

Binomial Theorem

Binomial Expression :

An expression containing two terms, is called a binomial expression. For example $a + b/x$, $x + 1/y$, $a - y^2$ etc. are binomial expressions. Expansion of $(x + a)^n$ is called Binomial Theorem. Expression containing three terms is called trinomial. For example $x + y + z$. And in general expression containing more than two terms is called multinomial.

Terminology used in binomial theorem :

Factorial notation : n or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)\dots\dots\dots 3.2.1 & ; \text{ if } n \in \mathbb{N} \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n \cdot (n-1)!$; $n \in \mathbb{N}$

Mathematical meaning of nC_r : The term nC_r denotes number of combinations of r things chosen

from n distinct things mathematically, ${}^nC_r = \frac{n!}{(n-r)!r!}$,
 $n \in \mathbb{N}$, $r \in \mathbb{W}$, $0 \leq r \leq n$

Note : Other symbols of nC_r are $\binom{n}{r}$ and $C(n, r)$.

Properties related to nC_r :

(i) ${}^nC_r = {}^nC_{n-r}$

Note : If ${}^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} {}^{n-2}C_{r-2}$
 $= \dots\dots\dots = \frac{n(n-1)(n-2)\dots\dots\dots(n-(r-1))}{r(r-1)(r-2)\dots\dots\dots 2.1}$

(v) If n and r are relatively prime, then nC_r is divisible by n . But converse is not necessarily true.

Statement of binomial theorem :

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + {}^nC_n a^0 b^n$$

where $n \in \mathbb{N}$

or $(a + b)^n = \sum_{r=0}^n {}^nC_r a^{n-r} b^r$

Note : If we put $a = 1$ and $b = x$ in the above binomial expansion, then

or $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

or $(1 + x)^n = \sum_{r=0}^n {}^nC_r x^r$

(i) $(x-3)^5$

(ii) $\left(1 - \frac{3x^2}{2}\right)^4$

$$\begin{aligned} \text{(ii)} \quad & \left(1 - \frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1 \left(-\frac{3x^2}{2}\right) + {}^4C_2 \left(-\frac{3x^2}{2}\right)^2 \\ & + {}^4C_3 \left(-\frac{3x^2}{2}\right)^3 + {}^4C_4 \left(-\frac{3x^2}{2}\right)^4 \\ & = 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8 \end{aligned}$$

Index of the binomial **The binomial coefficient**

0 1

1 1 1

2 1 2 1

3 1 3 3 1

4 1 4 6 4 1

5 5 10 10 5 1

- Each row of the triangle begins with 1 and ends with 1.
- Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Solved Examples

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **61**.

The r^{th} term from the end is equal to the $(n-r+2)^{\text{th}}$ term from the beginning, i.e. ${}^nC_{n-r+1} x^{r-1} y^{n-r+1}$

Solved Examples

$$\begin{aligned} T_{6+1} &= {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(-\frac{5}{2x}\right)^6 \\ &= \frac{9!}{3!6!} \left(\frac{4x}{5}\right)^3 \left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3} \end{aligned}$$

Solved Examples**Ex.4** The 7th term from the end in the expansion of $\left(x - \frac{2}{x^2}\right)^{10}$ is equal to-

[1] ${}^{10}C_4 2^4 \left(\frac{1}{x^2}\right)$

[2] ${}^{10}C_4 2^4$

[3] $-{}^{10}C_3 2^3 x$

[4] None of these

Sol.[1] The 7th term from the end = 5th term from beginning

$$T_5 = {}^{10}C_4 x^6 \left(-\frac{2}{x^2}\right)^4 = {}^{10}C_4 \cdot 2^4 \left(\frac{1}{x^2}\right)$$

Ex.5. The ratio of the coefficient of x^{15} to the termindependent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is

(1) 12 : 32

(2) 1 : 32

(3) 32 : 12

(4) 32 : 1

Sol. [2] General term in the expansion is

$${}^{15}C_r (x^2)^{15-r} \left(\frac{2}{x}\right)^r \text{ i.e., } {}^{15}C_r x^{30-3r} \cdot 2^r$$

Coefficient of x^{15} is ${}^{15}C_5 2^5$ ($r = 5$)

Coefficient of constant term is ${}^{15}C_{10} 2^{10}$ ($r = 10$)

Ratio is 1 : 32. (${}^{15}C_5 = {}^{15}C_{10}$)

Ex.6. The term independent of x in the expansion of $\left(\sqrt[6]{x} - \frac{1}{\sqrt[3]{x}}\right)^9$ is equal to-

[1] $-{}^9C_3$

[2] 9C_4

[3] 9C_2

[4] none of these

Sol. [1] $T_{r+1} = {}^9C_r (\sqrt[6]{x})^{9-r} \left(-\frac{1}{\sqrt[3]{x}}\right)^r$

$$= {}^9C_r (-1)^r x^{\frac{9-r}{6} - \frac{r}{3}} = {}^9C_r (-1)^r x^{\left(\frac{9-3r}{6}\right)}$$

Now, $\frac{9-3r}{6} = 0 \Rightarrow r = 3 \quad T = -{}^9C_3$

Ex.7 If the second, third and fourth terms in the expansion of $(a + b)^n$ are 135, 30 and $10/3$ respectively, then-

[1] $n = 3$

[2] $n = 2$

[3] $n = 7$

[4] $n = 5$

Sol. [4] $T_2 = {}^nC_1 ab^{n-1} = 135 \quad \dots(i)$

$T_3 = {}^nC_2 a^2 b^{n-2} = 30 \quad \dots(ii)$

$T_4 = {}^nC_3 a^3 b^{n-3} = \frac{10}{3} \quad \dots(iii)$

Dividing (i) by (ii)

$$\frac{{}^nC_2 ab^{n-1}}{{}^nC_1 a^2 b^{n-2}} = \frac{135}{30}$$

$$\frac{n}{2} \cdot \frac{b}{a} = \frac{9}{2}$$

$\dots(iv)$

$$\frac{b}{a} = \frac{9}{4}(n-1)$$

$\dots(v)$

Dividing (ii) and (iii)

$$\frac{\frac{n(n-1)}{2} \cdot \frac{b}{a} = \frac{10}{3} = 9 \quad \dots(vi)$$

Eliminating a and b from (v) and (vi)

