Sequence & Series

Sequence

A succession of numbers $a_1, a_2, a_3, ..., a_n$ formed according to some definite rule is called a sequence. A sequence is a function whose domain is the set N of natural numbers and range a subset of real numbers or complex numbers.

The different terms of a sequence are usually denoted by $a_1, a_2, a_3, ...$ or by $t_1, t_2, t_3, ...$ The subscript (always a natural number) denotes the position of the term in the sequence. The term at the nth place of a sequence. i.e., t_n is called the general term of the sequence.

Real sequence :

A sequence whose range is a subset of R is called a real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

(iii) 3, –9, 27, – 81, ...

Types of sequence :

On the basis of the number of terms there are two types of sequence.

(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences : A sequence is said to be infinite if it has infinitely many terms.

Solved Examples

Ex.1 Write down the sequence whose n^{th} term is

(i) $\frac{2^n}{n}$ (ii) $\frac{3+(-1)^n}{3^n}$

Sol. (i) Let $t_n = \frac{2^n}{n}$

put n = 1, 2, 3, 4, ... we get

$t_1 = 2,\ t_2 = 2,\ t_3 = \frac{8}{3},\ t_4 = 4$

so the sequence is $2, 2, \frac{8}{3}, 4, ...$

(ii) Let $t_n = \frac{3+(-1)^n}{3^n}$

put n = 1, 2, 3, 4, ...

so the sequence is $\frac{2}{3}, \frac{4}{9}, \frac{2}{27}, \frac{4}{81}, ...$

Series :

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ...a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

e.g. (i) 1 + 2 + 3 + 4 + ... + n

(ii) 2 + 4 + 8 + 16 + ...

(iii) – 1 + 3 – 9 + 27 – ....

Progression

The sequence which obey the definite rule and its general term is always expressible in terms of n, is called progression.

Arithmetic progression (A.P.) :

A.P. is a sequence whose successive terms are obtained by adding a fixed number 'd' to the preceding terms. This fixed number 'd' is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... , a + (n – 1) d,........

e.g. – 4, – 1, 2, 5 ....

n^{th} term of an A.P. :

Let 'a' be the first term and 'd' be the common difference of an A.P., then

$t_n = a + (n - 1)\, d$, where $d = t_n - t_{n-1}$

Solved Examples

Ex.2 If t_{54} of an A.P. is – 61 and $t_4 = 64$, find t_{10}.

Sol. Let a be the first term and d be the common difference

so $t_{54} = a + 53d = -61$ ....(i)

and $t_4 = a + 3d = 64$ ....(ii)

equation (i) – (ii) we get

$\Rightarrow 50d = -125$

$d = -\frac{5}{2}$

$\Rightarrow a = \frac{143}{2}$ so $t_{10} = \frac{143}{2} + 9\left(-\frac{5}{2}\right) = 49$

Ex.3 Find the number of terms in the sequence 4, 12, 20, ...,108.

Sol. a = 4, d = 8 so 108 = 4 + (n – 1)8

$\Rightarrow n = 14$

The sum of first n terms of an A.P. :

If a is first term and d is common difference, then sum of the first n terms of AP is

$S_n = \frac{n}{2}\,[2a + (n - 1)\, d]$

$= \frac{n}{2}[a + \ell\,] \equiv n t_{\left(\frac{n+1}{2}\right)}$, for n is odd. (Where ℓ is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.)

Note : For any sequence $\{t_n\}$, whose sum of first r terms is S_r, r^{th} term, $t_r = S_r - S_{r-1}$.

Solved Examples

Ex.4 Find the sum of all natural numbers divisible by 5, but less than 100.

Sol. All those numbers are 5, 10, 15, 20, ..., 95.

Here a = 5, n = 19 & ℓ = 95

so $S = \frac{19}{2}\,(5 + 95) = 950.$

Ex.5 Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Sol. All these numbers are 101, 108, 115, ..., 997

997 = 101 + (n – 1) 7

$\Rightarrow n = 129$

so $S = \frac{129}{2}\,[101 + 997] = 70821.$

Ex.6 The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively, then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11^{th} terms

$\frac{n-1}{2} = 10 \quad \Rightarrow \quad n = 21$

so ratio of 11^{th} terms is $= \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$

Ex.7 If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50^{th} term.

Sol. Let t_n is n^{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= 2n^2 + 3n - 2(n-1)^2 - 3(n-1)$

$= 4n + 1$

so $t_{50} = 201.$

Useful Formulae:

Sum of natural numbers ($\sum n$)

$\Sigma n = \frac{n(n+1)}{2}$ where, $n \in N$.

Remarks :

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(iii) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$; five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$; six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) Middle term: If the number of terms is n, and

* n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ term is the middle term.

* n is even, then $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ terms are middle terms.

Solved Examples

Ex.8 The sum of three numbers in A.P. is 27 and the sum of their squares is 293, find them.

Sol. Let the numbers be $a - d, a, a + d$

so $3a = 27 \Rightarrow a = 9$

Also $(a - d)^2 + a^2 + (a + d)^2 = 293$.

$3a^2 + 2d^2 = 293$

$d^2 = 25 \Rightarrow d = \pm 5$

therefore numbers are 4, 9, 14.

Ex.9 If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Sol. As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10$.

Properties of A.P.:

(a) If $t_n = an + b$, then the series so formed is an A.P.

(b) If $S_n = an^2 + bn + c$, then series so formed is an A.P.

(c) If every term of an A.P. is increased or decreased by the same quantity, the resulting terms will also be in A.P.

(d) If every term of an A.P. is multiplied or divided by the same non-zero quantity, the resulting terms will also be in A.P.

(e) If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P. Then sum of these terms will be equal to $(2n + 1)a_{n+1}$. Here total number of terms in the series is $(2n + 1)$ and middle term is a_{n+1}.

(f) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.

(g) Sum and difference of corresponding terms of two A.P.'s will form a series in A.P.

(h) If terms $a_1, a_2, ..., a_{2n-1}, a_{2n}$ are in A.P. The sum of these terms will be equal to $(2n)\left(\frac{a_n + a_{n+1}}{2}\right)$, where $\frac{a_n + a_{n+1}}{2}$ = A.M. of middle terms.

