Work, Energy and Power

INTRODUCTION

The term ‘work’ as understood in everyday life has a different meaning in scientific sense. If a coolie is carrying a load on his head and waiting for the arrival of the train, he is not performing any work in the scientific sense. In the present study, we shall have a look into the scientific aspect of this most commonly used term i.e., work.

WORK DONE BY CONSTANT FORCE

The physical meaning of the term work is entirely different from the meaning attached to it in everyday life. In everyday life, the term ‘work’ is considered to be synonym of ‘labour’, ‘toil’, effort’ etc. In physics, there is a specific way of defining work.

Work is said to be done by a force when the force produces a displacement in the body on which it acts in any direction except perpendicular to the direction of the force.

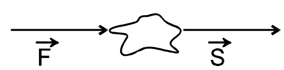
For work to be done, following two conditions must be fulfilled.

(i) A force must be ap plied.

(ii) The applied force must produce a displacement in any direction except perpendicular to the direction of the force.

Suppose a force $\vec{F}$ is applied on a body in such a way that the body suffers a displacement $\vec{S}$ in the direction of the force. Then the work done is given by

W = FS



However, the displacement does not always take place in the direction of the force. Suppose a constant force $\vec{F}$, applied on a body, produces a displacement $\vec{S}$ in the body in such a way that $\vec{S}$ is inclined to $\vec{F}$ at an angle θ. Now the work done will be given by the dot product of force and displacement.

$$\mathbf{W} = \vec{F} \cdot \vec{S}$$

Since work is the dot product of two vectors therefore it is a scalar quantity.

$$W = FS \cos\theta$$

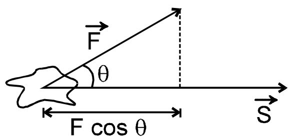
or $W = (F\cos\theta)S$

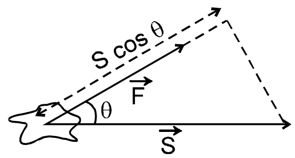
∴ W = component of force in the direction of displacement × magnitudes of displacement.

So work is the product of the component of force in the direction of displacement and the magnitude of the displacement.

Also, $W = F(S\cos\theta)$

or work is product of the component of displacement in the direction of the force and the magnitude of the displacement.





<u>**Special Cases :**</u>

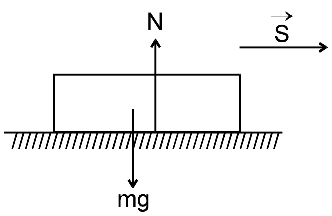
Case (i)

When θ = 90º, then W = FS cos 90º = 0

So, work done by a force is zero if the body is displaced in a direction perpendicular to the direction of the force.

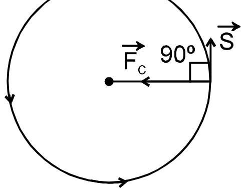
Examples :

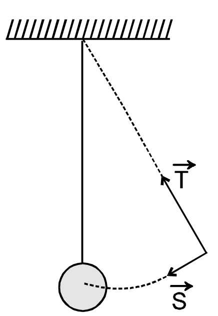
1. Consider a body sliding over a horizontal surface. The work done by the force of gravity and the reaction of the surface will be zero. This is because both the force of gravity and the reaction act normally to the displacement.



The same argument can be applied to a man carrying a load on his head and walking on a railway platform.

2. Consider a body moving in a circle with constant speed. At every point of the circular path, the centripetal force and the displacement are mutually perpendicular (Figure). So, the work done by the centripetal force is zero. The same argument can be applied to a satellite moving in a circular orbit. In this case, the gravitational force is always perpendicular to displacement. So, work done by gravitational force is zero.





3. The tension in the string of a simple pendulum is always perpendicular to displacement. (Figure). So, work done by the tension is zero.

Case (ii) :

When S = 0, then W = 0.

So, work done by a force is zero if the body suffers no displacement on the application of a force.

Example :

A person carrying a load on his head and standing at a given place does no work.

Case (iii) :

When 0º ≤ θ < 90º [Figure], then cos θ is positive. Therefore.

W (= FS cos θ) is positive.

Work done by a force is said to be positive if the applied force has a component in the direction of the displacement.

Examples :

1. When a horse pulls a cart, the applied force and the displacement are in the same direction. So, work done by the horse is positive.

2. When a load is lifted, the lifting force and the displacement act in the same direction. So, work done by the lifting force is positive.

3. When a spring is stretched, both the stretching force and the displacement act in the same direction. So, work done by the stretching force is positive.

Case (iv) :

When 90º < θ ≤ 180º (Figure), then cosθ is negative. Therefore W (= FS cos θ) is negative.

Work done by a force is said to be negative if the applied force has component in a direction opposite to that of the displacement.

Examples :

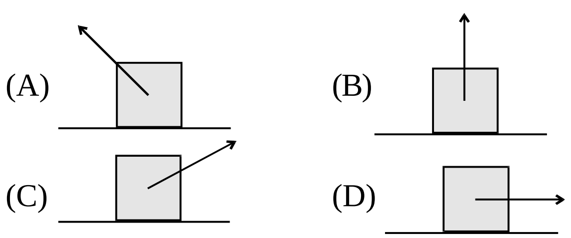
1. When brakes are applied to a moving vehicle, the work done by the braking force is negative. This is because the braking force and the displacement act in opposite directions.

2. When a body is dragged along a rough surface, the work done by the frictional force is negative. This is because the frictional force acts in a direction opposite to that of the displacement.

3. When a body is lifted, the work done by the gravitational force is negative. This is because the gravitational force acts vertically downwards while the displacement is in the vertically upwards direction.

Solved Examples

Ex.1 Figure shows four situations in which a force acts on a box while the box slides rightward a distance d across a frictionless floor. The magnitudes of the forces are identical, their orientations are as shown. Rank the situations according to the work done on the box during the displacement, from most positive to most negative.



Ans. D, C, B, A

Explanation :

In (D), $\theta = 0^o$, $\cos\theta = 1$ (maximum value). So, work done is maximum.

In (C), $\theta = 90^o$, $\cos\theta$ is positive. Therefore, W is positive.

In (B), $\theta = 90^o$, $\cos\theta$ is zero. W is zero.

In (A), θ is obtuse, $\cos\theta$ is negative. W is negative.

