GRAVITATION

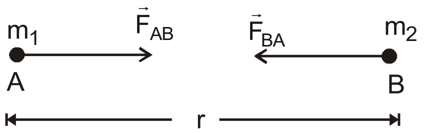
1. Newtons's Law of Universal Gravitation

All physical bodies are subjected to the action of the forces of mutual gravitational attraction. The basic law describing the gravitational forces was stated by Sir Issac Newton and it is called Newton's Law. of Universal gravitation.

The law is stated as : "Between any two particles of masses m_I and m_2 at separation r from each other there exist attractive forces $\vec{F}_{AB}$ and $\vec{F}_{BA}$ directed from one body to the other and equal in magnitude which is directly proportional to the product of the masses of the bodies and inversely proportional to the square of the distance between the two". Thus we can write

$$F_{AB} = F_{BA} = G\frac{m_1 m_2}{r^2} \qquad ...(1)$$

Where G is called universal gravitational constant. Its value is equal to 6.67×10^{-11} Nm^2/kg. The law of gravitation can be applied to the bodies whose dimensions are small as compared to the separation between the two or when bodies can be treated as point particles.



If the bodies are not very small sized, we can not directly apply the expression in equation-(1) to find their natural gravitational attraction. In this case we use the following procedure to find the same. The bodies are initially split into small parts or a large number of point masses. Now using equation-(1) the force of attraction exerted on a particle of one body by a particle of another body can be obtained. Now we add all forces vectorially which are exerted by all independent particles of second body on the particle of first body. Finally the resultants of these forces is summed over all particles of the first body to obtain the net force experinced by the bodies. In general we use integration or basic summation of these forces.

⇒ Gravitational force is a conservative force.
⇒ Gravitational force is a central force.
⇒ Gravitational force is equal in magnitude & opposite in direction
⇒ Gravitational forces are action - reaction pair.
⇒ Gravitational force acts along the line joining the two masses.
⇒ Gravitational force doesn't depend upon the medium
⇒ Gravitational force is an attractive force.

$$\vec{F} = \frac{-Gm_1m_2\vec{r}}{|\vec{r}|^3}$$

[Head of $\vec{r}$ is placed at that position where we have to evaluate force]

2. GRAVITATIONAL FIELD

We can state by Newton's universal law of gravitation that every mass M produces, in the region around it, a physical situation in which, whenever any other mass is placed, force acts on it, is called gravitational field. This field is recognized by the force that the mass M exerts another mass, such as *m*, brought into the region.

2.1 Strength of Gravitational Field

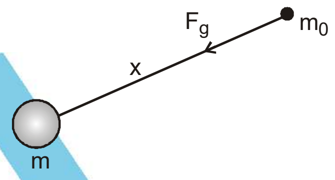
We define gravitational field strength at any point in space to be the gravitational force per unit mass on a test mass (mass brought into the field for experimental observation). Thus for a point in space if a test mass m_0, experiences a force $\vec{F}$, then at that point in space, gravitational field strength which is denoted by $\vec{g}$, is given as $\vec{g} = \frac{\vec{F}}{m_0}$

Gravitational field strength $\vec{g}$ is a vector quantity and has same direction as that of the force on the test mass in field.

Generally magnitude of test mass is very small such that its gravitational field does not modify the field that is being measured. It should be also noted that gravitational field strength is just the acceleration that a unit mass would experience at that point in space.

2.2 Gravitational Field Strength of Point Mass

As per our previous discussion we can state that every point mass also produces a gravitational field in its surrounding. To find the gravitational field strength due to a point mass, we put a test mass m_0 at a point P at distance x from a point mass m then force on m_0 is given as



$$F_g = \frac{Gmm_0}{x^2}$$

Now if at point P, gravitational field strength due to m is g_p then it is given as

$$g_p = \frac{F_g}{m_0} = \frac{Gm}{x^2}$$

The expression written in above equation gives the gravitational field strength at a point due to a point mass.

It should be noted that the expression in equation written above is only applicable for gravitational field strength due to point masses. It should not be used for extended bodies.

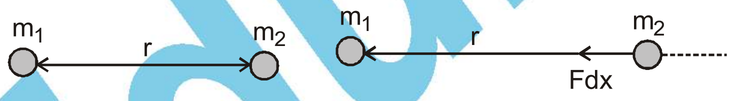
However, the expression for the gravitational field strength produced by extended masses has already been derived in electrostatics section.

3. INTERACTION ENERGY

This energy exists in a system of particles due to the interaction forces between the particles of system. Analytically this term is defined as the work done against the interaction of system forces in assembling the given configuration of particles. To understand this we take a simple example of interaction energy of two points masses.