$\Rightarrow n = 5$

Ex.8 Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.**Sol.** The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{1/4}\right)^{1000-r} \left(8^{1/6}\right)^r = {}^{1000}C_r 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integersThe possible set of values of r is $\{0, 2, 4, \dots, 1000\}$

Hence, number of rational terms is 501

Ex.9 Find the coefficient of x^{32} and x^{-17} in $\left(x^4 - \frac{1}{x^3}\right)^{15}$.**Sol.** Let $(r+1)^{\text{th}}$ term contains x^m

$$T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r = {}^{15}C_r x^{60-7r} (-1)^r$$

(i) for x^{32} , $60 - 7r = 32$

$\Rightarrow 7r = 28 \Rightarrow r = 4$, so 5th term.

$T_5 = {}^{15}C_4 x^{32} (-1)^4$

Hence, coefficient of x^{32} is 1365

(ii) for x^{-17} , $60 - 7r = -17$

$\Rightarrow r = 11$, so 12th term.

$T_{12} = {}^{15}C_{11} x^{-17} (-1)^{11}$

Hence, coefficient of x^{-17} is -1365

Middle term(s) :

- (a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{\text{th}}$ term.
- (b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+1}{2} + 1\right)^{\text{th}}$ terms.

Note : In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a+b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$ (Values of both these coefficients are equal)

Solved Examples

Ex.10 Find the middle term(s) in the expansion of

(i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

Sol. (i) $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{\text{th}}$ term.

It means T_8 is middle term

$$T_8 = {}^{14}C_7 \left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16} x^{14}.$$

(ii) $\left(3a - \frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{\text{th}}$

& $\left(\frac{9+1}{2} + 1\right)^{\text{th}}$.

It means T_5 & T_6 are middle terms

$$T_5 = {}^9C_4 (3a)^{9-4} \left(-\frac{a^3}{6}\right)^4 = \frac{189}{8} a^{17}$$

$$T_6 = {}^9C_5 (3a)^{9-5} \left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16} a^{19}.$$

Ex.11 If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924x^6$, then $n =$

- [1] 10 [2] 12
[3] 14 [4] None of these

Sol. [2] Since n is even therefore $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term is middle term, hence ${}^nC_{n/2} (x^2)^{n/2} \left(\frac{1}{x}\right)^{n/2} = 924x^6$
 $\Rightarrow x^{n/2} = x^6 \Rightarrow n = 12.$

Ex. 12 The greatest coefficient in the expansion of $(1+x)^{2n}$ is

- [1] $\frac{1.3.5 \dots (2n-1)}{n!} \cdot 2^n$ [2] ${}^{2n}C_{n-1}$
[3] ${}^{2n}C_{n+1}$ [4] None of these

Sol. [1] The greatest coefficient
 = the coefficient of the middle term
 = ${}^{2n}C_n = \frac{1.3.5 \dots (2n-1)}{n!} \cdot 2^n$

To Determine a Particular Term in the Expansion

In the expansion of $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1} , then r is given by

$$n\alpha - r(\alpha + \beta) = m \quad \Rightarrow \quad r = \frac{n\alpha - m}{\alpha + \beta}$$

Thus in above expansion if constant term i.e., the term which is independent of x , occurs in T_{r+1} then r is determined by

$$n\alpha - r(\alpha + \beta) = 0 \quad \Rightarrow \quad r = \frac{n\alpha}{\alpha + \beta}$$

Solved Examples

Ex.13 The term independent of x in the expansion of

$$\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9 \text{ is } -$$

- (1) 5^{th} (2) 6^{th}
(3) 7^{th} (4) 8^{th}

Sol. Here comparing $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$ with $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$
We get $\alpha = 2$, $\beta = 1$, $n = 9$

$$r = \frac{9(2)}{2+1} = 6$$

$\therefore (6+1) = 7^{\text{th}}$ term is independent of x .

Ex.14 The co-efficient of x^{39} in the expansion of $(x^4 - 1/x^3)^{15}$ is

- (1) 455 (2) -455
(3) 105 (4) None of these

Sol. From above formula

$$r = \frac{15(4) - 39}{4+3} = 3$$

$$\therefore \text{The required term} = T_4 = {}^{15}C_3 (x^4)^{12} (-1/x^3)^3$$

$$= -455 x^{39}$$

$$\therefore \text{coefficient of } x^{39} = -455$$

Numerically greatest term in the expansion of $(a+b)^n$, $n \in \mathbb{N}$

Binomial expansion of $(a+b)^n$ is as follows :-

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + {}^nC_n a^0 b^n$$

If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r+1)^{\text{th}}$ terms respectively

$$T_r = {}^nC_{r-1} a^{n-(r-1)} b^{r-1}$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\text{Now, } \left| \frac{T_{r+1}}{T_r} \right| = \left| \frac{{}^nC_r a^{n-r} b^r}{{}^nC_{r-1} a^{n-r+1} b^{r-1}} \right| = \frac{n-r+1}{r} \cdot \left| \frac{b}{a} \right|$$

$$\text{Consider } \left| \frac{T_{r+1}}{T_r} \right| \geq 1$$

$$\left(\frac{n-r+1}{r} \right) \left| \frac{b}{a} \right| \geq 1$$

$$\frac{n+1}{r} - 1 \geq \left| \frac{a}{b} \right| \quad r \leq \frac{n+1}{1 + \left| \frac{a}{b} \right|}$$

$$\text{So greatest term will be } T_{r+1} \text{ where } r = \left[\frac{n+1}{1 + \left| \frac{a}{x} \right|} \right]$$

[.] denotes greatest integer function.

