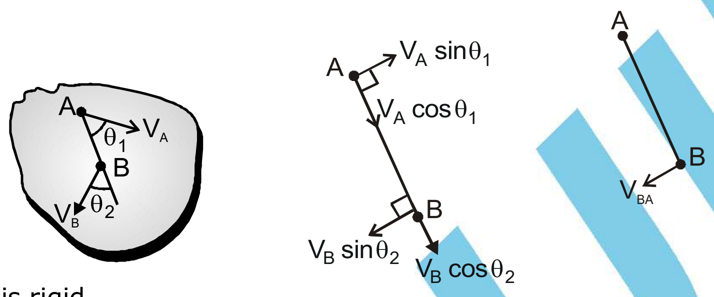
ROTATIONAL MOTION & CENTER OF MASS

1. RIGID BODY

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time) that means the shape and size do not change, during the motion. Eg. Fan, Pen, Table, stone and so on.

Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body. For every pair of particles in a rigid body, there is no velocity of seperation or approach between the particles.

In the figure shown velocities of A and B with respect to ground are $\vec{V}_A$ and $\vec{V}_B$ respectively

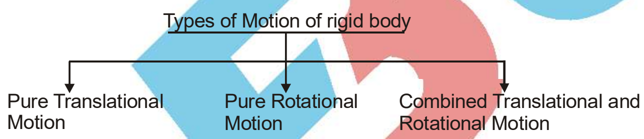


If the above body is rigid

$V_A \cos\theta_1 = V_B \cos\theta_2$

Note : With respect to any particle of rigid body the motion of any other particle of that rigid body is circular.

V_{BA} = relative velocity of B with respect to A.

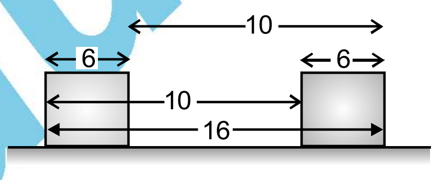


1.1. Pure Translational Motion :

A body is said to be in pure translational motion if the displacement of each particle is same during any time interval however small or large. In this motion all the particles have same $\vec{s}, \vec{v}$ & $\vec{a}$ at an instant.

Example.

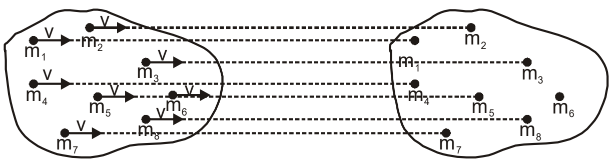
A box is being pushed on a horizontal surface.



$\vec{V}_{cm} = \vec{V}$ of any particle, $\vec{a}_{cm} = \vec{a}$ of any particle

$\Delta\vec{S}_{cm} = \Delta\vec{S}$ of any particle

For pure translational motion :-



$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots\ldots\ldots$

Where m_1, m_2, m_3, ... are the masses of different particles of the body having accelerations $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots\ldots\ldots\ldots\ldots$ respectively.

But acceleration of all the particles are same So, $\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots\ldots\ldots = \vec{a}$

$\vec{F}_{ext} = M\vec{a}$

Where M = Total mass of the body

$\vec{a}$ = acceleration of any particle or of centre of mass of body

$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots\ldots\ldots\ldots$

Where m_1, m_2, m_3 ...... are the masses of different particles of the body having velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots\ldots\ldots\ldots$ respectively

But velocities of all the particles are same so $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \ldots\ldots\ldots = \vec{v}$

$\vec{P} = M\vec{v}$

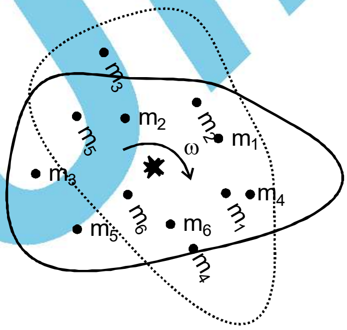
Where $\vec{v}$ = velocity of any particle or of centre of mass of the body.

Total Kinetic Energy of body = $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots\ldots\ldots\ldots = \frac{1}{2}Mv^2$

1.2. Pure Rotational Motion :

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constant from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion all the particles have same $\vec{\theta}, \vec{\omega}$ and $\vec{\alpha}$ at an instant. Eg. : - a rotating ceiling fan, arms of a clock.

For pure rotation motion :-



$\theta = \frac{s}{r}$ Where θ = angle rotated by the particle

s = length of arc traced by the particle.

r = distance of particle from the axis of rotation.

$\omega = \frac{d\theta}{dt}$ Where ω = angular speed of the body.

$\alpha = \frac{d\omega}{dt}$ Where α = angular acceleration of the body.

All the parameters θ, ω and α are same for all the particles. Axis of rotation is perpendicular to the plane of rotation of particles.

Special case : If α = constant,

$\omega = \omega_0 + \alpha t$ Where ω_0 = initial angular speed

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ t = time interval

$\omega^2 = \omega_0^2 + 2\alpha\theta$

Total Kinetic Energy $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$= \frac{1}{2}[m_1r_1^2 + m_2r_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots]\omega^2$

$= \frac{1}{2}I\omega^2$ Where I = Moment of Inertia = $m_1r_1^2 + m_2r_2^2 + \ldots\ldots.$

ω = angular speed of body.