Figure (a) shows a system of two point masses m_1 and m_2 placed at a distance r apart in space. here if we wish to find the interaction potential energy of the two masses, this must be the work done in bringing the two masses from infinity (zero interaction state) to this configuration. For this we first fix m_1 at its position and bring m_2 slowly from infinity to its location. If in the process m_2 is at a distance x from m_1 then force on it is

$$F = -\frac{Gm_1m_2}{x^2}\hat{i}$$



(a) (b)

This force is applied by the gravitational field of m_1 to m_2. If it is further displaced by a distance dx towards m_1 then work done by the field is

$$dW = \vec{F}.\vec{dx} = \frac{Gm_1m_2}{x^2}dx$$

Now in bringing m_2 from infinity to a position at a distance r from $m_{1,}$ the total work done by the field is

$$W = \int dW - \int_{\infty}^{r} \frac{Gm_1m_2}{x^2}dx = -G\,m_1\,m_2\left[-\frac{1}{x}\right]_{\infty}^{r}$$

$$W = +\frac{Gm_1m_2}{r}$$

Thus during the process field of system has done $\frac{Gm_1m_2}{r}$ amount of work. The work is positive because the displacment of body is in the direction of force. Initially when the separation between m_1 and m_2 was very large (at infinity) there was no interaction between them. We conversely say that as a reference when there is no interaction the interaction energy of the system is zero and during the process system forces (gravitational forces) are doing work so system energy will decrease and becomes negative (as initial energy was zero). As a consequence we can state that in general if system forces are attractive, in assembling a system of particles work will be done by the system and it will spend energy in assembling itself. Thus finally the interaction energy of system will be negative. On the other hand if in a given system of particles, the system forces are repulsive, then in assembling a system some external forces have to be work against the system forces and in this case some work must be done by external forces on the system hence finally the interaction energy of the system of particles must be positive.

In above example as work is done by the gravitaional forces of the system of two masses, the interaction energy of system must be negative and it can be given as

$$U_{12} = -\frac{Gm_1m_2}{r} \qquad ...(1)$$

As gravitational forces are always attractive, the gravitational potential energy is always taken negative.

3.1 Interaction Energy of a System of Particles

If in a system there are more than two particles then we can find the interaction energy of particle in pairs using equation (1) and finally sum up all the results to get the total energy of the system. For example in a system of N particles with masses m_1, m_2........m_n separated from each other by a distance r_{12}........ where r_{12} is the separation between m_1 and m_2 and so on.

In the above case the total interaction energy of system is given as

$$U = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_im_j}{r_{ij}}$$

In this expression the factor $\frac{1}{2}$ is taken because the interaction energy for each possible pair of system is taken twice during summation as for mass m_1 and m_3

$$U = -\frac{Gm_1m_3}{r_{13}} = -\frac{Gm_3m_1}{r_{31}}$$

Now to understand the applications of interaction energy we take few examples.

4 GRAVITATIONAL POTENTIAL

The gravitational potential at a point in gravitational field is the gravitional potenial energy per unit mass placed at that point in gravitational field. Thus at a certain point in gravitational field, a mass m_0 has a potential energy U then the gravitational potential at that point is given as

$$V = \frac{U}{m_0}$$

or if at a point in gravitational field gravitational potential V is known then the interaction potential energy of a point mass m_0 at that point in the field is given as

$$U = m_0V$$

Interaction energy of a point mass m_0 in a field is defined as work done in bringing that mass from infinity to that point. In the same fashion we can define gravitational potential at a point in field, alternatively as "*Work done in bringing a unit mass from infinity to that point against gravitational forces.*"

When a unit mass is brought to a point in a gravitational field, force on the unit mass is $\vec{g}$ at a point in the field. Thus the work done in bringing this unit mass from infinity to a point P in gravitational field or gravitational potential at point P is given as

$$V_P = -\int_{\infty}^{P}\vec{g}\cdot\vec{dx}$$

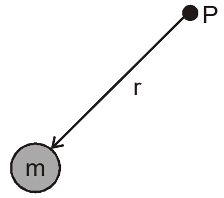
Here negative sign shown that V_P is the negative of work done by gravitation field or it is the external required work for the purpose against gravitational forces.

4.1 Gravitational Potential due to a Point Mass

We know that in the surrounding of a point mass it produces its gravitational field. If we wish to find the gravitational potential at a point P situated at a distance r from it as shown in figure, we place a test mass m_0 at P and we find the interaction energy of m_0 with the field of m, which is given as

$$U = -\frac{Gmm_0}{r}$$

Now the gravitational potential at P due to *m* can be written as



$$V = \frac{U}{m_0} = -\frac{Gm}{r}$$

The expression of gravitational potential in equation is a standard result due to a point mass which can be used as an elemental form to find other complex results, we'll see later.

The same thing can also be obtained by using equation

$$V_P = \int_{\infty}^{P} \vec{g}.\vec{dx} \quad \text{or} \quad V_P = \int_{\infty}^{r} \frac{Gm}{x^2}dx \quad \text{or} \quad V_P = -\frac{Gm}{r}$$

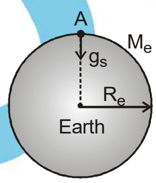
4.2 Gravitational Field Strength of Earth:

We can consider earth to be a very large sphere of mass M_e and radius R_e. Gravitational field strength due to earth is also regarded as acceleration due to gravity or gravitational acceleration. Now we find values of g at different points due to earth.