(i) nth term of a series is $a_n = S_n - S_{n-1}$ $(n \geq 2)$

Solved Examples

Ex.10 If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., prove that a^2, b^2, c^2 are also in A.P.

Sol. $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$

$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b}$

$\Rightarrow b^2 - a^2 = c^2 - b^2$

$\Rightarrow a^2, b^2, c^2$ are in A.P.

Ex.11 If $\frac{b+c-a}{a}$, $\frac{c+a-b}{b}$, $\frac{a+b-c}{c}$ are in A.P., then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Sol. Given $\frac{b+c-a}{a}$, $\frac{c+a-b}{b}$, $\frac{a+b-c}{c}$ are in A.P.

Add 2 to each term

$\Rightarrow \frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

divide each by a + b + c $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Ex.12 If a, b, c in A.P. and

$x=\sum_{n=0}^{\infty} a^n, y=\sum_{n=0}^{\infty} b^n, z=\sum_{n=0}^{\infty} c^n$ then x,y,z are in

(1) AP (2) GP

(3) HP (4) None of these

Sol. [3]

Here a, b, c in A.P., given

Also $x=\frac{1}{1-a}, y=\frac{1}{1-b}, z=\frac{1}{1-c}$

Now a, b, c in AP

$\Rightarrow 1-a, 1-b, 1-c$ in A.P.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ in HP $\Rightarrow x, y, z$ in HP

Ex.13 If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the p[th], q[th], r[th] terms respectively of an A.P. then $ab(p-q)+bc(q-r)+ca(r-p)$ equals to

(1) 1 (2) -1

(3) 0 (4) None of these

Sol. [3]

Let x be the first term and y be the c.d. of corresponding A.P., then

$\frac{1}{a}=x+(p-1)y$ ..... (1)

$\frac{1}{b}=x+(q-1)y$ ..... (2)

$\frac{1}{c}=x+(r-1)y$ ..... (3)

Multiplying (1), (2) and (3) respectively by $abc(q-r), abc(r-p), abc(p-q)$ and then adding we get $bc(q-r)+ca(r-p)+ab(p-q)=0$

Ex.14 If the sum of the first n terms of a sequence is of the form An^2+Bn where A, B are constants independent of n, then the sequence is

(1) an A.P. (2) a G.P.

(3) an H.P. (4) None of these

Sol. [1]

We have, $S_n = An^2+Bn$

$\therefore S_{n-1} = A(n-1)^2 + B(n-1)$

$= A(n^2-2n+1)+B(n-1) = An^2-2An+A+Bn-B$

$\therefore a_n = S_n - S_{n-1}$

$= An^2+Bn-(An^2-2An+A+B_n-B)$

$= An^2+Bn-An^2+2An-A-Bn+B = 2An+B-A$

$\Rightarrow a_{n-1} = 2A(n-1)+B-A$

Now $a_n - a_{n-1} = 2An+B-A-2A(n-1)+B-A$

= 2A (a constant)

Hence the sequence is an A.P.

Ex.15 If the sum of the first n even natural numbers is equal to k times the sum of first n odd natural numbers, then k is equal to

(1) $\frac{1}{n}$ (2) $\frac{n-1}{n}$ (3) $\frac{n+1}{2n}$ (4) $\frac{n+1}{n}$

Sol. [4]

Let S_1 denotes the sum of the first n even natural numbers

$\therefore S_1 = 2+4+6+....\text{to n terms}$

$= \frac{n}{2}[2\times 2+(n-1)\times 2] [\because a_1 = 2, d = 2]$

$\therefore S_1 = \frac{n}{2}(4+2n-2) = \frac{n}{2}(2n+2) = n(n+1)$ .....(1)

Let S_2 denotes the sum of the first n odd natural numbers

$\therefore S_2 = 1+3+5+....\text{to n terms}$

$= \frac{n}{2}[2\times 1+(n-1)2] [\because a_1 = 1, d = 2]$

$\therefore S_2 = \frac{n}{2}(2+2n-2) = \frac{n}{2}(2n) = n^2$ .... (2)

Dividing (1) by (2),

$\frac{S_1}{S_2} = \frac{n(n+1)}{n^2} = \frac{n+1}{n} = 1+\frac{1}{n}$

Hence $S_1 = \left(1+\frac{1}{n}\right)S_2$ $\therefore k = \frac{n+1}{n}$

Arithmetic mean (mean or average) (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n numbers $a_1, a_2,..., a_n$ is;

$$A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}.$$

n - Arithmetic means between two numbers :

If a, b are any two given numbers & a, $A_1, A_2,....,A_n$, b are in A.P., then $A_1, A_2,... A_n$ are the n A.M.'s between a & b.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}, ..., A_n = a + \frac{n(b-a)}{n+1}$$

Note : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$, where A is the single A.M. between a & b i.e. $A = \frac{a+b}{2}$

Solved Examples

Ex.16 Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

Sol. Let a and b be two numbers and 2n A.M.s are inserted between a and b, then

$$\frac{2n}{2}(a+b) = 2n+1.$$

$$n\left(\frac{13}{6}\right) = 2n+1.$$

$$\left[\text{given } a+b = \frac{13}{6}\right]$$

$\Rightarrow$ n = 6.

$\therefore$ Number of means = 12.

Ex.17 Insert 20 A.M. between 2 and 86.

Sol. Here 2 is the first term and 86 is the 22^{nd} term of A.P. so 86 = 2 + (21)d $\Rightarrow$ d = 4

so the series is 2, 6, 10, 14,......., 82, 86

$\therefore$ required means are 6, 10, 14,...,82.

Ex.18 If m arithmetic means are inserted between 1 and 31 so that the ratio of the 7th and (m–1)th means is 5:9, then the value of m is

(1) 9 (2) 11 (3) 13 (4) 14

Sol. [4] Let the means be $x_1, x_2,..., x_m$ so that $1, x_1, x_2,...., x_m, 31$ is an A.P. of $(m+2)$ terms.

