_3, y_3) = (1m, 0)

and m_4 = 4 kg, (x_4, y_4) = (0, 0)

Co-ordinates of their COM are

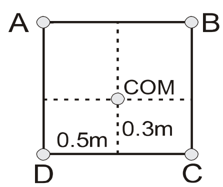
$$x_{COM} = \frac{m_1x_1 + m_2x_2 + m_3m_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1+2+3+4}$$

$$= \frac{5}{10} = \frac{1}{2}\text{m} = 0.5\text{ m}$$

Similarly, $$y_{COM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1+2+3+4}$$

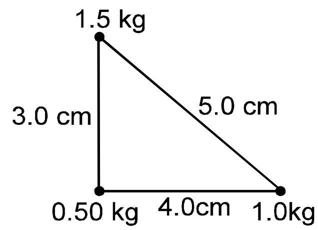


$$= \frac{3}{10}\text{m} = 0.3\text{ m}$$

$\therefore$ x_{COM}, y_{COM}) = (0.5 m, 0.3 m) **Ans.**

Thus, position of COM of the four particles is as shown in figure.

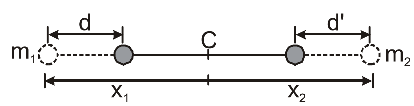
Ex.4 Three particles of masses 0.5 kg, 1.0 kg and 1.5 kg are placed at the three corners of a right angled triangle of sides 3.0 cm, 4.0 cm and 5.0 cm as shown in figure. Locate the centre of mass of the system.



Ans. The centre of mass is 1.3 cm to the right and 1.5 cm above the 0.5 kg particle.

Ex.5 Consider a two-particle system with the particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved so as to keep the centre of mass at the same position?

Sol. Consider figure. Suppose the distance of m_1 from the centre of mass C is x_1 and that of m_2 from C is x_2. Suppose the mass m_2 is moved through a distance d′ towards C so as to keep the centre of mass at C.



Then, $m_1x_2 = m_2x_2$ ....(i)

and $m_1(x_1 - d) = m_2(x_2 - d')$ ....(ii)

Subtracting (ii) from (i) $m_1d = m_2d'$ or, $d' = \dfrac{m_1}{m_2}d$,

CENTRE OF MASS OF A CONTINUOUS MASS DISTRIBUTION

For continuous mass distribution the centre of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation

$$x_{cm} = \frac{\int x\,dm}{\int dm},\ y_{cm} = \frac{\int y\,dm}{\int dm},\ z_{cm} = \frac{\int z\,dm}{\int dm}$$

$\int dm$ = M (mass of the body)

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm.$$

Note: If an object has symmetric uniform mass distribution about x axis than y coordinate of COM is zero and vice-versa

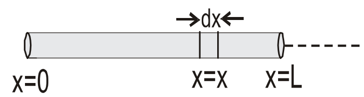
CENTRE OF MASS OF A UNIFORM ROD

Suppose a rod of mass M and length L is lying along the x-axis with its one end at x = 0 and the other at x = L. Mass per unit length of the rod = $\frac{M}{L}$

Hence, dm, (the mass of the element dx situated at x = x is) = $\frac{M}{L}dx$

The coordinates of the element PQ are (x, 0, 0). Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int dm}$$



$$= \frac{\int_0^L (x)\left(\frac{M}{L}dx\right)}{M}$$

$$= \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0$$

Similarly, $z_{COM} = 0$

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$. Or it lies at the centre of the rod.

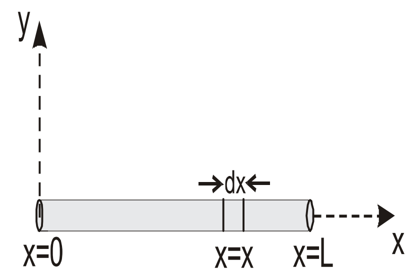
Solved Examples

Ex.6 A rod of length L is placed along the x-axis between x = 0 and x = L. The linear density (mass/length) λ of the rod varies with the distance x from the origin as $\lambda = Rx$. Here, R is a positive constant. Find the position of centre of mass of this rod.

Sol. Mass of element dx situated at x = x is dm = λ dx = Rx dx

The COM of the element has coordinates (x, 0, 0). Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm}$$



$$= \frac{\int_0^L (x)(Rx)\,dx}{\int_0^L (Rx)\,dx}$$

$$= \frac{R\int_0^L x^2 dx}{R\int_0^L x\,dx} = \frac{\left[\frac{x^3}{3}\right]_0^L}{\left[\frac{x^2}{2}\right]_0^L} = \frac{2L}{3}$$

The y-coordinate of COM of the rod is y_{COM}

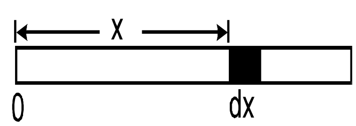
$$= \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $z_{COM} = 0$

Hence, the centre of mass of the rod lies at

$\left[\frac{2L}{3}, 0, 0\right]$ **Ans.**

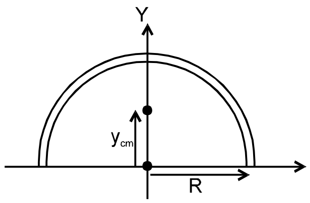
Ex. 7 The density of a straight rod of length L varies as $\rho = A + Bx$ where x is the distance from the left end. Locate the centre of mass.

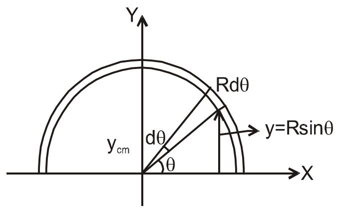


Sol. $\frac{2AL + 2BL^2}{3(2A + BL)}$

CENTRE OF MASS OF A SEMICIRCULAR RING

Figure shows the object (semi circular ring). By observation we can say that the x-coordinate of the centre of mass of the ring is zero as the half ring is symmetrical on both sides of the origin. Only we are required to find the y-coordinate of the centre of mass.





To find y_{cm} we use $y_{cm} = \frac{1}{M}\int dm\, y$ ...(i)

Here for dm we consider an elemental arc of the ring at an angle θ from the x-direction of angular width dθ. If radius of the ring is R then its y coordinate will be R sinθ, here dm is given as

$$dm = \frac{M}{\pi R} \times R\, d\theta$$

So from equation ---(i), we have

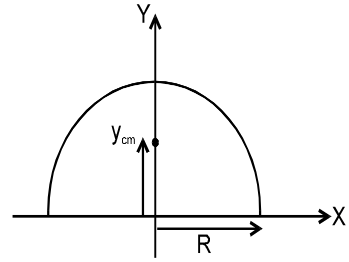
$$y_{cm} = \frac{1}{M}\int_0^{\pi} \frac{M}{\pi R} R d\theta\, (R \sin\theta)$$

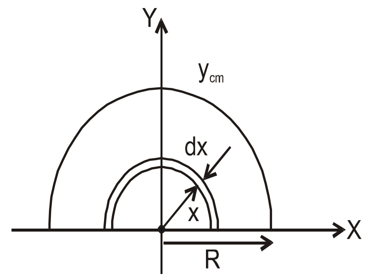
$$= \frac{R}{\pi}\int_0^{\pi} \sin\theta\, d\theta$$

$$y_{cm} = \frac{2R}{\pi} \qquad \text{...(ii)}$$

CENTRE OF MASS OF SEMICIRCULAR DISC

Figure shows the half disc of mass M and radius R. Here, we are only required to find the y-coordinate of the centre of mass of this disc as centre of mass will be located on its half vertical diameter. Here to find y_{cm}, we consider a small elemental ring of mass dm of radius x on the disc (disc can be considered to be made up such thin rings of increasing radii) which will be integrated from 0 to R. Here dm is given as $dm = \frac{2M}{\pi R^2}(\pi x)dx$





Now the y-coordinate of the element is taken as $\frac{2x}{\pi}$, as in previous section, we have derived that the centre of mass of a semi circular ring is concentrated at $\frac{2R}{\pi}$

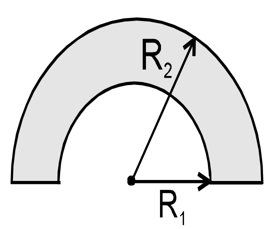
Here y_{cm} is given as $y_{cm} = \frac{1}{M}\int_0^R dm\, \frac{2x}{\pi}$

$$= \frac{1}{M}\int_0^R \frac{4M}{\pi R^2} x^2\, dx$$

$$y_{cm} = \frac{4R}{3\pi}$$

Solved Examples

Ex.8 Find the centre of mass of an annular half disc shown in figure.



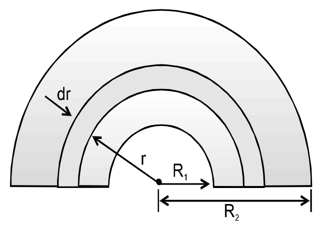
Sol. Let ρ be the mass per unit area of the object. To find its centre of mass we consider an element as a half ring of mass dm as shown in figure of radius r and width dr and there we have

Now, $\quad dm = \rho\pi r\, dr$

Centre of mass of this half ring will be at height $\frac{2r}{\pi}$

$$y_{cm} = \frac{1}{M}\int_{R_1}^{R_2}(\rho.\pi r\, dr).\frac{2r}{\pi}$$

$$y_{cm} = \frac{2\rho}{\rho\frac{\pi}{2}(R_2^2 - R_1^2)}$$



$$\int_{R_1}^{R_2} r^2\, dr = \frac{4(R_2^3 - R_1^3)}{3\pi(R_2^2 - R_1^2)}$$

Alternative solution :

We can also find the centre of mass of this object by considering it to be complete half disc of radius R_2 and a smaller half disc of radius R_1 cut from it. If y_{cm} be the centre of mass of this disc we have from the mass moments.

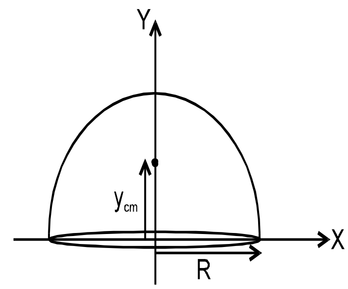
$$\left(\rho.\frac{\pi R_1^2}{2}\right)\times\left(\frac{4R_1}{3\pi}\right)+\left(\rho.\frac{\pi}{2}(R_2^2 - R_1^2)\right)$$

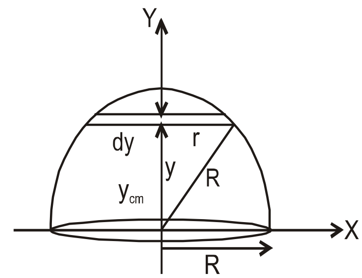
$$(y_{cm}) = \left(\rho\cdot\frac{\pi R_2^2}{2}\right)\times\left(\frac{4R_2}{3\pi}\right)$$

$$y_{cm} = \frac{4(R_2^3 - R_1^3)}{3\pi(R_2^2 - R_1^2)}$$

CENTRE OF MASS OF A SOLID HEMISPHERE

The hemisphere is of mass M and radius R. To find its centre of mass (only y-coordinate), we consider an element disc of width dy, mass dm at a distance y from the centre of the hemisphere. The radius of this elemental disc will be given as $r = \sqrt{R^2 - y^2}$





The mass dm of this disc can be given as

$$dm = \frac{3M}{2\pi R^3}\times\pi r^2\, dy$$

$$= \frac{3M}{2R^3}(R^2 - y^2)\, dy$$

y_{cm} of the hemisphere is given as

$$y_{cm} = \frac{1}{M}\int_0^R dm\, y$$

$$= \frac{1}{M}\int_0^R \frac{3M}{2R^3}(R^2 - y^2)\, dy\, y$$

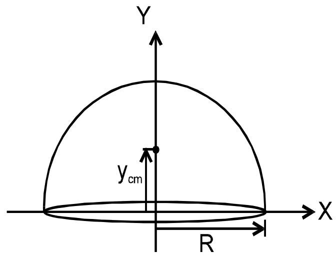
$$= \frac{3}{2R^3}\int_0^R (R^2 - y^2)\, y\, dy$$

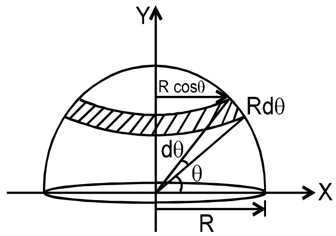
$$y_{cm} = \frac{3R}{8}$$

CENTRE OF MASS OF A HOLLOW HEMISPHERE

A hollow hemisphere of mass M and radius R. Now we consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere. This strip will have an area.

$dS = 2\pi R \cos\theta\, R d\theta$





Its mass dm is given as dm = $\frac{M}{2\pi R^2}$ $2\pi R \cos\theta\, Rd\theta$

Here y-coordinate of this strip of mass dm can be taken as R sinθ. Now we can obtain the centre of mass of the system as.

$$y_{cm} = \frac{1}{M}\int_0^{\frac{\pi}{2}} dm\, R\sin\theta$$

$$= \frac{1}{M}\int_0^{\frac{\pi}{2}}\left(\frac{M}{2\pi R^2}2\pi R^2\cos\theta\, d\theta\right) R\sin\theta$$

$$= R\int_0^{\frac{\pi}{2}}\sin\theta\ \cos\theta\ d\theta$$

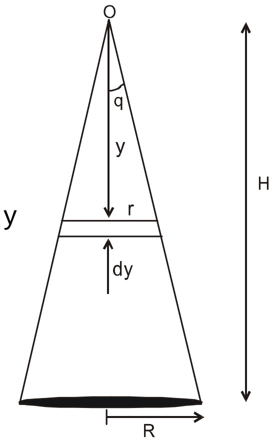
$$y_{cm} = \frac{R}{2}$$

CENTRE OF MASS OF A SOLID CONE

A solid cone has mass M, height H and base radius R. Obviously the centre of mass of this cone will lie somewhere on its axis, at a height less than H/2. To locate the centre of mass we consider an elemental disc of width dy and radius r, at a distance y from the apex of the cone. Let the mass of this disc be dm, which can be given as

$$dm = \frac{3M}{\pi R^2 H} \times \pi r^2\, dy$$

here y_{cm} can be given as



$$y_{cm} = \frac{1}{M}\int_0^H y\,dm$$

$$= \frac{1}{M}\int_0^R\left(\frac{3M}{\pi R^2 H}\pi\left(\frac{Ry}{H}\right)^2 dy\right)y$$

$$= \frac{3}{H^3}\int_0^H y^3\,dy$$

$$= \frac{3H}{4}$$

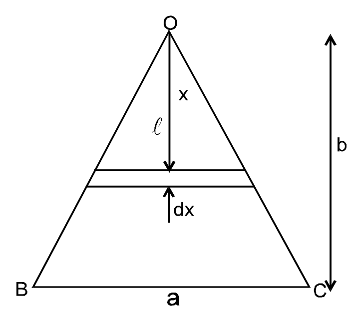
Solved Examples

Ex.9 Find out the centre of mass of an isosceles triangle of base length a and altitude b. Assume that the mass of the triangle is uniformly distributed over its area.

Sol. To locate the centre of mass of the triangle, we take a strip of width dx at a distance x from the vertex of the triangle. Length of this strip can be evaluated by similar triangles as $\ell = x.\ (a/b)$

Mass of the strip is $dm = \frac{2M}{ab}\ell dx$

Distance of centre of mass from the vertex of the triangle is



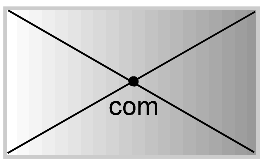
$$x_{CM} = \frac{1}{M}\int x\,dm$$

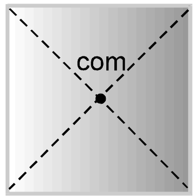
$$= \int_0^b \frac{2x^2}{b^2}dx$$

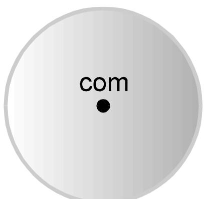
$$= \frac{2}{3}b$$

Proceeding in the similar manner, we can find the COM of certain rigid bodies. Centre of mass of some well known rigid bodies are given below :

1. Centre of mass of a uniform rectangular, square or circular plate lies at its centre. Axis of symmetry plane of symmetry.







2. For a laminar type (2-dimensional) body with uniform negligible thickness the formulae for finding the position of centre of mass are as follows :

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...}{m_1 + m_2 + ...} = \frac{\rho A_1 t\vec{r}_1 + \rho A_2 t\vec{r}_2 + ...}{\rho A_1 t + \rho A_2 t + ...}$$

($\because$ m = ρAt) or $\vec{r}_{COM} = \dfrac{A_1\vec{r}_1 + A_2\vec{r}_2 + ...}{A_1 + A_2 + ...}$

Here, A stands for the area,

3. If some mass of area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(i) $\vec{r}_{COM} = \dfrac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$ or $\vec{r}_{COM} = \dfrac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$

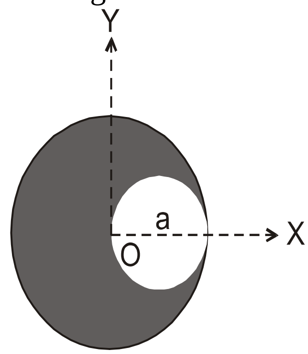
(ii) $x_{COM} = \dfrac{m_1x_1 - m_2x_2}{m_1 - m_2}$ or $x_{COM} = \dfrac{A_1x_1 - A_2x_2}{A_1 - A_2}$

$y_{COM} = \dfrac{m_1y_1 - m_2y_2}{m_1 - m_2}$ or $y_{COM} = \dfrac{A_1y_1 - A_2y_2}{A_1 - A_2}$

and $z_{COM} = \dfrac{m_1z_1 - m_2z_2}{m_1 - m_2}$ or $z_{COM} = \dfrac{A_1z_1 - A_2z_2}{A_1 - A_2}$

Here, m_1, A_1, $\vec{r}_1$, x_1, y_1 and z_1 are the values for the whole mass while m_2, A_2, $\vec{r}_2$, $\vec{x}_2$, y_2 and z_2 are the values for the mass which has been removed. Let us see two examples in support of the above theory.

Ex.10 Find the position of centre of mass of the uniform lamina shown in figure.



Sol. Here,

A_1 = area of complete circle = πa^2

A_2 = area of small circle = $\pi\left(\dfrac{a}{2}\right)^2 = \dfrac{\pi a^2}{4}$

(x_1, y_1) = coordinates of centre of mass of large circle = (0, 0) and (x_2, y_2) = coordinates of centre of mass of small circle = $\left(\dfrac{a}{2}, 0\right)$

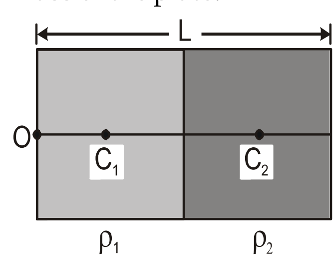
Using $x_{COM} = \dfrac{A_1x_1 - A_2x_2}{A_1 - A_2}$

we get $x_{COM} = \dfrac{-\dfrac{\pi a^2}{4}\left(\dfrac{a}{2}\right)}{\pi a^2 - \dfrac{\pi a^2}{4}} = \dfrac{-\left(\dfrac{1}{8}\right)}{\left(\dfrac{3}{4}\right)} a = -\dfrac{a}{6}$

and $y_{COM} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\dfrac{a}{6}, 0\right)$ **Ans.**

Ex.11 Half of the rectangular plate shown in figure is made of a material of density ρ_1 and the other half of density ρ_2. The length of the plate is L. Locate the centre of mass of the plate.



Ans. $X = \dfrac{(\rho_1 + 3\rho_2)}{4(\rho_1 + \rho_2)} L$

Ex.12 The centre of mass of rigid body always lie inside the body. Is this statement true or false?

Ans. False

Ex.13 The centre of mass always lie on the axis of symmetry if it exists. Is this statement true of false?

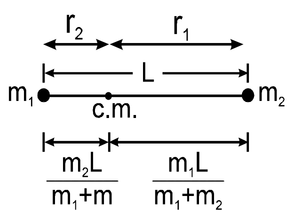
Ans. True

Ex.14 If all the particles of a system lie in y-z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or not?

Ans. True

CENTRE OF MASS OF SOME COMMON SYSTEMS

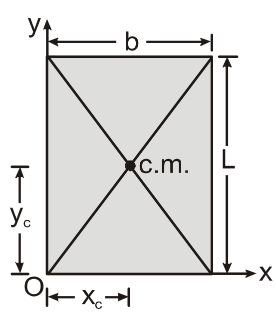
⇒ A system of two point masses $m_1 r_1 = m_2 r_2$



The centre of mass lies closer to the heavier mass.