- Earth behaves as a non conducting solid sphere

4.3 Value of g on Earth's Surface :

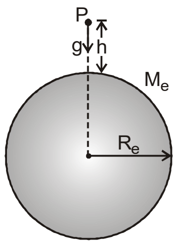
If g_s be the gravitational field strength at a point A on the surface of earth, then it can be easily obtained by using the result of a solid sphere. Thus for earth, value of g_s can be given as



$$g_s = \frac{GM_e}{R_e^2} \qquad \text{...(1)}$$

4.4 Value of g at a Height *h* Above the Earth's Surface:

If we wish to find the value of g at a point P as shown in figure shown at a height h above the Eath's surface. Then the value can be obtained as



$$g_s = \frac{GM_e}{(R_e + h)^2} \quad \text{or} \quad g_s = \frac{GM_e}{R_e^2\left(1 + \frac{h}{R_e}\right)^2} = \frac{g_s}{\left(1 + \frac{h}{R_e}\right)^2}$$

If point P is very close to Earth's surface then for h << R_e we can rewrite the expression in given equation as

$$g_h = g_s\left(1 + \frac{h}{R_e}\right)^{-2} \simeq g_s\left(1 - \frac{2h}{R_e}\right) \quad \text{[Using binomial approximation]} \qquad \text{...(2)}$$

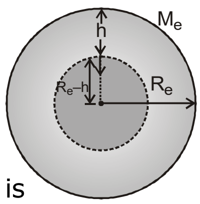
4.5 Value of g at a Depth *h* Below the Earth's Surface

If we find the value of g inside the volume of earth at a depth *h* below the earth's surface at point P as shown in figure, then we can use the result of *g* inside a solid sphere as

$$g_{in} = \frac{GM_e x}{R_e^3}$$

Here x, the distance of point from centre of earth is given as

$x = R_e - h$



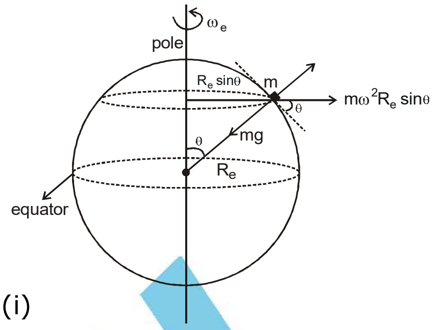
Thus we have $$g_h = \frac{GM_e(R_e - h)}{R_e^3} = g_s\left(1 - \frac{h}{R_e}\right) \qquad \text{...(3)}$$

From equation (1), (2) and (3) we can say that the value of *g* at eath's surface is maximum and as we move above the earth's surface or we go below the surface of earth, the value of *g* decrease.

4.6 Effect of Earth's Rotation on Value of *g*

Let us consider a body of mass *m* placed on Earth's surface at a latitude θ as shown in figure. This mass experiences a force mg_s towards the centre of earth and a centrifugal force $mw_e^2 R_e \sin\theta$ relative to Earth's surface as shown in figure.

If we consider g_{eff} as the effective value of g on earth surface at a latitude θ then we can write



$$g_{eff} = \frac{\vec{F}_{net}}{m} = g_{eff} = \sqrt{(\omega_e^2 R_e \sin\theta)^2 + g^2 + 2\omega_e^2 R_e \sin\theta . g \cos(90+\theta)}$$

ω_e^4 is very very small

So we can write $\quad g_{eff} = \sqrt{g^2 - 2\omega_e^2 R_e \sin^2\theta g}$

$$g_{eff} = g\left(1 - \frac{2\omega_e^2 R_e \sin^2\theta}{g}\right)^{1/2} \approx g - \omega_e^2 R \sin^2\theta \qquad ...(i)$$

From equation (1) we can find the value of effective gravity at poles and equatorial points on Earth as

At poles $\theta = 0 \Rightarrow \quad g_{poles} = g_s = 9.83$ m/s²

At equator $\theta = \frac{\pi}{2} \Rightarrow \quad g_{equator} = g_s - \omega^2 R_e = 9.78$ m/s²

Thus we can see that the body if placed at poles of Earth, it will only have a spin, not circular motion so there is no reduction in value of g at poles due to rotation of earth. Thus at poles value of g on Earth surface is maximum and at equator it is minimum. But an average we take 9.8 m/s², the value of g everywhere on earth's surface.

4.7 Effect of Shape of Earth on Value of g

Till now we considered that earth is spherical in its shape but this is not actually true. Due to some geological and astromonical reasons, the shape of earth is not exact spherical. It is ellipsoidal.

As we've discussed that the value of g at a point on earth surface depends on radius of Earth.It is observed that the approximate difference in earth's radius at different points on equator and poles is

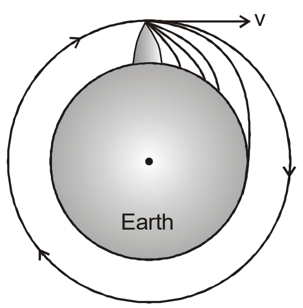
$r_e - r_p \simeq 21$ to 34 km. Due to this the difference in value of g at poles and equatorial points is approximately

$g_p - g_e \simeq 0.02$ to 0.04 m/s², which is very small. So for numerical calculations, generally, we ignore this factor while taking the value of g and we assume Earth is spherical in shape.

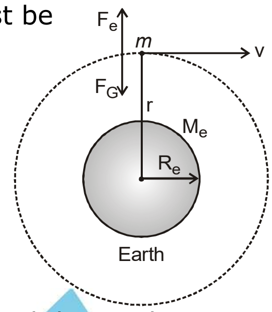
5. SATELLITE AND PLANETARY MOTION

5.1 Motion of a Satellite in a Circular Orbit

To understand how a satellite continously moves in its orbit, we consider the projection of a body horizontally from the top of a high mountain on earth as shown in figure. Here till our discussion ends we neglect air friction. The distance the projectile travels before hitting the ground depends on the launching speed. The greater the speed, the greater the distance. The distance the projectile travels before hitting the ground is also affected by the curvature of earth as shown in figure shown. This figure was given by newton in his explanantion of laws of gravitation. it shows different trajectories for diferent launching speeds. As the launching speed is made greater, a speed is reached where by the projectile's path follow the curvature of the earth. This is the launching speed which places the projectile in a circular orbit. Thus an object in circular orbit may be regarded as falling, but as it falls its path is concentric with the earth's spherical surface and the object maintains a fixed distance from the earth's centre. Since the motion may continue indefinitely, we may say that the orbit is stable.