WORK DONE BY MULTIPLE FORCES

If several forces act on a particle, then we can replace $\vec{F}$ in equation $W = \vec{F}\,.\,\vec{S}$ by the net force $\Sigma\vec{F}$ where

$\Sigma\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \quad .....$

$$\therefore\ W = [\Sigma\vec{F}]\,.\,\vec{S} \qquad ...(i)$$

This gives the work done by the net force during a displacement $\vec{S}$ of the particle.

We can rewrite equation (i) as :

$$W = \vec{F}_1\,.\,\vec{S} + \vec{F}_2\,.\,\vec{S} + \vec{F}_3\,.\,\vec{S} + ...$$

or $\quad W = W_1 + W_2 + W_3 + ...$

So, the work done on the particle is the sum of the individual works done by all the forces acting on the particle.

Important points about work:

1. Work is defined for an interval or displacement. There is no term like instantaneous work similar to instantaneous velocity.
2. For a particular displacement, work done by a force is independent of type of motion i.e. whether it moves with constant velocity, constant acceleration or retardation etc.
3. For a particular displacement work is independent of time. Work will be same for same displacement whether the time taken is small or large.
4. When several forces act, work done by a force for a particular displacement is independent of other forces.
5. A force is independent from reference frame. Its displacement depends on frame so work done by a force is frame dependent therefore work done by a force can be different in different reference frame.
6. Effect of work is change in kinetic energy of the particle or system.
7. Work is done by the source or agent that applies the force.

UNITS OF WORK

1. Unit of work :

I. In cgs system, the unit of work is erg.

One erg of work is said to be done when a force of one dyne displaces a body through one centimetre in its own direction.

$$\therefore\ 1\text{ erg} = 1\text{ dyne} \times 1\text{ cm} = 1\text{g cm s}^{-2} \times 1\text{ cm}$$
$$= 1\text{ g cm}^2\text{ s}^{-2}$$

Note, Erg is also called dyne centimetre.

II. In SI i.e., International System of units, the unit of work is joule (abbreviated as J). It is named after the famous British physicist James Personal Joule (1818 – 1869).

One joule of work is said to be done when a force of one newton displaces a body through one metre in its own direction.

$$1\text{ joule} = 1\text{ newton} \times 1\text{ metre} = 1\text{ kg} \times 1\text{ m/s}^2 \times 1\text{ m}$$
$$= 1\text{kg m}^2\text{ s}^{-2}$$

Note Another name for joule is newton metre.

Relation between joule and erg

1 joule = 1 newton × 1 metre

1 joule = 10^5 dyne × 10^2 cm = 10^7 dyne cm

1 joule = 10^7 erg

1 erg = 10^{-7} joule

DIMENSIONS OF WORK

[Work] = [Force] [Distance]

= $[MLT^{-2}]$ [L]

= $[ML^2T^{-2}]$

Work has one dimension in mass, two dimensions in length and '–2' dimensions in time,

On the basis of dimensional formula, the unit of work is kg m^2 s^{-2}.

Note that 1 kg m^2 s^{-2} = (1 kg m s^{-2}) m = 1 N m = 1 J.

Solved Examples

Ex.2 A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposite to the motion.

(a) How much work does the road do on the cycle?

(b) How much work does the cycle do on the road?

Sol. (a) The work done on the cycle by the road is the work done by the frictional force exerted by the road on the cycle.

Now, $W = \vec{F} \cdot \vec{S} = FS \cos 180^o$

or $W = -FS$

or $W = -200 \text{ N} \times 10 \text{ m}$

or W = **– 2000 J**

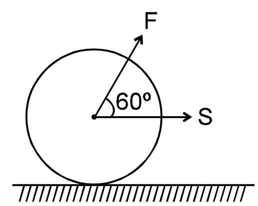
It is this negative work which brings the cycle to rest. This is clearly in accordance with work-energy theorem.

(b) The displacement of the road is zero. So, work done by the cycle on the road is zero.

(Using Newton's third law of motion, an equal and opposite force acts on the road due to the cycle. The magnitude of this force is 200 N.)

Ex.3 A gardener moves a lawn roller through a distance of 100 metre with a force of 50 newton. Calculate his wages if he is to be paid 10 paise for doing 25 joule of work. It is given that the applied force is inclined at 60º to the direction of motion.

Sol. Force, F = 50 N ; Distance, S = 100 m ; Angle, $\theta = 60^o$



$W = FS \cos\theta = 50 \times 100 \times \cos 60^o$ joule

$W = 50 \times 100 \times \frac{1}{2}$ J = 2500 J

$[\because \cos 60^o = \frac{1}{2}]$

Wages = $\frac{2500}{25} \times 10$ paise = **10 rupees**

Ex.4 Calculate the work done in raising a stone of mass 5 kg and specific gravity 3 lying at the bed of a lake through a height of 5 metre.

Sol. When a body is immersed in water, its apparent weight is decreased in accordance with the Archimedes' principle.

Loss of weight in water = $\frac{\text{weight in air}}{\text{specific gravity}} = \frac{5 \text{ kg wt}}{3}$

$\therefore$ Weight of stone in water = $\left(5 - \frac{5}{3}\right)$ kg wt = $\frac{10}{3}$ kg wt

Force, F = $\frac{10}{3}$ kg wt = $\frac{10}{3} \times 9.8$ N = $\frac{98}{3}$ N

Work done, W = $\frac{98}{3} \times 5$ J = 163.3 J.

WORK IN TERMS OF RECTANGULAR COMPONENTS

In terms of rectangular components, $\vec{F}$ and $\vec{S}$ may be written as :

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $\vec{S} = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$

$\vec{F} \cdot \vec{S} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (S_x\hat{i} + S_y\hat{j} + S_z\hat{k})$

$= F_xS_x(\hat{i}\cdot\hat{i}) + F_xS_y(\hat{i}\cdot\hat{j}) + F_xS_z(\hat{i}\cdot\hat{k}) + F_yS_x(\hat{j}\cdot\hat{i}) + F_yS_y(\hat{j}\cdot\hat{j}) + F_yS_z(\hat{j}\cdot\hat{k}) + F_zS_x(\hat{k}\cdot\hat{i}) + F_zS_y(\hat{k}\cdot\hat{j}) + F_zS_z(\hat{k}\cdot\hat{k})$

But $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \therefore [\vec{F}\cdot\vec{S} = F_xS_x + F_yS_y + F_zS_z]$

Solved Examples

Ex.5 A body constrained to move along the z-axis of a coordinate system is subjected to a constant force $\vec{F}$ given by

$\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ N

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x, y and z-axis of the system respectively. What is the work done by this force in moving the body a distance of 4m along the z-axis?