1.3 Combined translation and rotational Motion

A body is said to be in translation and rotational motion if all the particles rotates about an axis of rotation and the axis of rotation moves with respect to the ground.

2. MOMENT OF INERTIA

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating. The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotaional motion. If a body is at rest, the larger the moment of inertia of a body the more difficuilt it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis).

Moment of inertia depends on :

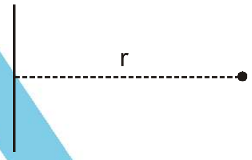
(i) density of the material of body
(ii) shape & size of body
(iii) axis of rotation

In totality we can say that it depends upon distribution of mass relative to axis of rotation.

Note :

Moment of inertia does not change if the mass :
(i) is shifted parallel to the axis of the rotation
(ii) is rotated with constant radius about axis of rotation

2.1 Moment of Inertia of a Single Particle



For a very simple case the moment of inertia of a single particle about an axis is given by,

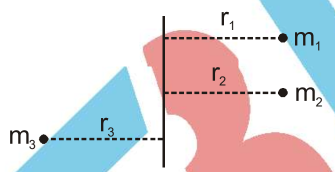
$I = mr^2$ ...(i)

Here, m is the mass of the particle and r its distance from the axis under consideration.

2.2 Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by,

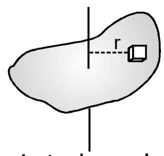
$$I = \sum_i m_i r_i^2 \quad ...(ii)$$



where r_i is the perpendicular distance from the axis to the ith particle, which has a mass m_i.

2.3 Moment of Inertia of Rigid Bodies

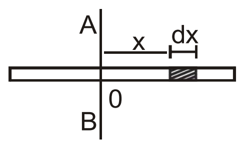
For a continuous mass distribution such as found in a rigid body, we replace the summation of $I = \sum_i m_i r_i^2$ by an integral. If the system is divided into infinitesimal element of mass dm and if r is the distance from a mass element to the axis of rotation, the moment of inertia is, $I = \int r^2\, dm$



where the integral is taken over the system.

(A) Uniform rod about a perpendicular bisector

Consider a uniform rod of mass M and length l figure and suppose the moment of inertia is to be calculated about the bisector AB. Take the origin at the middle point O of the rod. Consider the element of the rod between a distance x and x + dx from the origin. As the rod is uniform,



Mass per unit length of the rod = M/l

so that the mass of the element = $(M/l)dx$.

The perpendicular distance of the element from the line AB is x.

The moment of inertia of this element about AB is

$$dI = \frac{M}{l} dx\, x^2 .$$

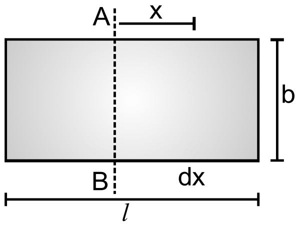
When $x = -l/2$, the element is at the left end of the rod. As x is changed from $-l/2$ to $l/2$, the elements cover the whole rod.

Thus, the moment of inertia of the entire rod about AB is

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx = \left[\frac{M x^3}{l\ 3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

(B) Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre

The situation is shown in figure. Draw a line parallel to AB at a distance x from it and another at a distance x + dx. We can take the strip enclosed between the two lines as the small element.



It is "small" because the perpendiculars from different points of the strip to AB differ by not more than dx. As the plate is uniform,

its mass per unit area = $\frac{M}{bl}$

Mass of the strip = $\frac{M}{bl}b\,dx = \frac{M}{l}dx$.

The perpendicular distance of the strip from AB = x.

The moment of inertia of the strip about AB = dI = $\frac{M}{l}dx\,x^2$. The moment of inertia of the given plate is, therefore,

$$I = \int_{-l/2}^{l/2}\frac{M}{l}x^2dx = \frac{Ml^2}{12}$$

The moment of inertia of the plate about the line parallel to the other edge and passing through the centre may be obtained from the above formula by replacing l by b and thus,

$$I = \frac{Mb^2}{12}.$$

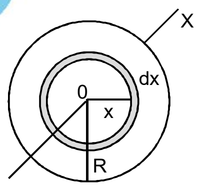
(C) *Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre)*

Suppose the radius of the ring is R and its mass is M. As all the elements of the ring are at the same perpendicular distance R from the axis, the moment of inertia of the ring is

$$I = \int r^2\,dm = \int R^2\,dm = R^2\int dm = MR^2.$$

(D) *Moment of inertia of a uniform circular plate about its axis*

Let the mass of the plate be M and its radius R. The centre is at O and the axis OX is perpendicular to the plane of the plate.



Draw two concentric circles of radii x and x + dx, both centred at O and consider the area of the plate in between the two circles.