Now, $31 = T_{m+2} = a + (m+1)d = 1 + (m+1)d$

$\therefore d = \frac{30}{m+1}$ Given: $\frac{x_7}{x_{m-1}} = \frac{5}{9}$

$$\therefore \frac{T_8}{T_m} = \frac{a+7d}{a+(m-1)d} = \frac{5}{9}$$

$$\Rightarrow 9a + 63d = 5a + (5m-5)d \Rightarrow 4.1 = (5m-68)\frac{30}{m+1}$$

$$\Rightarrow 2m + 2 = 75m - 1020 \Rightarrow 73m = 1022$$

$$\therefore m = \frac{1022}{73} = 14$$

Geometric Progression (G.P.)

The sequence $\{a_n\}$ in which $a_1 \neq 0$ is termed a geometric progression if there is a number $r \neq 0$ such that $\frac{a_n}{a_{n-1}} = r$ for all n, then r is called common ratio.

1. Useful Formulae:

If a = first term, r = common ratio and n is the number of term, then

(a) n^{th} term denoted by t_n is given by

$t_n = ar^{n-1}$

(b) Sum of first n terms denoted by S_n is given by

$S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ corresponding to r < 1 (or) r > 1, (or) $S_n = \frac{a - r\ell}{1-r}$

where ℓ is the last term in the series.

(c) Sum of infinite terms (S_∞)

$S_\infty = \frac{a}{1-r}$ (For |r| <1)

Note : If $r \geq 1$ then $S_\infty \to \infty$.

Solved Examples

Ex19. If the first term of G.P. is 7, its n^{th} term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Sol. Given a = 7

$t_n = ar^{n-1} = 7(r)^{n-1} = 448.$

$\Rightarrow 7r^n = 448\ r$

Also $S_n = \frac{a(r^n-1)}{r-1} = \frac{7(r^n-1)}{r-1}$

$\Rightarrow 889 = \frac{448r-7}{r-1} \quad \Rightarrow \quad r = 2$

Hence $T_5 = ar^4 = 7(2)^4 = 112.$

Ex.20 The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Sol. Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow r = \frac{r^2}{1-r} \Rightarrow r = \frac{1}{2}$,

Hence series is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\infty$

Ex.21 Let $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$, find the sum of

(i) first 20 terms of the series

(ii) infinite terms of the series.

Sol. (i) $S_{20} = \frac{\left(1-\left(\frac{1}{2}\right)^{20}\right)}{1-\frac{1}{2}} = \frac{2^{20}-1}{2^{19}}.$

(ii) $S_\infty = \frac{1}{1-\frac{1}{2}} = 2.$

Remarks :

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if $a_1, a_2, a_3, a_4, ... a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_2a_{n-1} = a_3\ a_{n-2} = ...$

(ii) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non–zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3,........$ and $b_1, b_2, b_3,.........$ are two G.P's with common ratio r_1 and r_2 respectively, then the sequence $a_1b_1, a_2b_2, a_3b_3, ...$ is also a G.P. with common ratio $r_1 r_2$.

(vi) If $a_1, a_2, a_3,..........$are in G.P. where each $a_i > 0$, then $\log a_1, \log a_2, \log a_3,..........$are in A.P. and its converse is also true.

Solved Examples

Ex.22 Find three numbers in G.P. having sum 19 and product 216.

Sol. Let the three numbers be $\frac{a}{r}$, a, ar

so $a\left[\frac{1}{r}+1+r\right] = 19$ ....(i)

and $a^3 = 216 \Rightarrow a = 6$

so from (i) $6r^2 - 13r + 6 = 0.$

$\Rightarrow r = \frac{3}{2}, \frac{2}{3}$

Hence the three numbers are 4, 6, 9.

Ex.23 Find the product of 11 terms in G.P. whose 6^{th} term is 5.

Sol. Using the property

$a_1a_{11} = a_2a_{10} = a_3a_9 = ... = a_6^2 = 25$

Hence product of terms = 5^{11}

Ex.24 Using G.P. express $0.\overline{3}$ and $1.2\overline{3}$ as $\frac{p}{q}$ form.

Sol. Let $x = 0.\overline{3} = 0.3333$ ....

$= 0.3 + 0.03 + 0.003 + 0.0003 + ...$

$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...$

$= \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{9} = \frac{1}{3}.$

Let $y = 1.2\overline{3} = 1.233333$

$= 1.2 + 0.03 + 0.003 + 0.0003 + ...$

$= 1.2 + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + ...$

$= 1.2 + \frac{\frac{3}{10^2}}{1-\frac{1}{10}} = 1.2 + \frac{1}{30} = \frac{37}{30}.$

Ex.25 Evaluate 7 + 77 + 777 + ... upto n terms.

Sol. Let S = 7 + 77 + 777 + ...upto n terms.

$= \frac{7}{9}$ [9 + 99 + 999 + ...]

$= \frac{7}{9}$ [(10 – 1) + (10² – 1) + (10³ – 1) + ... + upto n terms]

$= \frac{7}{9}$ [$10 + 10^2 + 10^3 + ...+ 10^n – n$]

$= \frac{7}{9}\left(\frac{10(10^n-1)}{9}-n\right)$

$= \frac{7}{81}$ [$10^{n+1} – 9n – 10$]

Geometric mean (G)

(i) $G = \sqrt{ab}$ where a, b are two positive numbers.

(ii) $G = (a_1 a_2 ...... a_n)^{1/n}$ is geometric mean of n positive numbers $a_1, a_2, a_3,...... a_n$.

* **n GM's between two given numbers**

If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2...... G_n$ then a, $G_1, G_2 ..... G_n$, b will be in GP. The series consist of (n+2) terms and the last term is b and first term is a.

$\Rightarrow ar^{n+2-1} = b$

$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$G_1 = ar$, $G_2 = ar^2$ ..... $G_n = ar^n$ or $G_n = b/r$

Note : Product of n GM's inserted between 'a' and 'b' is equal to n^th^ power of the single GM between 'a' and 'b'

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where $G = \sqrt{ab}$

Solved Examples

Ex.26 Insert 4 G.M.s between 2 and 486.