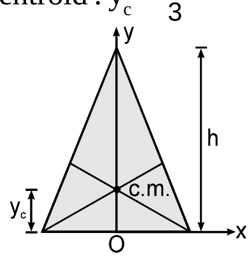
⇒ Rectangular plate (By symmetry)

$x_c = \frac{b}{2}$ $\qquad y_c = \frac{L}{2}$

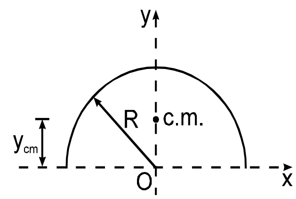


⇒ A triangular plate (By qualitative argument)

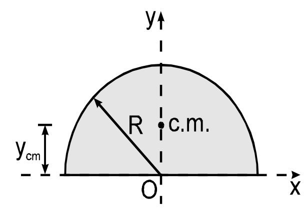
at the centroid : $y_c = \frac{h}{3}$



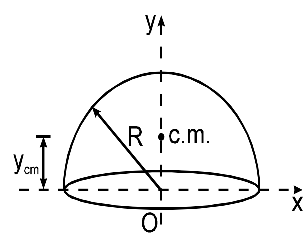
⇒ A semi-circular ring $\quad y_c = \frac{2R}{\pi} \qquad x_c = O$



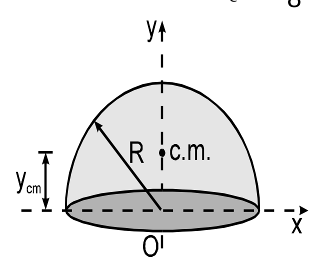
⇒ A semi-circular disc $\quad y_c = \frac{4R}{3\pi} \qquad x_c = O$



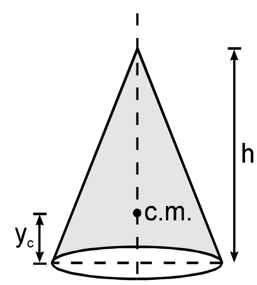
⇒ A hemispherical shell $\quad y_c = \frac{R}{2} \qquad x_c = O$



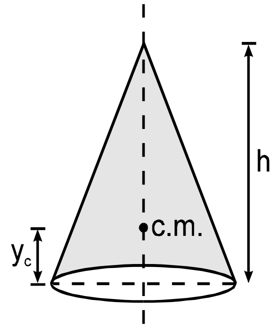
⇒ A solid hemisphere $\quad y_c = \frac{3R}{8} \qquad x_c = O$



⇒ A circular cone (solid) $\quad y_c = \frac{h}{4}$



$\Rightarrow$ A circular cone (hollow) $y_c = \frac{h}{3}$



MOTION OF CENTRE OF MASS AND CONSERVATION OF MOMENTUM

Velocity of centre of mass of system

$$\vec{v}_{cm} = \frac{m_1 \frac{\overrightarrow{dr_1}}{dt} + m_2 \frac{\overrightarrow{dr_2}}{dt} + m_3 \frac{\overrightarrow{dr_3}}{dt} .... + m_n \frac{\overrightarrow{dr_n}}{dt}}{M}$$

$$= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 .... + m_n \vec{v}_n}{M}$$

Here numerator of the right hand side term is the total momentum of the system i.e., summation of momentum of the individual component (particle) of the system

Hence velocity of centre of mass of the system is the ratio of momentum of the system per unit mass of the system.

Acceleration of centre of mass of system

$$\vec{a}_{cm} = \frac{m_1 \frac{\overrightarrow{dv_1}}{dt} + m_2 \frac{\overrightarrow{dv_2}}{dt} + m_3 \frac{\overrightarrow{dv_3}}{dt} .... + m_n \frac{\overrightarrow{dv_n}}{dt}}{M}$$

$$= \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 .... + m_n\vec{a}_n}{M}$$

$$= \frac{\text{Net force on system}}{M}$$

$$= \frac{\text{Net External Force} + \text{Net internal Force}}{M}$$

$$= \frac{\text{Net External Force}}{M}$$

($\because$ action and reaction both of an internal force must be within the system. Vector summation will cancel all internal forces and hence net internal force on system is zero)

$\therefore \quad \vec{F}_{ext} = M\,\vec{a}_{cm}$

where $\vec{F}_{ext}$ is the sum of the 'external' forces acting on the system. The internal forces which the particles exert on one another play absolutely no role in the motion of the centre of mass.

If no external force is acting on a system of particles, the acceleration of centre of mass of the system will be zero. If $a_c = 0$, it implies that v_c must be a constant and if v_{cm} is a constant, it implies that the total momentum of the system must remain constant. It leads to the principal of conservation of momentum in absence of external forces.

If $\vec{F}_{ext} = 0$ then $\vec{v}_{cm}$ = constant

"If no external force is acting on the system, net momentum of the system must remain constant".

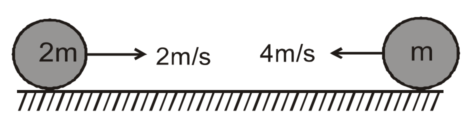
Motion of COM in a moving system of particles:

(1) COM at rest :

If $F_{ext} = 0$ and $V_{cm} = 0$, then COM remains at rest. Individual components of the system may move and have non-zero momentum due to mutual forces (internal), but the net momentum of the system remains zero.

(i) All the particles of the system are at rest.

(ii) Particles are moving such that their net momentum is zero. example:



(iii) A bomb at rest suddenly explodes into various smaller fragments, all moving in different directions then, since the explosive forces are internal & there is no external force on the system for explosion therefore, the COM of the bomb will remain at the original position and the fragment fly such that their net momentum remains zero.

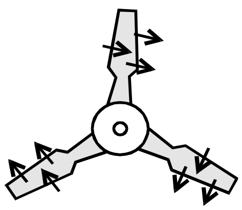
(iv) Two men standing on a frictionless platform, push each other, then also their net momentum remains zero because the push forces are internal for the two men system.

(v) A boat floating in a lake, also has net momentum zero if the people on it changes their position, because the friction force required to move the people is internal of the boat system.

(vi) Objects initially at rest, if moving under mutual forces (electrostatic or gravitation)also have net momentum zero.

(vii) A light spring of spring constant k kept copressed between two blocks of masses m_1 and m_2 on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions, such that the net momentum is zero.

(viii) In a fan, all particles are moving but com is at rest

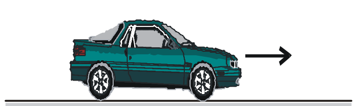


(2) COM moving with uniform velocity :

If $F_{ext} = 0$, then V_{cm} remains constant therefore, net momentum of the system also remains conserved. Individual components of the system may have variable velocity and momentum due to mutual forces (internal), but the net momentum of the system remains constant and COM continues to move with the initial velocity.

(i) All the particles of the system are moving with same velocity.

Example: A car moving with uniform speed on a straight road, has its COM moving with a constant velocity.



(ii) Internal explosions / breaking does not change the motion of COM and net momentum remains conserved. A bomb moving in a straight line suddenly explodes into various smaller fragments, all moving in different directions then, since the explosive forces are internal & there is no external force on the system for explosion therefore, the COM of the bomb will continue the original motion and the fragment fly such that their net momentum remains conserved.

(iii) Man jumping from cart or buggy also exert internal forces therefore net momentum of the system and hence, Motion of COM remains conserved.

(iv) Two moving blocks connected by a light spring of spring constant on a smooth horizontal surface. If the acting forces is only due to spring then COM will remain in its motion and momentum will remain conserved.

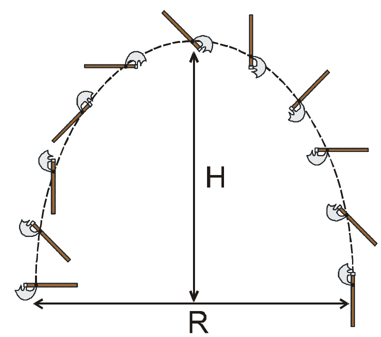
(v) Particles colliding in absence of external impulsive forces also have their momentum conserved.

(3) COM moving with acceleration :

If an external force is present then COM continues its original motion as if the external force is acting on it, irrespective of internal forces.

Example:

Projectile motion : An axe is thrown in air at an angle θ with the horizontal will perform a complicated motion of rotation as well as parabolic motion under the effect of gravitation

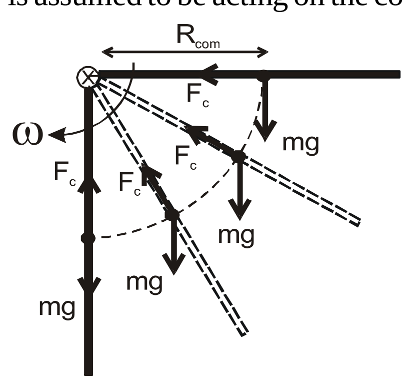


The motion of axe is complicated but the COM is moving in a parabolic motion.

$$H_{com} = \frac{u^2 \sin^2 \theta}{2g} \qquad R_{com} = \frac{u^2 \sin^2 \theta}{g} \qquad T = \frac{2u \sin \theta}{g}$$

example:

Circular Motion : A rod hinged at an end, rotates, than its COM performs circular motion. The centripetal force (F_c) required in the circular motion is assumed to be acting on the com.



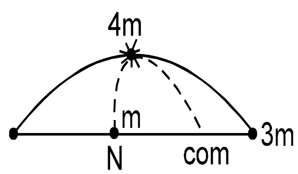
$$F_c = \frac{m\omega^2}{R_{com}}$$

Solved Examples

Ex.15 A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1 : 3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Sol. Internal force do not effect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is,

$$x_{COM} = \frac{2u^2 \sin\theta\cos\theta}{g} = \frac{2\times10^4 \times \frac{3}{5}\times\frac{4}{5}}{10}\ \text{m}$$



$= 960$ m

The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at $x = 480$ m. If the heavier block hits the ground at x_2, then

$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

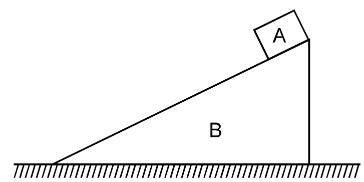
$$960 = \frac{(m)(480) + (3m)(x_2)}{(m + 3m)}$$

$x_2 = 1120$ m **Ans.**

Ex.16 In a boat of mass 4 M and length ℓ on a frictionless water surface. Two men A (mass = M) and B (mass 2M) are standing on the two opposite ends. Now A travels a distance $\ell/4$ relative to boat towards its centre and B moves a distance $3\ell/4$ relative to boat and meet A. Find the distance travelled by the boat on water till A and B meet.

Ans. $5\ell/28$

Ex.17 A block A (mass = 4M) is placed on the top of a wedge B of base length ℓ (mass = 20 M) as shown in figure. When the system is released from rest. Find the distance moved by the wedge B till the block A reaches ground. Assume all surfaces are frictionless.



Ans. $\ell/6$

Ex.18 An isolated particle of mass m is moving in a horizontal xy plane, along x-axis, at a certain height above ground. It suddenly explodes into two fragments of masses m/4 and 3m/4. An instant later, the smaller fragment is at y = + 15 cm. Find the position of heaver fragment at this instant.

Ans. y = – 5 cm

Momentum Conservation :

The total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass. $\vec{P} = M\,\vec{v}_{cm}$

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

If $\vec{F}_{ext} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0$; $\vec{P}$ = constant

When the vector sum of the external forces acting on a system is zero, the total linear momentum of the system remains constant.

$$\vec{P}_1 + \vec{P}_1 + \vec{P} + \ldots\ldots\ldots\ldots\ldots + \vec{P}_n = \text{constant.}$$

Ex.19 A shell is fired from a cannon with a speed of 100 m/s at an angle 60° with the horizontal (positive x-direction). At the highest point of its trajectory, the shell explodes into two equal fragments. One of the fragments moves along the negative x-direction with a speed of 50 m/s. What is the speed of the other fragment at the time of explosion?

Sol. As we know in absence of external force the motion of centre of mass of a body remains unaffected. Thus, here the centre of mass of the two fragment will continue to follow the original projectile path. The velocity of the shell at the highest point of trajectory is

$v_M = u\cos\theta = 100 \times \cos 60^o = 50$ m/s.

Let v_1 be the speed of the fragment which moves along the negative x-direction and the other fragment has speed v_2,. which must be along + ve x-direction. Now from momentum conservation, we have

$$mv = \frac{-m}{2} v_1 + \frac{m}{2} v_2$$

or $\quad 2v = v_2 - v_1$

or $\quad v_2 = 2v + v_1 = (2 \times 50) + 50 = 150$ m/s

Ex.20 A shell is fired from a cannon with a speed of 100 m/s at an angle 30° with the vertical (y-direction). At the highest point of its trajectory, the shell explodes into two fragments of masses in the ratio 1 : 2. The lighter fragments moves vertically upwards with an initial speed of 200 m/s. What is the speed of the heavier fragment at the time of explosion.

Ans. 125 m/sec

Ex.21 A shell at rest at origin explodes into three fragments of masses 1 kg, 2 kg and mkg. The fragments of masses 1 kg and 2 kg fly off with speeds 12 m/s along x-axis and 8 m/s along y-axis respectively. If mkg files off with speed 40 m/s then find the total mass of the shell.

Ans. 3.5 kg

Ex.22 A block moving horizontally on a smooth surface with a speed of 20 m/s bursts into two equal parts continuing in the same direction. If one of the parts moves at 30 m/s, with what speed does the second part move and what is the fractional change in the kinetic energy?

Ans. v = 10 m/s, $\frac{1}{4}$.

Ex.23 A block at rest explodes into three equal parts. Two parts starts moving along X and Y axes respectively with equal speeds of 10 m/s. Find the initial velocity of the third part.

Ans. $10\sqrt{2}$ m/s 135° below the X-axis.

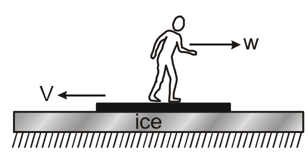
Ex.24 A boy of mass 25 kg stands on a board of mass 10 kg which in turn is kept on a frictionless horizontal ice surface. The boy makes a jump with a velocity component 5 m/s in horizontal direction with respect to the ice. With what velocity does the board recoil? With what rate are the boy and the board separating from each other ?

Sol. v = 12.5 m/s; 17.5 m/s.

Ex.25 A man of mass m is standing on a platform of mass M kept on smooth ice. If the man starts moving on the platform with a speed v relative to the platform, with what velocity relative to the ice does the platform recoil ?

Sol. Consider the situation shown in figure. Suppose the man moves at a speed w towards right and the platform recoils at a speed V towards left, both relative to the ice. Hence, the speed of the man relative to the platform is V + w. By the question,

$$V + w = v, \text{ or } w = v - V \qquad \text{.............(i)}$$



Taking the platform and the man to be the system, there is no external horizontal force on the system. The linear momentum of the system remains constant. Initially both the man and the platform were at rest. Thus,

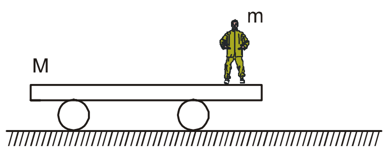
$0 = MV - mw$

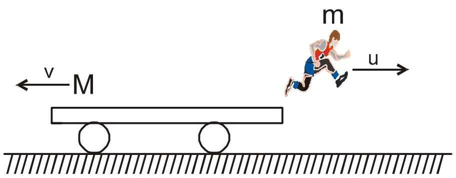
or, $\quad MV = m(v - V) \qquad$ [Using (i)]

or, $\quad V = \frac{mv}{M + m}$.

Ex.26 A flat car of mass M is at rest on a frictionless floor with a child of mass m standing at its edge. If child jumps off from the car towards right with an initial velocity u, with respect to the car, find the velocity of the car after its jump.

Sol. Let car attains a velocity v, and the net velocity of the child with respect to earth will be u – v, as u is its velocity with respect to car.





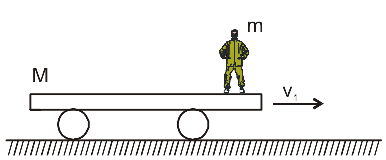
Initially, the system was at rest, thus according to momentum conservation, momentum after jump must be zero, as

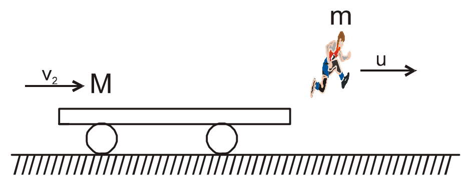
$$m\,(u - v) = M\,v$$

$$v = \frac{mu}{m + M}$$

Ex.27 A flat car of mass M with a child of mass m is moving with a velocity v_1. The child jumps in the direction of motion of car with a velocity u with respect to car. Find the final velocities of the child and that of the car after jump.

Sol. This case is similar to the previous example, except now the car is moving before jump. Here also no external force is acting on the system in horizontal direction, hence momentum remains conserved in this direction. After jump car attains a velocity v_2 in the same direction, which is less than v_1, due to backward push of the child for jumping. After jump child attains a velocity $u + v_2$ in the direction of motion of car, with respect to ground.





According to momentum conservation

$$(M + m)v_1 = Mv_2 + m\,(u + v_2)$$

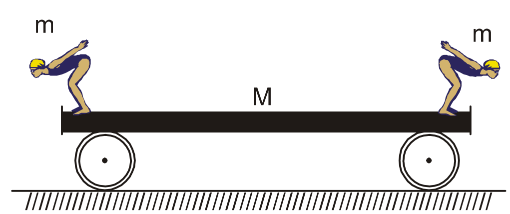
Velocity of car after jump is

$$v_2 = \frac{(M+m)v_1 - mu}{M+m}$$

Velocity of child after jump is

$$u + v_2 = \frac{(M+m)v_1 + (M)u}{M+m}$$

Ex.28 Two persons A and B, each of mass m are standing at the two ends of rail-road car of mass M. The person A jumps to the left with a horizontal speed u with respect to the car. Thereafter, the person B jumps to the right, again with the same horizontal speed u with respect to the car. Find the velocity of the car after both the persons have jumped off.

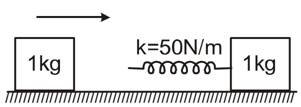


Ans. $\dfrac{m^2u}{(M + 2m)(M + m)}$

Ex.29 Two identical buggies move one after the other due to inertia (without friction) with the same velocity v_0. A man of mass m jumps into the front buggy from the rear buggy with a velocity u relative to his buggy. Knowing that the mass of each huggy is equal to M, find the velocities with which the buggies will move after that.

Ans. $v_F = v_0 + \dfrac{Mmu}{(M+m)^2}$; $v_A = v_0 - \dfrac{mu}{(M+m)}$

Ex.30 Each of the blocks shown in figure has mass 1 kg. The rear block moves with a speed of 2 m/s towards the front block kept at rest. The spring attached to the front block is light and has a spring constant 50 N/m. Find the maximum compression of the spring.



Sol. Maximum compression will take place when the blocks move with equal velocity. As no net external horizontal force acts on the system of the two blocks, the total linear momentum will remain constant. If V is the common speed at maximum compression, we have,

(1 kg) (2 m/s) = (1 kg)V + (1 kg)V

or, V = 1 m/s.

Initial kinetic energy = $\frac{1}{2}$ (1 kg) (2 m/s)^2 = 2 J.

Final kinetic energy

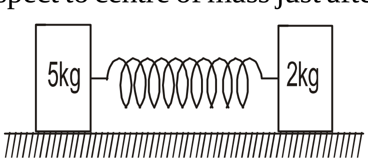
= $\frac{1}{2}$ (1 kg) (1m/s)^2 + $\frac{1}{2}$ (1 kg) (1 m/s)^2 = 1 J

The kinetic energy lost is stored as the elastic energy in the spring.

Hence, $\frac{1}{2}$(50 N/m) x^2 = 2J – 1J = 1 J

or, x = 0.2 m.

Ex.31 Figure shows two blocks of masses 5 kg and 2 kg placed on a frictionless surface and connected with a spring. An external kick gives a velocity 14 m/s to the heavier block in the direction of lighter one. Deduce (a) velocity gained by the centre of mass and (b) the separate velocities of the two blocks with respect to centre of mass just after the kick.



Sol. (a) Velocity of centre of mass is

$$v_{cm} = \frac{5\times14+2\times0}{5+2} = 10 \text{ m/s}$$

(b) Due to kick on 5 kg block, it starts moving with a velocity 14 m/s immediately, but due to inertia 2 kg block remains at rest, at that moment. Thus Velocity of 5 kg block with respect to the centre of mass is v_1 = 14 – 10 = 4 m/s

and the velocity of 2 kg block w.r.t. to centre of mass is v_2 = 0 – 10 = –10 m/s

Ex.32 A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions.

The spring loses contact with the blocks when it acquires natural length. If the spring was initially compressed through a distance x, find the final speeds of the two blocks.

Sol. Consider the two blocks plus the spring to be the system. No external force acts on this system in horizontal direction. Hence, the linear momentum will remain constant. Suppose, the block of mass M moves with a speed V and the other block with a speed v after losing contact with the spring. From conservation of linear momentum in horizontal direction we have

$$MV - mv = 0 \quad \text{or} \quad V = \frac{m}{M}v, \quad ....(i)$$

Initially, the energy of the system = $\frac{1}{2}kx^2$

Finally, the energy of the system

$$= \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

As there is no friction, mechanical energy will remain conserved.

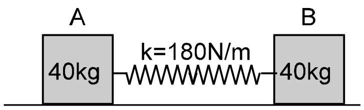
Therefore, $\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}kx^2$ ...(ii)

Solving Eqs. (i) and (ii), we get

or, $V = \left[\frac{kM}{m(M+m)}\right]^{1/2} x$

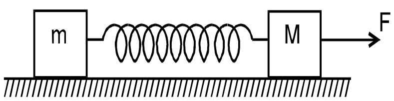
and $V = \left[\frac{km}{M(M+m)}\right]^{1/2} x$ **Ans.**

Ex.33 Blocks A and B have masses 40 kg and 60 kg respectively. They are placed on a smooth surface and the spring connected between them is stretched by 2m. If they are released from rest, determine the speeds of both blocks at the instant the spring becomes unstretched.