This part of the plate may be considered to be a circular ring of radius x. As the periphery of the ring is $2\pi x$ and its width is dx, the area of this elementary ring is $2\pi x dx$. The area of the plate is πR^2. As the plate is uniform,

Its mass per unit area = $\frac{M}{\pi R^2}$

Mass of the ring $= \frac{M}{\pi R^2}2\pi x\,dx = \frac{2Mx\,dx}{R^2}$

Using the result obtained above for a circular ring, the moment of inertia of the elementary ring about OX is

$$dI = \left[\frac{2Mx\,dx}{R^2}\right]x^2.$$

The moment of inertia of the plate about OX is

$$I = \int_0^R \frac{2M}{R^2} x^3 \, dx = \frac{MR^2}{2}.$$

(E) Moment of inertia of a hollow cylinder about its axis

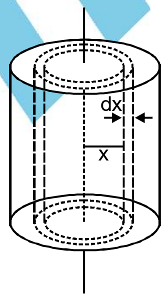
Suppose the radius of the cylinder is R and its mass is M. As every element of this cylinder is at the same perpendicular distance R from the axis, the moment of inertia of the hollow cylinder about its axis is

$$I = \int r^2 dm = R^2 \int dm = MR^2$$

(F) Moment of inertia of a uniform solid cylinder about its axis

Let the mass of the cylinder be M and its radius R. Draw two cylindrical surface of radii x and x + dx coaxial with the given cylinder. Consider the part of the cylinder in between the two surface. This part of the cylinder may be considered to be a hollow cylinder of radius x. The area of cross-section of the wall of this hollow cylinder is 2π x dx. If the length of the cylinder is l, the volume of the material of this elementary hollow cylinder is 2π x dx$l$.

The volume of the solid cylinder is $\pi R^2 l$ and it is uniform, hence its mass per unit volume is



$$\rho = \frac{M}{\pi R^2 l}$$

The mass of the hollow cylinder considered is

$$\frac{M}{\pi R^2 l} 2\pi x \, dx \, l = \frac{2M}{R^2} x \, dx.$$

As its radius is x, its moment of inertia about the given axis is

$$dI = \left[\frac{2M}{R^2} x \, dx\right] x^2.$$

The moment of inertia of the solid cylinder is, therefore,

$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{MR^2}{2}.$$

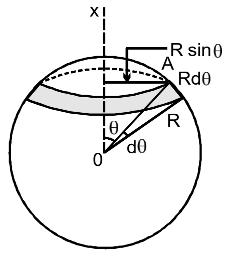
Note that the formula does not depend on the length of the cylinder.

(G) Moment of inertia of a uniform hollow sphere about a diameter

Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis (figure). The mass is spread over the surface of the sphere and the inside is hollow.

Let us consider a radius OA of the sphere at an angle θ with the axis OX and rotate this radius about OX. The point A traces a circle on the sphere. Now change θ to $\theta + d\theta$ and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius R sinθ. The width of this ring is R$d\theta$ and its periphery is 2πR sinθ. Hence,

the area of the ring = (2πR sinθ) (R$d\theta$).



Mass per unit area of the sphere $= \frac{M}{4\pi R^2}$.

The mass of the ring $= \frac{M}{4\pi R^2}(2\pi R \sin\theta)(R d\theta) = \frac{M}{2}\sin\theta \, d\theta.$

The moment of inertia of this elemental ring about OX is

$$dI = \left(\frac{M}{2}\sin\theta \, d\theta.\right)(R\sin\theta)^2 \quad = \frac{M}{2}R^2 \sin^3\theta \, d\theta$$

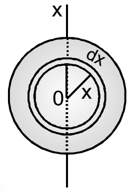
As θ increases from 0 to π, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$I = \int_0^\pi \frac{M}{2}R^2 \sin^3\theta \, d\theta = \frac{MR^2}{2}\left[\int_0^\pi (1-\cos^2\theta)\sin\theta \, d\theta\right] \qquad = \frac{MR^2}{2}\left[\int_{\theta=0}^{\pi} -(1-\cos^2\theta) d(\cos\theta)\right]$$

$$= \frac{-MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{2}{3}MR^2$$

(H) Moment of inertia of a uniform solid sphere about a diameter

Let M and R be the mass and radius of the given solid sphere. Let O be centre and OX the given axis. Draw two spheres of radii x and x + dx concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius x.



The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The thin hollow sphere considered above has a surface area $4\pi x^2$ and thickness dx. Its volume is $4\pi x^2\,dx$ and hence its mass is

$$= \left(\frac{3M}{4\pi R^3}\right)(4\pi x^2\,dx) = \frac{3M}{R^3}x^2\,dx$$

Its moment of inertia about the diameter OX is, therefore,

$$dI = \frac{2}{3}\left[\frac{3M}{R^3}x^2 dx\right]x^2 \quad = \frac{2M}{R^3}x^4 dx$$

If x = 0, the shell is formed at the centre of the solid sphere. As x increases from 0 to R, the shells cover the whole solid sphere.

The moment of inertia of the solid sphere about OX is, therefore,

$$I = \int_0^R \frac{2M}{R^3}x^4 dx = \frac{2}{5}MR^2 .$$

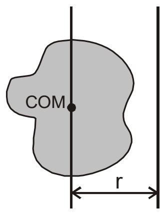
3. THEOREMS OF MOMENT OF INERTIA

There are two important theorems on moment of inertia, which, in some cases enable the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

3.1 Theorem of parallel axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes

through the centre of mass.