Sol. Common ratio of the series is given by $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$= (243)^{1/5} = 3$

Hence four G.M.s are 6, 18, 54, 162.

Ex.27 If x, y, z are in G.P. and $a^x = b^y = c^z$ then

(1) $\log_b a = \log_a c$ (2) $\log_c b = \log_a c$

(3) $\log_b a = \log_c b$ (4) None of these

Sol. [3]

x, y, z are in G.P. $\Rightarrow y^2 = xz$

We have, $a^x = b^y = c^z = \lambda(\text{say})$

$\Rightarrow x\log a = y\log b = z\log c = \log\lambda$

$\Rightarrow x = \frac{\log\lambda}{\log a}, y = \frac{\log\lambda}{\log b}, z = \frac{\log\lambda}{\log c}$

putting x, y, z in (i), we get

$\left(\frac{\log\lambda}{\log b}\right)^2 = \frac{\log\lambda}{\log a}.\frac{\log\lambda}{\log c}$

$(\log b)^2 = \log a.\log c$ or

$\log_a b = \log_b c \Rightarrow \log_b a = \log_c b$

Ex.28 If a, b, c, d are in G.P., then

$(a^3+b^3)^{-1}, (b^3+c^3)^{-1}, (c^3+d^3)^{-1}$ are in

(1) A.P. (2) G.P.

(3) H.P. (4) None of these

Sol. [2]

Let $b - ar, c = ar^2$ and $d = ar^3$. Then,

$\frac{1}{a^3+b^3} = \frac{1}{a^3(1+r^3)}, \frac{1}{b^3+c^3} = \frac{1}{a^3r^3(1+r^3)}$ and

$\frac{1}{c^3+d^3} = \frac{1}{a^3r^3(1+r^3)}$

Clearly, $(a^3+b^3)^{-1}, (b^3+c^3)^{-1}$ and $(c^3+d^3)^{-1}$ are in G.P. with common ration $\frac{1}{r^3}$

Ex.29 If r > 1 and $x = a + \frac{a}{r} + \frac{a}{r^2} + ...\text{to } \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - .....\text{to } \infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + ...\text{to } \infty$, then $\frac{xy}{z} =$

(1) $\frac{ab}{c}$ (2) $\frac{ac}{b}$

(3) $\frac{bc}{a}$ (4) None of these

Sol. [1]

Since $r > 1, \frac{1}{r} < 1$ $\quad \therefore x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$

Similarly, $y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{r+1}$ and

$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1} \quad \text{......... (1)}$$

$$\therefore xy = \frac{ar}{r-1} \times \frac{br}{r+1} = \frac{abr^2}{r^2-1} \quad \text{......... (2)}$$

Dividing (2) by (1), we get $\frac{xy}{z} = \frac{abr^2}{r^2-1} \times \frac{r^2-1}{cr^2} = \frac{ab}{c}$

Harmonic Progression (H.P.)

A sequence is said to be a harmonic progression, if and only if the reciprocal of its terms form an arithmetic progression.

For example: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ .... form a H.P. because 2, 4, 6,........... are in A.P.

If a, b, c are in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ forms an A.P.

1. Some Useful Formulae & Properties:

(a) If a, b are first two terms of an H.P. then

$$t_n = \frac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$$

(b) There is no formula for sum of a H.P.

(c) Harmonic mean H of any two numbers a and b is given by

$H = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}$ where a, b are two non–zero numbers.

Also $H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} = \frac{n}{\sum_{j=1}^{n} \frac{1}{a_j}}$

the harmonic mean of n non–zero numbers $a_1, a_2, a_3, \ldots, a_n$.

(d) If terms are given in H.P. then the terms could be picked up in the following way

* For three terms

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

For four terms

$$\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$$

* For five terms

$$\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$$

Note : In general, if we are to take (2r + 1) terms in H.P. we take them as

$$\frac{1}{a-rd}, \frac{1}{a-(r-1)d}, \ldots\ldots, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \ldots\ldots, \frac{1}{a+rd}$$

(e) Harmonical Mean (H.M.)

If three or more than three terms are in H.P, then all the numbers lying between first and last term are called Harmonical Means between them. i.e.

The H.M. between two given quantities a and b is H so that a, H , b are in H.P.

i.e. $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow H = \frac{2ab}{a+b}$$

* n H.M's between two given numbers

To find n HM's between a and b we first find n AM's between 1/a and 1/b then their reciprocals will be required HM's

Solved Examples

Ex.30 If p^{th}, q^{th} and r^{th} terms of H.P. are u, v, w respectively, then the value of the expression $(q-r)vw + (r-p)wu + (p-q)uv$ is

(1) 1 (2) 0 (3) -2 (4) -1

Sol. [2]

Let H.P. be

$$\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \ldots..$$

$$\therefore u = \frac{1}{a+(p-1)d}, v = \frac{1}{a+(q-1)d}, w = \frac{1}{a+(r-1)d}$$

$$\Rightarrow a+(p-1)d=\frac{1}{u},\ a+(q-1)d=\frac{1}{v},\ a+(r-1)d=\frac{1}{w}$$

$$\Rightarrow (q-r)\{a+(p-1)d\}+(r-p)\{a+(q-1)d\}$$

$$+.....=\frac{1}{u}(q-r)+\frac{1}{v}(r-p)+.....$$

$$\Rightarrow (q-r)vw+.....=0$$

Ex.31 If $H_1, H_2, H_3, ...H_n$ be n harmonic means between a and b then $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=$

(1) 0 (2) n (3) 2n (4) 1

Sol. [3]

Here $H_1=\frac{ab(n+1)}{b(n+1)-(b-a)}=\frac{ab(n+1)}{bn+a}$

Similarly $H_n=\frac{ab(n+1)}{an+b}$ (interchange a and b)

Hence $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}$

$$=\frac{(2n+1)b+a}{b-a}+\frac{(2n+1)a+b}{a-b}$$

$$=\frac{2nb+b+a-2na-a-b}{b-a}=2n$$

Ex.32 If $\frac{a_2a_3}{a_1a_4}=\frac{a_2+a_3}{a_1+a_4}=3\left(\frac{a_2-a_3}{a_1-a_4}\right)$ then a_1, a_2, a_3, a_4 are in

(1) A.P. (2) G.P.