Sol. Since the body is displaced 4 m along z-axis only,

$\therefore \quad \vec{S} = 0\hat{i} + 0\hat{j} + 4\hat{k}$

Also, $\quad \vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$

Work done, $W = \vec{F}.\vec{S} = (-\hat{i} + 2\hat{j} + 3\hat{k}).(0\hat{i} + 0\hat{j} + 4\hat{k})$

$= 12(\hat{k} \cdot \hat{k})$ joule = **12 joule.**

Ex. 6 An object is displaced from point A(2m, 3m, 4m) to a point B(1m, 2m, 3m) under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} + 4\hat{k})N$. Find the work done by this force in this process.

Sol.
$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \bullet d\vec{r}$$

$$= \int_{(2m,3m,4m)}^{(1m,2m,3m)} (2\hat{i} + 3\hat{j} + 4\hat{k}) \bullet (dx\hat{i} + dy\hat{i} + dz\hat{k})$$

$$= [2x + 3y + 4z]_{(2m,3m4m)}^{(1m,2m,3m)}$$

$= -9$ J **Ans.**

WORK DONE BY A VARIABLE FORCE

When the magnitude and direction of a force vary in three dimensions, it can be expressed as a function of the position. For a variable force work is calculated for infinitely small displacement and for this displacement force is assumed to be constant

$$\mathbf{dW} = \vec{F}.d\vec{s}$$

The total work done will be sum of infinitely small work

$$W_{A\to B} = \int_A^B \vec{F}.d\vec{s} = \int_A^B (\vec{F}\cos\theta)d\vec{s}$$

In terms of rectangular components,

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \quad d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$W_{A\to B} = \int_{x_A}^{x_B} F_x\,dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Solved Examples

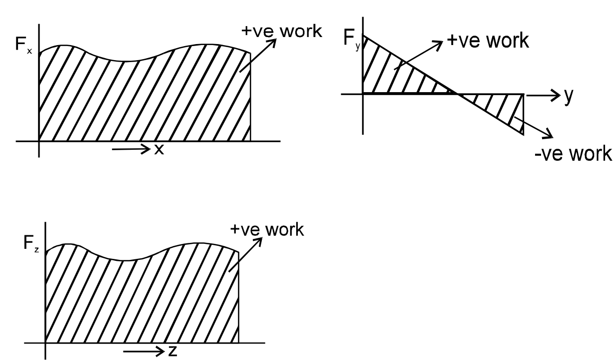
Ex. 7 An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})m$ to $\vec{r}_2 = (4\hat{j} + 6\hat{k})m$ under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})N$. Find the work done by this force.

Sol. $W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \bullet d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \bullet (dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2dx + 2ydy) = [x^3 + y^2]_{(2,\,3)}^{(4,\,6)} = 83 \text{ J}$$

AREA UNDER FORCE DISPLACEMENT CURVE

Graphically area under the force-displacement is the work done



The work done can be positive or negative as per the area above the x-axis or below the x-axis respectively.

Solved Examples

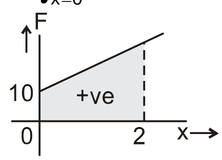
Ex.8 A force F = 0.5x + 10 acts on a particle. Here F is in newton and x is in metre. Calculate the work done by the force during the displacement of the particle from x = 0 to x = 2 metre.

Sol. Small amount of work done dW in giving a small displacement $\vec{dx}$ is given by $dW = \vec{F} \cdot \vec{dx}$

or dW = Fdx cos 0º or dW = Fdx [$\because$ cos 0º = 1]

$$\text{Total work done, } W = \int_{x=0}^{x=2} F dx = \int_{x=0}^{x=2} (0.5x + 10) dx$$

$$= \int_{x=0}^{x=2} 0.5x\, dx + \int_{x=0}^{x=2} 10\, dx = 0.5 \left|\frac{x^2}{2}\right|_{x=0}^{x=2} + 10 \left|x\right|_{x=0}^{x=2}$$

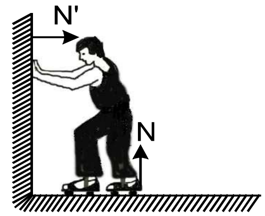


$$= \frac{0.5}{2}[2^2 - 0^2] + 10[2 - 0] = (1 + 20) = 21 \text{ J}$$

INTERNAL WORK

Suppose that a man sets himself in motion backward by pushing against a wall. The forces acting on the man are his weight 'W', the upward force N exerted by the ground and the horizontal force N´ exerted by the wall. The works of 'W' and of N are zero because they are perpendicular to the motion. The force N´ is the unbalanced horizontal force that imparts to the system a horizontal acceleration. The work of N´, however, is zero because there is no motion of its point of application. We are therefore confronted with a curious situation in which a force is responsible for acceleration, but its work, being zero, is not equal to the increase in kinetic energy of the system.

The new feature in this situation is that the man is a composite system with several parts that can move in relation to each other and thus can do work on each other, even in the absence of any interaction with externally applied forces. Such work is called internal work. Although internal forces play no role in acceleration of the composite system, their points of application can move so that work is done; thus the man's kinetic energy can change even though the external forces do no work.



"Basic concept of work lies in following lines"

Draw the force at proper point where it acts that give proper importance to the point of application of force.

Think independently for displacement of point of application of force, Instead of relating the displacement of applicant point with force relate it with the observer or reference frame in which work is calculated.

$$W = (\textit{Force} \text{ vector}) \times \begin{pmatrix} \text{displacement vector of point of} \\ \text{application of} \\ \text{force as seen by observer} \end{pmatrix}$$

ENERGY

Definition: Energy is defined as internal capacity of doing work. When we say that a body has energy we mean that it can do work.

Energy appears in many forms such as mechanical, electrical, chemical, thermal (heat), optical (light), acoustical (sound), molecular, atomic, nuclear etc., and can change from one form to the other.

KINETIC ENERGY

Definition:

Kinetic energy is the internal capacity of doing work of the object by virtue of its motion.