Let's find the speed of a satellite of mass m in a circular orbit around the earth. Consider a satellite revolving around the earth in a circular orbit of radius r as shown in figure.



If its orbit is stable during its motion, the net gravitational force on it must be balanced by the centrifugal force on it
relative to the rotating frame as

$$\frac{GM_e m}{r^2} = \frac{mv^2}{r} \quad \text{or} \quad v = \sqrt{\frac{GM_e}{r}}$$

Expression in above equation gives the speed of a statellite in a stable circular orbit of radius r.

5.2 Energies of a Satellite in a Circular Orbit

When there is a satellite revolving in a stable circular orbit of radius *r* around the earth, its speed is given by above equation. During its motion the kinetic energy of the satellite can be given as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GM_e m}{r} \qquad \text{...(1)}$$

As gravitational force on satellite due to earth is the only force it experiences during motion, it has gravitational interaction energy in the field of earth, which is given as

$$U = -\frac{GM_e m}{r} \qquad \text{...(2)}$$

Thus the total energy of a satellite in an orbit of radius r can be given as

Total energy E = Kinetic energy K + Potential Energy U

$$= \frac{1}{2}\frac{GM_e m}{r} - \frac{GM_e m}{r} \quad \text{or} \quad E = -\frac{1}{2}\frac{GM_e m}{r} \qquad \text{...(3)}$$

From equation (1), (2) and (3) we can see that $|k| = \left|\frac{U}{2}\right| = |E|$

The above relation in magnitude of total, kinetic and potential energies of a satelline is very useful in numerical problem so it is advised to keep this relation in mind while handing satellite problems related to energy.

Now to understand satellite and planetary motion in detail, we take few example.

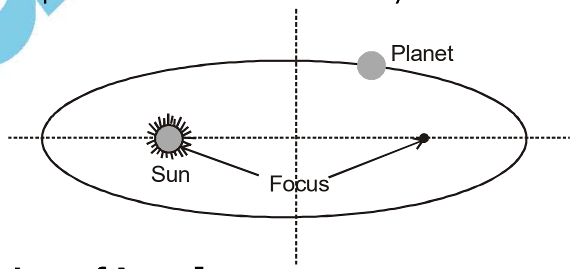
6. KEPLER'S LAWS OF PLANETARY MOTION

The motions of planet in universe have always been a puzzle. In 17^{th} century Johannes Kepler, after a life time of study worded out some empirical laws based on the analysis of astronomical measurements of Tycho Brahe. Kepler formulated his laws, which are kinematical description of planetary motion. Now we discuss these laws step by step.

6.1 Kepler's First Law [The Law of Orbits]

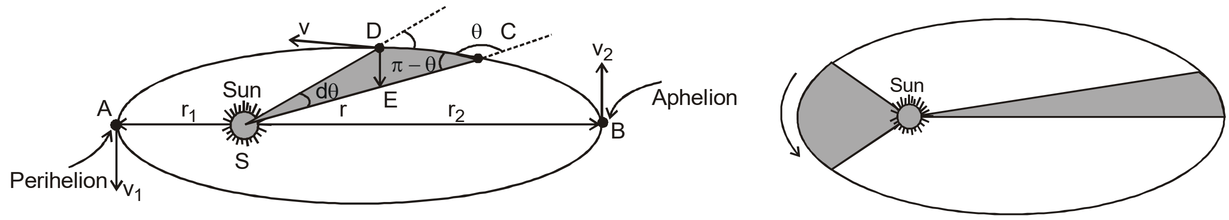
Kepler's first law is illustrated in the image shown in figure. It states that "All the planets move around the sun in ellipitcal orbits with sun at one of the focus not at centre of orbit."

It is observed that the orbits of planets around sun are very less ecentric or approximately circular



6.2 Kepler's Second Law [The Law of Areas]

Kepler's second Law is basically an alternative statement of law of conservation of momentum. It is illustrated in the image shown in figure(a). We know from angular momentum conservation, in elliptical orbit plane will move faster when it is nearer to the sun. Thus when a planet executes elliptical orbit its angular speed changes continuously as it moves in the orbit. The point of nearest approach of the planet to the sun is termed perihelion. The point of greatest seperation is termed aphelion. Hence by angular momentum conservation we can state that the planet moves with maximum speed when it is near perihelion and moves with slowest speed when it is near aphelion.



(a) (b)

Kepler's second law states that "The line joining the sun and planet sweeps out equal areas in equal time or the rate of sweeping area by the position vector of the planet with respect to sun remains constant. "This is shown in figure (b).