(3) H.P. (4) None of these

Sol. [3]

$\frac{a_1+a_4}{a_1a_4}=\frac{a_2+a_3}{a_2a_3}$, so $\frac{1}{a_4}+\frac{1}{a_1}=\frac{1}{a_3}+\frac{1}{a_2}$ or

$\frac{1}{a_4}-\frac{1}{a_3}=\frac{1}{a_2}-\frac{1}{a_1}$ ...... (1)

Also $\frac{3(a_2-a_3)}{a_2a_3}=\frac{a_1-a_4}{a_1a_4}$;

so $3\left(\frac{1}{a_3}-\frac{1}{a_2}\right)=\frac{1}{a_4}-\frac{1}{a_1}$ ...... (2)

Clearly, (1) and (2) $\Rightarrow \frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}=\frac{1}{a_4}-\frac{1}{a_3}$;

so $\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3}$ are in A.P.

Ex.33 If m^{th} term of H.P. is n, while n^{th} term is m, find its $(m+n)^{th}$ term.

Sol. Given $T_m=n$ or $\frac{1}{a+(m-1)d}=n$; where a is the first term and d is the common difference of the corresponding A.P.

so $a+(m-1)d=\frac{1}{n}$

and $a+(n-1)d=\frac{1}{m}$ $\Rightarrow (m-n)d=\frac{m-n}{mn}$

or $d=\frac{1}{mn}$ so $a=\frac{1}{n}-\frac{(m-1)}{mn}=\frac{1}{mn}$

Hence $T_{(m+n)}=\frac{1}{a+(m+n-1)d}=\frac{mn}{1+m+n-1}$

$=\frac{mn}{m+n}$.

Ex.34 Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Sol. Let 'd' be the common difference of corresponding A.P..

so $d=\frac{\frac{13}{2}-\frac{3}{2}}{5}=1$.

$\therefore \frac{1}{H_1}=\frac{3}{2}+1=\frac{5}{2}$ or $H_1=\frac{2}{5}$

$\frac{1}{H_2}=\frac{3}{2}+2=\frac{7}{2}$ or $H_2=\frac{2}{7}$

$\frac{1}{H_3}=\frac{3}{2}+3=\frac{9}{2}$ or $H_3=\frac{2}{9}$

$\frac{1}{H_4}=\frac{3}{2}+4=\frac{11}{2}$ or $H_4=\frac{2}{11}$.

Ex.35 If p^{th}, q^{th}, r^{th} terms of an H.P. be a, b, c respectively, prove that

$(q-r)bc+(r-p)ac+(p-q)ab=0$

Sol. Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $\frac{1}{a}=x+(p-1)d$ ....(i)

$\frac{1}{b}=x+(q-1)d$ ....(ii)

$\frac{1}{c}=x+(r-1)d$ ....(iii)

(i) - (ii) $\Rightarrow$ $ab(p-q)d=b-a$ ....(iv)

(ii) - (iii) $\Rightarrow$ $bc(q-r)d=c-b$ ....(v)

(iii) - (i) $\Rightarrow$ $ac(r-p)d=a-c$ ....(vi)

(iv) + (v) + (vi) gives

$bc(q-r)+ac(r-p)+ab(p-q)=0$.

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.) and $A \ge G \ge H$.

Solved Examples

Ex.36 The A.M. of two numbers exceeds the G.M. by $\frac{3}{2}$ and the G.M. exceeds the H.M. by $\frac{6}{5}$; find the numbers.

Sol. Let the numbers be a and b, now using the relation

$G^2 = AH$

$= \left(G + \frac{3}{2}\right)\left(G - \frac{6}{5}\right) = G^2 + \frac{3}{10}G - \frac{9}{5}$

$\Rightarrow G = 6$

i.e. ab = 36

also $a + b = 15$

Hence the two numbers are 3 and 12.

A.M. ≥ G.M. ≥ H.M.

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. $= \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. $= (a_1 a_2 a_3 ....a_n)^{1/n}$ and their

H.M. $= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$.

It can be shown that

A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff

$a_1 = a_2 = a_3 = ...= a_n$

Solved Examples

Ex.37 If a, b, c > 0, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$

Sol. Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \ge \left(\frac{a}{b}.\frac{b}{c}.\frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$$

Ex.38 If x,y,z are positive, then prove that $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$

Sol. Using the relation A.M. ≥ H.M.

$$\frac{x+y+z}{3} \ge \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$\Rightarrow (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$$

Ex.39 If $a_i > 0 \ \forall \ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1 + a_1)(1 + a_2)(1 + a_3) ....(1 + a_n) \ge 2^n$

Sol. Using A.M. ≥ G.M.

$1 + a_1 \ge 2\sqrt{a_1}$

$1 + a_2 \ge 2\sqrt{a_2}$

$\vdots$

$1 + a_n \ge 2\sqrt{a_n}$

$\Rightarrow (1 + a_1)(1 + a_2) ....(1 + a_n)$

$\ge 2^n (a_1 a_2 a_3 ......a_n)^{1/2}$

As $a_1 a_2 a_3 ..... a_n = 1$

Hence $(1 + a_1)(1 + a_2) .... (1 + a_n) \ge 2^n$.

Ex.40 If n > 0, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Sol. Using the relation A.M. ≥ G.M. on the numbers 1, 2, $2^2, 2^3, ..., 2^{n-1}$, we have

$$\frac{1+2+2^2+.......+2^{n-1}}{n} > (1.2.\ 2^2.\ 2^3.\ ....\ .2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \frac{2^n - 1}{2 - 1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}}$$

$$\Rightarrow 2^n - 1 > n.2^{\frac{(n-1)}{2}} \quad \Rightarrow \quad 2^n > 1 + n.2^{\frac{(n-1)}{2}}$$

Ex.41 Find the greatest value of xyz for positive value of x, y, z subject to the condition xy + yz + zx = 12.

