BINOMIAL

1. BINOMIAL EXPRESSIONS

An algebraic expression containing two terms is called a **binomial expression**.

For example , $2x + 3$, $x^2 - x/3$, $x + a$ etc. are Binomial Expressions.

2. BINOMIAL THEOREM

The rule by which any power of a binomial can be expanded is called the **Binomial Theorem**.

3. BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

If x and a are two real numbers and n is a positive integer then

$(x + a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots\ldots + {}^nC_r x^{n-r} a^r + \ldots\ldots + {}^nC_n x^0 a^n$.

Where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$,......${}^nC_r$..... are called **binomial coefficients** which can be denoted by $C_0, C_1, C_2, C_3, \ldots\ldots C_r \ldots\ldots$

General Term : In the expansion of $(x+a)^n$, $(r+1)^{th}$ term is called the **general term** which can be represented by T_{r+1}.

$$\mathbf{T_{r+1} = {}^nC_r\, x^{n-r} a^r}$$

$= {}^nC_r$(first term)$^{n-r}$ (second term)^r.

Characteristics of the expansion of $(x + a)^n$

Observing to the expansion of $(x + a)^n$, $n \in N$, we find that-

(i) The total number of terms in the expansion = (n + 1) i.e. one more than the index n.

(ii) In every successive term of the expansion the power of x (first term) decreases by 1and the power of (second term) increases by 1. Thus in every term of the expansion, the sum of the powers of x and a is equal to n (index).

(iii) The binomial coefficients of the terms which are at equidistant from the beginning and from the end are always equal i.e.

$${}^nC_r = {}^nC_{n-r}$$

Thus ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$,

${}^nC_2 = {}^nC_{n-2}$ etc.

(iv) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

Some deduction of Binomial Theorem :

(i) Expansion of $(x-a)^n$.

$(x-a)^n = {}^nC_0\, x^n a^0 - {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \ldots\ldots + (-1)^r\, {}^nC_r x^{n-r} a^r + \ldots\ldots + (-1)^n\, {}^nC_n\, x^0 a^n$

This expansion can be obtained by putting (–a) in place of a in the expansion of $(x+a)^n$.

General term = (r + 1)th term

$$\mathbf{T_{r+1} = {}^nC_r(-1)^r.\, x^{n-r} a^r}$$

(ii) By putting x = 1 and a = x in the expansion of $(x + a)^n$, we get the following result

$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots\ldots + {}^nC_n\, x^n$

which is the standard form of binomial expansion.

General term = (r + 1)th term

$$\mathbf{T_{r+1} = {}^nC_r\, x^r}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} \cdot x^r$$

(iii) By putting (–x) in place of x in the expansion of $(1 + x)^n$

$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - {}^nC_3 x^3 + \ldots\ldots + (-1)^r\, {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$.

General term = (r + 1)th term

$$\mathbf{T_{r+1} = (-1)^r.\, {}^nC_r\, x^r}$$

$$= (-1)^r . \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} . x^r$$

4. NUMBER OF TERMS IN THE EXPANSION OF $(x + y + z)^n$

$(x + y + z)^n$ can be expanded as-

$(x + y + z)^n = \{(x + y) + z\}^n$

$= (x + y)^n + {}^nC_1 (x + y)^{n-1}.z + {}^nC_2 (x + y)^{n-2} z^2 + \ldots\ldots + {}^nC_n\, z^n$.

= (n + 1) terms + n terms + (n–1) terms + ...+ 1 term

$\therefore$ Total number of terms = (n +1) + n + (n –1) + ...+ 1

$$= \frac{(n+1)(n+2)}{2}$$

5. MIDDLE TERM IN THE EXPANSION OF $(x + a)^n$

(a) If n is even, then the number of terms in the expansion i.e. (n+1) is odd, therefore, there will be only one middle term which is $\left(\frac{n+2}{2}\right)^{th}$ term.

i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

so middle term = $\left(\frac{n}{2}+1\right)^{th}$ term.

(b) If n is odd, then the number of terms in the expansion i.e. (n +1) is even, therefore there will be two middle terms which are

$$= \left(\frac{n+1}{2}\right)^{th} \text{ and } \left(\frac{n+3}{2}\right)^{th} \text{ term.}$$

Note : (i) When there are two middle terms in the expansion then their Binomial coefficients are equal.
(ii) Binomial coefficient of middle term is the greatest Binomial coefficient.

6. TO DETERMINE A PARTICULAR TERM IN THE EXPANSION

In the expansion of $\left(x^{\alpha} \pm \frac{1}{x^{\beta}}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$$n\alpha - r(\alpha + \beta) = m$$

$$\Rightarrow \quad r = \frac{n\alpha - m}{\alpha + \beta}$$

Thus in above expansion if constant term i.e. the term which is independent of x, occurs in T_{r+1} then r is determined by

$$n\alpha - r(\alpha + \beta) = 0$$

$$\Rightarrow \quad r = \frac{n\alpha}{\alpha + \beta}$$

7. TO FIND A TERM FROM THE END IN THE EXPANSION OF $(x+a)^n$:

It can be easily seen that in the expansion of $(x+a)^n$.