Note : If $\frac{n+1}{1 + \left| \frac{a}{x} \right|}$ itself is a natural number, then $T_r = T_{r+1}$

and both the terms are numerically greatest.

Solved Examples

Ex.15 If the sum of the coefficients in expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion

for $x = \frac{1}{2}$ is

- [1] 4^{th} [2] 5^{th} [3] 6^{th} [4] 7^{th}

Sol. [2]

Sum of the coefficient in the expansion of $(1+2x)^n$ = 6561

$$\Rightarrow (1+2x)^n = 6561, \text{ when } x = 1$$

$$\Rightarrow 3^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n = 8$$

$$\text{Now, } \frac{T_{r+1}}{T_r} = \frac{{}^8C_r (2x)^r}{{}^8C_{r-1} (2x)^{r-1}} = \frac{9-r}{r} \times 2x$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{9-r}{r} \quad \left[\because x = \frac{1}{2} \right]$$

$$\therefore \frac{T_{r+1}}{T_r} > 1 \Rightarrow \frac{9-r}{r} > 1 \Rightarrow 9-r > r \Rightarrow 2r < 9 \Rightarrow r < 4\frac{1}{2}$$

Hence, 5^{th} term is the greatest term.

Ex.16 The greatest term in the expansion of $(1+\sqrt{2})^{12}$ is -

- [1] 7th [2] 8th [3] 6th [4] 5th

Sol. [2]

$$\frac{T_{r+1}}{T_r} = \frac{12-r+1}{r} (\sqrt{2}) \geq 1$$

$$13-r \geq \frac{r}{\sqrt{2}} \Rightarrow 13 \geq \left(\frac{\sqrt{2}+1}{\sqrt{2}} \right) r$$

$$r \leq \left(\frac{13\sqrt{2}}{\sqrt{2}+1} \right) \Rightarrow r \leq 13\sqrt{2} (\sqrt{2}-1)$$

$$r \leq 13(2-\sqrt{2})$$

$$r \leq 13(0.586)$$

$$r \leq 7.616 \Rightarrow r = 7$$

So, 8th term is greatest.

Ex.17 Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$.

Sol. Let r^{th} and $(r + 1)^{\text{th}}$ be two consecutive terms in the expansion of $(3 - 5x)^{15}$

$$T_{r+1} \geq T_r$$

$${}^{15}C_r 3^{15-r} (-5x)^r \geq {}^{15}C_{r-1} 3^{15-(r-1)} (-5x)^{r-1}$$

$$\frac{(15)!}{(15-r)!r!} |-5x| \geq \frac{3 \cdot (15)!}{(16-r)!(r-1)!}$$

$$5 \cdot \frac{1}{5} (16-r) \geq 3r$$

$$16-r \geq 3r$$

$$4r \leq 16$$

$$r \leq 4$$

Result :

If $(\sqrt{A} + B)^n = I + f$, where I and n are positive integers, n being odd and $0 < f < 1$, then $(I + f)f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$. If n is an even integer, then $(I + f)(1 - f) = k^n$

Ex.18 If n is positive integer, then prove that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ (i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

$$\text{Now, } 0 < 7 - 4\sqrt{3} < 1$$

$$0 < (7 - 4\sqrt{3})^n < 1$$

$$\text{Let } (7 - 4\sqrt{3})^n = f' \text{(ii)}$$

$$\Rightarrow 0 < f' < 1$$

Adding (i) and (ii)

$$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$= 2 [{}^nC_0 7^n + {}^nC_2 7^{n-2} (4\sqrt{3})^2 + \dots]$$

$$I + f + f' = \text{even integer}$$

$$\Rightarrow (f + f' \text{ must be an integer})$$

$$0 < f + f' < 2 \Rightarrow f + f' = 1$$

$$I + 1 = \text{even integer therefore } I \text{ is an odd integer.}$$

Ex.19 Show that the integer just above $(\sqrt{3} + 1)^{2n}$ is divisible by 2^{n+1} for all $n \in \mathbb{N}$.

Sol. Let $(\sqrt{3} + 1)^{2n} = (4 + 2\sqrt{3})^n = 2^n (2 + \sqrt{3})^n = I + f$ (i)

where I and f are its integral & fractional parts respectively

$$0 < f < 1.$$

$$\text{Now } 0 < \sqrt{3} - 1 < 1$$

$$0 < (\sqrt{3} - 1)^{2n} < 1$$

$$\text{Let } (\sqrt{3} - 1)^{2n} = (4 - 2\sqrt{3})^n$$

$$= 2^n (2 - \sqrt{3})^n = f'. \text{(ii)}$$

$$0 < f' < 1$$

adding (i) and (ii)

$$I + f + f' = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$$

$$= 2^n [(2 + \sqrt{3})^n + (2 - \sqrt{3})^n] = 2 \cdot 2^n [{}^nC_0 2^n + {}^nC_2 2^{n-2} (\sqrt{3})^2 + \dots]$$

$$I + f + f' = 2^{n+1} k \text{ (where } k \text{ is a positive integer)}$$

$$0 < f + f' < 2 \Rightarrow f + f' = 1$$

$$I + 1 = 2^{n+1} k.$$

$I + 1$ is the integer just above $(\sqrt{3} + 1)^{2n}$ and which is divisible by 2^{n+1} .

Ex.20. Show that $9^n + 7$ is divisible by 8, where n is a positive integer.

$$\text{Sol. } 9^n + 7 = (1 + 8)^n + 7$$

$$= {}^nC_0 + {}^nC_1 \cdot 8 + {}^nC_2 \cdot 8^2 + \dots + {}^nC_n 8^n + 7.$$

$$= 8 \cdot C_1 + 8^2 \cdot C_2 + \dots + C_n \cdot 8^n + 8.$$

$$= 8\lambda, \text{ where } \lambda \text{ is a positive integer}$$

Hence, $9^n + 7$ is divisible by 8.