Sol. Using the relation A.M. ≥ G.M.

$$\frac{xy + yz + zx}{3} \ge (x^2 y^2 z^2)^{1/3} \Rightarrow \quad 4 \ge (x\,y\,z)^{2/3}$$

$\Rightarrow xyz \le 8$

Ex. 42 If a, b, c are in H.P. and they are distinct and positive, then prove that $a^n + c^n > 2b^n$

Sol. Let a^n and c^n be two numbers

then $\frac{a^n + c^n}{2} > (a^n c^n)^{1/2}$

$a^n + c^n > 2 (ac)^{n/2}$ ....(i)

Also G.M. > H.M.

i.e. $\sqrt{ac} > b$, $(ac)^{n/2} > b^n$ ....(ii)

hence from (i) and (ii), we get $a^n + c^n > 2b^n$

Ex.43 If a, b and c are distinct positive real numbers and $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ is

(1) less than 1 (2) equal to 1

(3) greater than 1 (4) any real number

Sol. [1]

Since a and b are unequal, $\frac{a^2 + b^2}{2} > \sqrt{a^2b^2}$

(A.M. > G.M. for unequal numbers)

$\Rightarrow a^2 + b^2 > 2ab$

Similarly $b^2 + c^2 > 2bc$ and $c^2 + a^2 > 2ca$

Hence $2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca < 1$

Ex.44 a, b, c are three positive numbers and abc^2 has the greatest value $\frac{1}{64}$. Then

(1) $a = b = \frac{1}{2}, c = \frac{1}{4}$ (2) $a = b = \frac{1}{4}, c = \frac{1}{2}$

(3) $a = b = c = \frac{1}{3}$ (4) None of these

Sol. [2]

$\frac{a + b + \frac{c}{2} + \frac{c}{2}}{4} \ge \sqrt[4]{a.b.\frac{c}{2}.\frac{c}{2}}$ or $\frac{a+b+c}{4} \ge \sqrt[4]{\frac{abc^2}{4}}$

$\therefore \frac{(a+b+c)^4}{4^4} \ge \frac{abc^2}{4}$; or $abc^2 \le \frac{1}{64}(a+b+c)^4$

$\therefore$ the greatest value of $abc^2 = \frac{1}{64}(a+b+c)^4$

Also for the greatest value of abc^2 the numbers have to be equal, i.e., $a = b = \frac{c}{2}$

Also, given the greatest value $= \frac{1}{64}$. So $a + b + c = 1$

Arithmetic-Geometric Series

The series whose each term is formed by multiplying corresponding terms of an A.P. and G.P. is called the Arithmetic–geometric series.

For example –

* $1 + 2x + 4x^2 + 6x^3 + ...$

* $a + (a + d)r + (a + 2d)r^2 + ...$

1. Summation of n terms of Arithmetic–Geometric Series :

Let $S = a + (a + d)r + (a + 2d)r^2 + ...$

(i) $t_n = [a + (n - 1)d].r^{n-1}$

(ii) $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n - 1)d]r^{n-1}$

Multiply by 'r' and rewrite the series in following way.

$rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + ...+ [a + (n - 2)d]r^{n-1} + [a + (n - 1)d]r^n$

On subtraction,

$S_n(1 - r) = a + d(r + r^2 + ... + r^{n-1}) - [a + (n - 1)d]r^n$

or, $S_n(1 - r) = a + \frac{dr(1 - r^{n-1})}{1 - r} - [a + (n - 1)d].r^n$

or, $S_n = \frac{a}{1-r} + \frac{dr(1 - r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]}{1-r}.r^n$

2. Summation of Infinite Terms Series :

$S = a + (a + d)r + (a + 2d)r^2 + ... \infty$

$rS = a r + (a + d)r^2 + ...\infty$

———————————————————————

On subtraction

$S(1 - r) = a + d(r + r^2 + r^3 + ...\infty)$

$S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Solved Examples

Ex.45 If r^{th} term of a series is $(2r+1)2^{-r}$, then sum of its infinite terms is

(1) 10 (2) 8 (3) 5 (4) 0

Sol. [3]

Here $T_r = (2r+1)2^{-r}$

$\therefore$ Series is: $\frac{1}{2}\left[3+\frac{5}{2}+\frac{7}{2^2}+.....\right]$

Obviously the series in the bracket is Arithmetic-Geometrical Series. Therefore by the formula

$$S_\infty = \frac{a}{1-r}+\frac{r}{(1-r)^2}$$

We have $S_\infty = \frac{1}{2}\left[\frac{3}{1-\frac{1}{2}}+\frac{2\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}\right] = 5$

Ex.46 Find the sum of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3}$ + ... to n terms.

Sol. Let $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ... + \frac{3n-2}{5^{n-1}}$ ....(i)

$\left(\frac{1}{5}\right) S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + ... + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$ ....(ii)

(i) – (ii) $\Rightarrow$

$$\frac{4}{5} S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + ... + \frac{3}{5^{n-1}} - \frac{3n-2}{5^n}.$$

$$\frac{4}{5} S = 1 + \frac{\frac{3}{5}\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{1-\frac{1}{5}} - \frac{3n-2}{5^n}$$

$$= 1 + \frac{3}{4} - \frac{3}{4} \times \frac{1}{5^{n-1}} - \frac{3n-2}{5^n}$$

$$= \frac{7}{4} - \frac{12n+7}{4.5^n} \quad \therefore \quad S = \frac{35}{16} - \frac{(12n+7)}{16 . 5^{n-1}}.$$

Ex.47 Evaluate $1+2x+3x^2+4x^3+$......... upto infinity, where $|x|<1$.

Sol. Let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ......(ii)

(i) - (ii) $\Rightarrow$ $(1 - x) S = 1 + x + x^2 + x^3 + ...$

or $S = \frac{1}{(1-x)^2}$

Ex.48 Evaluate : $1+(1+b)r+(1+b+b^2)r^2+$......... to infinite terms for $|br|<1$.