The above statement of Kepler's second law can be verified by the law of conservation of angular momentum. To verify this consider the moving planet around the sun at a general point C in the orbit at speed v. Let at this instant the distance of planet from sun is *r*. If θ be the angle between position vector $\vec{r}$ of planet and its velocity vector then the angular momentum of planet at this instant is

$$L = mvr\sin\theta \qquad ...(1)$$

In an elemental time the planet will cover a small distance CD = dl and will travel to another adjacent point D as shown in figure (a), thus the distance CD = vdt. In this duration dt, the position vector $\vec{r}$ sweeps out an area equal to that of triangle SCD, which is calculated as

Area of triangle SCD is $dA = \frac{1}{2} \times r \times vdt\sin(\pi - \theta) = \frac{1}{2} rv\sin\theta.\, dt$

Thus the rate of sweeping area by the position vector $\vec{r}$ is

$$\frac{dA}{dt} = \frac{1}{2} rv\sin\theta$$

Now from equation (1)

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant} \qquad ...(2)$$

The expression in equation (2) verifies the statement of Kepler' II law of planetary motion.

6.3 Kepler's Third law [The Law of Periods]

Kepler's Third Law is concerned with the time period of revolution of planets. It states that "The time period of revolution of a planet in its orbit around the sun is directly proportional to the cube of semi-major axis of the elliptical path around the sun"

If 'T' is the period of revolution and 'a' be the semi-major axis of the path of planet then according to Kepler's III law, we have

$$T^2 \propto a^3$$

For circular orbits, it is a special case of ellipse when its major and minor axis are equal. If a planet is in a circular orbit of radius r around the sun then its revolution speed must be given as

$$v = \sqrt{\frac{GM_s}{r}}$$

Where M_s is the mass of sun. Here you can recall that this speed is independent from the mass of planet. Here the time period of revolution can be given as

$$T = \frac{2\pi r}{v} \quad \text{or} \quad T = \frac{2\pi r}{\sqrt{\frac{GM_s}{r}}}$$

Squaring equation written above, we get

$$T^2 = \frac{4\pi^2}{GM_s} r^3 \qquad ...(1)$$

Equation (1) verifies the statement of Kepler's third law for circular orbits. Similarly we can also verify it for elliptical orbits. For this we start from the relation we've derived earlier for rate of sweeping area by the position vector of planet with respect to sun which is given as

$$\frac{dA}{dt} = \frac{L}{2m}$$

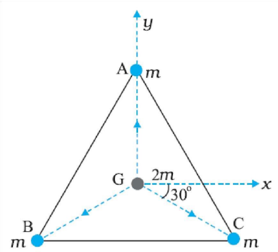
SOLVED EXAMPLE

Ex.1 Let the speed of the planet at the perihelion P be v_P and the Sun-planet distance SP be rP. Relate {rP, vP} to the corresponding quantities at the aphelion {rA, vA}. Will the planet take equal times to traverse BAC and CPB ?

Ans. The magnitude of the angular momentum at P is Lp = mp rp υp, since inspection tells us that **r**p and **v**p are mutually perpendicular. Similarly, LA = mp rA υA. From angular momentum conservation mp rp υp = mp rA υA or $\frac{\upsilon_P}{\upsilon_A} = \frac{r_A}{r_P}$

Since rA > rp, υp > υA . The area SBAC bounded by the ellipse and the radius vectors SB and SC is larger than SBPC in. From Kepler's second law, equal areas are swept in equal times. Hence the planet will take a longer time to traverse BAC than CPB.

Ex.2 Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.
(a) What is the force acting on a mass 2m placed at the centroid G of the triangle?
(b) What is the force if the mass at the vertex A is doubled ? Take AG = BG = CG = 1m



Ans. (a) The angle between GC and the positive x-axis is 30° and so is the angle between GB and the negative x-axis. The individual forces in vector notation are

$$F_{GA} = \frac{Gm(2m)}{1}\hat{j}$$

$$F_{GB} = \frac{Gm(2m)}{1}\left(-\hat{i}\cos 30^o - \hat{j}\sin 30^o\right)$$

$$F_{GC} = \frac{Gm(2m)}{1}\left(+\hat{i}\cos 30^o - \hat{j}\sin 30^o\right)$$

From the principle of superposition and the law of vector addition, the resultant gravitational force **F**R on (2m) is **F**R = **F**GA + **F**GB + **F**GC

$$F_R = 2Gm^2\hat{j} + 2Gm^2\left(-\hat{i}\cos 30^o - \hat{j}\sin 30^o\right)$$

$$+ 2Gm^2\left(\hat{i}\cos 30^o - \hat{j}\sin 30^o\right) = 0$$

Alternatively, one expects on the basis of symmetry that the resultant force ought to be zero.

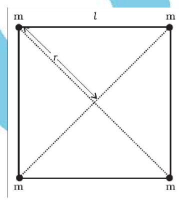
(b) By symmetry the x-component of the force cancels out. The y-component survives

$$F_R = 4Gm^2\hat{j} - 2Gm^2\hat{j} = 2Gm^2\hat{j}$$

Ex.3 Find the potential energy of a system of four particles placed at the vertices of a square of side l. Also obtain the potential at the centre of the square.