Ex.21 What is the remainder when 5^{99} is divided by 13.

$$\text{Sol. } 5^{99} = 5 \cdot 5^{98} = 5 \cdot (25)^{49} = 5 (26 - 1)^{49}$$

$$= 5 [{}^{49}C_0 (26)^{49} - {}^{49}C_1 (26)^{48} + \dots + {}^{49}C_{48} (26)^1 - {}^{49}C_{49} (26)^0]$$

$$= 5 [{}^{49}C_0 (26)^{49} - {}^{49}C_1 (26)^{48} + \dots + {}^{49}C_{48} (26)^1 - 1]$$

$$= 5 [{}^{49}C_0 (26)^{49} - {}^{49}C_1 (26)^{48} + \dots + {}^{49}C_{48} (26)^1 - 13] + 60$$

$$= 13(k) + 52 + 8 \text{ (where } k \text{ is a positive integer)}$$

$$= 13(k + 4) + 8$$

Hence, remainder is 8.

Ex.22 Find the last two digits of the number $(17)^{10}$.

Sol. $(17)^{10} = (289)^5 = (290 - 1)^5$
 $= {}^5C_0 (290)^5 - {}^5C_1 (290)^4 + \dots + {}^5C_4 (290)^1 - {}^5C_5 (290)^0$
 $= {}^5C_0 (290)^5 - {}^5C_1 \cdot (290)^4 + \dots + {}^5C_3 (290)^2 + 5 \times 290 - 1$
 $= \text{A multiple of } 1000 + 1449$
Hence, last two digits are 49

Note : We can also conclude that last three digits are 449.

Ex.23 Which number is larger $(1.01)^{1000000}$ or 10,000 ?

Sol. By Binomial Theorem

$$\begin{aligned} (1.01)^{1000000} &= (1 + 0.01)^{1000000} \\ &= 1 + {}^{1000000}C_1 (0.01) + \text{other positive terms} \\ &= 1 + 1000000 \times 0.01 + \text{other positive terms} \\ &= 1 + 10000 + \text{other positive terms} \end{aligned}$$

Hence $(1.01)^{1000000} > 10,000$

Some standard expansions :

(i) Consider the expansion

$$\begin{aligned} (x + y)^n &= \sum_{r=0}^n {}^nC_r x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 \\ &+ \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n x^0 y^n \dots (i) \end{aligned}$$

(ii) Now replace $y \rightarrow -y$ we get

$$\begin{aligned} (x - y)^n &= \sum_{r=0}^n {}^nC_r (-1)^r x^{n-r} y^r = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y^1 \\ &+ \dots + {}^nC_r (-1)^r x^{n-r} y^r + \dots + {}^nC_n (-1)^n x^0 y^n \dots (ii) \end{aligned}$$

(iii) Adding (i) & (ii), we get

$$(x + y)^n + (x - y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + \dots]$$

(iv) Subtracting (ii) from (i), we get

$$\begin{aligned} (x + y)^n - (x - y)^n &= 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 \\ &+ \dots] \end{aligned}$$

Binomial Coefficient & Their Properties

In the expansion of $(1 + x)^n$; i.e. $(1 + x)^n = {}^nC_0 + {}^nC_1 x + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

The coefficients ${}^nC_0, {}^nC_1, {}^nC_n$ of various powers of x , are called binomial coefficients and they are written as $C_0, C_1, C_2, \dots, C_n$

Hence

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_r x^r + \dots + C_n x^n \dots (1)$$

$$\text{Where } C_0 = 1, C_1 = n, C_2 = \frac{n(n-1)}{2!}$$

$$C_r = \frac{n(n-1)\dots(n-r+1)}{r!}, C_n = 1$$

Now, we shall obtain some important expressions involving binomial coefficients-

$$\text{* sum of coefficient : putting } x = 1 \text{ in (1), we get } C_0 + C_1 + C_2 + \dots + C_n = 2^n \dots (2)$$

$$\text{* Sum of coefficients with alternate signs : putting } x = -1 \text{ in (1)}$$

$$\text{We get } C_0 - C_1 + C_2 - C_3 + \dots = 0 \dots (3)$$

$$\text{* Sum of coefficients of even and odd terms : from (3), we have}$$

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots (4)$$

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$$\Rightarrow C_0 + C_2 + \dots = C_1 + C_3 + \dots = 2^{n-1}$$

$$\text{* Sum of products of coefficients : Replacing } x \text{ by } 1/x \text{ in (1) We get}$$

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} + \dots (5)$$

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1 x + C_2 x^2 + \dots)$$

$$(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots)$$

Now, comparing coefficients of x^r on both the sides, we get

$$\begin{aligned} C_0 C_r + C_1 C_{r+1} + \dots + C_{n-r} C_n &= {}^{2n}C_{n-r} \\ &= \frac{2n!}{(n+r)!(n-r)!} \dots (6) \end{aligned}$$

* **Sum of squares of coefficients :** putting $r=0$ in (6)

$$\text{we get } C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{2n!}{n! n!}$$

* putting $r=1$ in (6), we get

$$C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n = {}^{2n}C_{n-1} \\ = \frac{2n!}{(n+1)!(n-1)!} \quad \dots(7)$$

* putting $r=2$ in (6), we get

$$C_0 C_2 + C_1 C_3 + C_2 C_4 + \dots + C_{n-2} C_n = {}^{2n}C_{n-2} \\ = \frac{2n!}{(n+2)!(n-2)!} \quad \dots(8)$$

* Differentiating both sides of (1) w.r.t. x , we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$$