Two such axes are shown in figure for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them, these moments are related by,

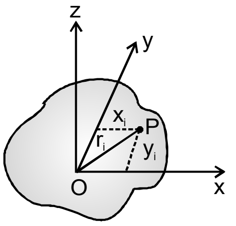
$$I = I_{COM} + Mr^2$$

* Theorem of parallel axis is applicable for any type of rigid body whether it is a two dimensional or three dimensional

3.2 Theorem of perpendicular axes

The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it.

Let x and y axes be chosen in the plane of the body and z-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that.



$$I_z = I_x + I_y$$

Important point in perpendicular axis theorem

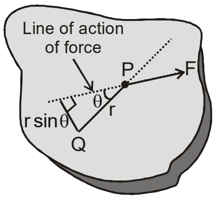
(i) This theorem is applicable only for the plane bodies (two dimensional).

(ii) In theorem of perpendicular axes, all the three axes (x, y and z) intersect each other and this point may be any point on the plane of the body (it may even lie outside the body).

(iii) Intersection point may or may not be the centre of mass of the body.

4. TORQUE :

Torque represents the capability of a force to produce change in the rotational motion of the body



4.1 Torque about point :

Torque of force $\vec{F}$ about a point $\vec{\tau} = \vec{r} \times \vec{F}$

where $\vec{F}$ = force applied

P = point of application of force

Q = point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force from the point about which we want to determine the torque.

$|\vec{\tau}| = rF\sin\theta$ $= r_\perp F$ $= rF_\perp$

where θ = angle between the direction of force and the position vector of P wrt. Q.

$r_\perp$ = perpendicular distance of line of action of force from point Q.

$F_\perp$ = force arm

SI unit to torque is Nm

Torque is a vector quantity and its direction is determined using right hand thumb rule.

5. BODY IS IN EQUILIBRIUM : -

We can say rigid body is in equillibrium when it is in

(a) Translational equilibrium

i.e. $\vec{F}_{net} = 0$, $F_{net\,x}$ = 0 and $F_{net\,y}$ = 0 and

(b) Rotational equillibrium

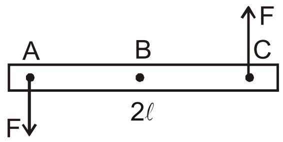
$\vec{\tau}_{net} = 0$ i.e., torque about any point is zero

Note :

(i) If net force on the body is zero then net torque of the forces may or may not be zero.

example.

A pair of forces each of same magnitude and acting in opposite direction on the rod.



$\tau_A = 2F\ell \odot$

(2) If net force on the body is zero then torque of the forces about each and every point is same

τ about B $\tau_B = F\ell + F\ell \odot$

$\tau_B = 2F\ell \odot$

τ about C $\tau_C = 2F\ell \odot$

6. RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}$$

Thus, for a rigid body we have the rotational analog of Newton's second law ;

$$\Sigma\tau = I\alpha \qquad ...(iii)$$

Following two points are important regarding the above equation.

(i) The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration α is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

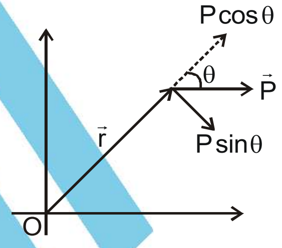
7. ANGULAR MOMENTUM

7.1 Angular momentum of a particle about a point.

$\vec{L} = \vec{r} \times \vec{P} \quad \Rightarrow \quad L = r\,p \sin\theta$

$|\vec{L}| = r_{\perp} \times P$

$|\vec{L}| = P_{\perp} \times r$



Where $\vec{P}$ = momentum of partilcle

$\vec{r}$ = position of vector of particle with respect to point about which angular momentum is to be calculated.

θ = angle between vectors $\vec{r}$ & $\vec{p}$

$r_{\perp}$ = perpendicular distance of line of motion of particle from point O.

$P_{\perp}$ = perpendicular component of momentum.

SI unit of angular momentum is kgm^2/sec.

7.2 Angular Momentum of a rigid body rotating about a fixed axis

Suppose a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case $\vec{r}$ and $\vec{P}$ are perpendicular to each other and $\vec{r} \times \vec{P}$ is along the axis. Thus, component of $\vec{r} \times \vec{P}$ along the axis is *mvr* itself. The angular momentum of the whole rigid body about AB is the sum of components of all particles, i.e.,

$$L = \sum_i m_i r_i v_i$$

Here, $v_i = r_i\,\omega$

$\therefore \quad L = \sum_i m_i r_i^2 \omega_i \quad$ or $L = \omega \sum_i m_i r_i^2$

or $\quad L = I\omega$

Here, I is the moment of inertia of the rigid body about AB.

Note : Angular momentum about axis is the component of $I\vec{\omega}$ along the axis. In most of the cases angular momentum about axis is $I\omega$.