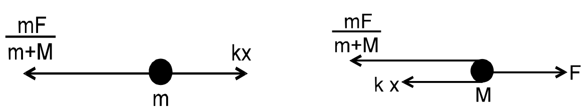


Ans. 3.2 m/s, 2.19 m/s

Ex.34 A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane and are at rest. The spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.



Sol. We solve the situation in the reference frame of centre of mass. As only F is the external force acting on the system, due to this force, the acceleration of the centre of mass is F/(M + m). Thus with respect to centre of mass there is a Pseudo force on the two masses in opposite direction, the free body diagram of m and M with respect to centre of mass (taking centre of mass at rest) is shown in figure.



Taking centre of mass at rest, if m moves maximum by a distance x_1 and M moves maximum by a distance x_2, then the work done by external forces (including Pseudo force) will be

$$W = \frac{mF}{m+M} \cdot x_1 + \left(F - \frac{MF}{m+M}\right) \cdot x_2$$

$$= \frac{mF}{m+M} \cdot (x_1 + x_2)$$

This work is stored in the form of potential energy of the spring as

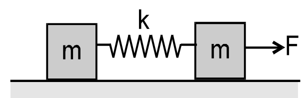
$$U = \frac{1}{2} k(x_1 + x_2)^2$$

Thus on equating we get the maximum extension in the spring, as after this instant the spring starts contracting.

$$\frac{1}{2} k(x_1 + x_2)^2 = \frac{mF}{m+M} \cdot (x_1 + x_2)$$

$$x_{max} = x_1 + x_2 = \frac{2mF}{k(m+M)}$$

Ex.35 Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure (a) Find the displacement of the centre of mass at time t (b) if the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.



Sol. (a) The acceleration of the centre of mass is

$$a_{COM} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2} a_{COM} t^2 = \frac{Ft^2}{4m} \quad \textbf{Ans.}$$

(b) Suppose the displacement of the first block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m}$$

or, $$\frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or, $$x_1 + x_2 = \frac{Ft^2}{2m} \quad \text{...(i)}$$

Further, the extension of the spring is $x_1 - x_2$. Therefore, $x_1 - x_2 = x_0$ ...(ii)

From Eqs. (i) and (ii), $$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right)$$

and $$x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right) \quad \textbf{Ans.}$$

IMPULSE

Impulse of a force F action on a body is defined as :-

$$\vec{J} = \int_{t_i}^{t_f} F dt$$

$$\vec{J} = \int F dt = \int m \frac{dv}{dt} dt = \int m\, dv$$

$$\vec{J} = m(v_2 - v_1)$$

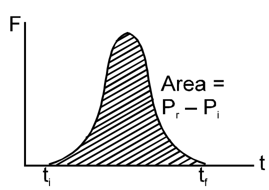
It is also defined as change in momentum

$\vec{J} = \Delta\vec{P}$ **(impulse - momentum theorem)**

Instantaneous Impulse :

There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write.

$$\vec{J} = \int \vec{F} dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$



Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force time (F-t) graph in the same time interval.

Important Points :

(1) It is a vector quantity.

(2) Dimensions = [MLT^{-1}]

(3) SI unit = kg m/s

(4) Direction is along change in momentum.

(5) Magnitude is equal to area under the F-t. graph.

(6) $J = \int F dt = F_{av} \int dt = F_{av} \Delta t$

(7) It is not a property of any particle, but it is a measure of the degree, to which an external force changes the momentum of the particle.

Solved Examples

Ex.36 The hero of a stunt film fires 50 g bullets from a machine gun, each at a speed of 1.0 km/s. If he fires 20 bullets in 4 seconds, what average force does he exert against the machine gun during this period?

Sol. The momentum of each bullet = (0.050 kg) (1000 m/s) = 50 kg-m/s.

The gun is imparted this much of momentum by each bullet fired. Thus, the rate of change of momentum

of the gun $= \frac{(50\ kg - m/s) \times 20}{4\,s}$ = 250 N.

In order to hold the gun, the hero must exert a force of 250 N against the gun.

Impulsive force :

A force, of relatively higher magnitude and acting for relatively shorter time, is called impulsive force.

An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval. **Impulsive force** is a relative term. There is no clear boundary between an impulsive and Non-Impulsive force.

Note: Usually colliding forces are impulsive in nature.

Since, the application time is very small, hence, very little motion of the particle takes place.

Important points :

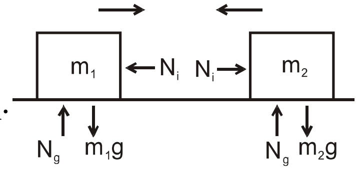
1. Gravitational force and spring force are always non-Impulsive.

2. Normal, tension and friction are case dependent.

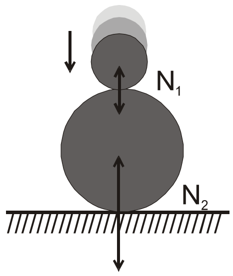
3. An impulsive force can only be balanced by another impulsive force.

1. **Impulsive Normal :** In case of collision, normal forces at the surface of collision are always impulsive

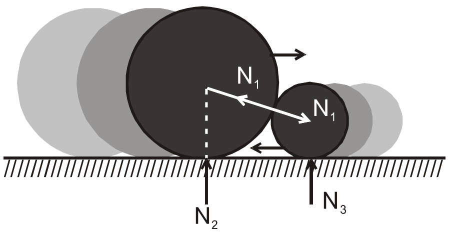
eq.



N_i = Impulsive; N_g = Non-impulsive

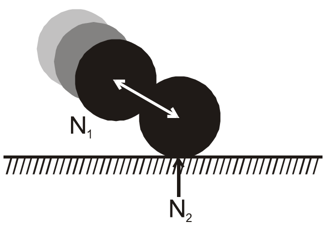


Both normals are Impulsive

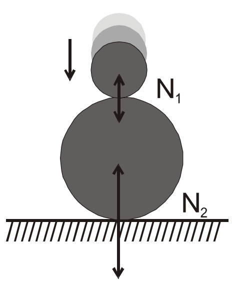


N_1, N_3 = Impulsive; N_2 = non-impulsive

Both normal are impulsive

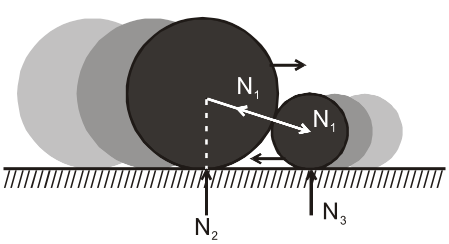


2. Impulsive Friction : If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



Friction at both surfaces is impulsive

Both normals are Impulsive

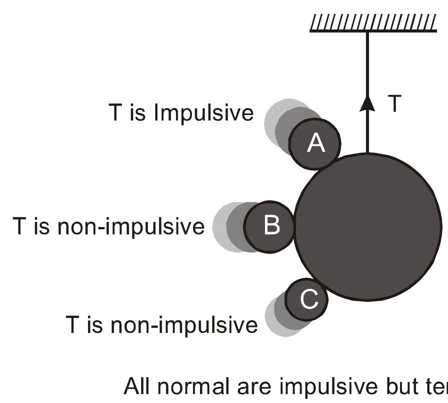


Friction due to N_2 is non-impulsive and due to N_3 is impulsive

3. Impulsive Tensions : When a string jerks, equal and opposite tension act suddenly at each end. Consequently equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

(a) One end of the string is fixed : The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object there. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

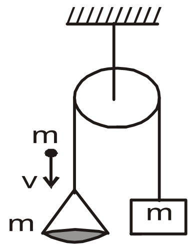
(b) Both ends of the string attached to movable objects : In this case equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string however, no impulse acts and the momentum of each particle in this direction is unchanged.



All normal are impulsive but tension
T is impulsive only for the ball A

Note : In case of rod Tension is always impulsive In case of spring Tension is always non-impulsive.

Ex.37 A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v, find the speed with which the system moves just after the collision.



Sol. Let the required speed is V.

Further,

let J_1 = impulse between particle and pan

and J_2 = impulse imparted to the block and the pan by the string

Using impulse = change in momentum

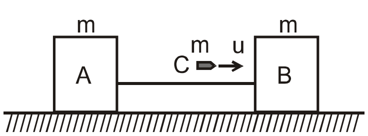
For particle $J_1 = mv - mV$ ....(i)

For pan $J_1 - J_2 = mV$ .....(ii)

For block $J_2 = mV$ .....(iii)

Solving, these three equation, we get $V = \frac{v}{3}$

Ex.38 Two identical block A and B, connected by al massless string are placed on a frictionless horizontal plane. A bullet having same mass, moving with speed u strikes block B from behind as shown. If the bullet gets embedded into the block B then find:



(a) The velocity of A,B,C after collision.

(b) Impulse on A due to tension in the string

(c) Impulse on C due to normal force of collision.

(d) Impulse on B due to normal force of collision.

Sol. (a) By Conservation of linear momentum $v = \frac{u}{3}$

(b) $\int T\,dt = \frac{mu}{3}$ (c) $\int N\,dt = m\left(\frac{u}{3} - u\right) = \frac{-2mu}{3}$

(d) $\int (N - T)\,dt = \int N dt - \int T\,dt = \frac{mu}{3}$

$\Rightarrow \int N\,dt = \frac{2mu}{3}$

COLLISION OR IMPACT

Collision is an isolated event in which a strong force acts between two or more bodies for a short time, which results in change of their velocities.

Note :

(a) In collision particles may or may not come in physical contact.

(b) The duration of collision, Δt is negligible as compared to the usual time intervals of observation of motion.

(c) In a collision the effect of external non impulsive forces such as gravity are not taken into a account as due to small duration of collision (Δt) average impulsive force responsible for collision is much larger than external forces acting on the system.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles.

Line of Impact:

The line passing through the common normal to the surfaces in contact during impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of Line of impact can be determined by:

(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

If one particle is stationary before the collision then the line of impact will be along its motion after collision.

Classification of collisions

(a) On the basis of line of impact

(i) Head-on collision : If the velocities of the particles are along the same line before and after the collision.

(ii) Oblique collision : If the velocities of the particles are along different lines before and after the collision.

(b) On the basis of energy :

(i) Elastic collision : In an elastic collision, the particle regain their shape and size completely after collision. i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus, kinetic energy of system after collision is equal to kinetic energy of system before collision. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

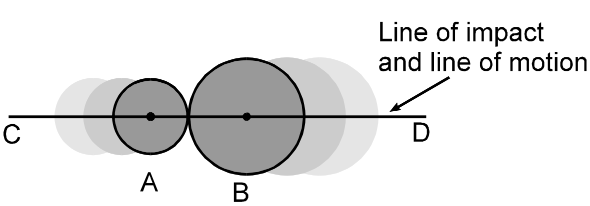
(ii) Inelastic collision : In an inelastic collision, the particle do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

(iii) Perfectly inelastic : If velocity of separation just after collision becomes zero then the collision is perfectly inelastic. Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity,

Note : Actually collision between all real objects are neither perfectly elastic nor perfectly inelastic, its inelastic in nature.

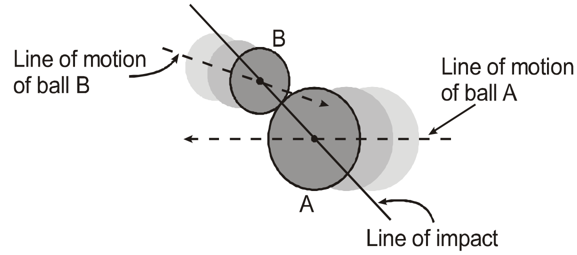
Illustrations of line of impact and collisions based on line of impact

(i) Two balls A and B are approaching each other such that their centres are moving along line CD.



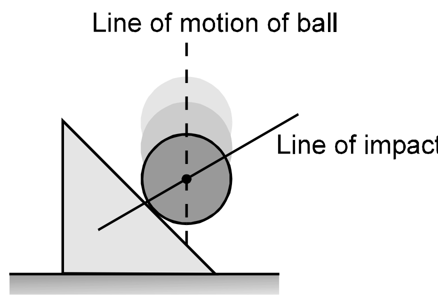
Head on Collision

(ii) Two balls A and B are approaching each other such that their centre are moving along dotted lines as shown in figure.



Oblique Collision

(iii) Ball is falling on a stationary wedge.



Oblique Collision

COEFFICIENT OF RESTITUTION (e)

The coefficient of restitution is defined as the ratio of the impulses of recovery and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r\, dt}{\int F_d\, dt}$$

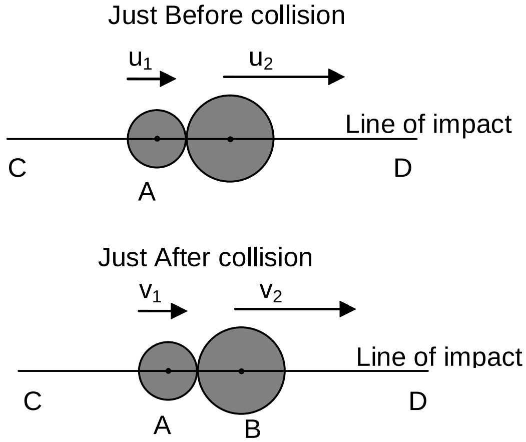
$$= \frac{\text{Velocity of seperation along line of impact}}{\text{Velocity of approach along line of impact}}$$

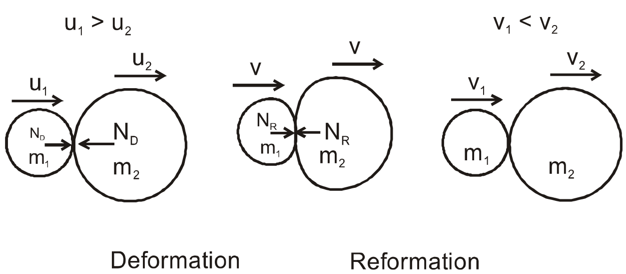
The most general expression for coefficient of restitution is

$$e = \frac{\text{velocity of separation of points of contact along line of impact}}{\text{velocity of approach of point of contact along line of impact}}$$

Illustration for calculation of e

Two smooth balls A and B approaching each other such that their centres are moving along line CD in absence of external impulsive force. The velocities of A and B just before collision be u_1 and u_2 respectively. The velocities of A and B just after collision be v_1 and v_2 respectively.





$\because$ $F_{ext} = 0$ momentum is conserved for the system.

$\Rightarrow$ $m_1u_1 + m_2u_2 = (m_1 + m_2)v = m_1v_1 + m_2v_2$

$$\Rightarrow \quad v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \qquad \text{......(1)}$$

Impulse of Deformation :

J_D = change in momentum of any one body during deformation.

$= m_2(v - u_2)$ for m_2

$= m_1(-v + u_1)$ for m_1

Impulse of Reformation :

J_R = change in momentum of any one body during Reformation.

$= m_2(v_2 - v)$ for m_2

$= m_1(v - v_1)$ for m_1

$$e = \frac{\textbf{Impulse of Reformation } (\vec{J}_R)}{\textbf{Impulse of Deformation } (\vec{J}_D)} = \frac{v_2 - v}{v - v_1}$$

$$= \frac{v_2 - v_1}{u_1 - u_2} \qquad \text{(substituting v from (1))}$$

$$= \frac{\textbf{Velocity of separation along line of impact}}{\textbf{Velocity of approach along line of impact}}$$

Note : e is independent of shape and mass of object but depends on the material.

The coefficient of restitution is constant for two particular objects.

(a) e = 1

$\Rightarrow$ Impulse of Reformation

= Impulse of Deformation

$\Rightarrow$ Velocity of separation = Velocity of approach

$\Rightarrow$ Kinetic Energy may be conserved

$\Rightarrow$ Elastic collision.

(b) e = 0

$\Rightarrow$ Impulse of Reformation = 0

$\Rightarrow$ Velocity of separation = 0

$\Rightarrow$ Kinetic Energy is not conserved

$\Rightarrow$ Perfectly Inelastic collision.

(c) 0 < e < 1

$\Rightarrow$ Impulse of Reformation

< Impulse of Deformation

$\Rightarrow$ Velocity of separation < Velocity of approach

$\Rightarrow$ Kinetic Energy is not conserved

$\Rightarrow$ Inelastic collision.

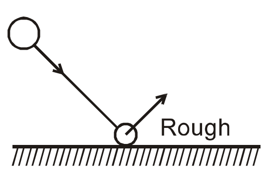
Note : In case of contact collisions e is always less than unity.

$\therefore \quad 0 \le e \le 1$

Important Point :

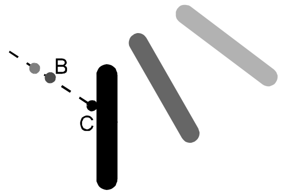
In case of elastic collision, if rough surface is present then

$k_f < k_i$ (because friction is impulsive)

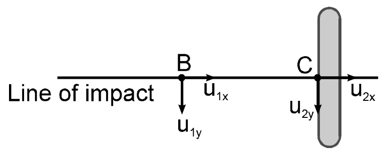


A particle 'B' moving along the dotted line collides with a rod also in state of motion as shown in the figure. The particle B comes in contact with point C on the rod.

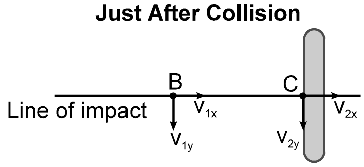
To write down the expression for coefficient of restitution e, we first draw the line of impact. Then we resolve the components of velocities of points of contact of both the bodies along line of impact just before and just after collision.



Just Before Collision

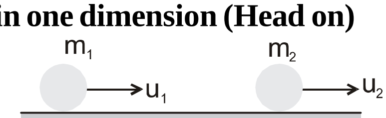


Just After Collision



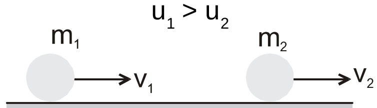
Then $\quad e = \dfrac{v_{2x} - v_{1x}}{u_{1x} - u_{2x}}$

Collision in one dimension (Head on)



(a)
Before Collision

$u_1 > u_2$



(b)
After Collision

$v_2 > v_1$

$e = \dfrac{v_2 - v_1}{u_1 - u_2} \quad \Rightarrow \quad (u_1 - u_2)e = (v_2 - v_1)$

By momentum conservation,

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$v_2 = v_1 + e(u_1 - u_2)$

and $v_1 = \dfrac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$

$v_2 = \dfrac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2}$

Special Case :

(1) $\quad e = 0 \Rightarrow \quad v_1 = v_2$

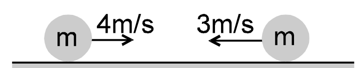
$\Rightarrow$ for perfectly inelastic collision, both the bodies, move with same vel. after collision.

(2) $\quad e = 1$

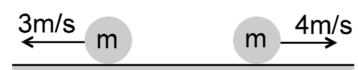
and $m_1 = m_2 = m$,

we get $v_1 = u_2$ and $v_2 = u_1$

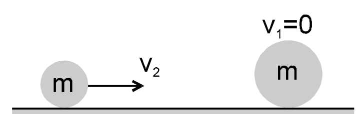
i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., e.g.



Before Collision



After Collision



Before Collision

After Collision

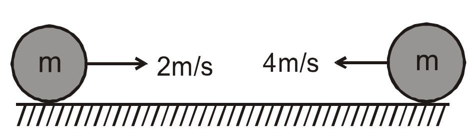
(3) $\quad m_1 >>>> m_2 \qquad m_1 + m_2 \approx m_1$ and $\dfrac{m_2}{m_1} \approx 0$

$\Rightarrow \quad v_1 = u_1 \qquad$ No change

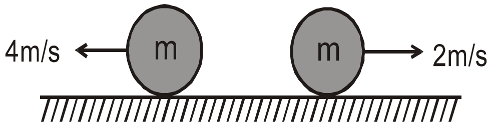
and $\qquad v_2 = u_1 + e(u_1 - u_2)$

Solved Examples

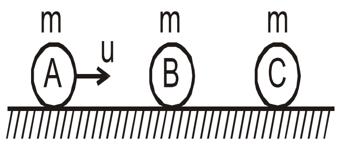
Ex.39 Two identical balls are approaching towards each other on a straight line with velocity 2 m/s and 4 m/s respectively. Find the final velocities, after elastic collision between them.



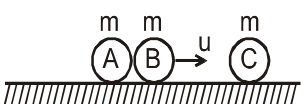
Sol. The two velocities will be exchanged and the final motion is reverse of initial motion for both.



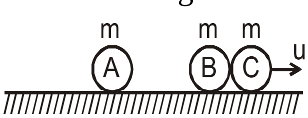
Ex.40 Three balls A, B and C of same mass 'm' are placed on a frictionless horizontal plane in a straight line as shown. Ball A is moved with velocity u towards the middle ball B. If all the collisions are elastic then, find the final velocities of all the balls.