Kinetic energy is a scalar property that is associated with state of motion of an object. An aeroplane in straight and level flight has kinetic energy of translation and a rotating wheel on a machine has kinetic energy of rotation. If a particle of mass m is moving with speed 'v' much less than the speed of the light than the kinetic energy 'K' is given by

$$K = \frac{1}{2}mv^2$$

Important Points for K.E.

1. As mass m and v^2 ($\vec{v}.\vec{v}$) are always positive, kinetic energy is always positive scalar i.e, kinetic energy can never be negative.

2. The kinetic energy depends on the frame of reference,

$$K = \frac{p^2}{2m} \text{ and } P = \sqrt{2mK} \text{ ; P = linear momentum}$$

The speed v may be acquired by the body in any manner. The kinetic energy of a group of particles or bodies is the sum of the kinetic energies of the individual particles. Consider a system consisting of n particles of masses $m_1, m_2, ..., m_n$. Let $\vec{v_1}, \vec{v_2}$, ..., $\vec{v_n}$ be their respective velocities. Then, the total kinetic energy E_k of the system is given by

$$E_k = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... + \frac{1}{2}m_nv_n^2$$

If n is measured in gram and v in cm s^{-1}, then the kinetic energy is measured in erg. If m is measured in kilogram and v in m s^{-1}, then the kinetic energy is measured in joule. It may be noted that the units of kinetic energy are the same as those of work. Infact, this is true of all forms of energy since they are inter-convertible.

Typical kinetic energies (K)

S.No.	Object	Mass (kg)	Speed (m s^{-1})	K(J)
1	Air molecule	$\approx 10^{-26}$	500	$\approx 10^{-21}$
2	Rain drop at terminal speed	3.5×10^{-5}	9	1.4×10^{-3}
3	Stone dropped from 10 m	1	14	10^2
4	Bullet	5×10^{-5}	200	10^3
5	Running athlete	70	10	3.5×10^3
6	Car	2000	25	6.3×10^5

RELATION BETWEEN MOMENTUM AND KINETIC ENERGY

Consider a body of mass m moving with velocity v. Linear momentum of the body, p = mv

Kinetic energy of the body, $E_k = \frac{1}{2}mv^2$

$$E_k = \frac{1}{2m}(m^2v^2) \quad \text{or} \quad E_k = \frac{p^2}{2m} \text{ or } p = \sqrt{2mE_k}$$

Solved Examples

Ex.9 The kinetic energy of a body is increased by 21%. What is the percentage increase in the magnitude of linear momentum of the body?

Sol. $E_{k2} = \frac{121}{100}E_{k1}$ or $\frac{1}{2}mv_2^2 = \frac{121}{100}\frac{1}{2}mv_1^2$

or $v_2 = \frac{11}{10}v_1$ or $mv_2 = \frac{11}{10}mv_1$

or $p_2 = \frac{11}{10}p_1$ or $\frac{p_2}{p_1} - 1 = \frac{11}{10} - 1 = \frac{1}{10}$

or $\frac{p_2 - p_1}{p_1} \times 100 = \frac{1}{10} \times 100 = 10$

So, the percentage increase in the magnitude of linear momentum is 10%.

Ex.10 The linear momentum of a body is increased by 10%. What is the percentage change in its kinetic energy?

Ans. Percentage increase in kinetic energy = 21%]

$$\left[\text{Hint.} mv_2 = \frac{110}{100}mv_1, v_2 = \frac{11}{10}v_1, \frac{E_1}{E_1} = \left(\frac{11}{10}\right)^2 = \frac{121}{100}\right.$$

$$\left.\text{Percentage increase in kinetic energy} = \frac{E_1 - E_1}{E_1} \times 100\right]$$

= 21%

POTENTIAL ENERGY

Definition:

Potential energy is the internal capacity of doing work of a system by virtue of its configuration.

In case of conservative force (field) potential energy is equal to negative of work done by the conservative force in shifting the body from some reference position to given position.

Therefore, in case of conservative force

$$\int_{U_1}^{U_2} dU = -\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \text{ i.e., } U_2 - U_1 = -\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = -W$$

Whenever and wherever possible, we take the reference point at ∞ and assume potential energy to be zero there, i.e., If we take $r_1 = \infty$ and $U_1 = 0$ then

$$U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = -W$$

Important Points for P.E. :

1. Potential energy can be defined only for conservative forces. It has no relevance for non-conservative forces.
2. Potential energy can be positive or negative, depending upon choice of frame of reference.
3. Potential energy depends on frame of reference but change in potential energy is independent of reference frame.
4. Potential energy should be considered to be a property of the entire system, rather than assigning it to any specific particle.
5. It is a function of position and does not depend on the path.

Types of Potential Energy

(a) Elastic Potential Energy:

It is the energy associated with state of compression or expansion of an elastic (spring like) object and is given by:

$$U = \frac{1}{2}ky^2$$

where k is force constant and 'y' is the stretch or compression. Elastic potential energy is always positive.

(b) Electric Potential Energy:

It is the energy associated with charged particles that interact via electric force. For two point charges q_1 and q_2 separated by a distance 'r',

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}$$

As charge can be positive or negative, therefore electric potential energy can also be positive or negative.

(c) Gravitational Potential Energy:

It is due to gravitational force. For two particles of masses m_1 and m_2 separated by a distance 'r', it is given by:

$$U = -G\frac{m_1m_2}{r}$$

which for a body of mass 'm' at height 'h' relative to surface of the earth reduces to U = mgh

Gravitational potential energy can be positive or negative.

MECHANICAL ENERGY

Definition

Mechanical energy 'E' of an object or a system is defined as the sum of kinetic energy 'K' and potential energy 'U', i.e.,

E = K + U

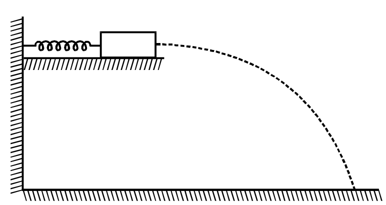
Important Points for M.E.:

1. It is a scalar quantity having dimensions $[ML^2T^{-2}]$ and SI units joule.
2. It depends on frame of reference.
3. A body can have mechanical energy without having either kinetic energy or potential energy. However, if both kinetic and potential energies are zero, mechanical energy will be zero. The converse may or may not be true, i.e., if E = 0 either both PE and KE are zero or PE may be negative and KE may be positive such that KE + PE = 0.
4. As mechanical energy E = K + U, i.e., E - U = K. Now as K is always positive, E - U $\geq$ 0,i.e., for existence of a particle in the field, E $\geq$ U.
5. As mechanical energy E = K + U and K is always positive, so, if 'U' is positive 'E' will be positive. However, if potential energy U is negative, 'E' will be positive if K > |U| and E will be negative if K < |U|

   i.e., mechanical energy of a body or system can be negative, and negative mechanical energy means that potential energy is negative and in magnitude it is more than kinetic energy. Such a state is called bound state, e.g., electron in an atom or a satellite moving around a planet are in bound state.