Sol. Let $S = 1 + (1 + b)r + (1 + b + b^2) r^2 +...$ ....(i)

$rS = r + (1 + b) r^2 + ...$ ......(ii)

(i) - (ii)

$\Rightarrow (1 - r)S = 1 + br + b^2r^2 + b^3r^3 + ...$

$\Rightarrow S = \frac{1}{(1-br)(1-r)}$

Ex.49 $1+2.2+3.2^2+4.2^3+.....+100.2^{99}$ equals

(1) 99.2^{100} (2) 100.2^{100}

(3) $1+99.2^{100}$ (4) none of these

Sol. [3]

Let $S = 1+2.2+3.2^2+4.2^3+....+100.2^{99}$ .... (1)

$\Rightarrow 2S = 2+2.2^2+3.2^3+....+99.2^{99}+100.2^{100}$ ... (2)

Subtracting (1) from (2) we get

$-S = \left(1+2+2^2+2^3+.....+2^{99}\right) - 100.2^{100}$

$\Rightarrow S = 100.2^{100} - \frac{2^{100}-1}{2-1}$

$= 100.2^{100} - 2^{100} + 1$

$= 1+99.2^{100}$

Results :

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$.

(ii) $\sum_{r=1}^{n} k\, a_r = k \sum_{r=1}^{n} a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k +$..............n times $= nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2$

$= \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

Solved Examples

Ex.50 Find the sum of the series to n terms whose general term is 2n + 1.

Sol. $S_n = \Sigma T_n = \Sigma(2n+1)$

$= 2\Sigma n + \Sigma 1$

$= \frac{2(n+1)n}{2} + n = n^2 + 2n$

Ex.51 $T_k = k^2 + 2^k$, then find $\sum_{k=1}^{n} T_k$.

Sol. $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$

$= \frac{n(n+1)(2n+1)}{6} + \frac{2(2^n-1)}{2-1}$

$= \frac{n(n+1)(2n+1)}{6} + 2^{n+1} - 2.$

Ex.52 Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Sol. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j$

$= \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}.$

Method of difference for finding n^th term :

Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1$, $u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ....(i)

$S = u_1 + u_2 + ... + u_{n-1} + u_n$ ....(ii)

(i) – (ii) $\Rightarrow u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n.

So sum of series $S = \sum_{r=1}^{n} u_r$

Note : The above method can be generalised as follows :

Let u_1, u_2, u_3, ... be a given sequence.

The first differences are $\Delta_1 u_1, \Delta_1 u_2, \Delta_1 u_3$, ...

where $\Delta_1 u_1 = u_2 - u_1$, $\Delta_1 u_2 = u_3 - u_2$ etc.

The second differences are $\Delta_2 u_1, \Delta_2 u_2, \Delta_2 u_3$, ..., where $\Delta_2 u_1 = \Delta_1 u_2 - \Delta_1 u_1$, $\Delta_2 u_2 = \Delta_1 u_3 - \Delta_1 u_2$ etc.

This process is continued untill the k^{th} differences $\Delta_k u_1$, $\Delta_k u_2$, ... are obtained, where the k^{th} differences are all equal or they form a GP with common ratio different from 1.

Case - 1 : The k^{th} differences are all equal.

In this case the n^{th} term, u_n is given by

$u_n = a_0 n^k + a_1 n^{k-1} + ... + a_k$, where a_0, a_1, ..., a_k are calculated by using first 'k + 1' terms of the sequence.

Case - 2 : The k^{th} differences are in GP with common ratio r $(r \neq 1)$

The n^{th} term is given by

$u_n = \lambda r^{n-1} + a_0 n^{k-1} + a_1 n^{k-2} + ... + a_{k-1}$

Solved Examples

Ex.53 Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Sol. Let $S = 3 + 7 + 13 + 21 + ... + T_n$ ....(i)

$S = \quad 3 + 7 + 13 + ... + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow T_n = 3 + 4 + 6 + 8 + ... + (T_n - T_{n-1})$

$= 3 + \frac{n-1}{2}[8 + (n-2)2]$

$= 3 + (n-1)(n+2) \quad = n^2 + n + 1$

Hence $S = \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$

Ex.54 Find the sum to n-terms $1 + 4 + 10 + 22 + \ldots\ldots..$

Sol. Let $S = 1 + 4 + 10 + 22 + \ldots\ldots\ldots + T_n$ ....(i)

$S = \quad 1 + 4 + 10 + \ldots\ldots\ldots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $T_n = 1 + (3 + 6 + 12 + \ldots\ldots\ldots + T_n - T_{n-1})$

$$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right)$$

$$T_n = 3 \,.\, 2^{n-1} - 2$$

So $\quad S = \sum T_n = 3 \sum 2^{n-1} - \sum 2$

$$= 3 \,.\, \left(\frac{2^n - 1}{2 - 1}\right) - 2n \quad = 3.2^n - 2n - 3$$

Difference Method

Let $T_1, T_2, T_3 \ldots\ldots T_n$ are the terms of sequence, then

(i) If $(T_2 - T_1), (T_3 - T_2) \ldots\ldots\ldots\ldots\ldots (T_n - T_{n-1})$ are in A.P. then, the sum of the such series may be obtained by using summation formulae in nth term,

(ii) If $(T_2 - T_1), (T_3 - T_2) \ldots\ldots\ldots\ldots\ldots\ldots (T_n - T_{n-1})$ are found in G.P. then the sum of the such series may be obtained by using summation formulae of a G.P.

Solved Examples

Ex.55 Sum of the series

$3 + 7 + 14 + 24 + 37 + \ldots..10$ terms , is

(1) 560 (2) 570

(3) 580 (4) None of these

Sol. [2]

Here the given series is not A.P., G.P., or H.P.