Ans. Consider four masses each of mass m at the corners of a square of side l ; See Fig. We have four mass pairs at distance l and two diagonal pairs at distance $\sqrt{2}\,\ell$ Hence

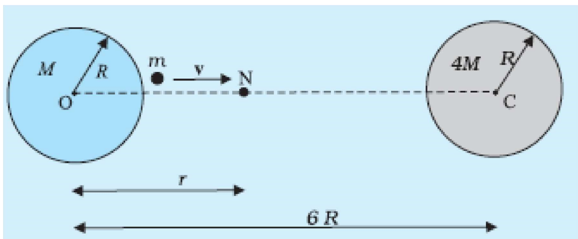
$$W(r) = -4\frac{Gm^2}{\ell} - 2\frac{Gm^2}{\sqrt{2}\ell}$$



$$= -\frac{2Gm^2}{\ell}\left(2 + \frac{1}{\sqrt{2}}\right) = -5.41\frac{Gm^2}{\ell}$$

The gravitational potential at the centre of the square $\left(r = \sqrt{2}\,\ell / 2\right)$ is $U(r) = -4\sqrt{2}\frac{Gm}{\ell}$

Ex.4 Two uniform solid spheres of equal radii R, but mass M and 4 M have a center to centre separation 6 R, as shown in Fig. 8.10. The two spheres are held fixed. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.



Ans. The projectile is acted upon by two mutually opposing gravitational forces of the two spheres. The neutral point N (see Fig. 8.10) is defined as the position where the two forces cancel each other exactly. If

ON = r, we have $\frac{GMm}{r^2} = \frac{4GMm}{(6R-r)^2}$

$(6R - r)2 = 4r^2$

$6R - r = \pm 2r$

$r = 2R$ or $-6R$

The neutral point r = – 6R does not concern us in this example. Thus ON = r = 2R. It is sufficient to project the particle with a speed which would enable it to reach N. Thereafter, the greater gravitational pull of 4M would suffice. The mechanical energy at the surface of M is $E_t = \frac{1}{2}m\upsilon^2 - \frac{GMm}{R} - \frac{4GMm}{5/R}$

At the neutral point N, the speed approaches zero. The mechanical energy at N is purely potential.

$$E_N = -\frac{GMm}{2R} - \frac{4GMm}{4R}$$

From the principle of conservation of mechanical energy

$$\frac{1}{2}\upsilon^2 - \frac{GM}{R} - \frac{4GM}{5R} = -\frac{GM}{2R} - \frac{GM}{R}$$

or $$\upsilon^2 = \frac{2GM}{R}\left(\frac{4}{5} - \frac{1}{2}\right)$$

$$\upsilon = \left(\frac{3GM}{5R}\right)^{1/2}$$

Ex.5 The planet Mars has two moons, phobos and delmos. (i) phobos has a period 7 hours, 39 minutes and an orbital radius of 9.4 ~ 103 km. Calculate the mass of mars. (ii) Assume that earth and mars move in circular orbits around the sun, with the martian orbit being 1.52 times the orbital radius of the earth. What is the length of the martian year in days ?

Ans. (i) The sun's mass replaced by the martian mass

$$M_m \quad T^2 = \frac{4\pi^2}{GM_m}R^3$$

$$M_m = \frac{4\pi^2 R^3}{G\,T^2}$$

$$= \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times 10^{-11} \times (459 \times 60)^2}$$

$$M_m = \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times (459 \times 6)^2 \times 10^{-5}} = 6.48 \times 10^{23}\ \text{kg}$$

(ii)Once again Kepler's third law comes to our aid,

$$\frac{T_M^2}{T_E^2} = \frac{R_{MS}^3}{R_{ES}^3}$$

where RMS is the mars -sun distance and RES is the earth-sun distance.

$\therefore$ TM = $(1.52)^{3/2} \times 365$

= 684 days

We note that the orbits of all planets except Mercury, Mars and Pluto are very close to being circular.

Ex.6 Weighing the Earth : You are given the following data: g = 9.81 ms^{-2}, R_E = 6.37 × 10^6 m, the distance to the moon R = 3.84 × 10^8 m and the time period of the moon's revolution is 27.3 days. Obtain the mass of the Earth M_E in two different ways.

Ans. we have $M_E = \frac{gR_E^2}{G}$

$$= \frac{9.81 \times (6.37 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.97 \times 10^{23}\ \text{kg}$$

The moon is a satellite of the Earth. From the derivation of Kepler's third law

$$T^2 = \frac{4\pi^2 R^3}{G\,M_E}$$

$$M_E = \frac{4\pi^2 R^3}{G\,T^2}$$

$$= \frac{4 \times 3.14 \times 3.14 \times (3.84)^3 \times 10^{24}}{6.67 \times 10^{-11} \times (27.3 \times 24 \times 60 \times 60)^2}$$

$= 6.02 \times 10^{24}$ kg

Ex.7 Express the constant k of Eq. (8.38) in days and kilometres. Given k = 10^{-13} s^2 m^{-3}. The moon is at a distance of 3.84 × 10^5 km from the earth. Obtain its time-period of revolution in days.

Ans. Given k = 10^{-13} s^2 m^{-3}

$$=10^{-13}\left[\frac{1}{(24 \times 60 \times 60)^2}d^2\right]\left[\frac{1}{(1/1000)^3 km^3}\right]$$

$= 1.33 \times 10^{-14}$ d^2 km^{-3}

$$K = \frac{4\pi^2}{GMg}, T^2$$

= K $[R_E = h]^3$ and the given value of k, the time period of the moon is

$T^2 = (1.33 \times 10^{-14})(3.84 \times 10^5)^3$

T = 27.3 d

Exercise - I UNSOLVED PROBLEMS

Q.1 Answer the following : **NCERT XI_8.1**

(a) Among the known types of forces in nature, the gravitational force is the weakest. Why then does it play a dominant role for motion of bodies on the terrestrial, astronomical and cosmological scale

(b) Do the forces of friction and other contact forces arise due to gravitational attraction ? If not, what is the origin of these forces ?