8. CONSERVATION OF ANGULAR MOMENTUM :

The time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it.

$$\text{or} \quad \vec{\tau}_{net} = \frac{d\vec{L}}{dt} \qquad ....(i)$$

Now, suppose that $\vec{\tau}_{net} = 0$, then $\frac{d\vec{L}}{dt} = 0$, so that $\vec{L}$ = constant.

"When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega$$

It is possible for the moment of inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then L_z must remains constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed.

$I\omega$ = constant.

9. ANGULAR IMPULSE

The angular impulse of a torque in a given time interval is defined as $\int_{t_1}^{t_2} \vec{\tau}\, dt$

Here, $\vec{\tau}$ is the resultant torque acting on the body. Further, since

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \therefore \quad \vec{\tau}\, dt = d\vec{L}$$

or $$\int_{t_1}^{t_2} \vec{\tau}\, dt = \text{angular impulse} = \vec{L}_2 - \vec{L}_1$$

Thus, the angular impulse of the resultant torque is equal to the change in angular momentum. Let us take few examples based on the angular impulse.

9.2 Kinetic Energy of a Rolling Body

If a body of mass M is rolling on a plane such that velocity of its centre of mass is V and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

I is moment of inertia of body about axis passing through centre of mass.

In case of rolling without slipping.

$$KE = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\therefore v = \omega R]$$

$$= \frac{1}{2}\left[MR^2 + I\right]\omega^2 \quad = \frac{1}{2}I_c\,\omega^2$$

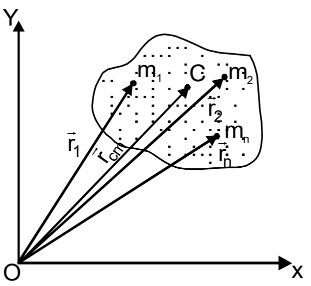
I_c is moment of inertia of the body about the axis passing through point of contact.

10.CENTRE OF MASS :

Every physical system has associated with it a certain point whose motion characterises the motion of the whole system. When the system moves under some external forces, then this point moves as if the entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is called the centre of mass of the system.

10.1 Centre of Mass of a System of 'N' Discrete Particles :

Consider a system of N point masses m_1, m_2, m_3, ... m_n whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3$ .... $\vec{r}_n$ respectively. Then the position vector of the centre of mass C of the system is given by.



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...+ m_n\vec{r}_n}{m_1 + m_2 + ...+ m_n} \; ; \; \vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} \qquad \vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

where, $m_i\vec{r}_i$ is called the **moment of mass** of particle with respect to origin.

$M = \left(\sum_{i=1}^{n} m_i\right)$ is the total mass of the system.

Further, $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ and $\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j} + z_{COM}\hat{k}$

So, the cartesian co-ordinates of the COM will be

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + ...+ m_nx_n}{m_1 + m_2 + ...m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$$

or $$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}$$

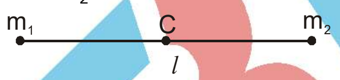
Similarly, $$y_{COM} = \frac{\sum_{i=1}^{n} m_iy_i}{M} \quad \text{and} \quad z_{COM} = \frac{\sum_{i=1}^{n} m_iz_i}{M}$$

Note :

- If the origin is taken at the centre of mass then $\sum_{i=1}^{n} m_i\vec{r}_i = 0$. hence, the COM is the point about which the sum of "mass moments" of the system is zero.
- If we change the origin then $\vec{r}_1, \vec{r}_2, \vec{r}_3$...... changes. So $\vec{r}_{cm}$ also changes but exact location of center of mass does not change.

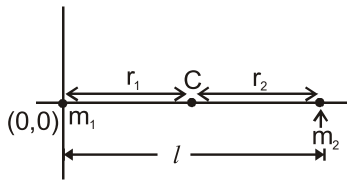
10.2 Position of COM of two particles : -

Consider two particles of masses m_1 and m_2 separated by a distance l as shown in figure.



Let us assume that m_1 is placed at origin and m_2 is placed at position (l, 0) and the distance of centre of mass from m_1 & m_2 is r_1 & r_2 respectively.

So $$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$



$$r_1 = \frac{0 + m_2l}{m_1 + m_2} = \frac{m_2l}{m_1 + m_2} \quad ...(1)$$

$$r_2 = l - \frac{m_2l}{m_1 + m_2} = \frac{m_1l}{m_1 + m_2} \quad ...(2)$$

From the above discussion, we see that

$r_1 = r_2 = \frac{l}{2}$ if $m_1 = m_2$, i.e., COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_2 < m_1$ i.e., COM is nearer to the particle having larger mass.

From equation (1) & (2)

$m_1r_1 = m_2r_2$ ...(3)

Centre of mass of two particle system lie on the line joining the centre of mass of two particle system.

For continuous mass distribution the centre of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation

$$x_{cm} = \frac{\int x\,dm}{\int dm},\ y_{cm} = \frac{\int y\,dm}{\int dm},\ z_{cm} = \frac{\int z\,dm}{\int dm} \quad ...(i)$$

$\int dm$ = M (mass of the body)

here x,y,z in the numerator of the eq. (i) is the coordinate of the centre of mass of the dm mass.