Sol. A collides elastically with B and comes to rest but B starts moving with velocity u

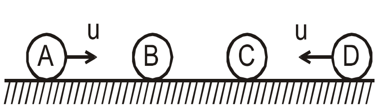


After a while B collides elastically with C and comes to rest but C starts moving with velocity u

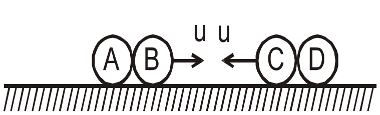


$\therefore$ Final velocities $V_A = 0$; $V_B = 0$ and $V_C = u$ **Ans.**

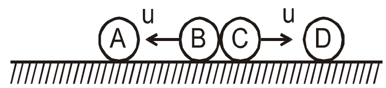
Ex.41 Four identical balls A, B, C and D are placed in a line on a frictionless horizontal surface. A and D are moved with same speed 'u' towards the middle as shown. Assuming elastic collisions, find the final velocities.



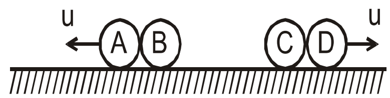
Sol. A and D collides elastically with B and C respectively and come to rest but B and C starts moving with velocity u towards each other as shown



B and C collides elastically and exchange their velocities to move in opposite directions

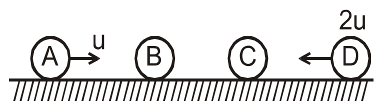


Now, B and C collides elastically with A and D respectively and come to rest but A and D starts moving with velocity u away from each other as shown

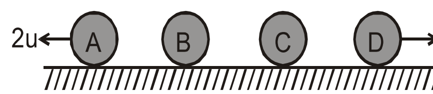


$\therefore$ Final velocities $V_A = u\ (\leftarrow)$; $V_B = 0$; $V_C = 0$ and $V_D = u\ (\rightarrow)$ **Ans.**

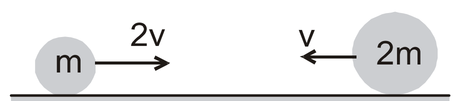
Ex.42 If A is moved with velocity u and D is moved with 2u as shown. What will be the final velocities now be?



Ans.



Ex.43 Two particles of mass m and 2m moving in opposite directions collide elastically with velocity v and 2v respectively. Find their velocities after collision.



Sol. Let the final velocities of m and 2m be v_1 and v_2 respectively as shown in the figure:

By conservation of momentum:

$$m(2v) + 2m(-v) = m(v_1) + 2m\,(v_2)$$

or $\quad 0 = mv_1 + 2mv_2$

or $\quad v_1 + 2v_2 = 0 \qquad\qquad ....(1)$

and since the collision is elastic:

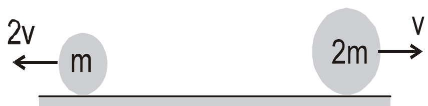
$$v_2 - v_1 = 2v - (-v)$$

or $\quad v_2 - v_1 = 3v \qquad$ ....(2)

Solving the above two equations, we get,

$$v_2 = v \text{ and } v_1 = -2v \qquad \text{Ans.}$$

i.e., the mass 2m returns with velocity v while the mass m returns with velocity 2v in the direction shown in figure:



Ex.44 Find the fraction of kinetic energy lost by the colliding particles after collision in the above situation.

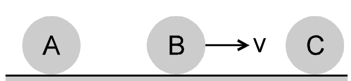
Ans. The collision was elastic therefore, no kinetic energy is lost,

KE loss = KE_i - KE_f

$$\left(\frac{1}{2}m(2v)^2 + \frac{1}{2}(2m)(-v)^2\right) - \left(\frac{1}{2}m(-2v)^2 + \frac{1}{2}(2m)v^2\right)$$

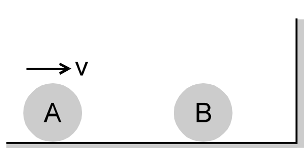
$= 0$

Ex.45 Three balls A, B and C are placed on a smooth horizontal surface. Given that $m_A = m_C = 4m_B$. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls. All collisions are elastic.



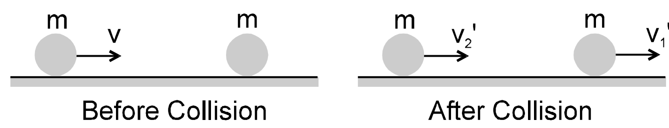
Ans. 2 collisions

Ex.46 Two balls shown in figure are identical. Ball A is moving towards right with a speed v and the second ball is at rest. Assume all collisions to be elastic. Show that the speeds of the balls remains unchanged after all the collisions have taken place.



Ex.47 A ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4th of the original. Find the coefficient of restitution.

Sol. As we have seen in the above discussion, that under the given conditions :



$$v_1' = \left(\frac{1+e}{2}\right)v \qquad \text{and} \qquad v_2' = \left(\frac{1-e}{2}\right)v$$

Given that $\qquad K_f = \frac{3}{4}K_i$

or $\qquad \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $\qquad (1 + e)^2 + (1 - e)^2 = 3$

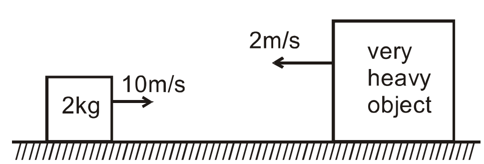
or $\qquad 2 + 2e^2 = 3$

or $\qquad e^2 = \frac{1}{2} \qquad$ or $\qquad e = \frac{1}{\sqrt{2}}$ **Ans.**

Ex.48 A block of mass m moving at speed v collides with another block of mass 2 m at rest. The lighter block comes to rest after the collision. Find the coefficient of restitution.

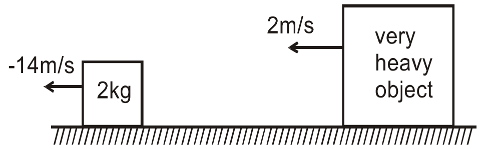
Ans. $\frac{1}{2}$

Ex.49 A block of mass 2 kg is pushed towards a very heavy object moving with 2 m/s closer to the block (as shown). Assuming elastic collision and frictionless surfaces, find the final velocities of the blocks.

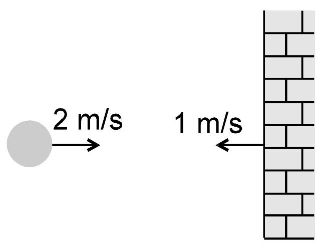


Sol. Let v_1 and v_2 be the final velocities of 2kg block and heavy object respectively then,

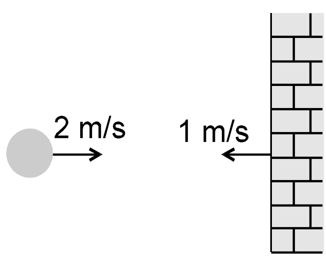
$$v_1 = u_1 + 1\,(u_1 - u_2) = 2u_1 - u_2$$
$$= -14 \text{ m/s}$$
$$v_2 = -2\text{m/s}$$



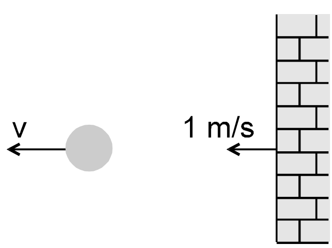
Ex.50 A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1m/s as shown in fig. Assuming collision to be elastic, find the velocity of the ball immediately after the collision.



Sol. The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since collision is elastic (e = 1),



Before Collision



After Collision

separation speed = approach sped

or $\quad v - 1 = 2 + 1$

or $\quad v = 4$ m/s **Ans.**

Ex.51 Two balls of masses 2 kg and 4 kg are moved towards each other with velocities 4 m/s and 2 m/s respectively on a frictionless surface. After colliding the 2 kg balls returns back with velocity 2m/s. Then find:

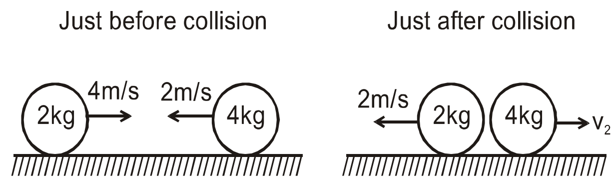
(a) velocity of 4 kg ball after collision

(b) coefficient of restitution e.

(c) Impulse of deformation J_D.

(d) Maximum potential energy of deformation.

(e) Impulse of reformation J_R.



Sol. (a) By momentum conservation,

$$2(4) - 4(2) = 2(-2) + 4(v_2)$$
$$\Rightarrow \quad v_2 = 1 \text{ m/s}$$

(b) $$e = \frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{1-(-2)}{4-(-2)} = \frac{3}{6}$$
$$= 0.5$$

(c) At maximum deformed state, by conservation of momentum, common velocity is v = 0.

$$J_D = m_1(v - u_1) = m_2(v - u_2)$$
$$= 2(0 - 4) = -8 \text{ N -s}$$
$$= 4(0 - 2) = -8 \text{ N - s}$$
or $$= 4(0 - 2) = -8 \text{ N - s}$$

(d) Potential energy at maximum deformed state U = loss in kinetic energy during deformation.

$$\text{or } U = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \left(\frac{1}{2}2(4)^2 + \frac{1}{2}4(2)^2\right) - \frac{1}{2}(2 + 4)(0)^2$$

or U = 24 Joule

(e) $J_R = m_1(v_1 - v) = m_2(v - v_2)$

$= 2(-2 - 0) = -4$ N-s

or $= 4(0 - 1) = -4$ N-s

or $e = \dfrac{J_R}{J_D}$

$\Rightarrow J_R = eJ_D$

$= (0.5)(-8)$

$= -4$ N-s

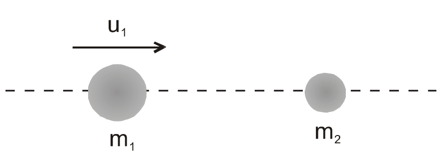
Ex.52 A block of mass m moving at a speed v collides with another block of mass 2m at rest. The lighter block comes to rest after the collision. Find the coefficient of restitution.

Ans. $\dfrac{1}{2}$.

Ex.53 A block of mass 1.2 kg moving at a speed of 20 cm/s collides head-on with a similar block kept at rest. The coefficient of restitution is 3/5. Find the loss of the kinetic energy during the collision.

Ans. 7.7×10^{-3} J.

Ex.54 The sphere of mass m_1 travels with an initial velocity u_1 directed as shown and strikes the stationary sphere of mass m_2 head on. For a given coefficient of restitution e, what condition on the mass ratio $\dfrac{m_1}{m_2}$ ensures that the final velocity of m_2 is greater than u_1?



Ans. $\dfrac{m_1}{m_2} > \dfrac{1}{e}$

Collision in two dimension (oblique)

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

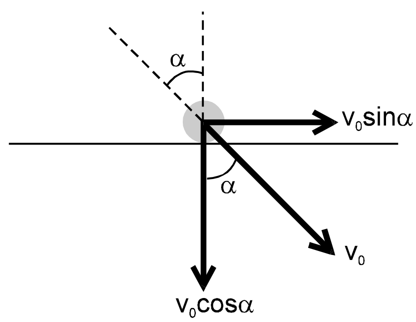
3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

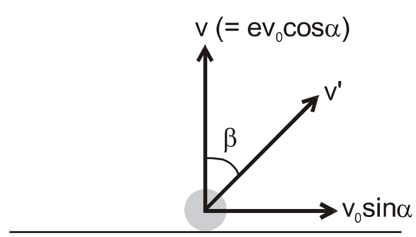
4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply

Relative speed of separation = e (relative speed of approach)

Ex.55 A ball of mass m hits a floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.

Sol. The component of velocity v_0 along common tangent direction $v_0 \sin \alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying





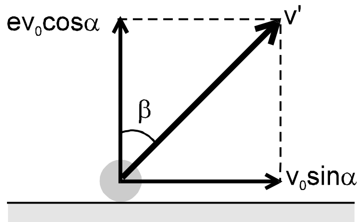
Relative speed of separation = e (relative speed of approach) along common normal direction, we get

$$v = ev_0 \cos \alpha$$

Thus, after collision components of velocity v' are $v_0 \sin\alpha$ and $ev_0 \cos\alpha$

$\therefore \quad v' = \sqrt{(v_0 \sin\alpha)^2 + (ev_0 \cos\alpha)^2}$ **Ans.**

and $\quad \tan\beta = \dfrac{v_0 \sin\alpha}{ev_0 \cos\alpha}$



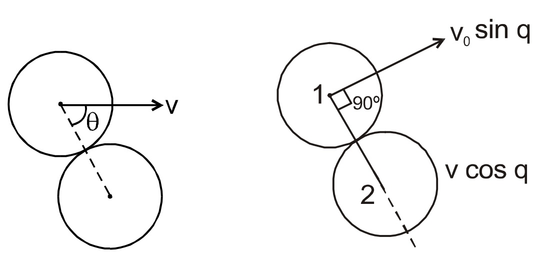
or $\quad \tan\beta = \dfrac{\tan\alpha}{e}$ **Ans.**

Note : For elastic collision, e = 1

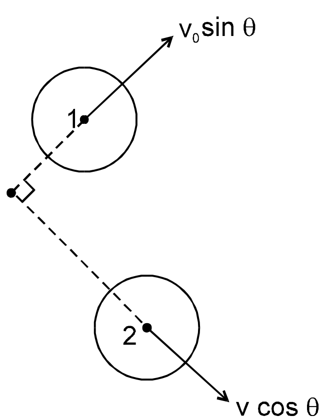
$\therefore \quad v' = v_0$ and $\beta = \alpha$

Ex.56 A ball of mass m makes an elastic collision with nother identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.

Sol. In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, v cos θ



Before collision



After collision

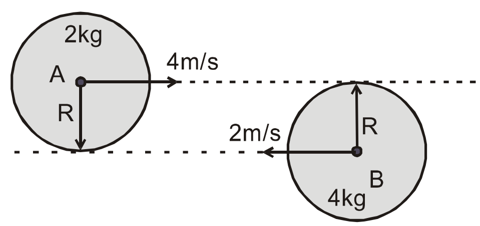
becomes zero after collision, while that of 2 becomes v cos θ. While the components along common tangent direction of both the particles remain unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given below.

Ball	Component along common tangent direction		Compoent along common normal direction	
	Before collision	After collision	Before collision	After collision
1	v sin θ	v sin θ	v cos θ	0
2	0	0	0	v cos θ

From the above table and figure, we see that both the balls move at right angle after collision with velocities v sin θ and v cos θ.

Note : When two identical bodies have an oblique elastic collision, with one particle at rest before collision, then the two particles will go in ⊥ directions.

Ex.57 Two spheres are moving towards each other. Both have same radius but their masses are 2 kg and 4 kg. If the velocities are 4 m/s and 2 m/s respectively and coefficient of restitution is e = 1/3, find.



(a) The common velocity along the line of impact.

(b) Final velocities along line of impact.

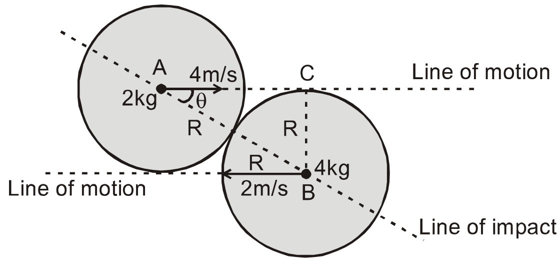
(c) Impulse of deformation.

(d) impulse of reformation.

(e) Maximum potential energy of deformation.

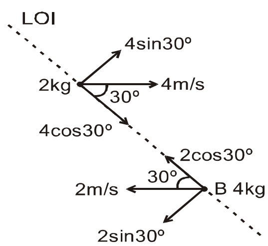
(f) Loss in kinetic energy due to collision.

Sol.

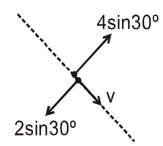


In ΔABC $\sin\theta = \frac{BC}{AB} = \frac{R}{2R} = \frac{1}{2}$ or $\theta = 30^o$

(a) By conservation of momentum along line of impact.



Just Before Collision Along LOI

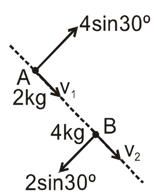


Maximum Deformed State

$$2(4\cos 30^o) - 4(2\cos 30^o) = (2+4)v$$

or $v = 0$ (common velocity along LOI)

(b)



Just After Collision Along LOI

Let v_1 and v_2 be the final velocity of A and B respectively then, by conservation of momentum along line of impact,

$$2(4\cos 30^o) - 4(2\cos 30^o) = 2(v_1) + 4(v_2)$$

or $0 = v_1 + 2v_2$ .... (1)

By coefficient of restitution,

$$e = \frac{\text{velocity of separation along LOI}}{\text{veloctiy of approach along LOI}}$$

or $\frac{1}{3} = \frac{v_2 - v_1}{4\cos 30^o + 2\cos 30^o}$

or $v_2 - v_1 = \sqrt{3}$ .... (2)

from the above two equations,

$$v_1 = \frac{2}{\sqrt{3}} m/s \text{ and } v_2 = \frac{1}{\sqrt{3}} m/s.$$

(c) $J_D = m_1(v - u_1)$

$= 2(0 - 4\cos 30^o) = -4\sqrt{3}$ N-s

(d) $J_R = eJ_D = \frac{1}{3}(-4\sqrt{3}) = -\frac{4}{\sqrt{3}}$ N-s

(e) Maximum potential energy of deformation is equal to loss in kinetic energy during deformation upto maximum deformed state,

$$U = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \frac{1}{2}(m_1+m_2)v^2$$

$$= \frac{1}{2}2(4\cos 30^o)^2 + \frac{1}{2}4(-2\cos 30^o)^2 - \frac{1}{2}(2+4)(0)^2$$

or $U = 18$ Joule.

(f) Loss in kinetic energy,

$$\Delta KE = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= \frac{1}{2}2(4\cos 30^o)^2 + \frac{1}{2}4(-2\cos 30^o)^2 - \left(\frac{1}{2}2\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}4\left(\frac{1}{\sqrt{3}}\right)^2\right)$$

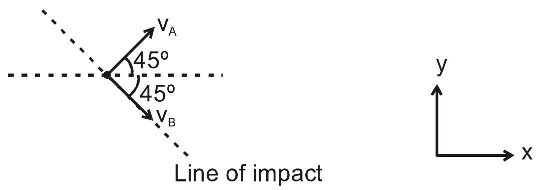
$\Delta KE = 16$ Joule

Ex.58 Two point particles A and B are placed in line on a friction less horizontal plane. If particle A (mass 1 kg) is moved with velocity 10 m/s towards stationary particle B (mass 2 kg) and after collision the two move at an angle of 45° with the initial direction of motion, then find :

(a) Find velocities of A and B just after collision.

(b) Coefficient of restitution.

Sol. The very first step to solve such problems is to find the line of impact which is along the direction of force applied by A on B, resulting the stationary B to move. Thus, by watching the direction of motion of B, line of impact can be determined. In this case line of impact is along the direction of motion of B. i.e. 45° with the initial direction of motion of A.



(a) By conservation of momentum, along x direction:

$m_A u_A = m_A v_A \cos 45° + m_B v_B \cos 45°$

or $1(10) = 1(v_A \cos 45°) + 2(v_B (\cos 45°)$

or $v_A + 2v_B = 10\sqrt{2}$ .... (1)

along y direction

$0 = m_A v_A \sin 45° + m_B v_B \sin 45°$

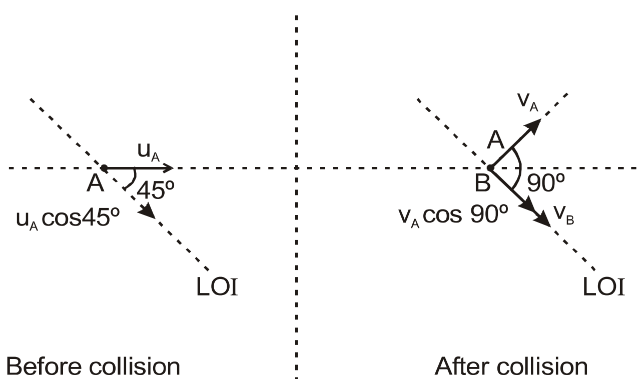
or $0 = 1(v_A \sin 45°) - 2(v_B \sin 45°)$

or $v_A = 2v_B$ .... (2)

solving the two equations,

$v_A = \frac{10}{\sqrt{2}}$ m/s and $v_B = \frac{5}{\sqrt{2}}$ m/s.

(b) $$e = \frac{\textbf{velocity of separation along LOI}}{\textbf{veloctiy of approach along LOI}}$$



or $$e = \frac{v_B - v_A \cos 90°}{u_A \cos 45°}$$

$$= \frac{\frac{5}{\sqrt{2}} - 0}{\frac{10}{\sqrt{2}}} = \frac{1}{2}$$ **Ans.**

Ex.59 A smooth sphere of mass m is moving on a horizontal plane with a velocity $3\hat{i} + \hat{j}$ when it collides with a vertical wall which is parallel to the vector $\hat{j}$. If the coefficient of restitution between the sphere and the wall is $\frac{1}{2}$, find

(a) the velocity of the sphere after impact,

(b) the loss in kinetic energy caused by the impact.

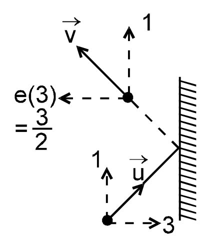
(c) the impulse $\vec{J}$ that acts on the sphere.

Sol. Let $\vec{v}$ be the velocity of the sphere after impact.

To find $\vec{v}$ we must separate the velocity components parallel and perpendicular to the wall.