Solved Examples

Ex.11 A small block of mass 100 g is pressed against a horizontal spring fixed at one end to compress the spring through 5.0 cm (figure). The spring constant is 100 N/m. When released, the block moves horizontally till it leaves the spring. Where will it hit the ground 2 m below the spring ?



Sol. When block released, the block moves horizontally with speed V till it leaves the spring.

By energy conservation $\frac{1}{2}kx^2 = \frac{1}{2}mv^2$

$$V^2 = \frac{kx^2}{m} \quad \Rightarrow \quad V = \sqrt{\frac{kx^2}{m}}$$

Time of flight $t = \sqrt{\frac{2H}{g}}$

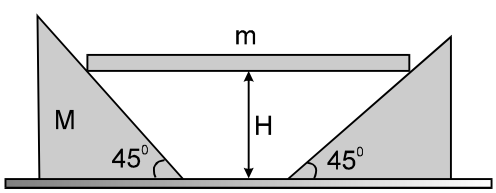
So. horizontal distance travelled from the free end of the spring is V × t

$$= \sqrt{\frac{kx^2}{m}} \times \sqrt{\frac{2H}{g}}$$

$$= \sqrt{\frac{100 \times (0.05)^2}{0.1}} \times \sqrt{\frac{2 \times 2}{10}} = 1\text{ m}$$

So, At a horizontal distance of 1 m from the free end of the spring.

Ex.12 A rigid body of mass m is held at a height H on two smooth wedges of mass M each of which are themselves at rest on a horizontal frictionless floor. On releasing the body it moves down pushing aside the wedges. The velocity of recede of the wedges from each other when rigid body is at a height h from the ground is



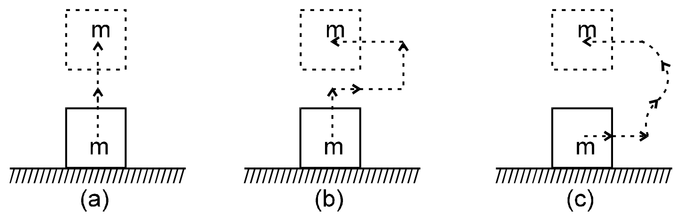
(A) $\sqrt{\frac{2mg(H-h)}{m+2M}}$ (B) $\sqrt{\frac{2mg(H-h)}{2m+M}}$

(C) $\sqrt{\frac{8mg(H-h)}{m+2M}}$ (D) $\sqrt{\frac{8mg(H-h)}{2m+M}}$

Ans. (C)

CONSERVATIVE FORCES

A force is said to be conservative if work done by or against the force in moving a body depends only on the initial and final positions of the body and not on the nature of path followed between the initial and final positions.



Consider a body of mass m being raised to a height h vertically upwards as show in above figure. The work done is *mgh*. Suppose we take the body along the path as in (b). The work done during horizontal motion is zero. Adding up the works done in the two vertical parts of the paths, we get the result *mgh* once again. Any arbitrary path like the one shown in (c) can be broken into elementary horizontal and vertical portions. Work done along the horizontal parts is zero. The work done along the vertical parts add up to *mgh*. Thus we conclude that the work done in raising a body against gravity is independent of the path taken. It only depends upon the initial and final positions of the body. We conclude from this discussion that the force of gravity is a conservative force.

Examples of Conservative forces.

(i) Gravitational force, not only due to the Earth but in its general form as given by the universal law of gravitation, is a conservative force.

(ii) Elastic force in a stretched or compressed spring is a conservative force.

(iii) Electrostatic force between two electric charges is a conservative force.

(iv) Magnetic force between two magnetic poles is a conservative forces.

Forces acting along the line joining the centres of two bodies are called central forces. Gravitational force and Electrostatic forces are two important examples of central forces. Central forces are conservative forces.

PROPERTIES OF CONSERVATIVE FORCES

(i) **Work done by or against a conservative force depends only on the initial and final positions of the body.**

(ii) Work done by or against a conservative force does no depend upon the nature of the path between initial and final positions of the body.

If the work done a by a force in moving a body from an initial location to a final location is independent of the path taken between the two points, then the force is conservative.

(iii) Work done by or against a conservative force in a round trip is zero.

If a body moves under the action of a force that does no total work during any round trip, then the force is conservative; otherwise it is non-conservative.

The concept of potential energy exists only in the case of conservative forces.

(iv) The work done by a conservative force is completely recoverable.

Complete recoverability is an important aspect of the work of a conservative force.

CONSERVATIVE FORCE & POTENTIAL ENERGY

$F_S = -\,\partial U/\partial s$,

i.e. the projection of the field force, the vector **F**, at a given point in the direction of the displacement dr equals the derivative of the potential energy U with respect to a given direction, taken with the opposite sign. The designation of a partial derivative $\partial/\partial s$ emphasizes the fact of deriving with respect to a definite direction.

So, having reversed the sign of the partial derivatives of the function U with respect to x, y, z, we obtain the projection F_x, F_y and F_z of the vector **F** on the unit vectors **i**, **j** and **k**. Hence, one can readily find the vector itself : $F = F_x\mathbf{i} + F_u\mathbf{j} + F_z\mathbf{k}$, or

$$F = -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}\right).$$

The quantity in parentheses is referred to as the scalar gradient of the function U and is denoted by grad U or ∇U. We shall use the second, more convenient, designation where ∇ ("nabla") signifies the symbolic vector or operator

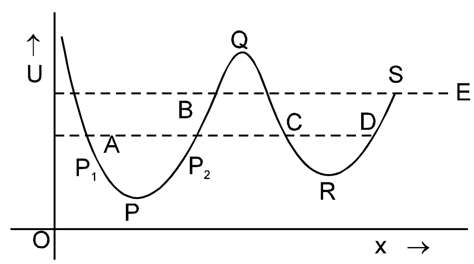
$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Potential Energy curve

(a) A graph plotted between the PE a particle and its displacement from the centre of force field is called PE curve.