Now putting $x=1$ and $x=-1$ respectively

$$C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1} \quad \dots(9)$$

$$\text{and } C_1 - 2C_2 + 3C_3 - \dots = 0 \quad \dots(10)$$

* adding (2) and (9)

$$C_0 + 2C_1 + 3C_2 + \dots + {}^{(n+1)}C_n = 2^{n-1} (n+2) \quad \dots(11)$$

* Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \left[C_0x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1} \right]_0^1$$

$$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \quad \dots(12)$$

* Integrating (1) w.r.t. x between the limits -1 to 0, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_{-1}^0 = \left[C_0x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1} \right]_{-1}^0$$

$$\Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + \frac{(-1)^n C_n}{n+1} = \frac{1}{(n+1)} \quad \dots(13)$$

Solved Examples

Ex.24 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then show that

$$(i) \quad C_0 + 3C_1 + 3^2C_2 + \dots + 3^n C_n = 4^n.$$

$$(ii) \quad C_0 + 2C_1 + 3C_2 + \dots + (n+1) C_n = 2^{n-1} (n+2).$$

$$(iii) \quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}.$$

Sol.

$$(i) \quad (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$$

put $x=3$

$$C_0 + 3 \cdot C_1 + 3^2 \cdot C_2 + \dots + 3^n \cdot C_n = 4^n$$

(ii) **I. Method : By Summation**

$$\text{L.H.S.} = {}^nC_0 + 2 \cdot {}^nC_1 + 3 \cdot {}^nC_2 + \dots + (n+1) \cdot {}^nC_n.$$

$$= \sum_{r=0}^n (r+1) \cdot {}^nC_r = \sum_{r=0}^n r \cdot {}^nC_r + \sum_{r=0}^n {}^nC_r$$

$$= n \sum_{r=0}^n {}^{n-1}C_{r-1} + \sum_{r=0}^n {}^nC_r = n \cdot 2^{n-1} + 2^n = 2^{n-1} (n+2).$$

RHS

II. Method : By Differentiation

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$$

Multiplying both sides by x ,

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \dots + C_nx^{n+1}.$$

Differentiating both sides

$$(1+x)^n + x n (1+x)^{n-1} = C_0 + 2 \cdot C_1x + 3 \cdot C_2x^2 + \dots + (n+1)C_nx^n.$$

putting $x=1$, we get

$$C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \dots + (n+1) C_n = 2^n + n \cdot 2^{n-1}$$

$$C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \dots + (n+1) C_n = 2^{n-1} (n+2)$$

Proved

(iii) **I Method : By Summation**

$$\text{L.H.S.} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + (-1)^n.$$

$$\frac{C_n}{n+1} = \sum_{r=0}^n (-1)^r \cdot \frac{{}^nC_r}{r+1}$$

$$= \frac{1}{n+1} \sum_{r=0}^n (-1)^r \cdot {}^{n+1}C_{r+1} \left\{ \frac{n+1}{r+1} \cdot {}^nC_r = {}^{n+1}C_{r+1} \right\}$$

$$= \frac{1}{n+1} [{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \dots + (-1)^n \cdot {}^{n+1}C_{n+1}]$$

$$= \frac{1}{n+1} [-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \dots + (-1)^n \cdot {}^{n+1}C_{n+1} + {}^{n+1}C_0]$$

$$= \frac{1}{n+1} = \text{R.H.S.}, \text{ since}$$

$$\{ -{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \dots + (-1)^n \cdot {}^{n+1}C_{n+1} = 0 \}$$

II Method : By Integration

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n.$$

Integrating both sides, within the limits -1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_{-1}^0$$

$$= \left[C_0x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + C_n \frac{x^{n+1}}{n+1} \right]_{-1}^0$$

$$\frac{1}{n+1} - 0 = 0 -$$

$$\left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \dots + (-1)^{n+1} \frac{C_n}{n+1} \right]$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$$

Proved

Ex.25 If $C_0, C_1, C_2, \dots, C_n$ are binomial coefficients then

$$\frac{1}{n!0!} + \frac{1}{(n-1)!1!} + \frac{1}{(n-2)!2!} + \dots + \frac{1}{0!n!} \text{ is equal to-}$$

$$[1] 2^n \quad [2] \frac{2^{n-1}}{n!}$$

$$[3] \frac{2^n}{n!} \quad [4] \text{none of these}$$

Sol. [3]

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$\frac{n!}{(n-1)!} + \frac{n!}{(n-1)!1!} + \frac{n!}{(n-1)!2!} + \dots + \frac{n!}{0!n!} = \frac{2^n}{n!}$$

(divide it by $n!$)

Ex.26 If $C_0, C_1, C_2, \dots, C_n$ are binomial coefficients,

then $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$ is equal to-

$$[1] \frac{2^{n+1}+1}{n+1} \quad [2] \frac{2^{n+1}-1}{n+1}$$

$$[3] \frac{2^{n+1}}{n+1} \quad [4] \text{none of these}$$

Sol. [2]

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$$

Integrate it with respect to x

$$\frac{(1+x)^{n+1}}{n+1} = C_0x + \frac{C_1x^2}{2} + C_2 \frac{x^3}{3} + \dots + C_n \frac{x^{n+1}}{n+1} + K$$

Put $x = 0$ then, $K = \frac{1}{n+1}$, Now put $x = 1$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{(n+1)}$$

Ex.27 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then prove that

$$(i) C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = 2^n C_n$$

$$(ii) C_0C_2 + C_1C_3 + C_2C_4 + \dots + C_{n-2}C_n = 2^n C_{n-2} \text{ or } 2^n C_{n+2}$$

$$(iii) 1. C_0^2 + 3 \cdot C_1^2 + 5 \cdot C_2^2 + \dots + (2n+1) \cdot C_n^2 = 2n \cdot 2^{n-1} C_n + 2^n C_n$$

Sol.

$$(i) (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \dots (i)$$

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_nx^0 \dots (ii)$$

Multiplying (i) and (ii)

$$(C_0 + C_1x + C_2x^2 + \dots + C_nx^n) (C_0x^n + C_1x^{n-1} + \dots + C_nx^0) = (1+x)^{2n}$$