$(r+1)^{th}$ term from end = $(n-r+1)^{th}$ term from beginning.

i.e. $T_{r+1}(E) = T_{n-r+1}$ (B)

$\therefore \quad T_r(E) = T_{n-r+2}$ (B)

8. BINOMIAL COEFFICIENTS & THEIR PROPERTIES :

In the expansion of $(1+x)^n$; i.e. $(1+x)^n = {}^nC_0 + {}^nC_1 x + \ldots\ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_n$ of various powers of x, are called binomial coefficients and they are written as

$$C_0, C_1, C_2, \ldots\ldots, C_n$$

Hence

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_r x^r + \ldots\ldots + C_n x^n \quad ..(1)$$

Where $C_0 = 1$, $C_1 = n$, $C_2 = \frac{n(n-1)}{2!}$

$$C_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}, \quad C_n = 1$$

Now, we shall obtain some important expressions involving binomial coefficients-

(a) Sum of Coefficient : putting x = 1 in (1), we get

$$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n \quad \ldots(2)$$

(b) Sum of coefficients with alternate signs : putting x = –1 in(1)

We get

$$C_0 - C_1 + C_2 - C_3 + \ldots\ldots = 0 \quad \ldots(3)$$

(c) Sum of coefficients of even and odd terms: from (3), we have

$$C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots \quad \ldots(4)$$

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$$\Rightarrow \quad C_0 + C_2 + \ldots\ldots = C_1 + C_3 + \ldots\ldots = 2^{n-1}$$

9. APPLICATIONS OF BINOMIAL THEOREM

(a) With the help of binomial theorem, we can find out the value of sq. root, cube root and 4th root etc. of the given number upto any decimal places.

(b) To find the sum of Infinite series :

We can compare the given infinite series with the expansion of $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ and by finding the value of x and n and putting in $(1 + x)^n$ the sum of series is determined.

SET AND RELATIONS

Ex.1 Find the first four terms of the expansion of $\left(ax-\frac{1}{bx^2}\right)^5$

Sol. $\left(ax-\frac{1}{bx^2}\right)^5$

$= {}^5C_0\ (ax)^5 + {}^5C_1\ (ax)^4 \left(-\frac{1}{bx^2}\right) +$

${}^5C_2\ (ax)^3\left(-\frac{1}{bx^2}\right)^2 + {}^5C_3 (ax)^2\left(-\frac{1}{bx^2}\right)^3 + \ldots$

$= a^5x^5 - 5\ \frac{a^4}{b} x^2 + 10\ \frac{a^3}{b^2x} -$

$10\ \frac{a^2}{b^3x^4} + \ldots\ldots..$

Ex.2 Find the sixth term in the expansion of $\left(3x^2-\frac{1}{2x}\right)^8$

Sol. $T_6 = {}^8C_5\ (3x^2)^3 \left(-\frac{1}{2x}\right)^5$

$= 56 \times (27x^6) \times \left(-\frac{1}{32x^5}\right) = -\frac{189}{4}x$

Ex.3 If in the expansion of $(1+y)^n$, the coefficient of 5^{th}, 6^{th} and 7^{th} terms are in A.P., then find n

Sol. As given ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in AP.

$\Rightarrow {}^nC_4 + {}^nC_6 = 2.\ {}^nC_5$

$\Rightarrow \frac{n!}{(n-4)!\,4!} + \frac{n!}{(n-6)!\,6!} = 2\ \frac{n!}{(n-5)!\,5!}$

$\Rightarrow$ 30 + (n – 5) (n – 4)

= 2.6 (n – 4)

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow$ (n – 7) (n – 14) = 0

$\therefore$ n = 7, 14

Ex.4 Find the sum of the coefficient of the terms of the expansion of polynomial $(1+x-3x^2)^{2143}$

Sol. We get the sum of the coefficients of terms by putting x = 1 in the polynomial

$(1+x-3x^2)^{2143}$

$\therefore (1+1-3)^{2143} = (-1)^{2143}$

$= (-1)^{2142}.\ (-1)$

$= [(-1)^2]^{1021}.\ (-1)$

$= 1 \times -1 = -1.$

Ex.5 Find the middle term of the expansion $\left(x-\frac{2}{x}\right)^8$

Sol. Since (n = 8) is even then there is only one middle term i.e. $T_{\frac{8+2}{2}} = T_5$

$\therefore T_5 = {}^8C_4(x)^4(-2/x)^4$

$= {}^8C_4.\ (-2)^4 = 16.{}^8C_4$

$= 1120$

Ex.6 Find the term independent from x in the expansion of $\left(\sqrt{x}-\frac{3}{x^2}\right)^{10}$

Sol. Since we require term independent from x

$\therefore n\alpha - r\ (\alpha+\beta) = 0$

$\Rightarrow 10 \times \frac{1}{2} - r\left(\frac{1}{2}+2\right) = 0$

$\Rightarrow$ r = 2 i.e. 3^{rd} term.