(b) Using graph, we can predict the rate of motion of a particle at various positions.

(c) Force on the particle is $F_{(x)} = -\frac{dU}{dx}$



Case : I On increasing x, if U increases, force is in (–) ve x direction i.e. attraction force.

Case : II On increasing x, if U decreases, force is in (+) ve x-direction i.e. repulsion force.

Different positions of a particle :-

Position of equilibrium :

If net force acting on a body is zero, it is said to be in equilibrium. For equilibrium $\frac{dU}{dx} = 0$. Points P, Q R and S are the states of equilibrium positions.

Types of equilibrium :

(a) **Stable equilibrium :** When a particle is displaced slightly from a position and a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position.

Necessary conditions : $-\frac{dU}{dx} = 0$, and $\frac{d^2U}{dx^2} = +ve$

(b) Unstable Equilibrium : When a particle is displaced slightly from a position and force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium.

Condition : $-\frac{dU}{dx} = 0$ potential energy is maximum

i.e. $= \frac{d^2U}{dx^2} = -$ ve

(c) Neutral equilibrium : In the neutral equilibrium potential energy is constant. When a particle is displaced from its position it does not experience any force acting on it and continues to be in equilibrium in the displaced position. This is said to be neutral equilibrium.

Solved Examples

Ex.13 The potential energy between two atoms in a molecule is given by, $U_{(x)} = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants and x is the distance between the atoms. The system is in stable equilibrium when -

(A) $x = 0$ (B) $x = \frac{a}{2b}$

(C) $x = \left(\frac{2a}{b}\right)^{1/6}$ (D) $x = \left(\frac{11a}{5b}\right)$

Sol. (C)

Given that, $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$

We, know $F = -\frac{du}{dx}$

$= (-12)\, a\, x^{-13} - (-6b)\, x^{-7} = 0$

or $\frac{-6b}{x^7} = \frac{12a}{x^{13}}$

or $x^6 = 12a/6b = 2a/b$

or $x = \left(\frac{2a}{b}\right)^{1/6}$

NON-CONSERVATIVE FORCES

A force is said to be non-conservative if work done by or against the force in moving a body depends upon the path between the initial and final positions.

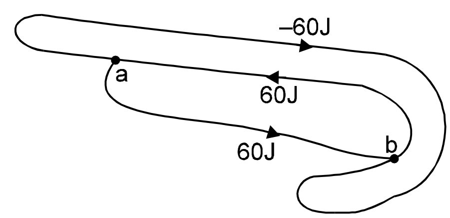
The frictional forces are non-conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does no depend only on the initial and final positions. Note that the work done by frictional force in a round trip is not zero.

The velocity-dependent forces such as air resistance, viscous force, magnetic force etc., are non conservative forces.

S.No.	Conservative forces	Non-Conservative forces
1	Work done does not depend u	Work done depends on path.
2	Work done in round trip is zero.	Work done in a round trip is not zero.
3	Central in nature.	Forces are velocity-dependent and retarding in nature.
4	When only a conservative force acts within a systrem, the kinetic enrgy and potential energy can change. However their sum, the mechanical	Work done against a non-conservative force may be dissipated as heat energy.
5	Work done is completely recoverable.	Work done in not completely recoverable.

Solved Examples

Ex. 14 The figure shows three paths connecting points a and b. A single force F does the indicated work on a particle moving along each path in the indicated direction. On the basis of this information, is force F conservative?



Ans. No

Explanation :

For a conservative force, the work done in a round trip should be zero.

Ex.15 The potential energy of a conservative system is given by $U = ax^2 - bx$ where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

Sol. In a conservative field $F = -\frac{dU}{dx}$

$\therefore\ F = -\frac{d}{dx}(ax^2 - bx) = b - 2ax$

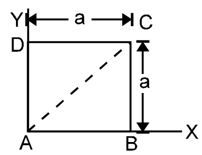
For equilibrium F = 0

or $b - 2ax = 0 \quad \therefore\ x = \frac{b}{2a}$

From the given equation we can see that $\frac{d^2U}{dx^2} = 2a$ (positive), i.e., U is minimum.

Therefore, $x = \frac{b}{2a}$ is the stable equilibrium position.

Ex.16 A force $\mathbf{F} = x^2y^2\mathbf{i} + x^2y^2\mathbf{j}$ (N) acts on a particle which moves in the XY plane.



(a) Determine if F is conservative and

(b) find the work done by **F** as it moves the particle from A to C (fig.) along each of the paths ABC, ADC, and AC.

[Ans. (b) $W_{ABC} = W_{ADC} = \frac{a^5}{3}$ (J), $W_{AC} = \frac{2a^5}{5}$ (J)]

WORK-ENERGY THEOREM

According to work-energy theorem, the work done by all the forces on a particle is equal to the change in its kinetic energy.

$W_C + W_{NC} + W_{PS} = \Delta K$

Where, W_C is the work done by all the conservative forces.

W_{NC} is the work done by all non-conservative forces.

W_{PS} is the work done by all psuedo forces.

Modified Form of Work-Energy Theorem

We know that conservative forces are associated with the concept of potential energy, that is

$W_C = -\Delta U$

So, Work-Energy theorem may be modified as

$W_{NC} + W_{PS} = \Delta K + \Delta U$

$W_{NC} + W_{PS} = \Delta E$

Solved Examples

Ex.17 A particle of mass 0.5 kg travels in a straight line with velocity $v = ax^{3/2}$ where $a = 5\ m^{-1/2}\ s^{-1}$. What is the work done by the net force during its displacement from x = 0 to x = 2m?

Sol. $m = 0.5$ kg, $v = ax^{3/2}$, $a = 5\ m^{-1/2}\ s^{-1}$, W = ?