Comparing coefficient of x^n ,

$$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = 2^n C_n$$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

$$C_0C_2 + C_1C_3 + C_2C_4 + \dots + C_{n-2}C_n = 2^n C_{n-2} \text{ or } 2^n C_{n+2}$$

(iii) **I Method : By Summation**

$$\text{L.H.S.} = 1 \cdot C_0^2 + 3 \cdot C_1^2 + 5 \cdot C_2^2 + \dots + (2n+1) C_n^2$$

$$= \sum_{r=0}^n (2r+1) C_r^2 = \sum_{r=0}^n 2r \cdot ({}^nC_r)^2 + \sum_{r=0}^n ({}^nC_r)^2$$

$$= 2 \sum_{r=1}^n n \cdot {}^{n-1}C_{r-1} \cdot {}^nC_r + 2^n C_n$$

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n \dots (i)$$

$$(x+1)^{n-1} = {}^{n-1}C_0x^{n-1} + {}^{n-1}C_1x^{n-2} + \dots + {}^{n-1}C_{n-1}x^0 \dots (ii)$$

Multiplying (i) and (ii) and comparing coefficients of x^n .

$${}^{n-1}C_0 \cdot {}^nC_1 + {}^{n-1}C_1 \cdot {}^nC_2 + \dots + {}^{n-1}C_{n-1} \cdot {}^nC_n = 2^{n-1} C_n$$

$$\sum_{r=0}^n {}^{n-1}C_{r-1} \cdot {}^nC_r = 2^{n-1} C_n$$

Hence, required summation is $2n \cdot 2^{n-1} C_n + 2^n C_n = \text{R.H.S.}$

II Method : By Differentiation

$$(1 + x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + C_3 x^6 + \dots + C_n x^{2n}$$

Multiplying both sides by x

$$x(1 + x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \dots + C_n x^{2n+1}$$

Differentiating both sides

$$x \cdot n(1 + x^2)^{n-1} \cdot 2x + (1 + x^2)^n = C_0 + 3C_1 x^2 + 5C_2 x^4 + \dots + (2n+1)C_n x^{2n} \quad \dots (i)$$

$$(x^2 + 1)^n = C_0 x^{2n} + C_1 x^{2n-2} + C_2 x^{2n-4} + \dots + C_n \quad \dots (ii)$$

Multiplying (i) & (ii)

$$(C_0 + 3C_1 x^2 + 5C_2 x^4 + \dots + (2n+1)C_n x^{2n})(C_0 x^{2n} + C_1 x^{2n-2} + \dots + C_n) = 2n x^2 (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$$

comparing coefficient of x^{2n} ,

$$C_0^2 + 3C_1^2 + 5C_2^2 + \dots + (2n+1)C_n^2 = 2n \cdot {}^{2n-1}C_{n-1} + {}^{2n}C_n$$

$$C_0^2 + 3C_1^2 + 5C_2^2 + \dots + (2n+1)C_n^2 = 2n \cdot {}^{2n-1}C_n + {}^{2n}C_n \quad \text{Proved}$$

Ex.28 The value of

$$3^n C_0 - 8^n C_1 + 13^n C_2 - 18^n C_3 + \dots + n \text{ terms is -}$$

[1] 0 [2] 3^n
[3] 5^n [4] none of these

Sol. [1]

$$3^n C_0 - 8^n C_1 + 13^n C_2 - 18^n C_3 + \dots - 3^n C_3 - \dots$$

$$\Rightarrow 3(0) - 5(0) = 0$$

Ex.29 If C_r stands for nC_r , then the value of

$$2C_0 - \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \dots + (-1)^{n+1} \frac{2^{n+1}}{(n+1)}$$

is equal to-

[1] 0, if n is even [2] 0, if n is odd
[3] $\frac{1}{n+1}$, if n is even [4] $\frac{1}{n+1}$, if n is odd

Sol. [2] $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + C_n \frac{x^{n+1}}{n+1}$$

Put $x = -2$ $\frac{(-1)^{n+1} - 1}{n+1}$

$$= -2C_0 + C_1 \frac{2^2}{2} - C_2 \frac{2^3}{3} + \dots + C_n \frac{(-2)^{n+1}}{(n+1)}$$

If n is odd, then L.H.S. = 0.

Ex.30 Find the summation of the following series –

(i) ${}^m C_m + {}^{m+1} C_m + {}^{m+2} C_m + \dots + {}^n C_m$
(ii) ${}^n C_3 + 2 \cdot {}^{n+1} C_3 + 3 \cdot {}^{n+2} C_3 + \dots + n \cdot {}^{2n-1} C_3$

Sol.

(i) **I Method :** Using property, ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$

$${}^m C_m + {}^{m+1} C_m + {}^{m+2} C_m + \dots + {}^n C_m$$

$$= \underbrace{{}^{m+1} C_{m+1} + {}^{m+1} C_m}_{= {}^{m+2} C_{m+1}} + {}^{m+2} C_m + \dots + {}^n C_m$$

$$\{ \because {}^m C_m = {}^{m+1} C_{m+1} \}$$

$$= \underbrace{{}^{m+2} C_{m+1} + {}^{m+2} C_m}_{= {}^{m+3} C_{m+1}} + \dots + {}^n C_m$$

$$= {}^{m+3} C_{m+1} + \dots + {}^n C_m = {}^n C_{m+1} + {}^n C_m = {}^{n+1} C_{m+1}$$

II Method

$${}^m C_m + {}^{m+1} C_m + {}^{m+2} C_m + \dots + {}^n C_m$$

The above series can be obtained by writing the coefficient of x^m in

$$(1+x)^m + (1+x)^{m+1} + \dots + (1+x)^n$$

Let $S = (1+x)^m + (1+x)^{m+1} + \dots + (1+x)^n$

$$= \frac{(1+x)^m [(1+x)^{n-m+1} - 1]}{x} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1} C_{m+1} + 0 = {}^{n+1} C_{m+1}$$