(c) You can shield a charge from electrical forces by putting it inside a hollow conductor. Can you shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means ?

(d) An astronaut inside a small space ship orbiting around the earth cannot detect gravity. If the space station orbiting around the Earth has a large size, can he hope to detect gravity ?

(e) If you compare the gravitational force on the Earth due to the Sun to that due to the moon, you would find that the Sun's pull is greater than the moon's pull is greater than the tidal effect of Sun. Why?

Q.2 Choose the correct alternative : **NCERT XI_8.2**

(a) Acceleration due to gravity increases/ decreases with increasing altitude.

(b) Acceleration due to gravity increases/ decreases with increasing depth (assume the Earth to be a sphere of uniform density).

(c) The effect of rotation on the effective value of acceleration due to gravity is greatest at the equator/poles.

(d) Acceleration due to gravity is independent of mass of the Earth/mass of the body.

(e) The formula – G Mm $(1/r_2 - 1/r_1)$ is more/less accurate than the formula mg $(r_2 - r_1)$ for the difference of potential energy between two points r_2 and r_1 distance away from the centre of the Earth.

Q.3 Choose the correct alternative : **NCERT XI_8.3**

(a) If the gravitational potential energy of two mass points infinite distance away is taken to be is (positive/negative/zero)

(b) The universe on the large scale is shaped by (gravitational/electromagnetic) forces, on the atomic scale by (gravitational/electromagnetic) forces, on the nuclear scale by (gravitational/ electromagnetic/strong nuclear) forces.

(c) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its kinetic/potential energy.

(d) The energy required to rocket an orbiting satellite out of Earth's gravitational influence is more/ less than the energy requires to project a stationary object at the same height (as the satellite) out of Earth's influence.

Q.4 Does the escape speed of a body from the Earth depend on (a) the mass of the body, (b) the location from where it is projected, (c) the direction of projection, (d) the height of the location from where the body is launched ? Explain your answer. **NCERT XI_8.4**

Q.5 A comet orbits the sun in a highly elliptical orbit. Does the comet have a constant (a) linear speed, (b) angular speed, (c) angular momentum, (d) kinetic energy, (e) potential energy, (f) total energy throughout its orbit ? Neglect any mass loss of the comet when it comes very close to the Sun.

Q.6 Which of the following observations point to the equivalence of inertial and gravitational mass :

(a) Two spheres of different masses dropped from the top of a long evacuated tube reach the bottom of the tube at the same time.

(b) The time-period of a simple pendulum is independent of its mass.

(c) The gravitational force on a particle inside a hollow isolated sphere is zero.

(d) For a man in closed cabin that is falling freely under gravity, gravity disappears'.

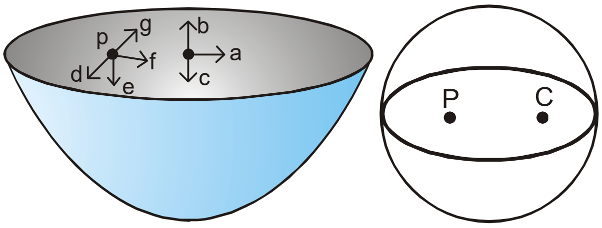
(e) An astronaut inside a spaceship orbiting around the earth feels weightless.

(f) Planets orbiting around the sun obey Kepler's Third Law (approximately).

(g) The gravitational force on a body due to the Earth is equal and opposite to the gravitational force on the Earth due to the body.

Q.7 Which of the following symptoms is likely to afflict an astronaut in space (a) swollen feet, (b) swollen face, (c) headache, (d) orientational problem.

Q.8 The gravitation intensity at the centre of the drumhead defined by a hemispherical shell has the direction indicated by the arrow (see figure) (i) a, (ii) a, (iii) C, (iv) zero.



Q.9 For the above problem, the direction of the gravitational intensity at an arbitrary point P is indicated by the arrow (i) d, (ii) e, (iii) f, (iv) g.

Q.10 A rocket is fired from the earth towards the Sun. At what distance from the Earth's centre is the gravitational force on the rocket zero ? Mass of the Sun = 2×10^{30} kg, mass of the Earth = 6×10^{24} kg. Neglect the effect of other planets etc. (orbital radius = 1.5×10^{11} m).

Q.11 How will you 'weight the Sun', that is estimate its mass ? You will need to know the period of one of its planets and the radius of the planetary orbit. The mean orbital radius of the Earth around the Sun is 1.5×10^{8} km. Estimate the mass of the sun.

Q.12 A Saturn year is 29.5 times the Earth year. How far is the Saturn from the Sun if the Earth is 1.50×10^{8} km away from the Sun ?

Q.13 A body weighs 63 N on the surface of the Earth. What is the gravitational force on it due to the Earth at a height equal to half the radius of the Earth ?

Q.14 Assuming the Earth to be a sphere of uniform mass density, how much would a body weigh half way down to the centre of the Earth if it weighed 250 N on the surface ?