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Note :

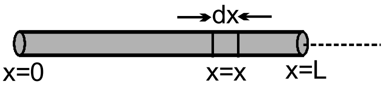
- If an object has symmetric mass distribution about x axis then y coordinate of COM is zero and vice-versa

(a) Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at x = 0 and the other at x = L. Mass per unit length of the rod $\lambda = \frac{M}{L}$

Hence, dm, (the mass of the element dx situated at x = x is) = λ dx

The coordinates of the element dx are (x, 0, 0). Therefore, x-coordinate of COM of the rod will be



$$x_{COM} = \frac{\int_0^L x dm}{\int_0^L dm} = \frac{\int_0^L (x)(\lambda dx)}{\int_0^L \lambda dx} = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y dm}{\int dm} = 0$$

Similarly, $z_{COM} = 0$

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$, i.e, it lies at the centre of the rod.

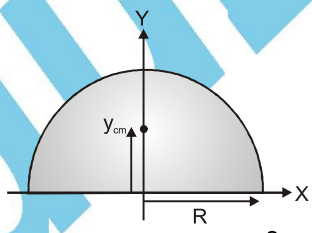
(c) Centre of mass of Semicircular Disc :

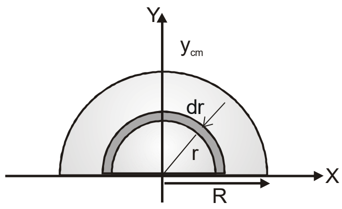
Figure shows the half disc of mass M and radius R. Here, we are only required to find the y-coordinate of the centre of mass of this disc as centre of mass will be located on its half vertical diameter. Here to find y_{cm}, we consider a small elemental ring of mass dm of radius r on the disc (disc can be considered to be made up such thin rings of increasing radii) which will be integrated from 0 to R. Here dm is given as

$$dm = \sigma\pi r dr$$

where σ is the mass density of the semi circular disc.

$$\sigma = \frac{M}{\pi R^2/2} = \frac{2M}{\pi R^2}$$





Now the y-coordinate of the element is taken as $\frac{2r}{\pi}$, (as in previous section, we have derived that the centre of mass of a semi circular ring is concentrated at $\frac{2R}{\pi}$)

$$y_{cm} = \frac{\int_0^R dm.y}{\int_0^R dm}$$

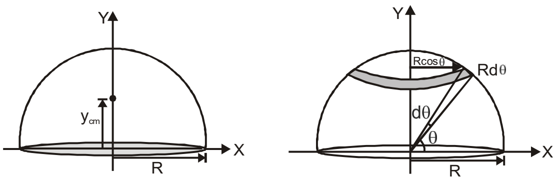
Here y is the position COM of dm mass.

Here y_{cm} is given as $y_{cm} = \frac{\int_0^R dm\frac{2r}{\pi}}{\int_0^R \sigma\pi r dr} = \int_0^R \frac{4}{\pi R^2} r^2 dr \Rightarrow y_{cm} = \frac{4R}{3\pi}$

(d) Centre of mass of a Hollow Hemisphere :

A hollow hemisphere of mass M and radius R. Now we consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere. This strip will have an area.

$dS = 2\pi R\cos\theta\, Rd\theta$



Its mass dm is given as $dm = \sigma 2\pi R\cos\theta Rd\theta$

Here σ is the mass density of a hollow hemisphere

$$\sigma = \frac{M}{2\pi R^2}$$

Here y-coordinate of this strip of mass dm can be taken as R sinθ. Now we can obtain the centre of mass of the system as.

$$y_{cm} = \frac{\int_0^{\frac{\pi}{2}} dm R\sin\theta}{\int_0^{\pi/2} dm} = \frac{\int_0^{\frac{\pi}{2}}\left(\sigma 2\pi R^2\cos\theta d\theta\right)R\sin\theta}{\int_0^{\pi/2}\left(\sigma 2\pi R^2\cos\theta d\theta\right)} = R\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta \quad y_{cm} = \frac{R}{2}$$

11.MOTION OF CENTRE OF MASS AND CONSERVATION OF MOMENTUM: -

The position of centre of mass is given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots} \quad \ldots(1)$$

Here m_1, m_2, m_3 ..... are the mass in the system and $\vec{r}_1, \vec{r}_2, \vec{r}_3$......... is the corresponding position vector of m_1, m_2, m_3 respectively

11.1Velocity of C.O.M of system :

To find the velocity of centre of mass we differentiate equation (1) with respect to time

$$\frac{d\vec{r}_{com}}{dt} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots}$$

$$\Rightarrow \vec{V}_{com} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots}$$

$$\vec{V}_{com} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots} \quad \ldots(2)$$

11.2 Acceleration of centre of mass of the system : -

To find the acceleration of C.O.M we differentiate equation (2)

$$\Rightarrow \frac{d\vec{V}_{com}}{dt} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots}$$

$$\vec{a}_{com} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots} \quad \ldots(3)$$

Now $(m_1 + m_2 + m_3)\ \vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ...$

$\vec{F}_{net(system)} = \vec{F}_{1net} + \vec{F}_{2net} + \vec{F}_{3net} + ...$

The internal forces which the particles exert on one another play absolutely no role in the motion of the centre of mass.