$\therefore T_3 = {}^{10}C_2\ (\sqrt{x})^8\ (-3/x^2)^2$

$= {}^{10}C_2\ .(-3)^2.x^0$

$= \frac{10.9}{2.1}\ .\ 9 = 405$

Ex.7 If in the expansion of $\left(x^3-\frac{3}{x^2}\right)^{15}$ the r^{th} term is independent of x, then find r

Sol. If r^{th} term is independent of x, then by the formula

15 × 3 – (r–1) (3+ 2) = 0

$\Rightarrow$ r – 1 = 9 $\Rightarrow$ r = 10.

Ex.8 If the coefficients of rth and $(r+1)^{th}$ terms in the expansion of $(3+7x)^{29}$ are equal, then find

Sol. We have

$T_{r+1} = {}^{29}C_r 3^{29-r}\ (7x)^r$

$= ({}^{29}C_r.3^{29-r}.7^r)\ x^r$

$\therefore$ a_r = coefficient of (r +1)th term

$= {}^{29}C_r . 3^{29-r} . 7^r$

Now,

$a_r = a_{r-1}$

$\Rightarrow {}^{29}C_r . 3^{29-r}.7^r$

$= {}^{29}C_{r-1} . 3^{30-r} . 7^{r-1}$

$$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7}$$

$$\Rightarrow \frac{30-r}{r} = \frac{3}{7} \Rightarrow r = 21.$$

Ex.9 If the fourth term in the expansion of $(px + 1/x)^n$ is 5/2 then find the value of n and p

Sol. The fourth term in expansion of $(px + 1/x)^n$

$T_4 = {}^nC_3 . (px)^{n-3} (1/x)^3 = 5/2.$

$\Rightarrow ({}^nC_3.p^{n-3}) . x^{n-6} = 5/2. x^0$

Compairing the coefficient of x and constant term $n - 6 = 0 \Rightarrow n = 6$

and ${}^nC_3 (p)^{n-3} = 5/2$

putting n = 6 in it

$6C_3 p^3 = 5/2$

$\Rightarrow p^3 = 1/8$

$\Rightarrow p^3 = (1/2)^3$

$\Rightarrow p = 1/2$

Ex.10 Find the coefficient of x^4 in the expansion of $(1+ x + x^2+ x^3)^n$

Sol. Exp. $= (1+ x)^n (1+ x^2)^n$

$= (1+ {}^nC_1x+ {}^nC_2x^2+{}^nC_3x^3+ {}^nC_4x^4 + ...+ x^n)$

$(1+ {}^nC_1x^2 + {}^nC_2x^4 + ...+ x^{2n})$

$\therefore$ Coefficient of x^4

$= {}^nC_4+ {}^nC_2.{}^nC_1+ {}^nC_2$

Ex.11 If x^m occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{2n}$, find the coefficient of x^m

Sol. The general term in the expansion of the given expression is

$$T_{r+1} = {}^{2n}C_r\, x^{2n-r} \left(\frac{1}{x^2}\right)^r$$

$= {}^{2n}C_r\, x^{2n-3r}$

For the coefficient of x^m, we must have

$$2n-3r = m \Rightarrow r = \frac{2n-m}{3}$$

So, coefficient of x^m

$$= {}^{2n}C_{\frac{2n-m}{3}} = \frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$$

Ex.12 Find the sum of the rational terms in the expansion of $\left(\sqrt{2}+3^{1/5}\right)^{10}$

Sol. Here $T_{r+1} = {}^{10}C_r (\sqrt{2})^{10-r} (3^{1/5})^r$, where r = 0, 1, 2, ...,10.

We observe that in general term T_{r+1} powers of 2 and 3 are $\frac{1}{2}(10-r)$ and $\frac{1}{5}$ r respectively and $0 \le r \le 10$. So both these powers will be integers together only when r = 0 or 10

$\therefore$ sum of required terms

$= T_1 + T_{11}$

$= {}^{10}C_0(\sqrt{2})^{10} + {}^{10}C_{10} (3^{1/5})^{10}$

$= 32 + 9 = 41$

EXERCISE

Using binomial theorem, expand each of the following :

Q.1 $(3x + 2y)^4$ **Q.2** $(2x - 3y)^4$

Q.3 $\left(x + \frac{1}{y}\right)^{11}$ **Q.4** $(\sqrt{x} + \sqrt{y})^{10}$

Q.5 Evaluate : $(\sqrt{2} + 1)^6 + (\sqrt{2} + 1)^6$

Q.6 Evaluate : $(2 + \sqrt{3})^7 + (2 - \sqrt{3})^7$

Q.7 Evaluate : $(1001)^5$

Q.8 (i) Find the 7^{th} term in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x}\right)^8$.