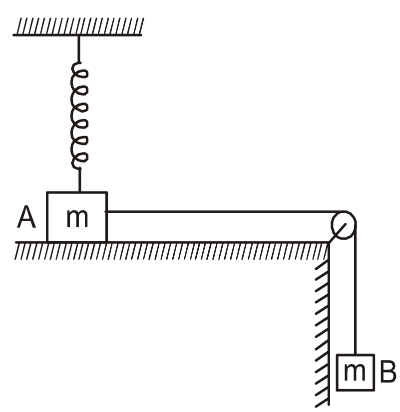
Initial velocity at x = 0, $v_0 = a \times 0 = 0$

Final velocity at x = 2, $v_2 = a \times 2^{3/2} = 5 \times 2^{3/2}$

Work done = Increase in kinetic energy = $\frac{1}{2}m$

$(v_2^2 - v_0^2) = \frac{1}{2} \times 0.5\,[(5 \times 2^{3/2})^2 - 0] = 50$ J.

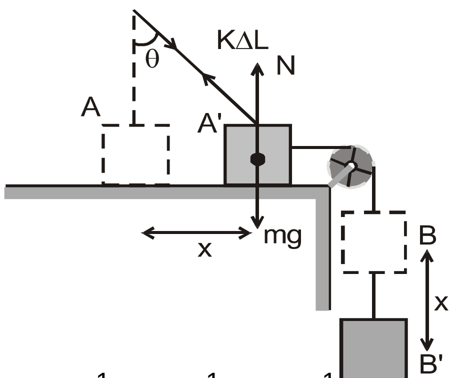
Ex.18 Figure shows two blocks A and B, each having a mass of 320 g connected by a light string passing over a smooth light pulley. The horizontal surface on which the block A can slide is smooth. The block A is attached to a spring of spring constant 40 N/m whose other end is fixed to a support 40 cm above the horizontal surface. Initially, the spring is vertical and unstretched when the system is released to move. Find the velocity of the block A at the instant it breaks off the surface below it. Take $g = 10\ m/s^2$.



Sol. Let the block A start loosing contact with the surface below it at A´ after travelling a distance x as shown in figure.

In this process the block B will shift from B to B' such that BB' = AA´ = x (as string is inextensible) and so there is a loss of gravitational potential energy = mgx.

This energy is partly stored as elastic potential energy in the spring which is stretched by ΔL and partly appears as kinetic energy of blocks A and B. So, by conservation of mechanical energy, we have



$$mgx = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}k(\Delta L)^2$$

or $\quad v^2 = gx - \frac{k}{2m}(\Delta L)^2 \quad$ ...(i)

Now, for vertical equilibrium of block A at A',

$N + F\cos\theta = mg$

But as for spring $F = k\Delta L$ and for breaking off N = 0 the above equation reduces to

$k\Delta L \cos\theta = mg \qquad$ ...(ii)

So, substituting the value of ΔL from Eq. (iii) in (ii) and solving for cos θ, we get

$$\cos\theta = 1 - \frac{mg}{kL} = 1 - \frac{0.32\times10}{40\times0.40} = \frac{4}{5}$$

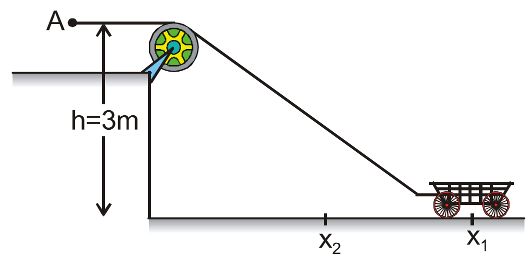
So, that $\Delta L = \left(\frac{L}{\cos\theta} - L\right) = \frac{0.4\times5}{4} - 0.4 = 0.1$ m

and $x = L\tan\theta = 0.4 \times \frac{3}{4} = 0.3$ m

Substituting these value of ΔL and x in Equation (ii),

$$v = \left[10\times0.3 - \frac{40\times(0.1)^2}{2\times0.32}\right]^{1/2} = \sqrt{3-0.625} = 1.54\,m/s$$

Ex.19 Figure shows a light, inextensible string attached to a cart that can slide along a frictionless horizontal rail aligned along an x axis. The left end of the string is pulled over a pulley, of negligible mass and friction and fixed at height h = 3m from the ground level. The cart slides from $x_1 = 3\sqrt{3}$ m to $x_2 = 4$ m and during the move, tension in the string is kept constant 50 N. Find change in kinetic energy of the cart in joules. (Use $\sqrt{3} = 1.7$)



Ans. 50

Sol. Displacement of the point of 'A' of the string

$= \sqrt{(3\sqrt{3})^2 + (3)^2} - \sqrt{4^2 + 3^2} = 6 - 5 = 1$ m

Δk = Work done by tension $= 50 \times 1 = 50$ Joule.

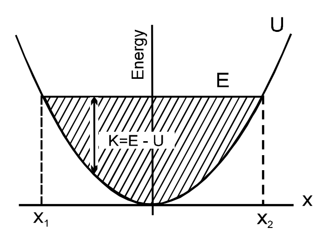
ENERGY DIAGRAMS

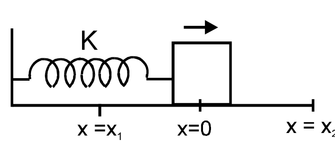
Diagram, in which the total energy E and the potential energy U are plotted as functions of positions is called as energy diagram.

The kinetic energy K = E – U is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where U ≤ E.

(a) Energy Diagram for a harmonic oscillator.

$$U = \frac{Kx^2}{2}$$





The potential energy of the block is $U = \frac{Kx^2}{2}$ is a parabola centered at the origin. Since, the total energy is constant for a conservative system, E is represented by a horizontal straight line. Motion is limited to the shaded region where $E \geq U$; the limits of the motion x_1 and x_2 in the sketch, are sometimes called the turning points. The kinetic energy, K = E - U, is greatest at the origin. The particle flies past to a complete rest at one of the turning points x_1, x_2; then moves towards the origin with increasing kinetic energy, and the cycle is repeated.

The harmonic oscillator provides a good example of bounded motion. As E increases, the turning point moves farther and farther away, but the particle can never move away freely. If E is decreased, the amplitude of motion decreases, until finally for E = 0 the particle lies at rest at x = 0.

POWER

Power is defined as the time rate of doing work.

When the time taken to complete a given amount of work is important, we measure the power of the agent doing work.

The average power ($\overline{P}$ or p_{av}) delivered by an agent is given by $\overline{P}$ or $p_{av} = \frac{W}{t}$

where W is the amount of work done in time t.