12. IMPULSE :

Impulse of a force $\vec{F}$ acting on a body for the time interval $t = t_1$ to $t = t_2$ is defined as

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt \quad \vec{I} = \int \vec{F}\,dt$$

$$= \int m\frac{d\vec{v}}{dt}dt \quad = \int m\,d\vec{v}$$

$\vec{I} = m(\vec{v}_2 - \vec{v}_1) = \Delta\vec{P} =$ change in momentum due to force $\vec{F}$

Also $\vec{I}_{Re} = \int_{t_1}^{t_2} \vec{F}_{Res}dt = \Delta\vec{P}$ **(impulse - momentum theorem)**

Note :

* Impulse applied to an object in a given time interval can also be calculated from the area under force time (F-t) graph in the same time interval.

12.1 Instantaneous Impulse :

There are many cases when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (i.e., impulse) can be known by measuring the initial and final momentum. Thus, we can write.

$$\vec{I} = \int \vec{F}\,dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

Important Points :

(1) It is a vector quantity.

(2) Dimensions = $[MLT^{-1}]$

(3) SI unit = kg m/s

(4) Direction is along change in momentum.

(5) Magnitude is equal to area under the F-t. graph.

(6) $\vec{I} = \int \vec{F}\,dt = \vec{F}_{av}\int dt = \vec{F}_{av}\,\Delta t$

(7) It is not a property of a particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

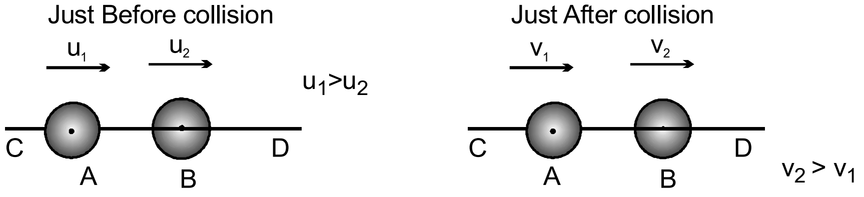
13. COEFFICIENT OF RESTITUTION (e)

The coefficient of restitution is defined as the ratio of the impulses of reformation and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r dt}{\int F_d dt} \qquad e = \frac{\text{Velocity of separation of point of contact}}{\text{Velocity of approach of point of contact}}$$

Example for calculation of e :

Two smooth balls A and B approaching each other such that their centres are moving along line CD in absence of external impulsive force. The velocities of A and B just before collision be u_1 and u_2 respectively. The velocities of A and B just after collision be v_1 and v_2 respectively.



$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

Note : Coefficient of restitution is a factor between two colliding bodies which is depends on the material of the body but independent of shape.

We can say e is a factor which relates deformation and reformation of the body.

$$0 \le e \le 1$$

13.1 Line of Motion

The line passing through the centre of the body along the direction of resultant velocity.

13.2 Line of Impact

The line passing through the common normal to the surfaces in contact during impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of Line of impact can be determined by :

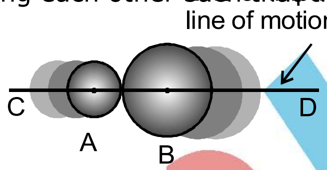
(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

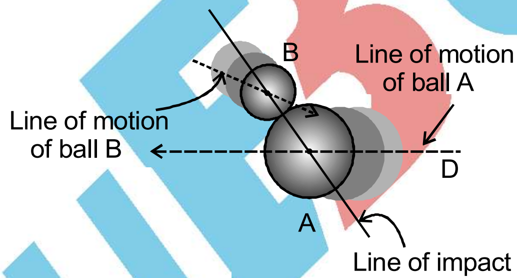
If one particle is stationary before the collision then the line of impact will be along its motion after collision.

Examples of line of impact

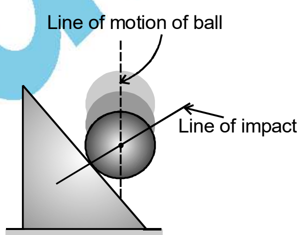
(i) Two balls A and B are approaching each other such that their centres are moving along line CD.



(ii) Two balls A and B are approaching each other such that their centre are moving along dotted lines as shown in figure.



(iii) Ball is falling on a stationary wedge.



Note : In previous discussed examples line of motion is same as line of impact. But in problems in which line of impact and line of motion is different then e will be

$$e = \frac{\text{velocity of seperation along line of impact}}{\text{velocity of approach along line of impact}}$$

14. COLLISION OR IMPACT

Collision is an event in which an impulsive force acts between two or more bodies for a short time, which results in change of their velocities.

Note :

- In a collision, particles may or may not come in physical contact.
- The duration of collision, Δt is negligible as compared to the usual time intervals of observation of motion.
- In a collision the effect of external non impulsive forces such as gravity are not taken into account as due to small duration of collision (Δt) average impulsive force responsible for collision is much larger than external forces acting on the system.

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The collision is in fact a redistribution of total momentum of the particle :

Thus law of conservation of linear momentum is indepensible in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

Two balls of masses m_1 and m_2 are moving with velocities v_1 and v_2 ($< v_1$) along the same straight line in a smooth horizontal surface. Now let us see what happens during the collision between two particles.