Using the law of restitution the component of velocity parallel to the wall remains unchanged while component perpendicular to the wall becomes e times in opposite direction.

Thus, $$\vec{v} = -\frac{3}{2}\hat{i} + \hat{j}$$



(a) Therefore, the velocity of the sphere after impact is $= -\frac{3}{2}\hat{i} + \hat{j}$ **Ans.**

(b) The loss in K.E. $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m$

$$(3^2 + 1^2) - \frac{1}{2}m\left(\left\{\frac{3}{2}\right\}^2 + 1^2\right) = \frac{27}{8}m$$ **Ans.**

(c) $\vec{J} = \Delta\vec{P} = \vec{P}_f - \vec{P}_i = m(\vec{v}) - m(\vec{u})$

$$= m\left(-\frac{3}{2}\hat{i} + \hat{j}\right) - m\left(3\hat{i} + \hat{j}\right) = -\frac{9}{2}m\hat{i}$$ **Ans.**

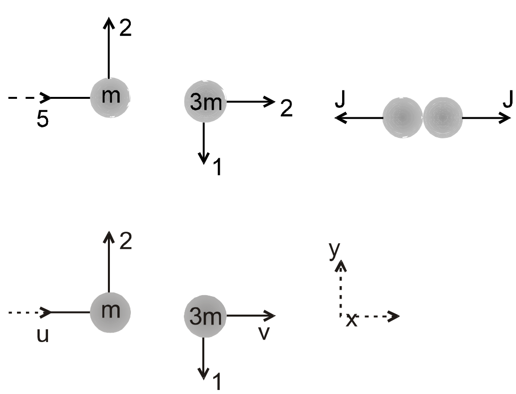
Ex.60 A sphere of mass m is moving with a velocity $4\hat{i}-\hat{j}$ when it hits a wall and rebounds with velocity $\hat{i}+3\hat{j}$. Find the impulse it receives. Find also the coefficient of restitution between the sphere and the wall.

Sol. $\vec{J} = m(-3\hat{i} + 4\hat{j})$ and $e = \frac{9}{16}$ **Ans.**

Ex.61 Two smooth spheres, A and B, having equal radii, lie on a horizontal table. A is of mass m and B is of mass 3m. The spheres are projected towards each other with velocity vector $5\hat{i}+2\hat{j}$ and $2\hat{i}-\hat{j}$ respectively and when they collide the line joining their centres is parallel to the vector $\hat{i}$.

If the coefficient of restitution between A and B is $\frac{1}{3}$, find the velocities after impact and the loss in kinetic energy caused by the collision. Find also the magnitude of the impulses that act at the instant of impact.

Sol. The line of centres at impact, is parallel to the vector $\hat{i}$, the velocity components of A and B perpendicular to $\hat{i}$ are unchanged by the impact.



Applying conservation of linear momentum and the law of restitution, we have

in x direction $5m + (3m)(2) = mu + 3mv$ .... (i)

and $\frac{1}{3}(5-2) = v - u$ .....(ii)

Solving these equations, we have $u = 2$ and $v = 3$

The velocities of A and B after impact are therefore,

$2\hat{i}+2\hat{j}$ and $3\hat{i}-\hat{j}$

respectively **Ans.**

Before impact the kinetic energy of A is

$$\frac{1}{2}m(5^2 + 2^2) = \frac{29}{2}\ m$$

and of B is $\frac{1}{2}(3m)(2^2 + 1^2) = \frac{15}{2}\ m$

After impact the kinetic energy of A is

$$\frac{1}{2}m(2^2 + 2^2) = 4m$$

and of B is $\frac{1}{2}(3m)(3^2 + 1^2) = 15\ m$

Therefore, the loss in K.E. at impact is

$$\frac{29}{2}m + \frac{15}{2}m - 4m - 15m = 3m$$

Ans.

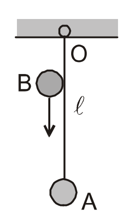
To find value of J, we consider the change in momentum along $\hat{i}$ for one sphere only.

For sphere B $J = 3m(3-2)$

or $J = 3m$

Ans.

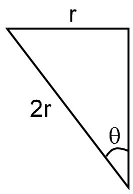
Ex.62 A small steel ball A is suspended by an inextensible thread of length $\ell = 1.5$ from O. Another identical ball is thrown vertically downwards such that its surface remains just in contact with thread during downward motion and collides elastically with the suspended ball. If the suspended ball just completes vertical circle after collision, calculate the velocity of the falling ball just before collision. ($g = 10\ ms^{-2}$)

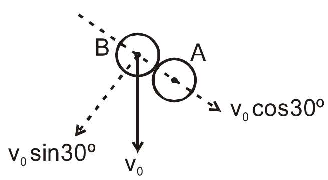


Sol. Velocity of ball A just after collision is $\sqrt{5gl}$

Let radius of each ball be r and the joining centres of the two balls makes an angle θ with the vertical at the instant of collision, then

$$\sin\theta = \frac{r}{2r} = \frac{1}{2} \text{ or } \theta = 30^\circ$$





Let velocity of ball B (just before collision) be v_0. This velocity can be resolved into two components, (i) $v_0 \cos 30^\circ$, along the line joining the centre of the two balls and (ii) $v_0 \sin 30^\circ$ normal to this line. Head -on collision takes place due to $v_0 \cos 30^\circ$ and the component $v_0 \sin 30^\circ$ of velocity of ball B remains unchanged.

Since, ball A is suspended by an inextensible string, therefore, just after collision, it can move along horizontal direction only. Hence, a vertically upward impulse is exerted by thread on the ball A. This means that during collision two impulses act on ball A simultaneously. One is impulsive interaction J between the balls and the other is impulsive reaction J' of the thread.

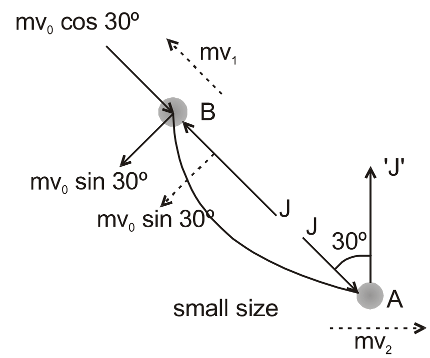
Velocity v_1 of ball B along line of collision is given by

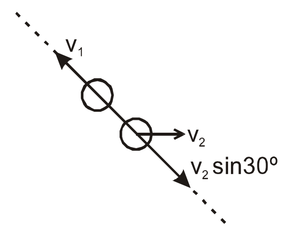
$$J - mv_0 \cos 30^\circ = mv_1$$

or $$v_1 = \frac{J}{m} - v_0 \cos 30^\circ \qquad \text{...(i)}$$

Horizontal velocity v_2 of ball A is given by $J \sin 30^\circ = mv_2$

or $$v_2 = \frac{J}{2m} \qquad \text{...(ii)}$$





Since, the balls collide elastically, therefore, coefficient of restitution is e = 1.

Hence,

$$e = \frac{v_2 \sin 30^\circ - (-v_1)}{v_0 \cos 30^\circ - 0} = 1 \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii), and (iii),

$$J = 1.6\, mv_0 \cos 30^\circ$$

$\therefore \quad v_1 = 0.6\, v_0 \cos 30^\circ$ and $v_2 = 0.8\, v_0 \cos 30^\circ$

Since, ball A just completes vertical circle, therefore

$v_2 = \sqrt{5g\ell}$

$\therefore \quad 0.8 v_0 \cos 30^\circ = \sqrt{5g\ell}$ or $v_0 = 12.5 \text{ ms}^{-1}$

VARIABLE MASS SYSTEM

If a mass is added or ejected from a system, at rate μ kg/s and relative velocity $\vec{v}_{rel}$ (w.r.t. the system), then the force exerted by this mass on the system has magnitude $\mu|\vec{v}_{rel}|$.

Thrust Force ($\vec{F}_t$)

$$\vec{F}_t = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

Suppose at some moment t = t mass of a body is m and its velocity is $\vec{v}$. After some time at t = t + dt its mass becomes (m – dm) and velocity becomes $\vec{v} + d\vec{v}$. The mass dm is ejected with relative velocity $\vec{v}_r$. Absolute velocity of mass 'dm' is therefore ($\vec{v} + \vec{v}_r$). If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\vec{P}_i = \vec{P}_f$$

or $$m = (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v} + \vec{v}_r)$$

or $$m\vec{v} = m\vec{v} + md\vec{v} - (dm)\vec{v} - (dm)(d\vec{v}) + (dm)\vec{v} + \vec{v}_r\, dm$$

The term (dm) ($d\vec{v}$) is too small and can be neglected.

$\therefore$ $$md\vec{v} = -\vec{v}_r\, dm$$

or $$m\left(\frac{d\vec{v}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$$

Here, $$m\left(-\frac{d\vec{v}}{dt}\right) = \text{thrust force } \left(\vec{F}_t\right)$$

and $$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

or $$\vec{F}_t = \vec{v}_r\left(\frac{dm}{dt}\right)$$

Problems related to variable mass can be solved in following four steps

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt}$$

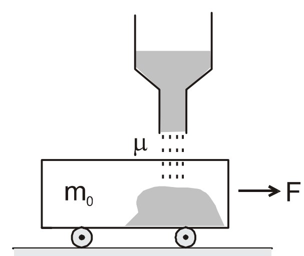
(m = mass at the particular instant)

4. Integrate it with proper limits to find velocity at any time t.

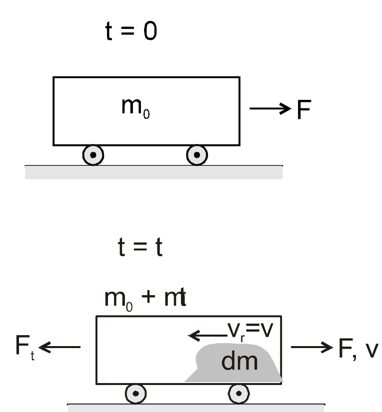
Note : Problems of one-dimensional motion (which are mostly asked in JEE) can be solved in easier manner just by assigning positive and negative signs to all vector quantities. Here are few example in support of the above theory.

Solved Examples

Ex.63 A flat car of mass m_0 starts moving to the right due to a constant horizontal force F. Sand spills on the flat car from a stationary hopper. The rate of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flat car in the process of loading. The friction is negligibly small.



Sol. Initial velocity of the flat car is zero. Let v be its velocity at time t and m its mass at that instant. Then



At $t = 0$, $v = 0$ and $m = m_0$ at $t = t$, $v = v$ and $m = m_0 + \mu t$

Here, $v_r = v$ (backwards)

$$\frac{dm}{dt} = \mu$$

$$\therefore \quad F_t = v_r \frac{dm}{dt} = \mu v \quad \text{(backwards)}$$

Net force on the flat car at time t is $F_{net} = F - F_t$

$$\text{or} \quad m\frac{dv}{dt} = F - \mu v \quad \text{....(i)}$$

$$\text{or} \quad (m_0 + \mu t)\frac{dv}{dt} = F - \mu v$$

$$\text{or} \quad \int_0^v \frac{dv}{F - \mu v} = \int_0^t \frac{dt}{m_0 + \mu t}$$

$$\therefore \quad -\frac{1}{\mu}[\ell n(F - \mu v)]_0^v = \frac{1}{\mu}[\ell n(m_0 + \mu t)]_0^t$$

$$\Rightarrow \quad \ell n\left(\frac{F}{F - \mu v}\right) = \ell n\left(\frac{m_0 + \mu t}{m_0}\right)$$

$$\therefore \quad \frac{F}{F - \mu v} = \frac{m_0 + \mu t}{m_0}$$

$$\text{or} \quad v = \frac{Ft}{m_0 + \mu t} \quad \textbf{Ans.}$$

From Eq. (i), $\frac{dv}{dt}$ = acceleration of flat car at time t

$$\text{or} \quad = \frac{F - \mu v}{m}$$

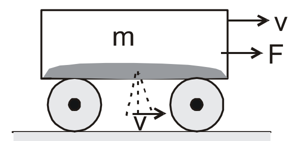
$$a = \left(\frac{F - \dfrac{F\mu t}{m_0 + \mu t}}{m_0 + \mu t}\right)$$

$$\text{or} \quad a = \frac{Fm_0}{(m_0 + \mu t)^2} \quad \textbf{Ans.}$$

Ex.64 A cart loaded with sand moves along a horizontal floor due to a constant force F coinciding in direction with the cart's velocity vector. In the process sand spills through a hole in the bottom with a constant rate μkg/s. Find the acceleration and velocity of the cart at the moment t, if at the initial moment t = 0 the cart with loaded sand had the mass m_0 and its velocity was equal to zero. Friction is to be neglected.

Sol. In this problem the sand through a hole in the bottom of the cart. Hence, the relative velocity of the sand v_r will be zero because it will acquire the same velocity as that of the cart at the moment.

$$V_r = 0$$



Thus, $F_t = 0$

$$\left(\text{as } F_t = v_r \frac{dm}{dt}\right)$$

and the net force will be F only.

$$\therefore \quad F_{net} = F$$

$$\text{or} \quad m\left(\frac{dv}{dt}\right) = F \quad \text{...(i)}$$

But here $m = m_0 - \mu t$

$$\therefore \quad (m_0 - \mu t)\frac{dv}{dt} = F$$

$$\text{or} \quad \int_0^v dv = \int_0^t \frac{F\,dt}{m_0 - \mu t}$$

$$\therefore \quad v = \frac{F}{-\mu}[\ln(m_0 - \mu t)]_0^t$$

$$\text{or} \quad v = \frac{F}{\mu}\ln\left(\frac{m_0}{m_0 - \mu t}\right) \quad \textbf{Ans.}$$

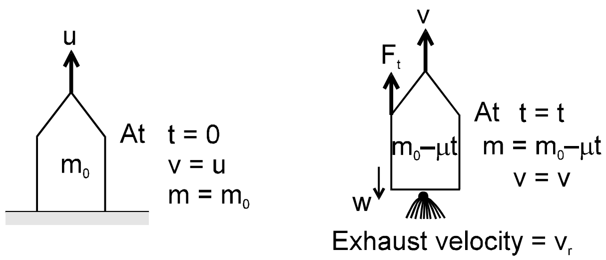
From Eq. (i), acceleration of the cart

$$a = \frac{dv}{dt} = \frac{F}{m}$$

$$\text{or} \quad a = \frac{F}{m_0 - \mu t} \quad \textbf{Ans.}$$

Rocket propulsion :

Let m_0 be the mass of the rocket at time t = 0. m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.



Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases with respect to rocket. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time t = t,

1. Thrust force on the rocket

$$F_t = v_r\left(\frac{-dm}{dt}\right) \qquad \text{(upwards)}$$

2. Weight of the rocket

$$W = mg \qquad \text{(downwards)}$$

3. Net force on the rocket

$$F_{net} = F_t - W \qquad \text{(upwards)}$$

or $$F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket

$$a = \frac{F}{m}$$

or $$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = \frac{v_r}{m}(-dm) - g\,dt$$

or $$\int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$$

Thus,

$$v = u - gt + v_r\,\ell n\left(\frac{m_0}{m}\right) \qquad ...(i)$$

Note : 1. $F_t = v_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.

2. If gravity is ignored and initial velocity of the rocket u = 0, Eq. (i) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$.

Ex.65 A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning after one minute. Find the maximum velocity of the rocket. (Take g as at 10 ms^{-2})

Sol. Using the velocity equation

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Here u = 0, t = 60s, g = 10 m/s^2, v_r = 2000 m/s, m_0 = 1000 kg

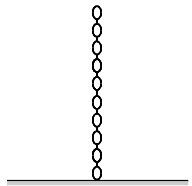
and $\qquad m = 1000 - 10 \times 60 = 400$ kg

We get $\qquad v = 0 - 600 + 2000 \ln\left(\frac{1000}{400}\right)$

or $\qquad v = 2000 \ln 2.5 - 600$

The maximum velocity of the rocket is 200(10 ln 2.5 – 3) = 1232.6 ms^{-1} **Ans.**

Ex.66 A uniform chain of mass m and length ℓ hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.

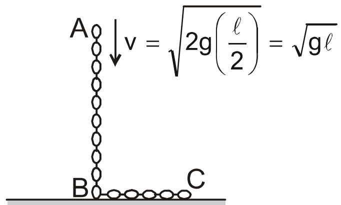


Sol.

1. Weight of the portion BC of the chain

lying on the table, $W = \frac{mg}{2}$ (downwards)

Using $v = \sqrt{2gh}$



2. Thrust force $F_t = v_r\left(\frac{dm}{dt}\right)$

$$v_r = v$$

$$\frac{dm}{dt} = \lambda v$$

$$F_t = \lambda v^2$$

(where, $\lambda = \frac{m}{\ell}$, is mass per unit length of chain)

$$v^2 = \left(\sqrt{g\ell}\right)^2 = g\ell$$

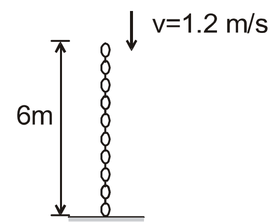
$\therefore$ $F_t = \left(\frac{m}{\ell}\right)(g\ell) = mg$ (downwards)

$\therefore$ Net force exerted by the chain on the table is

$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2}mg$$

So, from Newton's third law the force exerted by the table on the chain will be $\frac{3}{2}$ mg (vertically upwards).

Ex.67 If the chain is lowered at a constant speed v = 1.2 m/s, determine the normal reaction exerted on the floor as a function of time. The chain has a mass of 80 kg and a total length of 6 m.



Ans. (19.2 + 16 t) N

LINEAR MOMENTUM CONSERVATION IN PRESENCE OF EXTERNAL FORCE.

$$F_{ext} = \frac{dp}{dt}$$

$\Rightarrow$ $F_{ext}\,dt = dP$

$\Rightarrow$ dP = Fext) impulsive dt

$\therefore$ If $F_{ext})_{impulsive} = 0$

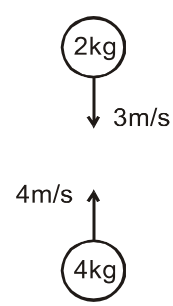
$\Rightarrow$ dP = 0

or P is constant

Note: Momentum is conserved if the external force present is non-Impulsive. eg. Gravitation or Spring force

Solved Examples

Ex.68 Two balls are moving towards each other on a vertical line collides with each other as shown. Find their velocities just after collision.



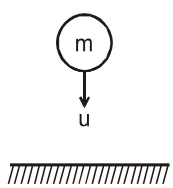
Sol. Let the final velocity of 4 kg ball just after collision be v. Since, external force is gravitational which is non - impulsive, hence, linear momentum will be conserved.

Applying linear momentum conservation:

$$2(-3) + 4(4) = 2(4) + 4(v)$$

or $v = \frac{1}{2}$ m/s

Ex.69 A ball is approaching ground with speed u. If the coefficient of restitution is e then find out:



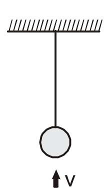
(a) the velocity just after collision.

(b) the impulse exerted by the normal due to ground on the ball.

Ans (a) v = eu;

(b) J = mu(1 + e)

Ex.70 A bullet of mass 50g is fired from below into the bob of mass 450g of a long simple pendulum as shown in figure. The bullet remains inside the bob and the bob rises through a height of 1.8 m. Find the speed of the bullet. Take g = 10 m/s^2.



Sol. Let the speed of the bullet be v. Let the common velocity of the bullet and the bob, after the bullet is embedded into the bob, is V. By the principle of conservation of the linear momentum,

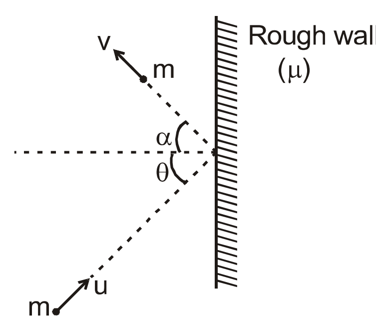
$$V = \frac{(0.05\text{ kg})\,v}{0.45\text{ kg} + 0.05\text{ kg}} = \frac{v}{10}$$

The string becomes loose and the bob will go up with a deceleration of g = 10 m/s^2. As it comes to rest at a height of 1.8 m, using the equation $v^2 = u^2 + 2ax$,

$$1.8\text{ m} = \frac{(v/10)^2}{2 \times 10\text{ m/s}^2}$$

or, v = 60 m/s.

Ex.71 A small ball of mass m collides with a rough wall having coefficient of friction μ at an angle θ with the normal to the wall. If after collision the ball moves with angle α with the normal to the wall and the coefficient of restitution is e then find the reflected velocity v of the ball just after collision.



Sol. $mv\cos\alpha - (m(-u\cos\theta)) = \int N dt$

$mv\sin\alpha - mu\sin\theta = -\mu \int N dt$

and $e = \frac{v\cos\alpha}{u\cos\theta} \Rightarrow v\cos\alpha = eu\cos\theta$

or $mv\sin\alpha - mu\sin\theta = -\mu(mv\cos\alpha + mu\cos\theta)$

or $v = \frac{u}{\sin\alpha}[\sin\theta - \mu\cos\theta(e+1)]$ **Ans.**

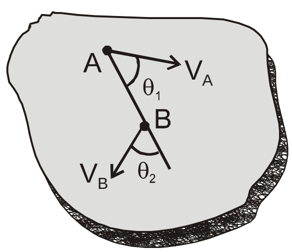
ROTATION OF MOTION

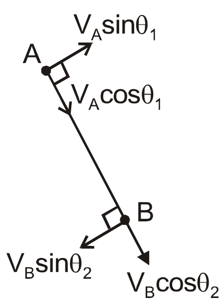
RIGID BODY

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time) that means the shape and size do not change, during the motion. Eg : Fan, Pen, Table, stone and so on.

Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body.

For every pair of particles in a rigid body, there is no velocity of seperation or approach between the particles. In the figure shown velocities of A and B with respect ground are V_A and V_B respectively.



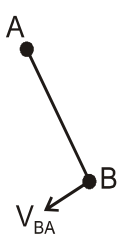


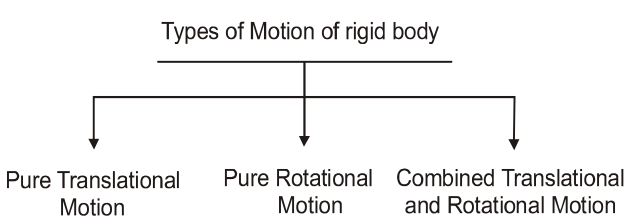
If the above body is rigid

$V_A \cos\theta_1 = V_B \cos\theta_2$

NOTE : With respect to any particle of rigid body the motion of any other particle of that rigid body is circular.

V_{BA} = relative velocity of B with respect to A.

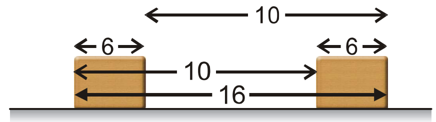




I. Pure Translational Motion :

A body is said to be in pure translational motion if the displacement of each particle is same during any time interval howsoever small or large. In this motion all the particles have same $\vec{s}$, $\vec{v}$ & , $\vec{a}$ at an instant.

Ex : A box is being pushed on a horizontal surface.



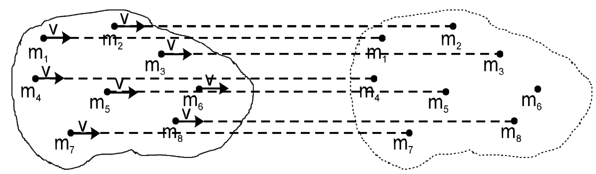
$\vec{V}_{cm} = \vec{V}$ of any particle

$\vec{a}_{cm} = \vec{a}$ of any particle

$\Delta\vec{S}_{cm} = \Delta\vec{S}$ of any particle

For pure translation motion :–

$$\vec{F}_{ext} = + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots\ldots\ldots$$



Where $m_1, m_2, m_3, \ldots\ldots$ are the masses of different particles of the body having accelerations

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots\ldots\ldots$ respectively

But acceleration of all the particles are same so

$$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots\ldots = \vec{a} \qquad = M\vec{a}$$

Where M = Total mass of the body

$\vec{a}$ = acceleration of any particle or of centre of mass

or of body = $m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots\ldots\ldots\ldots$

Where $m_1, m_2, m_3, \ldots\ldots$ are the masses of different particles of the body

having velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots\ldots\ldots$ respectively

But velocities of all the particles are same so

$$\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \ldots\ldots = \vec{v} \quad = M\vec{v}$$

Where $\vec{v}$ = velocity of any particle or of centre of mass.

Total Kinetic Energy of body = $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots\ldots\ldots\ldots = \frac{1}{2} Mv^2$

II. Pure Rotational Motion :

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constant from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion all the particles have same $\vec{\theta}, \vec{\omega}$ & , $\vec{\alpha}$ at an instant. Eg :– a rotating ceiling fan, arms of a clock.

For pure rotation motion : –

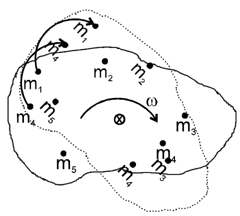
$\theta = \frac{s}{r}$ Where θ = angle rotated by the particle

s = length of arc traced by the particle.

r = distance of particle from the axis of rotation.

$\omega = \frac{d\theta}{dt}$ Where ω = angular speed of the body.

$\alpha = \frac{d\omega}{dt}$ Where α = angular acceleration of the body.



All the parameters θ, ω and α are same for all the particles.

Axis of rotation is perpendicular tothe plane of rotationof particles.

Special case :

If α = constant ,

$\omega = \omega_0 + \alpha t$ Where ω_0 = initial angular speed

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ t = time interval

$\omega^2 = \omega_0^2 + 2\alpha\theta$

Total Kinetic Energy

$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + ...$

$= \frac{1}{2} [m_1 r_1^2 + m_2 r_2^2 + ...] \omega^2$

$= \frac{1}{2} I\omega^2$ Where I = Moment of Inertia

$= m_1 r_1^2 + m_2 r_2^2 + ...$

ω = angular speed of body.

Solved Examples

Ex.72 A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that the rope does not slip on the pulley. Find the angular velocity and angular acceleration of the pulley at an instant when the bucket is going down at a speed of 20 cm/s and has an acceleration of 4.0 m/s^2.

Sol. Since the rope does not slip on the pulley, the linear speed v of the rim of the pulley is same as the speed of the bucket.

The angular velocity of the pulley is then

$$\omega = v/r = \frac{20\,cm/s}{10\,cm} = 2 \text{ rad/s}$$

and the angular acceleration of the pulley is

$$\alpha = a/r = \frac{4.0\,m/s^2}{10\,cm} = 40 \text{ rad/s}^2.$$

Ex.73 A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed of

(a) a point on the rim,

(b) the middle point of a radius

Sol. 2 m/s, 1 m/s

Ex.74 A wheel rotates with a constant acceleration of 2.0 rad/s^2. If the wheel starts from rest, how many revolutions will it make in the first 10 seconds?

Sol. The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0 \text{ rad/s}^2)(10 \text{ s})^2 = 100 \text{ rad.}$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in 10 s in

$$n = \frac{100}{2\pi} = 16.$$

Ex.75 The wheel of a motor, accelerated uniformly from rest, rotates through 2.5 radian during the first second. Find the angle rotated during the next second.

Sol. As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha \quad t^2 = \frac{1}{2}\alpha t^2.$$

Thus, $\quad 2.5 \text{ rad} = \frac{1}{2}\alpha \quad (1s)^2$

$\alpha = 5 \text{ rad/s}^2$ or $\alpha = 5 \text{ rad/s}^2$

The angle rotated during the first two seconds is

$$= \frac{1}{2} \times (5 \text{ rad/s}^2)(2s)^2 = 10 \text{ rad}.$$

Thus, the angle rotated during the 2nd second is

$$10 \text{ rad} - 2.5 \text{ rad} = 7.5 \text{ rad}.$$

Ex.76 A wheel is making revolutions about its axis with a uniform angular acceleration. Starting from rest, it reaches 100 rev/sec in 4 seconds. Find the angular acceleration. Find the angle rotated during these four seconds.

Ans. 25 rev/s^2, 400 π rad

Ex.77 Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm (revolution per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Sol. Let the angular acceleration be α. According to the question,

$400 \text{ rev/min} = 0 + \alpha\, 5 \text{ s}$ ....(i)

Let t be the time taken in attaining the speed of 200 rev/min which is half the maximum.

Then, $\quad 200 \text{ rev/min} = 0 + \alpha t$ ....(ii)

Dividing (i) by (ii), we get,

$2 = 5 \text{ s}/t$ or $t = 2.5$ s.

Ex.78 The motor of an engine is rotating about its axis with an angular velocity of 100 rev/minute. It comes to rest in 15 s, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

Sol. The initial angular velocity

$= 100$ rev/minute

$= (10\pi/3)$ rad/s.

Final angular velocity = 0.

Time invertial = 15 s.

Let the angular acceleration be α. Using the equation

$\omega = \omega_0 + at$, we obtain

$\alpha = (-2p/9) \text{ rad/s}^2$

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(\frac{10\pi}{3}\frac{\text{rad}}{\text{s}}\right)(15s) - \frac{1}{2}\left(\frac{2\pi}{9}\frac{\text{rad}}{\text{s}^2}\right)(15s)^2$$

$= 25\pi$ rad = 12.5 revolutions.

Hence the motor rotates through 12.5 revolutions before coming to rest.

III. Combined Translational and Rotational Motion

A body is said to be in translation and rotational motion if all the particles rotates about an axis of rotation and the axis of rotation moves with respect to the ground.

MOMENT OF INERTIA (I)

Definition : Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational motion of a body.

Moment of Inertia depends on :

(i) density of the material of body

(ii) shape & size of body

(iii) axis of rotation

Combinedly we can say that it depends upon distribution of mass relative to axis of rotation.

Note : Moment of inertia does not change if the mass :

(i) is shited parallel to the axis of the rotation.

(ii) is rotated with constant radius about axis of rotation.

Moment of Inertia is a scalar positive quantity.

$I = mr_1^2 + m_2 r_2^2 + \ldots\ldots\ldots\ldots\ldots$

$= I_1 + I_2 + I_3 + \ldots\ldots\ldots\ldots\ldots$

SI units of Moment of Inertia is Kgm^2.

Moment of Inertia of :

(I) A single particle : $I = mr^2$

where m = mass of the particle

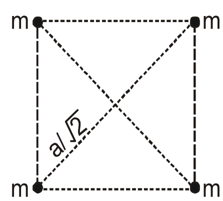
r = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

(II) For many particles (system of particles) :

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Solved Examples

Ex.79 Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



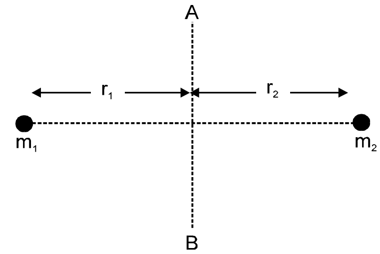
Sol. The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$.

Ex.80 Two heavy particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively.

(i) What is the moment of inertia of the system about axis AB ?

(ii) What is the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2 ?

(iii) What is the moment of inertia of the system about an axis passing through m_1 and m_2 ?



Sol. (i) Moment of inertia of particle on left is $I_1 = m_1 r_1^2$.

Moment of Inertia of particle on right is $I_2 = m_2 r_2^2$.

Moment of Inertia of the system about AB is

$I = I_1 + I_2 = m_1 r_2^2 + m_2 r_2^2$

(ii) Moment of inertia of particle on left is $I_1 = 0$

Moment of Inertia of particle on right is

$I_2 = m_2 (r_1 + r_2)^2$.

Moment of Inertia of the system about AB is $I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$

(iii) Moment of inertia of particle on left is $I_1 = 0$

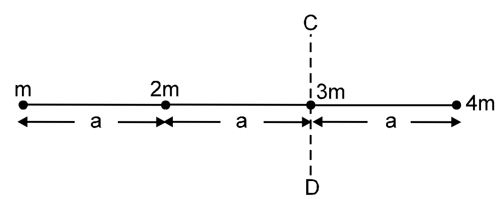
Moment of Inertia of particle on right is $I_2 = 0$

Moment of Inertia of the system about AB is

$I = I_1 + I_2 = 0 + 0$

Q.81 Four point masses are connected by a massless rod as shown in figure. Find out the moment of inertia of the system

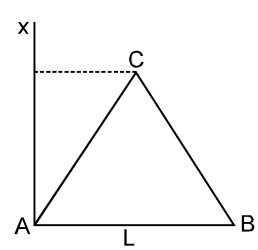
about axis CD ?



Ans. $10\ ma^2$

Ex.82 Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L (figure). Find the moment of inertia of

the system about the line AX perpendicular to AB in the plane of ABC.



Sol. Perpendicular distance of A from AX = 0

Perpendicular distance of B from AX = L

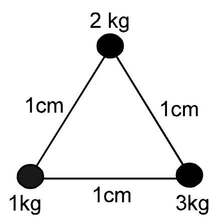
Perpendicular distance of C from AX = L/2

Thus, the moment of inertia of the particle at A = 0, of the particle at B - mL^2, and of the particle at C = $m(L/2)^2$. The moment of inertia of the three-particle system about AX is

$$0 + mL^2 + m(L/2)^2 = \frac{5\,mL^2}{4}$$

Note that the particles on the axis do not contribute to the moment of inertia.

Ex.83 Three point masses are located at the corners of an equilibrium triangle of side 1 cm. Masses are of1,2,&3kg respectively and kept as shown in figure. Calculate the moment of Inertia of system about an axis passing through 1 kg mass and perpendicular to the plane of triangle ?



Sol. 5×10^{-4} kgm^2

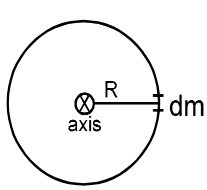
(III) For a continuous object :

$$I = \int dmr^2$$

where dm = mass of a small element

r = perpendicular distance of the particle from the axis

Ex.84 Calculate the moment of inertia of a ring having mass M, radius R and having uniform mass distribution about an axis passing through the centre of ring and perpendicular to the plane of ring ?



Sol. $I = \int (dm)r^2$

Because each element is equally distanced from the axis so r = R

$$= R^2 \int dm = MR^2$$

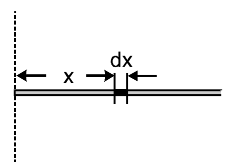
$I = MR^2$

(Note : Answer will remain same even if the mass is nonuniformly distributed because $\int dm = M$ always.)

Solved Examples

Ex.85 Calculate the moment of inertia of a uniform rod of mass M and length ℓ about an axis passing through an end and perpendicular to the rod.

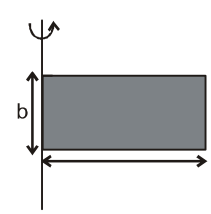
Sol. $I = \int (dm)r^2 = \int_0^{\ell} \left(\frac{M}{\ell}dx\right)x^2 = \frac{M\ell^2}{3}$



Ex.86 Using the above formula or other wise find the moment of inertia about an axis which is perpendicular to the rod and passing through centre of rod. Also calculate the moment of inertia of rod about an axis passing through rod and parallel to the rod.

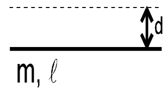
Ans. $\frac{M\ell^2}{12}$, 0

Ex.87 Using the above result $\left(I = \frac{m\ell^2}{3}\right)$ or otherwise determined the moment of inertia of a uniform rectangular plate of side 'b' and 'ℓ" about an axis passing through the edge 'b' and in the plane of plate.



Ans. $\frac{M\ell^2}{3}$

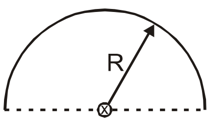
Ex.88 Find the moment of inertia of uniform rod of length 'ℓ' about the axis parallel to the rod and 'd' distance apart



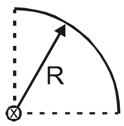
Ans. Md^2

Ex.89 Find out the moment of Inertia of figures shown each having mass M, radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane ?

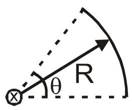
(a)



(b)



(c)



Ans. MR^2 (infact M.I. of any part of mass M of a ring of radius R about axis passing through geometrical centre and perpendicular to the plane of the ring is = MR^2)

Ex.90 Find out the moment of Inertia of a hollow cyllinder of mass M, radius R and having uniform mass distribution about its axis. Will the answer change if mass is nonuniformly distributed?

Ans. MR^2, No

(IV) For a larger object : $I = \int dI_{element}$

where dI = moment of inertia of a small element

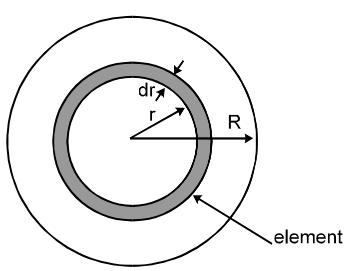
Element chosen should be :

(i) as larger as possible among all types of elements.

(ii) as much symmetric as possible

(iii) moment of inertia of element should be known earlier.

Ex.91 Determine the moment of Inertia of a uniform disc having mass M, radius R about an axis passing through centre & perpendicular to the plane of disc ?



Sol. $I = \int dI_{ring}$

element - ring $\quad dI = dmr^2$

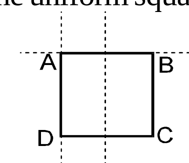
$$dm = \frac{M}{\pi R^2}\, 2\pi r dr$$

(here we have used the uniform mass distribution)

$$\therefore \quad I = \int_0^R \frac{M}{\pi R^2}\,.(2\pi r dr).r^2$$

$$\Rightarrow \quad I = \frac{MR^2}{2}$$

Ex.92 Find the moment of inertia of the uniform square plate of side 'a' and mass M about the axis AB.



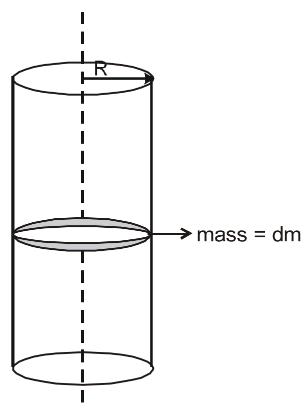
Ans. $\dfrac{ma^2}{3}$

Ex.93 Calculate the moment of inertia of a uniform hollow cylinder of mass M, radius R and length ℓ about its axis.

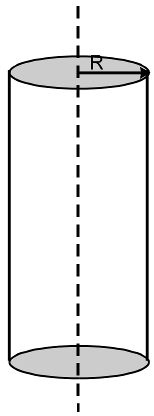
Sol. Moment of inertia of a uniform hollow cylinder is

$$I = \int (dm)R^2$$

$$= mR^2$$



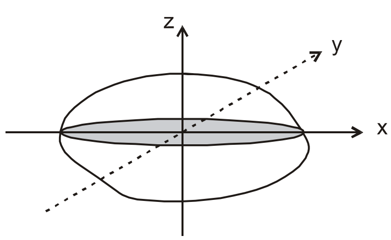
Ex.94 Calculate the moment of inertia of a uniform solid cylinder of mass M, radius R and length ℓ about its axis.



Ans. $I = \dfrac{MR^2}{2}$

Two Important Theorems on Moment of Inertia :

I. Perpendicular Axis Theorem [Only applicable to plane lamina (that means for 2-D objects only)].



Body is in x-y plane

$I_z = I_x + I_y$ (when object is in x-y plane).

Where I_z = moment of inertia of the body about z axis.

I_x = moment of inertia of the body about x axis.

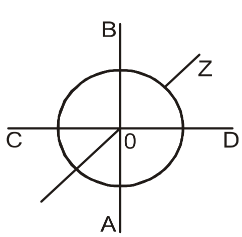
I_y = moment of inertia of the body about y axis.

$I_y = I_x + I_z$ (when object is in x-z plane)

$I_x = I_y + I_z$ (when object is in y-z plane)

Note : Defined for any 3 perpendicular concurrent axis out of which two lie in the plane of object.

Ex.95 Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.



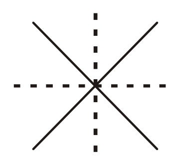
Sol. Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axes and the line perpendicular to the plane ofthe ring through the

centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = MR^2$. As the ring is uniform, all of its diameters are equivalent and so $I_x = I_y$, From perpendicular axes theorem,

$I_z = I_x + I_y$. Hence $I_x = \dfrac{I_z}{2} = \dfrac{MR^2}{2}$.

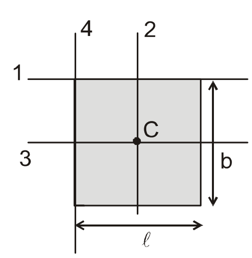
Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$.

Ex.96 Two uniform identical rods each of mass M and length ℓ are joined to form a cross as shown in figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.



Sol. Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\dfrac{M\ell^2}{12}$ and hence the moment of inertia of the cross is $\dfrac{M\ell^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\dfrac{M\ell^2}{12}$.

Ex.97 In the figure shown find moment of inertia of a plate having mass M, length ℓ and width b about axis 1,2,3 and 4. Assume that see is centre and mass is uniformly distributed



Sol. Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1) $I_1 = Mb^2 / 3$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2) $I_2 = M\ell^2 / 12$

Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3) $I_3 = Mb^2 / 12$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4) $I_4 = M\ell^2 / 3$

Ex. 98 Find the moment of Inertia of a uniform ring of mass M and radius R about a diameter.

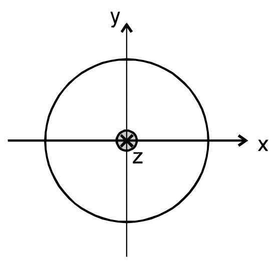
Sol. Consider x & y two mutually perpendicular diameters of the ring.

$I_x + I_y = I_z$

$I_x = I_y$ (due to symmetry)

$I_z = MR^2$

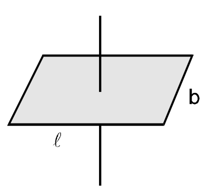
$$I_x = I_y = \frac{MR^2}{2}$$



Ex.99 Find the moment of inertia of a uniform disc having mass M, radius R about a diameter.

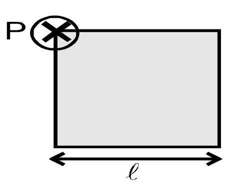
Ans. $\frac{MR^2}{4}$

Ex.100 Find the moment of inertia of a uniform rectangular plate of mass M, edges of length 'ℓ' and 'b' about its axis passing through centre and perpendicular to it.