Power is the ratio of two scalars- work and time. So, power is a scalar quantity. If time taken to complete a given amount of work is more, then power is less. For a short duration dt, if P is the power delivered during this duration, then

$$P = \frac{\vec{F}\cdot d\vec{S}}{dt} = \vec{F}\cdot\frac{d\vec{S}}{dt} = \vec{F}\,.\,\vec{v}$$

By definition of dot product, $P = Fv\cos\theta$

where θ is the smaller angle between $\vec{F}$ and $\vec{v}$.

This P is called as instantaneous power if dt is very small.

Solved Examples

Ex.20 A block moves in uniform circular motion because a cord tied to the block is anchored at the centre of a circle. Is the power of the force exerted on the block by the cord position, negative, or zero?

Ans. Zero

Explanation. $\vec{F}$ and $\vec{v}$ are perpendicular.

$\therefore$ Power = $\vec{F}\,.\,\vec{v}$ = Fv cos 90° = Zero.

UNIT OF POWER

A unit power is the power of an agent which does unit work in unit time.

The power of an agent is said to be one watt if it does one joule of work in one second.

1 watt = 1 joule/section = 10^7 erg/second

Also, $1 \text{ watt} = \frac{1\text{ newton}\times 1\text{ metre}}{1\text{ second}} = 1\text{ N m s}^{-1}$.

Dimensional formula of power

$$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]} = \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$$

Power has 1 dimension in mass, 2 dimensions in length and – 3 dimensions in time.

S.No.	Human Activity	Power (W)
1	Heart beat	1.2
2	Sleeping	83
3	Sitting	120
4	Riding in a car	140
5	Walking (4.8 km h^{-1})	265
6	Cycling (15 km h^{-1})	410
7	Playing Tennis	440
8	Swimming (breaststroke, 1.6 km h^{-1})	475
9	Skating	535
10	Climbing Stairs (116 steps min^{-1})	685
11	Cycling (21.3 km h^{-1})	700
12	Playing Basketball	800
13	Tube light	40
14	Fan	60

Solved Examples

Ex.21 What is represented by the slope of the work-time graph?

Ans. Instantaneous power.

Ex.22 What is represented by area under power-time graph?

Ans. Work.

Ex.23 What is the power of an engine which can lift 20 metric ton of coal per hour from a 20 metre deep mine?

Sol. Mass, m = 20 metric ton = 20 × 1000 kg; Distance, S = 20 m; Time, t = 1 hour = 3600 s

Power = $\frac{\text{Work}}{\text{Time}} = \frac{mg \times S}{t} = \frac{20 \times 1000 \times 9.8 \times 20}{3600}$ watt = $\mathbf{1.09 \times 10^3}$ **W**

Ex.24 A one kilowatt motor pumps out water from a well 10 metre deep. Calculate the quantity of water pumped out per second.

Sol. Power, P = 1 kilowatt = 10^3 watt

S = 10 m ; Time, t = 1 second; Mass of water, m = ?

Power = $\frac{mg \times S}{t}$ $\therefore 10^3 = \frac{m \times 9.8 \times 10}{1}$

or $m = \frac{10^3}{9.8 \times 10}$ kg = 10.204 kg

Ex.25 The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through in time t ? (b) What is the kinetic energy or the air? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that A = 30 m^2, v = 36 km h^{-1} and the density of air is 1.2 kg m^{-3}. What is the electrical power produced?

Sol. (a) Volume of wind flowing per second = Av

Mass of wind flowing per second = Avρ

Mass of air passing in t second = Avρt

(b) Kinetic energy of air = $\frac{1}{2}mv^2 = \frac{1}{2}(Av\rho t)v^2$

$= \frac{1}{2}Av^2\rho t$

(c) Electrical energy produced = $\frac{25}{100} \times \frac{1}{2}Av^3\rho t$

$= \frac{Av^3\rho t}{8}$

Electrical power $= \frac{Av^3\rho t}{8t} = \frac{Av^3\rho}{8}$

Now, A = 30 m^2, v = 36 km h^{-1} = 36 × $\frac{5}{18}$ m s^{-1} = 10 m s^{-1}, ρ = 1.2 kg ms^{-1}

∴ Electrical power = $\frac{30 \times 10 \times 10 \times 1.2}{8}$ W = 4500 W = **4.5 kW**

Ex.26 One coolie takes one minute to raise a box through a height of 2 metre. Another one takes 30 second for the same job and does the same amount of work. Which one of the two has greater power and which one uses greater energy?

Sol. Power of first coolie = $\frac{\text{Work}}{\text{Time}} = \frac{M \times g \times S}{t}$

$= \frac{M \times 9.8 \times 2}{60}$ J s^{-1}

Power of second coolie $= \frac{M \times 9.8 \times 2}{30}$ J s^{-1} = 2 $\left(\frac{M \times 9.8 \times 2}{60}\right)$ J s^{-1} = 2 × Power of first coolie

So, the power of the second coolie is double that of the first.

Both the coolies spend the same amount of energy.

Aliter, We know that W = Pt

For the same work, W = $p_1t_1 = P_2t_2$ or $\frac{P_2}{P_1} = \frac{t_1}{t_2}$

$= \frac{1\text{ minute}}{30\text{ s}} = 2$ or $P_2 = 2P_1$

Ex.27 A large family uses 8 kW of power. Direct solar energy is incident on the horizontal surface at an average rate of 200 W per square metre. If 20% of this energy can be converted to useful electrical energy, how large an area is needed to supply 8kW?

Sol. If A m^2 be the area, then power = 200 . A watts

Useful electrical energy produced/s = $\frac{20}{100}$ (200 A)

= 40 . A watts But 40 A = 8000 or A = 200 m^2

Ex.28 An elevator of total mass (elevator + passenger) 1800 kg is moving up with a constant speed of 2 ms^{-1}. A frictional force of 4000 N oppose its motion. Determine the minimum power delivered by the motor to the elevator. Take g = 10 m s^{-2}.

Sol. Weight of (elevator + passenger) = mg = 1800 × 10 N = 18000 N

Frictional force = 4000 N

Total downward force on the elevator = (18000 + 4000) N = 22000 N

Clearly, the motor must have enough power to balance this force.

Now, power, P = Fv = 2200 N × 2 m s^{-1} = 4400 W

= $\frac{44000}{746}$ hp = 58.98 hp