(ii) Find the 10^{th} term in the expansion of $\left(\frac{a}{b} - \frac{2b}{a^2}\right)^{12}$

(iii) Find the 16^{th} term in the expansion of $(\sqrt{x} - \sqrt{y})^{17}$

Q.9 Write the general term in the expansion of $(x^2 - y)^6$.

Q.10 Find the 5^{th} term from the end in the expansion of $\left(x - \frac{1}{x}\right)^{12}$.

Q.11 Find the coefficients of

(i) x^2 in the expansion of $\left(x^2 + \frac{1}{x}\right)^{11}$;

(ii) x^2 in the expansion of $\left(3x - \frac{1}{x}\right)^6$;

Q.12 Find the middle term in the expansion of;

(i) $(3 + x)^6$ (ii) $\left(1 - \frac{x^2}{2}\right)^{14}$

Q.13 Find the two middle terms in the expansion of

(i) $(x^2 + a^2)^5$ (ii) $\left(3x - \frac{2}{x^2}\right)^{15}$

Q.14 Show that the term containing x^3 does not exist in the expansion of $\left(3x - \frac{1}{2x}\right)^8$.

Q.15 Find the term independent of x in the expansion of;

(i) $\left(x + \frac{1}{x}\right)^{10}$ (ii) $\left(\sqrt{x} + \frac{1}{3x^2}\right)^{10}$

Q.16 Show that the ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ and the term independent of x in $\left(x - \frac{2}{x}\right)^{10}$ is (1 : 32).

Q.17 Simplify the following expressions:

$\left(x + \frac{1}{x}\right)^6 - \left(x - \frac{1}{x}\right)^6$

Q.18 Find the coefficient of x^5 in the expansion of $(1 + x)^3 (1 - x)^6$.

Q.19 If the coefficients of 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(1 + x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

Q.20 In the expansion of $(1 + x)^n$, the three successive coefficients are 462, 330 and 165 respectively. Find the values of n and r.

Q.21 The first three terms in the binomial expansion of $(x + y)^n$ are 1, 56 and 1372 respectively. Find the values of x and y.

Q.22 The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42. Find n.

Q.23 In the binomial expansion of $(a + b)^n$, the coefficients of 4^{th} and 13^{th} terms are equal to each other. Find n.

Q.24 In the expansion of $(1 + x)^{m+n}$, where m and n are positive integers , prove that the coefficients of x^m and x^n are equal.

ANSWER KEY

1. $81 x^4 + 216 x^3y + 216 x^2y^2 + 96xy^3 + 16y^4$

2. $16x^4 - 96x^3y + 216x^2y^3 - 216xy^2 + 81y^4$

3. $\left[x^{11} + \frac{11x^{10}}{y} + \frac{55x^2}{y^2} + \frac{165x^8}{y^3} + \frac{330x^7}{y^4} + \frac{462x^6}{y^5} + \frac{462x^5}{y^6} + \frac{330x^4}{y^7} + \frac{165x^3}{y^8} + \frac{55x^2}{y^9} + \frac{11x}{y^{10}} + \frac{1}{y^{11}}\right]$

4. $x^5 + 10x^{9/2}y^{1/2} + 45x^4y + 120x^{7/2}y^{3/2} + 210x^3y^2 + 252x^{5/2}y^{5/2} + 210x^2y^3 + 120x^{3/2}y^{7/2} + 45xy^4 + 10x^{1/2}y^{9/2} + y^5$

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8. (i) $\frac{4375}{x^4}$ (ii) $\frac{(-366080)b^5}{a^{14}}$ (iii) $-136\ xy^{15/2}$

9. $(-1)^r\ {}^6C_r\ x^{12-2r}\ y^r$ **10.** $495x^4$ **11.** (i) 462 (ii) 1215

12. (i) $540\ x^3$ (ii) $\frac{-429}{16}\ x^{14}$

13. (i) $10x^6a^4$ (ii) $\frac{-6435 \times 3^8 \times 2^7}{x^6}$ and $\frac{6435 \times 3^7 \times 2^8}{x^9}$

15. (i) 252 (ii) 5 **17.** $12\left(x^4 + \frac{1}{x^4} + \frac{10}{3}\right)$ **18.** –6

20. n = 11 and r = 7 **21.** x = 1, y = 7

22. 55 **23.** 15