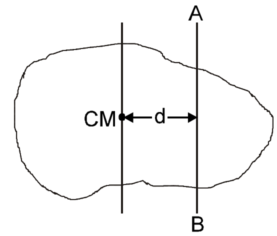
Ans. $\frac{M(\ell^2 + b^2)}{12}$

Ex.101 Find the moment of inertia of a uniform square plate of mass M, edge of length 'ℓ' about its axis passing through P and perpendicular to it.



Ans. $\frac{2m\ell^2}{3}$

II. Parallel Axis Theorem (Applicable to any type of object):



$I_{AB} = I_{cm} + Md^2$ Where

I_{cm} = Moment of Inertia of the object about an axis passing through centre of mass and parallel to axis AB

I_{AB} = Moment of Inertia of the object about axis AB

M = Total mass of object

d = perpendicular distance between axis about which moment of nertia is to be calculated & the one passing through the centre of mass

Ex.102 Find the moment of inertia of a solid cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.

Sol. The moment of inertia of the cylinder about its axis

$$= \frac{MR^2}{2}.$$

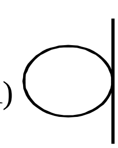
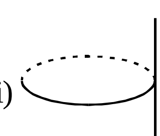
Using parallel axes theorem,

$$I = I_0 + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2.$$

Similarly, the moment of inertia of a solid sphere about a tangent is

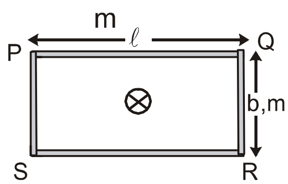
$$\frac{2}{5} MR^2 + MR^2 = \frac{7}{5} MR^2.$$

Ex.103 Find out the moment of inertia of a ring having uniform mass distribution of mass M & radius R about an axis which is tangent to the ring and (i) in the plane of the ring (ii) perpendicular to the plane of the ring.

(i)  (ii) 

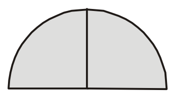
Ans. (i) $\frac{3MR^2}{2}$ (ii) $2MR^2$

Ex.104 Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in figure about an axis passing through its centre and perpendicular to the plane of frame ? Also find moment of inertia about an axis passing through PQ ?



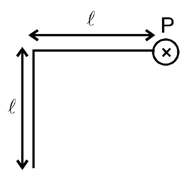
Ans. (i) $\frac{2m}{3}(\ell^2+b^2)$ (ii) $\frac{5mb^2}{3}$

Ex.105 Find out the moment of inertia of a semi circular disc about an axis passing through its centre of mass and perpendicular to the plane?



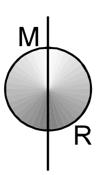
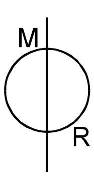
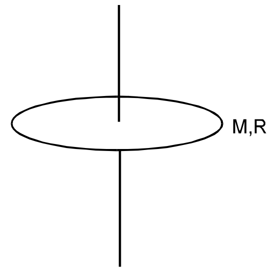
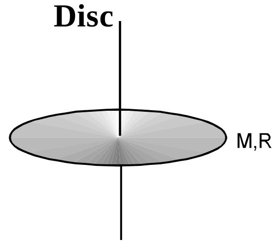
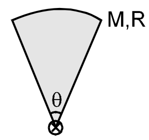
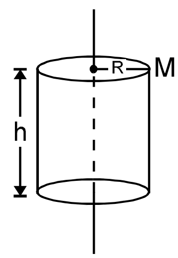
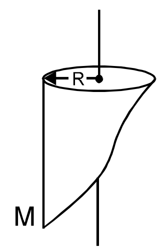
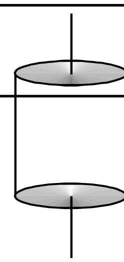
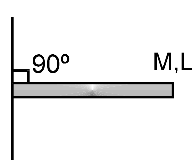
Ans. $\left[\frac{MR^2}{2}-M\left(\frac{4R}{3\pi}\right)^2\right]$

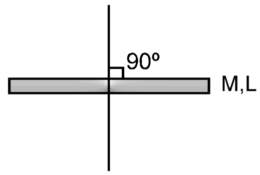
Ex.106 Find the moment of inertia of the two uniform joint rods about point P as shown in figure. Using parallel axis theorem.



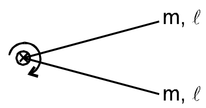
Ans. $\frac{10m\ell^2}{3}$

List of some useful formula :

Object	Moment of Inertia
Solid Sphere 	$\frac{2}{5}MR^2$ (Uniform)
Hollow Sphere 	$\frac{2}{3}MR^2$ (Uniform)
Ring. 	MR^2 (Uniform or Non Uniform)
Disc  	$\frac{MR^2}{2}$ (Uniform)
Hollow cylinder  	MR^2 (Uniform or Non Uniform)
Solid cylinder 	$\frac{MR^2}{2}$ (Uniform)
	$\frac{ML^2}{3}$ (Uniform)

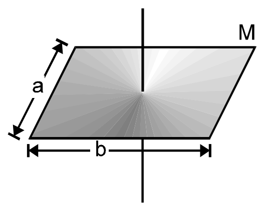


$\frac{ML^2}{12}$ (Uniform)



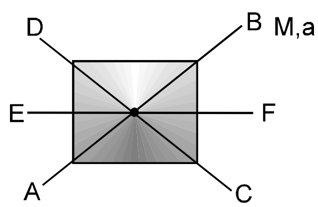
$\frac{2m\ell^2}{3}$ (Uniform)

Rectangular Plate



$I = \frac{M(a^2+b^2)}{12}$ (Uniform)

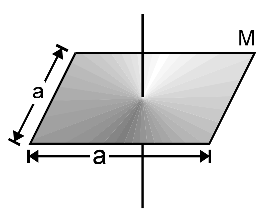
Square Plate



$I_{AB} = I_{CD} = I_{DF} = \frac{Ma^2}{12}$

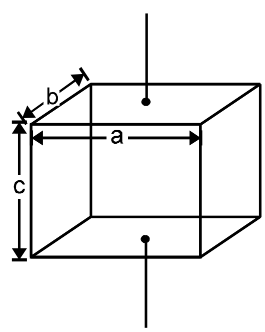
(Uniform)

Square Plate



$\frac{Ma^2}{6}$ (Uniform)

Cuboid



$\frac{M(a^2+b^2)}{12}$ (Uniform)

Radius of Gyration

As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation we can define a new parameter, the radius of gyration. It is related to the moment of intertia and total mass of the body.

$$I = MK^2$$

where I = Moment of Inertia of a body

M = Mass of a body

K = Radius of gyration

$$K = \sqrt{\frac{I}{M}}$$

Length K is the geometrical property of the body and axis of rotation.

S.I. Unit of K is meter.

Solved Examples

Ex.107 Find the radius of gyration of a solid uniform sphere of radius R about its tangent.

Sol. $I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2 \Rightarrow K = \sqrt{\frac{7}{5}}R$

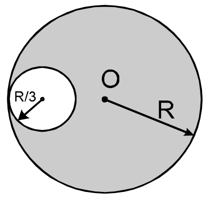
Ex.108 Find the radius of gyration of a hollow uniform sphere of radius R about its tangent.

Ans. $\sqrt{\frac{5}{3}}R$

Moment of inertia of Bodies with cut

Solved Examples

Ex.109 A uniform disc of radius R has a round disc of radius R/3 cut as shown in Fig. .The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.



Sol. Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$ and σ.

Now $\quad I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma \pi R^2)R^2/2$ = M.I. of σ about o

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2]$$

$(2R/3)^2$

= M.I. of $-\sigma$ about o

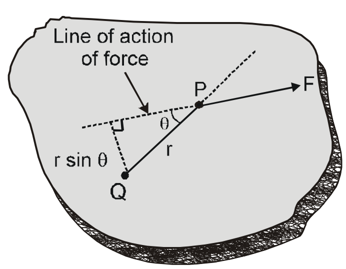
$\therefore I_0 = \frac{4}{9}\sigma\pi R^4$ **Ans.**

Ex.110 Find the moment of inertia of a uniform disc of radius R_1 having an empty symmetric annular region of radius R_2 in between, about an axis passing through geometrical centre and perpendicular to the disc.

Ans. $\frac{M(R_1^{\,2} + R_2^{\,2})}{2}$

TORQUE

Torque represents the capability of a force to produce change in the rotational motion of thebody.



Torque about point :

Torque of force $\vec{F}$ about a point $\quad \vec{\tau} = \vec{r} \times \vec{F}$

Where $\quad \vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force from the point about which we want to determine the torque.

$|\vec{\tau}| = r\,F \sin\theta \quad = r_\perp F \quad = rF_\perp$ Where

θ = angle between the direction of force and the position vector of P wrt. Q.

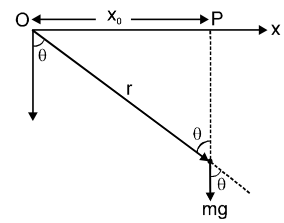
$r_\perp$ = perpendicular distance of line of action of force from point Q.

$F_\perp$ = force arm $\quad$ SI unit of torque is Nm

Torque is a vector quantity and its direction is determined using right hand thumb rule.

Solved Examples

Ex.111 A particle of mass M is released in vertical plane from a point P at $x = x_0$ on the x-axis it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about origin ?



Sol. Torque is produced by the force of gravity.

$$\vec{\tau} = r\,F \sin\theta\ \hat{k}$$

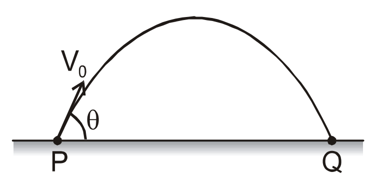
or $\quad \tau = r_\perp F = x_0 mg$

$$= r\,mg\,\frac{x_0}{r} = mgx_0\,\hat{k}$$

Ex.112 A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when

(a) it is at its maximum height ?

(b) It is just about to hit the ground back ?



Ans. (a) $\frac{mv_0^{\,2}\sin 2\theta}{2}$ $\quad$ (b) $mv_0^{\,2}\sin 2\theta$

Ex.113 In the previous question, during the motion of particle from P to Q. Torque of gravitational force about P is :

(A) increasing

(B) decreasing

(C) remains constant

(D) first increasing then decreasing

Ans. Increasing

Torque about axis :

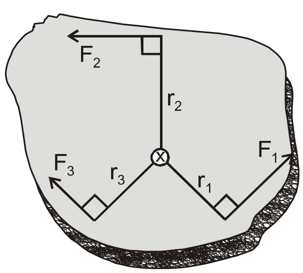
$\vec{\tau} = \vec{r} \times \vec{F}$

where $\vec{\tau}$ = torque acting on the body about the axis of rotation.

$\vec{r}$ = position vector of the point of application of force about the axis of rotation.

$\vec{F}$ = force applied on the body.

Note : The direction of torque is calculated using right hand thumb rule and it is always perpendicular to the plane of rotation of the body.



If F_1 or F_2 is applied body applied body revolves in anti-clockwise direction F_3 makes body revolve in clockwise direction. If all three are applied.

$\tau_{resultant} = F_1 r_1 + F_2 r_2 + F_3 r_3$

(in anti-clockwise direction)

Note :– Torque produced by a force can be zero in cases. If force vector :–

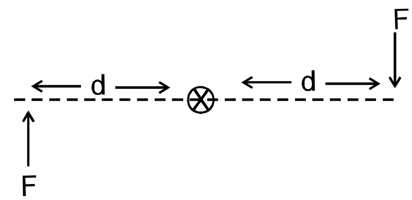
(i) is parallel to the axis of rotation.

(ii) passes through the axis of rotation.

Force Couple :

A pair of forces each of same magnitude and acting in opposite direction is called a force couple.

Torque due to couple = Magnitude of one force × distance between their lines of action.



Magnitude of torque = τ = F (2d)

A couple does not exert a net force on an object even though it exerts a torque.

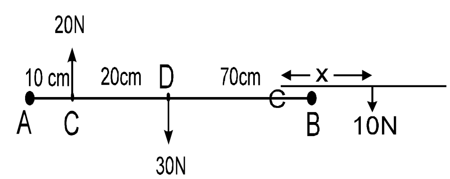
Point of Application of Force :

Point of Application of force is the point at which if nett force is assumed to be acting then it will produce same effect of both translation & rotational nature, as was produced earlier.

OR

If nett force is applied at the point of application in the opposite direction, then the body will be in equilibrium. (translational and rotational both)

Ex.114 Determine the point of application of force, when forces of 20 N & 30 N are acting on the rod as shown in figure.



Sol. Nett force acting on the rod F_{rel} = 10N

Nett torque acting on the rod about point C

τ_c = (20 × 0) + (30 × 20)

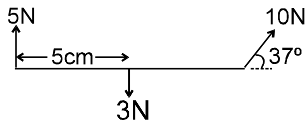
= 600 clockwise

Let the point of application be at a distance x from point C

600 = 10 x $\Rightarrow$ x = 60 cm

$\therefore$ 70 cm from A is point of Application

Ex.115 Determine the point of application of force, when forces are acting on the rod as shown in figure.



Ans. 4.375 cm left on the rod from the point where 10 N force is acting.

Note : (i) Point of application of gravitational force is known as the centre of gravity.

(ii) Centre of gravity coincides with the centre of mass if value of $\vec{g}$ is assumed to be constant.

(iii) Concept of point of application of force is imaginary, as in some cases it can lie outside the body.

Relation between 'τ' & 'α '

(for hinged object or pure rotation)

$$\vec{\tau}_{ext}\Big)_{Hinge} = I_{Hinge}\ \vec{\alpha}$$

Where $\vec{\tau}_{ext}\Big)_{Hinge}$ = nett external torque acting on the body about Hinge point

I_{Hinge} = moment of Inertia of body about Hinge point

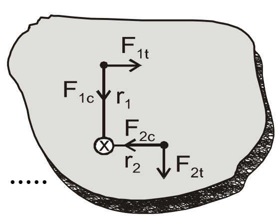
$F_{1t} = M_1 a_{1t} = M_1 r_1 \alpha$

$F_{2t} = M_2 a_{2t} = M_2 r_2 \alpha$

$t_{resultant} = F_{1t} r_1 + F_{2t} r_2 + ...$

$= M_1\ \alpha\ r_1^2 + M_2\ \alpha\ r_2^2 + ...$

$\tau_{resultant}\Big)_{external} = I\ \alpha$



Rotational Kinetic Energy $= \frac{1}{2}.I.\omega^2$

$$\vec{P} = M\vec{v}_{CM}$$

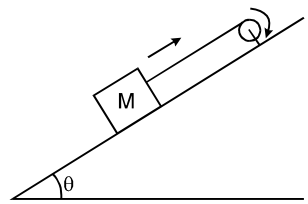
$$\vec{F}_{external} = M\vec{a}_{CM}$$

Net external force acting on the body has two parts tangential and centripetal.

$$\Rightarrow \quad F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$$

$$\Rightarrow \quad F_t = ma_t = m\alpha\, r_{CM}$$

Ex.116 A wheel of radius r and moment of inertia I about ts axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping ?



Sol. Suppose the deceleration of the block is a. The linear deceleration of the rim of the wheel is also a. The angular deceleration of the wheel is $\alpha = a/r$. If the tension in the string is T, the equations of motion are as follows:

$Mg \sin\theta - T = Ma$ and $Tr = I\alpha = Ia/r$.

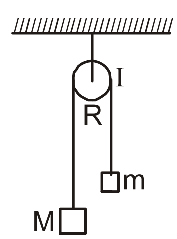
Eliminating T from these equations,

$$Mg\sin\theta - I\frac{a}{r^2} = Ma \qquad \text{giving,} \quad a = \frac{Mgr^2\sin\theta}{I + Mr^2}$$

The initial velocity of the block up the incline is $v = \omega r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2M\, r^2 \sin\theta} = \frac{(I + Mr^2)\omega^2}{2M\, g\sin\theta}$$

Ex.117 The pulley shown in figure has a moment of inertia I about its axis and its radius is R. Find the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.



Sol : Suppose the tension in the left string is T_1 and that in the right string in T_2. Suppose the block of mass M goes down with an acceleration α and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/R$. The equations of motion for the mass M, the mass m and the pulley are as follows :

$Mg - T_1 = Ma$ ....(i)

$T_2 - mg = ma$ ....(ii)

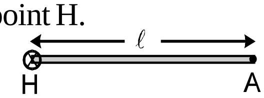
$T_1R - T_2R = I\alpha = I\alpha / R$ ....(iii)

Putting T_1 and T_2 from (i) and (ii) into (iii),

$$[(Mg - a) - m(g + a)]\, R = I\frac{a}{R}$$

which gives $a = \dfrac{(M-m)gR^3}{I + (M+m)R^2}$.

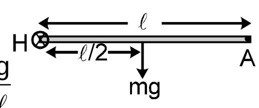
Ex.118 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smoot h horizontal axis hinged at point H.



(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

Sol. (i) $\tau_H = I_H\ \alpha$

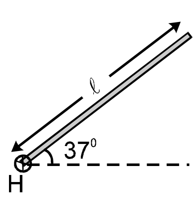


$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \quad \Rightarrow \alpha = \frac{3g}{2\ell}$$

(ii) $a_{tA} = \alpha\ell = \dfrac{3g}{2\ell}.\ell = \dfrac{3g}{2}$

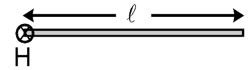
$a_{CA} = \omega^2 r = 0.\ell = 0$ ($\because$ $\omega = 0$ just after release)

Ex.119 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration α of the rod just after it is released from initial position making an angle of 37^0 with horizontal from rest ?



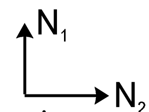
Ans. $6g / 5\ell$

Ex.120 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find force exerted by the hinge just after the rod is released from rest, from an initial horizontal position ?



Sol. Suppose hinge exerts normal reaction in component form as shown In vertical direction

$F_{ext} = ma_{CM}$



$\Rightarrow mg - N_1 = m.\dfrac{3g}{4}$

(we get the value of a_{CM} from previous example)

$\Rightarrow N_1 = \dfrac{mg}{4}$

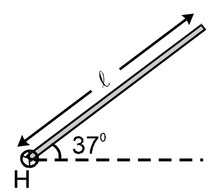
In horizontal direction

$F_{ext} = ma_{CM} \quad \Rightarrow \quad N_2 = 0$

($\because$ a_{CM} in horizontal = 0 as $\omega = 0$ just after release).

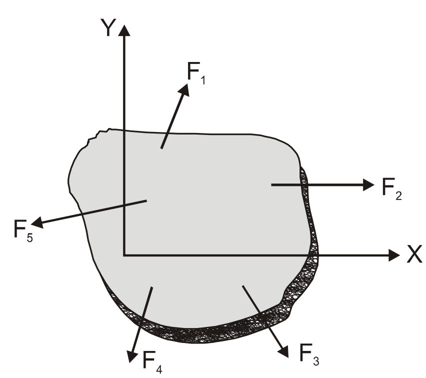
Ex.121 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find force exerted by the hinge just after the rod is released from rest, from an initial position making an angle of 37^0 with horizontal ?

Ans.



ROTATIONAL EQUILIBRIUM

If nett external torque acting on the body is zero, then the body is said to be in rotational equilibrium. The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero.



For translational equilibrium.

$$\Sigma F_x = 0 \quad ....\ (i)$$

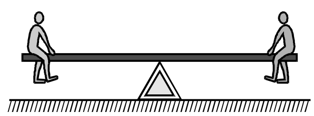
and $$\Sigma F_y = 0 \quad ....\ (ii)$$

The condition of rotational equilibrium is $\Sigma\Gamma_z = 0$

The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. It is called unstable if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in neutral equilibrium.

Solved Examples

Ex.122 Two small kids weighing 10 kg and 15 kg are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit ?



Sol. It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

(a) (15 kg) g downward by the 15 kg kid,

(b) (10 kg) g downward by the 10 kg kid,

(c) weight of the seesaw and

(d) the normal force by the fulcrum.

Taking torques about the fulcrum,

(15 kg)g x = (10 kg)g (2.5 m) or x = 1.7 m.

Ex.123 A uniform ladder of mass 10 kg leans against a smooth vertical wall making an angle of 53° with it. The other ends rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.

Sol. The forces acting on the ladder are shown in figure. They are

(a) Its weight W,

(b) normal force N_1 by the vertical wall,

(c) normal force N_2 by the floor and

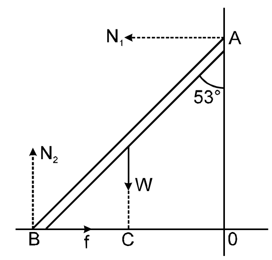
(d) frictional force f by the floor.