(ii) ${}^n C_3 + 2 \cdot {}^{n+1} C_3 + 3 \cdot {}^{n+2} C_3 + \dots + n \cdot {}^{2n-1} C_3$

The above series can be obtained by writing the coefficient of x^3 in

$$(1+x)^n + 2 \cdot (1+x)^{n+1} + 3 \cdot (1+x)^{n+2} + \dots + n \cdot (1+x)^{2n-1}$$

Let $S = (1+x)^n + 2 \cdot (1+x)^{n+1} + 3 \cdot (1+x)^{n+2} + \dots + n \cdot (1+x)^{2n-1} \quad \dots (i)$

$$(1+x)S = (1+x)^{n+1} + 2 \cdot (1+x)^{n+2} + \dots + (n-1) \cdot (1+x)^{2n-1} + n \cdot (1+x)^{2n} \quad \dots (ii)$$

Subtracting (ii) from (i)

$$\begin{aligned} -xS &= (1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \dots \\ &+ (1+x)^{2n-1} - n(1+x)^{2n} \\ &= \frac{(1+x)^n [(1+x)^n - 1]}{x} - n(1+x)^{2n} \end{aligned}$$

$$S = \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-^{2n}C_5 + ^nC_5 + n \cdot ^{2n}C_4$

Ex.31 The value of $^{4n}C_0 + ^{4n}C_4 + ^{4n}C_8 + \dots + ^{4n}C_{4n}$ is-

[1] 2^{4n} [2] 2^{4n-2}

[3] $2^{2n-1} + 2^{4n-2}$ [4] $2^{2n-1}(-1)^n + 2^{4n-2}$

Sol. [4]

$$(1+x)^{4n} = ^{4n}C_0 + ^{4n}C_1 x + ^{4n}C_2 x^2 + ^{4n}C_3 x^3 + ^{4n}C_4 x^4 + \dots + ^{4n}C_n x^{4n}$$

Put $x = 1$ and $x = -1$, then adding.

$$2^{4n-1} = ^{4n}C_0 + ^{4n}C_2 + ^{4n}C_4 + \dots + ^{4n}C_{4n} \quad \dots(i)$$

Now put, $x = i$

$$(1+i)^{4n} = ^{4n}C_0 + ^{4n}C_1 i - ^{4n}C_2 + ^{4n}C_3 i + ^{4n}C_4 + \dots + ^{4n}C_{4n}$$

Compare real and imaginary part, we get

$$(-1)^n (2)^{2n} = ^{4n}C_0 - ^{4n}C_2 + ^{4n}C_4 - ^{4n}C_6 + \dots + ^{4n}C_{4n} \quad \dots(ii)$$

Adding (i) and (ii), we get

$$\Rightarrow ^{4n}C_0 + ^{4n}C_4 + \dots + ^{4n}C_{4n} = (-1)^n (2)^{2n-1} + 2^{4n-2}$$

Ex.32 $aC_0 + (a+b)C_1 + (a+2b)C_2 + \dots + (a+nb)C_n$ is equal to-

[1] $(2a+nb)2^n$ [2] $(2a+nb)2^{n-1}$

[3] $(na+2b)2^n$ [4] $(na+2b)2^{n-1}$

Sol. [2]

$$a(C_0 + C_1 + C_2 + \dots + C_n) + b(C_1 + 2C_2 + \dots + ^nC_n)$$

$$= a \cdot 2^n + b \cdot n 2^{n-1} = 2^n \left(\frac{2a+nb}{2} \right) = (2a+nb) 2^{n-1}$$

Multinomial theorem :

As we know the Binomial Theorem –

$$(x+y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r = \sum_{r=0}^n \frac{n!}{(n-r)! r!} x^{n-r} y^r$$

putting $n-r = r_1, r = r_2$

$$\text{therefore, } (x+y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1! r_2!} x^{r_1} \cdot y^{r_2}$$

Total number of terms in the expansion of $(x+y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. $^{n+2-1}C_{2-1} = ^{n+1}C_1 = n+1$

In the same fashion we can write the multinomial theorem

$$(x_1 + x_2 + x_3 + \dots + x_k)^n$$

$$= \sum_{r_1+r_2+\dots+r_k=n} \frac{n!}{r_1! r_2! \dots r_k!} x_1^{r_1} \cdot x_2^{r_2} \dots x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + \dots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \dots + r_k = n$ i.e. $^{n+k-1}C_{k-1}$

Solved Examples

Ex.33 Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a-b-c+d)^{10}$

$$\text{Sol. } (a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1! r_2! r_3! r_4!}$$

$$(a)^{r_1} (-b)^{r_2} (-c)^{r_3} (d)^{r_4}$$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$$\therefore \text{coeff. of } a^2 b^3 c^4 d \text{ is } \frac{(10)!}{2! 3! 4! 1!} (-1)^3 (-1)^4 = -12600$$

Ex.34 In the expansion of $\left(1+x+\frac{7}{x}\right)^{11}$, find the term independent of x .

$$\text{Sol. } \left(1+x+\frac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1! r_2! r_3!} (1)^{r_1} (x)^{r_2} \left(\frac{7}{x}\right)^{r_3}$$

The exponent 11 is to be divided among the base variables 1, x and $\frac{7}{x}$ in such a way so that we get x^0 .