Let $S = 3 + 7 + 14 + 24 + 37 + \ldots. + T_n$

$S = \quad 3 + 7 + 14 + 24 + \ldots.. + T_n$

after subtracting

$$0 = 3 + \underbrace{4 + 7 + 10 + 13 + \ldots.. - T_n}_{\text{A.P.}}$$

$$\therefore T_n = 3 + \frac{(n-1)}{2}\left[2(4) + (n-2)3\right] \quad = \frac{1}{2}\left(3n^2 - n + 4\right)$$

$$\therefore S_n = \frac{1}{2}\left[3\Sigma n^2 - \Sigma n + 4n\right]$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

Putting $n = 10$

$$S_{10} = \frac{1}{2}\left[\frac{10 \times 11 \times 21}{2} - \frac{10 \times 11}{2} + 40\right]$$

$$= \frac{1}{2}\left[1155 - 55 + 40\right] = \frac{1140}{2} = 570$$

Ex.56 Find the sum of n-terms of the series $1.2 + 2.3 + 3.4 + \ldots\ldots\ldots..$

Sol. Let T_r be the general term of the series

So $T_r \quad = r(r + 1)$.

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r + 2) - (r - 1)]$

so $T_r \quad = \frac{r}{3}\,(r + 1)\,[(r + 2) - (r - 1)]$

$\quad = \frac{1}{3}\,[r\,(r + 1)\,(r + 2) - (r - 1)\,r\,(r + 1)]$.

Let $f(r) \quad = \frac{1}{3}\,r\,(r + 1)\,(r + 2)$

so $T_r \quad = [f(r) - f(r - 1)]$.

Now $S \quad = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$$T_1 = \frac{1}{3}\,[1 \,.\, 2 \,.\, 3 - 0]$$

$$T_2 = \frac{1}{3}\,[2 \,.\, 3 \,.\, 4 - 1 \,.\, 2 \,.\, 3]$$

$$T_3 = \frac{1}{3}\,[3 \,.\, 4 \,.\, 5 - 2 \,.\, 3 \,.\, 4]$$

$$\vdots$$

$$T_n = \frac{1}{3}\,[n(n+1)\,(n + 2) - (n - 1)n\,(n + 1)]$$

$$\therefore \quad S = \frac{1}{3}\,n\,(n + 1)\,(n + 2)$$

Hence sum of series is $f(n) - f(0)$.

Ex.57 Sum to n terms of the series $\frac{1}{(1+x)(1+2x)}$ $+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\ldots\ldots\ldots$

Sol. Let T_r be the general term of the series

$$T_r=\frac{1}{(1+rx)(1+(r+1)x)}$$

$$\text{So } T_r=\frac{1}{x}\left[\frac{(1+(r+1)x)-(1+rx)}{(1+rx)(1+(r+1)x)}\right]$$

$$=\frac{1}{x}\left[\frac{1}{1+rx}-\frac{1}{1+(r+1)x}\right]$$

$T_r = f(r) - f(r + 1)$

$\therefore\ S = \Sigma T_r = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$$=\frac{1}{x}\left[\frac{1}{1+x}-\frac{1}{1+(n+1)x}\right]$$

$$=\frac{n}{(1+x)[1+(n+1)x]}$$

Ex.58 Sum to n terms of the series $\frac{4}{1.2.3}+\frac{5}{2.3.4}$ $+\frac{6}{3.4.5}+\ldots\ldots\ldots$

Sol. Let $T_r=\frac{r+3}{r(r+1)(r+2)}$

$$=\frac{1}{(r+1)(r+2)}+\frac{3}{r(r+1)(r+2)}$$

$$=\left[\frac{1}{r+1}-\frac{1}{r+2}\right]+\frac{3}{2}\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$$

$$\therefore\quad S=\left[\frac{1}{2}-\frac{1}{n+2}\right]+\frac{3}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

$$=\frac{5}{4}-\frac{1}{n+2}\left[1+\frac{3}{2(n+1)}\right]=\frac{5}{4}-\frac{1}{2(n+1)(n+2)}[2n+5]$$

Ex.59 Find the nth term and the sum of n term of the series 2 + 12 + 36 + 80 + 150 + 252 + ...

Sol. Let S = 2 + 12 + 36 + 80 + 150 + 252 + ...+T_n ....(i)

S = 2 + 12 + 36 + 80 + 150 + 252 + ...+ $T_{n-1}+T_n$ ....(ii)

(i) – (ii) $\Rightarrow T_n$ = 2 + 10 + 24 + 44 + 70 + 102 + ... + $(T_n - T_{n-1})$ ....(iii)

T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + $(T_{n-1}-T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) $\Rightarrow T_n - T_{n-1}$ = 2 + 8 + 14 + 20 + 26 + ...

$$=\frac{n}{2}[4+(n-1)6]=n[3n-1]\ \Rightarrow T_n-T_{n-1}=3n^2-n$$

$\therefore$ general term of given series is $\Sigma(T_n - T_{n-1}) = \Sigma(3n^2 - n) = n^3 + n^2$.

Hence sum of this series is

$S = \Sigma n^3 + \Sigma n^2$

$$=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{12}(3n^2+7n+2)$$

$$=\frac{1}{12}n(n+1)(n+2)(3n+1)$$

Ex.60 Find the general term and sum of n terms of the series 9 + 16 + 29 + 54 + 103 + ...

Sol. Let S = 9 + 16 + 29 + 54 + 103 + ... + T_n ....(i)

S = 9 + 16 + 29 + 54 + 103 + ... + $T_{n-1}+T_n$ ....(ii)

(i) – (ii) $\Rightarrow T_n$ = 9 + 7 + 13 + 25 + 49 + ... + $(T_n - T_{n-1})$ ....(iii)

T_n = 9 + 7 + 13 + 25 + 49 + ... + $(T_{n-1}-T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) $\Rightarrow T_n - T_{n-1}$ = 9 + (–2) + $\underbrace{6+12+24+\ldots\ldots}_{(n-2)\text{ terms}}$ = 7 + 6 $[2^{n-2} - 1] = 6(2)^{n-2} + 1$.

$\therefore$ General term is $T_n = 6(2)^{n-1} + n + 2$

Also sum $S = \Sigma T_n$

$= 6\Sigma 2^{n-1} + \Sigma n + \Sigma 2$

$$=6\,.\,\frac{(2^n-1)}{2-1}+\frac{n(n+1)}{2}+2n$$

$$=6(2^n-1)+\frac{n(n+5)}{2}$$