Q.15 A rocket is fired vertically with a speed of 5 km s^{-1} from the Earth's surface. How far from the Earth does the rocket go before returning to the Earth ? Mass of the earth = 6.0×10^{24} kg; mean radius of the Earth = 6.4×10^{6} m ; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Q.16 The escape speed of a projectile on the Earth's surface is 11.2 km s^{-1}. A body is projected out with thrice this speed. What is the speed of the body far away from the Earth ? Ignore the presence of the Sun and other planets.

Q.17 A satellite orbits the Earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the Earth's gravitational influence ? Mass of the satellite = 200 kg; mass of the Earth = 6.0×10^{24} kg; radius of the Earth = 6.4×10^{6} m;
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Q.18 Two stars each of one solar mass (= 2×10^{30} kg) are approaching each other for a head on collision. When they are a distance 10^{9} km, their speeds are negligible. What is the speed with which they collide ? The radius of each star is 10^{4} km. Assume the stars to remain undistorted until they collide. (Use the known value of G).

Q.19 Two heavy spheres each of mass 100 kg and radius 0.10 m are placed 1.0 m apart on a horizontal table. What is the gravitational field and potential at the mid point of the line joining the centres of the spheres ? Is an object placed at that point in equilibrium ? If so, is the equilibrium stable or unstable? What are the factors on which it depend.

Q.20 As you have learnt in the text, a geostationary satellite orbits the Earth at a height of nearly 36,000 km from the surface of the Earth. What is the potential due to Earth's gravity at the site of this satellite ? (Take the potential energy at infinity to be zero).
Mass of the Earth = 6.0×10^{24} kg, radius = 6400 km.

Q.21 In a two-stage launch of a satellite, the first stage brings the satellite to a height of 150 km and the second stage gives it the necessary critical speed to put it in a circular orbit around the Earth. Which stage requires more expenditure of fuel ? (Neglect damping due to air resistance, especially in the first stage). Mass of the Earth = 6.0×10^{24} kg; radius = 6400 km; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Q.22 In an imaginary planetary system, the central star has the same mass as our Sun, but is much brighter so that only a planet twice the distance between the Earth and the Sun can support life. Assuming biological evolution (including aging processes etc.) on that planet similar to ours, what would be the average life span of a 'human' on that planet in terms of its natural year " The average life span of a human on the Earth may be taken to be 70 years.

Q.23 Imagine a tunnel dug along a diameter of the earth. Show that a particle dropped from one end of the tunnel executes simple harmonic motion. What is the time (equal to its known average density = 5520 kg m^{-3}) and G = 6.67 × 10^{-11} N m^2 kg^{-2}. Neglect all damping forces.

Q.24 If the Earth were a perfect sphere of radius 6.37 × 10^6 m, rotating about its axis with a period of one day (= 8.64 × 10^4 s) how much would the acceleration due to a gravity (g) differ from the poles to equator ?

Q.25 A star 2.5 times the mass of the sun and collapsed to a size of 12 km rotates with the speed of 1.2 rev. per second. (Extremely compact stars of this kind are known as neutron stars. Certain observed stellar objects called pulsars are believed to belong to this category). Will an object placed on its equator remain stuck to its surface due to gravity ? (Mass of the Sun = 2 × 10^{30} kg).

Q.26 A spaceship is stationed on Mars. How much energy must be expended on the spaceship to rocket it out of the solar system ? mass of the space ship = 1000 kg; mass of the sun = 2 × 10^{30} kg; mass of Mars = 6.4 × 10^{23} kg; radius of Mars = 3395 km; radius of the orbit of Mars = 2.28 × 10^8 km; G = 6.67 × 10^{-11} N m^2 kg^{-2}.

Q.27 A rocket is fired 'vertically' from the surface of Mars with a speed of 2 km s^{-1}. If 20% of its initial energy is lost due to Martian atmospheric resistance, how far will the rocket go from the surface of Mars before returning to it ? Mass of Mars = 6.4 × 10^{23} kg; radius of Mars = 3395 km; G = 6.67 × 10^{-11} N m^2 kg^{-2}.

Q.28 A non-homogeneous sphere of radius R has the following density variation :

Q.29 Light from a massive star suffers 'gravitational red-shift' i.e., its wavelength changes towards the red end due to the gravitational attraction of the star. Obtain the formula for this gravitational red-shift using the simple consideration that a photon of frequency ν has energy hν (h is the Planck's constant) and mass hν/c^2. Estimate the magnitude of the red-shift for light of wavelength 5000 Å from a star of mass 10^{32} kg and radius 10^6 km. Use the known values of G and c. (G = 6.67 × 10^{-11} N m^2 kg^{-2}, c = 3.00 × 10^8 m s^{-1}).

Q.30 The following problem is calculator/ computer based :

We have studied the depth dependence of the acceleration due to gravity of the Earth (g(d)), assuming the Earth to be a sphere of constant density. Dziewonski and Anderson have proposed a sophisticated five-tier model outlined in the table below. Plot the variation of g(r) with r where r is the distance from the centre of the Earth ($0 < r < R_E$).

Tier	Inner radius (km)	Outer radius (km)	Average density (10^5 kg/m^8)
Inner Core	0.0	1221.5	12.893
Outer Core	1221.5	3480.0	10.901
Lower Mantle	2480.0	5701.0	4.904
Upper Mantle	5701.0	6346.6	3.605
Crust/Ocean	6346.6	6371.0	2.395
Total	0.0	6371.0	5.513