Therefore, possible set of values of (r_1, r_2, r_3) are $(11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)$

Hence the required term is

$$\begin{aligned} & \frac{(11)!}{(11)!} (7^0) + \frac{(11)!}{9!1!1!} 7^1 + \frac{(11)!}{7!2!2!} 7^2 + \frac{(11)!}{5!3!3!} 7^3 \\ & + \frac{(11)!}{3!4!4!} 7^4 + \frac{(11)!}{1!5!5!} 7^5 \\ & = 1 + \frac{(11)!}{9!2!} \cdot \frac{2!}{1!1!} 7^1 + \frac{(11)!}{7!4!} \cdot \frac{4!}{2!2!} 7^2 + \frac{(11)!}{5!6!} \cdot \frac{6!}{3!3!} 7^3 \\ & + \frac{(11)!}{3!8!} \cdot \frac{8!}{4!4!} 7^4 + \frac{(11)!}{1!10!} \cdot \frac{(10)!}{5!5!} 7^5 \\ & = 1 + {}^{11}C_2 \cdot {}^2C_1 \cdot 7^1 + {}^{11}C_4 \cdot {}^4C_2 \cdot 7^2 + {}^{11}C_6 \cdot {}^6C_3 \cdot 7^3 \\ & + {}^{11}C_8 \cdot {}^8C_4 \cdot 7^4 + {}^{11}C_{10} \cdot {}^{10}C_5 \cdot 7^5 \\ & = 1 + \sum_{r=1}^5 {}^{11}C_{2r} \cdot {}^{2r}C_r \cdot 7^r \end{aligned}$$

Binomial Theorem for any Index

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

- (i) Its first term is 1, and
- (ii) Its second term is numerically less than 1.

Thus when $n \notin \mathbb{N}$ and $|x| < 1$, then it states

$$\begin{aligned} (1+x)^n &= 1 + nx + \\ & \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \\ & + \frac{n(n-1)(n-r+1)}{r!} x^r + \dots \infty \end{aligned}$$

1. General Term :

$$* T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

Note :

- (i) In this expansion the coefficient of different terms can not be expressed as ${}^nC_0, {}^nC_1, {}^nC_2, \dots$ because n is not a positive integer.
- (ii) In this case there are infinite terms in the expansion.

2. Some Important Expansions :

If $|x| < 1$ and $n \in \mathbb{Q}$ but $n \notin \mathbb{N}$, then

- (a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} x^r + \dots$
- (b) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} (-x)^r + \dots$
- (c) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!} x^r + \dots$
- (d) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!} (-x)^r + \dots$

By putting $n = 1, 2, 3$ in the above results (c) and (d), we get the following results-

- * $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots$
General term $T_{r+1} = x^r$
- * $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-x)^r + \dots$
General term $T_{r+1} = (-x)^r$
- * $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots$
General term $T_{r+1} = (r+1)x^r$
- * $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (r+1)(-x)^r + \dots$
General term $T_{r+1} = (r+1)(-x)^r$
- * $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{2!} x^r + \dots$

$$\text{General term} = \frac{(r+1)(r+2)}{2!} x^r$$

- * $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + \frac{(r+1)(r+2)}{2!} (-x)^r + \dots$

$$\text{General term} = \frac{(r+1)(r+2)}{2!} (-x)^r$$

Solved Examples

Ex.35 If x is very large and n is a negative integer or a proper fraction, then an approximate value of

$\left(\frac{1+x}{x}\right)^n$ is equal to-

- [1] $1 + \frac{x}{n}$ [2] $1 + \frac{n}{x}$
[3] $1 + \frac{1}{x}$ [4] $n \left(1 + \frac{1}{x}\right)$

Sol. [2] $\left(1 + \frac{1}{x}\right)^n = 1 + \frac{n}{x} + \frac{n(n-1)}{1.2} \left(\frac{1}{x}\right)^2 + \dots$

but as x is very large terms therefore after 2nd term other terms can be ignored.

Ex.36 If $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to $a + bx$ for small values of x , then $(a, b) =$

- [1] $\left(1, \frac{35}{24}\right)$ [2] $\left(1, -\frac{35}{24}\right)$
[3] $\left(2, \frac{35}{12}\right)$ [4] $\left(2, -\frac{35}{12}\right)$

Sol. [2] $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{2 \left[1 - \frac{x}{4}\right]^{1/2}}$

$$= \frac{\left[1 + \frac{1}{2}(-3x) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2 + \dots\right] + \left[1 + \frac{5}{3}(-x) + \frac{5 \cdot 2}{3 \cdot 2}(-x)^2 + \dots\right]}{2 \left[1 + \frac{1}{2}\left(-\frac{x}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2 + \dots\right]}$$

$$= \frac{\left[1 - \frac{19}{12}x + \frac{53}{144}x^2 - \dots\right]}{\left[1 - \frac{x}{8} - \frac{1}{8}x^2 - \dots\right]} = 1 - \frac{35}{24}x + \dots$$

Neglecting higher powers of x , then

$$a + bx = 1 - \frac{35}{24}x \Rightarrow a = 1, b = -\frac{35}{24}$$

Ex.37 Prove that the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$

Sol. $(r+1)^{\text{th}}$ term in the expansion of $(1-x)^{-n}$ can be written as

$$\begin{aligned} T_{r+1} &= \frac{-n(-n-1)(-n-2)\dots(-n-r+1)}{r!} (-x)^r \\ &= (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} (-x)^r \\ &= \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r \\ &= \frac{(n-1)! n(n+1)\dots(n+r-1)}{(n-1)! r!} x^r \end{aligned}$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)! r!} = {}^{n+r-1}C_r$ Proved

Ex.38 If x is so small such that its square and higher powers may be neglected, then find the value of

$$\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$$

Sol. $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1 - \frac{3}{2}x + 1 - \frac{5x}{3}}{2 \left(1 + \frac{x}{4}\right)^{1/2}}$

$$\begin{aligned} &= \frac{1}{2} \left(2 - \frac{19}{6}x\right) \left(1 + \frac{x}{4}\right)^{-1/2} \\ &= \frac{1}{2} \left(2 - \frac{19}{6}x\right) \left(1 - \frac{x}{8}\right) = \frac{1}{2} \left(2 - \frac{x}{4} - \frac{19}{6}x\right) \\ &= 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x \end{aligned}$$