Taking horizontal and vertical components,

$$N_1 = f \quad ....(i)$$

and $$N_2 = W \quad ....(ii)$$

Taking torque about B,



$$N_1(AO) = W(CB)$$

or, $N_1(AB)\cos 53°$

$$= W\,\frac{AB}{2}\sin 53°$$

or $$N_1\,\frac{3}{5} = \frac{W}{2}\,\frac{4}{5}$$

or, $$N_1 = \frac{2}{3}W \quad ....(iii)$$

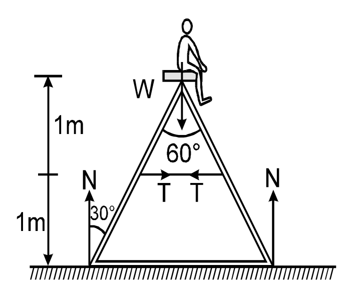
The normal force by the floor is

$N_2 = W = (10\text{ kg})\,(9.8\text{ m/s}^2) = 98\text{ N}.$

The frictional force is

$$f = N_1 = \frac{2}{3}W = 65\text{ N}.$$

Ex.124 The ladder shown in figure has negligible mass and rests on a frictionless floor. The crossbar connected the two legs of the ladder at the middle. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the constant forces exerted by the floor on each leg and the tension in the crossbar.



Sol . The forces acting on different parts are shown in figure. Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80 kg)g and the contact force N + N = 2 N due to the floor. Thus

2 N = (80 kg) g

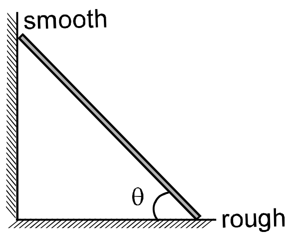
or N = (40 kg) (9.8 m/s^2) = 392 N.

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

N (2m) tan 30° = T(1 m)

or $T = N\dfrac{2}{\sqrt{3}} = (392\text{ N}) \times \dfrac{2}{\sqrt{3}} = 450$ N.

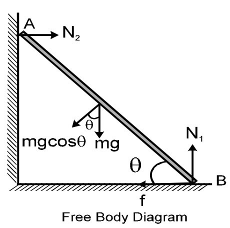
Ex.125 A stationary uniform rod of mass 'm' , length 'ℓ' leans against a smooth vertical wall making an angle θ with rough horizontal floor. Find the normal force & frictional force that is exerted by the floor on the rod ?



Sol . As the rod is stationary so the linear acceleration and angular acceleration of rod is zero.

i.e. $a_{cm} = 0$; $\alpha = 0$.

$$\left.\begin{matrix} N_2 = f \\ N_1 = mg \end{matrix}\right\} \because a_{cm} = 0$$



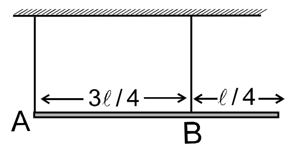
Torque about any point of the rod should also be zero. $\because \alpha = 0$

$$\tau_A = 0 \Rightarrow mg\cos\theta\,\frac{\ell}{2} + f\,\ell\sin\theta = N_1\cos\theta\,.\,\ell$$

$$N_1\cos\theta = \sin\theta\, f + \frac{mg\cos\theta}{2}$$

$$f = \frac{mg\cos\theta}{2\sin\theta} = \frac{mg\text{Cot}\theta}{2}$$

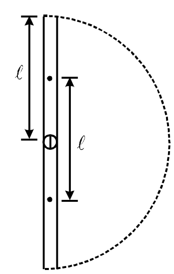
Q.126 A uniform rod of length ℓ, mass m is hung from two strings of equal length from a ceiling as shown in figure . Determine the tensions in the strings ?



Ans. $T_A = mg/3$, $T_B = 2mg/3$

Ex.127 A uniform rod of mass m and length ℓ is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant?

Sol. As the rod reaches its lowest position, the centre of mass is lowered by a distance ℓ. Its gravitational potential energy is decreased by mgℓ. As no energy is lost against friction, this should be equal to the increase in the kinetic energy. As the rotation occurs about the horizontal axis through the clamped end, the moment of inertia is $I = m\ell^2/3$. Thus,



$$\frac{1}{2}I\omega^2 = mg\ell \quad \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 = mg\ell \quad \text{or} \quad \omega = \sqrt{\frac{6g}{\ell}}\,.$$

The linear speed of the free end is $v = \ell\omega = \sqrt{6g\ell}$

ANGULAR MOMENTUM

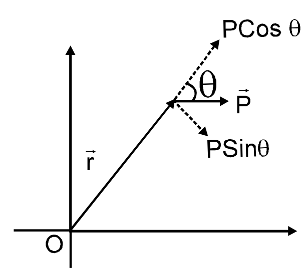
Angular momentum of a particle about a point.

$\vec{L} = \vec{r} \times \vec{P}$

$\Rightarrow L = rp\sin\theta$

$|\vec{L}| = r_\perp \times P$

$|\vec{L}| = P_\perp \times r$



Where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point about which angular momentum is to be calculated .

θ = angle between vectors $\vec{r}$ & $\vec{P}$

$r_\perp$ = perpendicular distance of line of motion of particle from point O.

$P_\perp$ = perpendicular component of momentum.

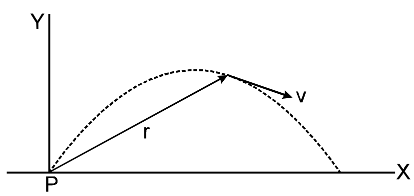
SI unit of angular momentum is kgm^2/sec.

Solved Examples

Ex.128 A particle is projected at time t = 0 from a point P with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point P at time t = v_0/g.

Sol. Let us take the origin at P, X-axis along the horizontal and Y-axis along the vertically upward direction as shown in figure.

For horizontal motion during the time 0 to t,



$$v_x = v_0 \cos 45° = v_0/\sqrt{2}$$

and $x = b_x t = \frac{v_0}{\sqrt{2}} . \frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}g}$.

For vertical motion,

$$v_y = v_0 \sin 45° = \frac{v_0}{\sqrt{2}} - v_0 = \frac{(1-\sqrt{2})}{\sqrt{2}} v_0$$

and $y = (v_0 \sin 45°)\, t - \frac{1}{2} gt^2$

$$= \frac{v_0^2}{\sqrt{2}\,g} - \frac{v_0^2}{2\,g} = \frac{v_0^2}{2\,g} (\sqrt{2} - 1).$$

The angular momentum of the particle at time t about the origin is

$$L = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$

$$= (\vec{i}\, x + \vec{j}\, y) \times (\vec{i}\, v_x + \vec{j}\, v_y)$$

$$= m\,(\vec{k}\, xu_u - \vec{k}\, yu_x)$$

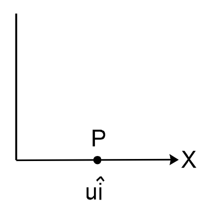
$$= m\, \vec{k}\left[\left(\frac{v_0^2}{\sqrt{2}\,g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right]$$

$$= -\,\vec{k}\, \frac{mv_0^3}{2\sqrt{2}\,g}.$$

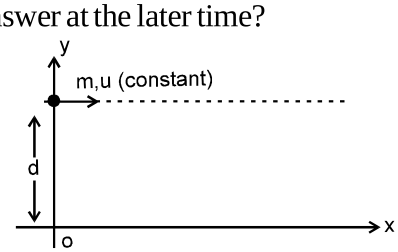
Thus, the angular momentum of the particle is $\frac{mv_0^3}{2\sqrt{2}\,g}$ in the negative Z-direction i.e., perpendicular to the plane of motion, going into the plane.

Q.129 A particle of mass m starts moving from origin with a constant velocity $u\hat{i}$ find out its angular momentum about origin at this moment. What will be the answer later time ?

Ans.



Ex.130 A particle of mass 'm' starts moving from point (o,d) with a constant velocity $u\hat{i}$. Find out its angular momentum about origin at this moment what will be the answer at the later time?



Sol. L = mud direction is always clockwise same.

Q.131 A particle of mass 'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal . Find out the angular momentum of particle about the point of projection when .

(i) it just starts its motion

(ii) it is at highest point of path.

(iii) it just strikes the ground.

Ans. (i) O ; (ii) mu cosθ $\frac{u^2 \sin^2\theta}{2g}$

; (iii) mu sin θ $\frac{u^2 \sin 2\theta}{g}$

For system of particles :

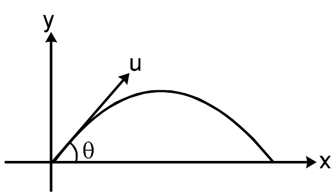
Considering a system of particles with both external and internal forces acting we can add the angular momentum of the indivizual particles to obtain the angular momentum L.

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \vec{r}_3 \times \vec{p}_3 + ...$$

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ...$$

about the same point.

Ex.132 A particle of mass'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal . Find out the angular momentum at any time t of particle p about :



(i) y axis (ii) z-axis

Sol. (i) O

(ii) – 1/2 u cos θ. gt^2

Angular momentum of a rigid body rotating about fixed axis :

$$\vec{L}_H = I_H \vec{\omega}$$

L_H = angular momentum of object about axis H.

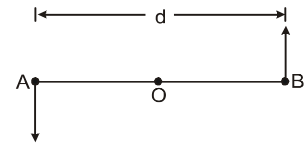
I_H = Moment of Inertia of rigid , object about axis H.

ω = angular velocity of the object.

Ex.133 Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.

Sol. Consider the situation shown in figure. The velocity of the ball A with respect to the centre O is v = $\frac{\omega d}{2}$.

The angular momentum of the ball with respect to the axis is

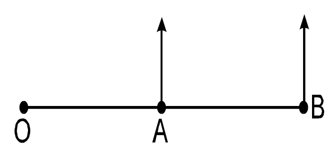


L_1 = mvr = m $\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}$ mωd^2. The same the angular momentum L_2 of the second ball. The angular momentum of the system is equal to sum of these two angular momenta i.e., L = 1/2 mωd^2.

Ex.134 Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end. The rod is fixed at the other end and is rotated in a plane at an angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.

Sol. The situation is shown in figure. The velocity of the particle A with respect to the fixed end O is v_A = ω $(d/2)$ and that of B with respect to O is v_B = ωd. Hence the velocity of B with respect to A is $v_B - v_A$ = ω $(d/2)$. The angular momentum of B with respect to A is, therefore,

L = mvr = mω $\left(\frac{d}{2}\right)$



$\frac{d}{2} = \frac{1}{4}$ mωd^2

along the direction perpendicular to the plane of rotation.

Ex.135 A uniform circular disc of mass 200 g and radius 4.0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

Sol. The moment of inertia of the circular disc about its diameter is

$$I = \frac{1}{4}Mr^2 = \frac{1}{4}(0.200\ \text{kg})(0.04\ \text{m})^2$$

$$= 8.0 \times 10^{-5}\ \text{kg-m}^2.$$

The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8.0 \times 10^{-5}\ \text{kg - m}^2)(100\ \text{rad}^2/\text{s}^2)$$

$$= 4.0 \times 10^{-3}\ \text{J}$$

and the angular momentum about the axis of rotation is

$$L = I\omega = (8.0 \times 10^{-5}\ \text{kg-m}^2)(10\ \text{rad/s})$$

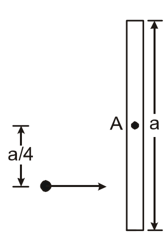
$$= 8.0 \times 10^{-4}\ \text{kg-m}^2/\text{s} = 8.0 \times 10^{-4}\ \text{J-s}.$$

Conservation of Angular Momentum

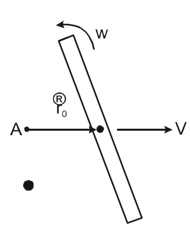
Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

Ex.136 A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance a/4 from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.

Sol. The situation is shown in figure. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remains constant. Also there is no resultant external torque on the system and so the angular momentum of the system about the any line will remain constant. Suppose the velocity of the centre of the rod is V and the angular velocity about the centre is ω.



(a)



(b)

(a) The linear momentum before the collision is mv and that after the collision is MV.

Thus, $mv = MV$,

or $V = \frac{m}{M}v.$

(b) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB. Initially the rod is at rest. The angular momentum of the particle about AB is $L = mv\,(a/4)$

After the collision, the particle comes to rest. The angular momentum of the rod about A is

$$\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{V}$$

As $\vec{r}_0 \parallel \vec{V}$, $\vec{r}_0 \times \vec{V} = 0$

Thus, $\vec{L} = \vec{L}_{cm}$

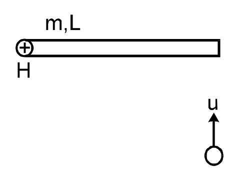
Hence the angular momentum of the rod about AB is

$$L = I\omega = \frac{M\alpha^2}{12}\omega.$$

Thus, $$\frac{mva}{4} = \frac{Ma^2}{12}\omega$$

or, $$\omega = \frac{3mv}{Ma}$$

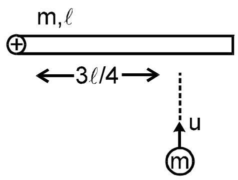
Ex.137 A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision ?



Sol . Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$mul = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \quad \Rightarrow \quad \omega = \frac{3a}{4\ell}$$

Q.138 A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at a distance of $3\ell/4$ from hinge point. Find out the angular velocity of the rod just after collision ?



Relation between Torque and Angular Momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Torque is change in angular momentum

Impulse of Torque :

$\int \tau dt = \Delta J$ $\Delta J \to$ d Charge in angular momentum.

Combined Translational and Rotational motion of a rigid body :

If the axis of rotation is moving w.r. to ground then the motion is combined translational and rotational motion.

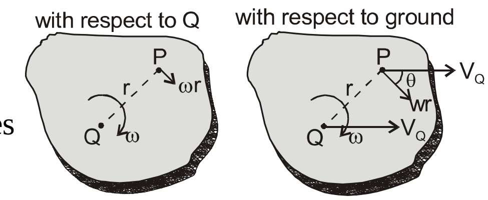
Kinematics : The most general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point . This is called **Chasle's Theorem**. A convenient choice of the point is the centre of mass of the body. One good example of the type of motion is rolling of a wheel.

The general motion of the body can be thought of as the result of a translation of the point Q and the motion of the body about Q. Let us choose another point P in the body with position vector $\vec{r}_P$. Let $\vec{r}_{P/Q}$ denote the position vector of P with respect to Q, then $\vec{r}_P = \vec{r}_Q + \vec{r}_{P/Q}$.

By differentiating we get, $\vec{\upsilon}_P = \vec{\upsilon}_Q + \vec{\upsilon}_{P/Q}$.

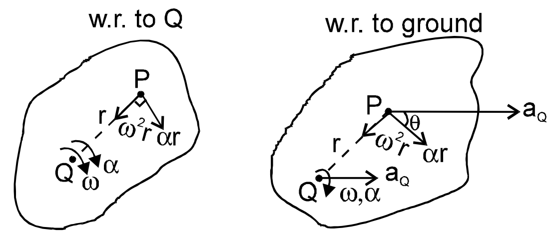
For a rigid body, the distance between the particles remain unchanged during its motion i.e. $r_{P/Q}$ = constant

For velocities



$$V_P = \sqrt{V_Q^2 + (\omega r)^2 + 2\, V_Q\, \omega r \cos\theta}$$

For acceleration :



θ, ω, α are same about every point of the body (or any other point outside which is rigidly attached to the body).

Dynamics :

$\vec{\tau}_{cm} = I_{cm}\vec{\alpha}$, $\vec{F}_{ext} = M\vec{a}_{cm}$ $\quad \vec{P}_{system} = M\vec{v}_{cm}$,

Total K.E. $= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

Angular momentum axis AB = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about AB $\vec{L}_{AB} = I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$

Ex.139 A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find its kinetic energy.

Sol. As the sphere rolls without slipping on the plane surface, its angular speed about the centre is $\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$$K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

$$= \frac{1}{2}\cdot\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv_{cm}^2$$

$$= \frac{1}{5}Mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{7}{10}Mv_{cm}^2$$

$$= \frac{7}{10}(0.200\text{ kg})(0.02\text{ m/s})^2 = 5.6 \times 10^{-5}\text{ J}.$$

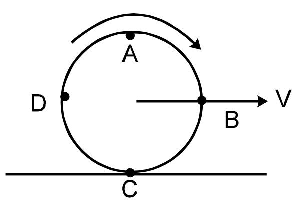
Ex.140 A wheel of perimeter 220 cm rolls on a level road at a speed of 9 km/h. How many revolutions does the wheel make per second ?

Sol. As the wheel rolls on the road, its angular speed ω about the centre and the linear speed v of the centre are related as $v = \omega r$.

$$\therefore \quad \omega = \frac{9 \text{ km/h}}{220 \text{ cm}/2\pi} = \frac{2\pi \times 9 \times 10^5}{220 \times 3600} \text{ rad/s.}$$

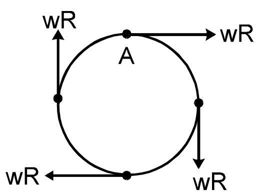
$$= \frac{900}{22 \times 36} \text{ rev/s} = \frac{25}{22} \text{ rev/s.}$$

Ex.141

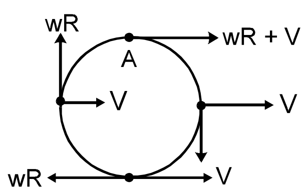


Find $\vec{V}_A$ $\vec{V}_B$ $\vec{V}_C$ $\vec{V}_D$

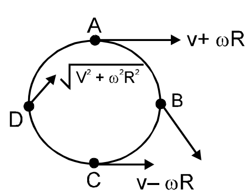
Sol . wrt. centre



wrt. ground



Result



PURE ROLLING

Solved Examples

Ex.142 A cylinder is released from rest from the top of an incline of inclination θ and length ℓ. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom ?

Sol. Let the mass of the cylinder be m and its radius r. Suppose the linear speed of the cylinder when it reaches the bottom is v. As the cylinder rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

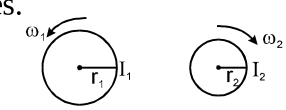
$$K = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{4} mv^2 + \frac{1}{2} mv^2 = \frac{3}{4} mv^2.$$

This should be equal to the loss of potential energy $mg\ell \sin\theta$. Thus,

$$\frac{3}{4} mv^2 = mg\ell \sin\theta \text{ or } \qquad v = \sqrt{\frac{4}{3} g\ell \sin\theta}\ .$$

Ex.143 Figure shows two cylinders of radii r_1 and r_2 having moments of inertia I_1 and I_2 about their respective axes. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the figure. The cylinders are moved closed to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.



Sol. When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω'_1 and ω'_2 be the respective angular speeds, we have

$\omega'_1 r_1$ and $\omega'_2 r_2$ ....(i)

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fr_1 and that on the second is fr_2. Assuming $\omega_1 > \omega_2$, the corresponding angular impulses are $-fr_1t$ and fr_2t, We, there fore, have

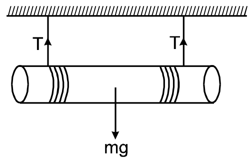
$$-f r_1 t = I_1(\omega'_1 - \omega_1) \quad \text{and} \quad fr_2 t = I_2(\omega'_2 - \omega_2)$$

$$\text{or, } -\frac{I_1}{r_1}(\omega'_1 - \omega_1) = \frac{I_2}{r_2}(\omega'_2 - \omega_2) \quad \text{................(ii)}$$

Solving (i) and (ii) $\omega'_1 = \dfrac{I_1 \omega_1 r_2 + I_2 \omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2$

and $\omega'_2 = \dfrac{I_1 \omega_1 r_2 + I_2 \omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1.$

Ex.144 A cylinder of mass m is suspended through two strings wrapped around it as shown in figure. Find (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance h.



Sol. The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings. The equation of motion for the centre of mass of the cylinder is

$mg - 2T - ma$ ....(i)

and for the motion about the centre of mass, it is

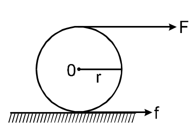
$2\,Tr\left(\frac{1}{2}mr^2\alpha\right) = \frac{1}{2}\,mra$ or $2T = \frac{1}{2} = ma.$

From (i) and (ii), $a = \frac{2}{3}g$ and $T = \frac{mg}{6}$.

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance h is given by

$v^2 = 2\left(\frac{2}{3}g\right)h$ or $v = \sqrt{\frac{4\,gh}{3}}$.

Ex.145 A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre of the sphere.



Sol. The situation is shown in figure. As the force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre,

$F + f = ma$ ....(i)

and for the rotational motion about the centre,

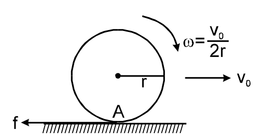
$$Fr - f\,r = I\,\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or, $F - f = \frac{2}{5}\,ma$, ...(iii)

From (i) and (ii), $2F = \frac{7}{5}\,ma$

or $a = \frac{10\,F}{7m}$.

Ex.146 A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the centre $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.



Sol. Velocity of the centre = v_0 and the angular velocity about the centre = $\frac{v_0}{2r}$. Thus $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is $\mu N = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$v(t) = v_0 - \frac{f}{M}t.$ ....(i)

This friction will also have a torque Γ = fr about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f \frac{r}{(2/5)Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time t will be

$$\omega(t) = \omega_0 + \frac{5f}{2Mr} t = \frac{v_0}{2r} + \frac{5f}{2Mr} t.$$

Pure rolling starts when v(t) = r ω(t)

i.e., $v(t) = \frac{v_0}{2} + \frac{5f}{2M} t.$ ....(ii)

Eliminating t from (i) and (ii),

$$\frac{5}{2} v(t) + v(t) = \frac{5}{2} v_0 + \frac{v_0}{2}$$

$$\text{or, } v(t) = \frac{2}{7} \times 3v_0 = \frac{6}{7} v_0.$$

Thus, the sphere rolls with translational velocity $6v_0/7$ in the forward direction.