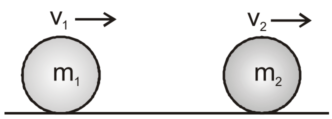
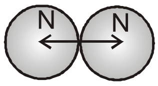
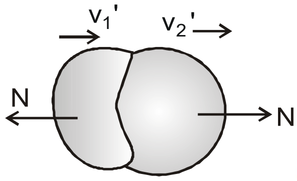


figure (a)

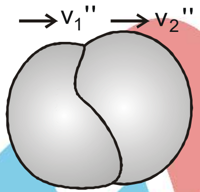


figure(b)



figure(c)

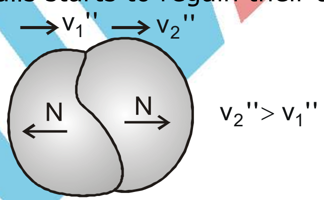
figure (a) : Balls of mass m_1 is behind m_2. Since $v_1 > v_2$, the balls will collide after some time.
figure (b) : During collision both the balls are a little bit deformed. Due to deformation two equal and opposite normal forces act on both the balls. These forces decreases the velocity of m_1 and increase the velocity of m_2
figure (c): Now velocity of ball m_1 is decrease from v_1 to v_1' and velocity of ball m_2 is increase from v_2 to v_2'. But still $v_1' > v_2'$ so both the ball are continuously deformed.
figure(d) : Contact surface of both the balls are deformed till the velocity of both the balls become equal. So at maximum deformation velocities of both the blocks are equal



figure(d)

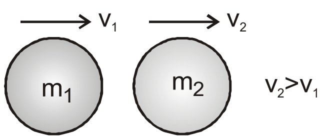
at maximum deformation $v_1'' = v_2''$

figure(e) : Normal force is still in the direction shown in figure i.e. velocity of m_1 is further decreased and that of m_2 increased. Now both the balls starts to regain their original shape and size.



figure(e)

figure (f) : These two forces redistributes their linear momentum in such a manner that both the blocks are separated from one another, Velocity of ball m_2 becomes more than the velocity of block m_1 i.e., $v_2 > v_1$



figure(f)

The collision is said to be elastic if both the blocks regain their original form, The collision is said to be inelastic. If the deformation is permanent, and the blocks move together with same velocity after the collision, the collision is said to be perfectly inelastic.

14.1 Classification of collisions

(a) On the basis of line of impact

(i) Head-on collision : If the velocities of the colliding particles are along the same line before and after the collision.

(ii) Oblique collision : If the velocities of the colliding particles are along different lines before and after the collision.

(b) On the basis of energy :

(i) Elastic collision :

(a) In an elastic collision, the colliding particles regain their shape and size completely after collision. i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies.

(b) Thus, kinetic energy of system after collision is equal to kinetic energy of system before collision.

(c) e = 1

(d) Due to F_{net} on the system is zero linear momentum remains conserved.

(ii) Inelastic collision :

(a) In an inelastic collision, the colliding particles do not regain their shape and size completely after collision.

(b) Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved.

(c) However, in the absence of external forces, law of conservation of linear momentum still holds good.

(d) $(\text{Energy loss})_{\text{Perfectly Inelastic}} > (\text{Energy loss})_{\text{Partial Inelastic}}$

(e) $0 < e < 1$

(iii) Perfectly Inelastic collision :

(i) In this the colliding bodies do not return to their original shape and size after collision i.e. both the particles stick together after collision and moving with same velocity

(ii) But due to F_{net} of the system is zero linear momentum remains conserved.

(iii) Total energy is conserved.

(iv) Initial kinetic energy > Final K.E. Energy

(v) Loss in kinetic energy goes to the deformation potential energy

(vi) e = 0

14.2 Value of Velocities after collision :

Let us now find the velocities of two particles after collision if they collide directly and the coefficient of restitution between them is given as e.

(a) Before Collision $\qquad$ (b) After Collision

$u_1 > u_2$ $\qquad$ $v_2 > v_1$

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

$\Rightarrow (u_1 - u_2)e = (v_2 - v_1)$ ...(i)

By momentum conservation

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ ...(ii)

$v_2 = v_1 + e(u_1 - u_2)$ ...(iii)

from above equation

$$v_1 = \frac{m_1u_1 + m_2u_2 + m_2e(u_2 - u_1)}{m_1 + m_2} \quad ...(iii)$$

$$v_2 = \frac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2} \quad ...(iv)$$

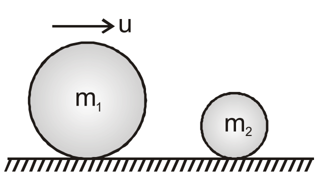
Special cases :

1. If $m_1 >> m_2$ and $u_2 = 0$ and $u_1 = u$

and e = 1

$m_1 = m_2$

from eq. (iii) & (iv)



$$v_1 = \frac{m_1u - m_2u}{m_1 + m_2} = \frac{u(m_1 - m_2)}{m_1 + m_2}$$

$v_1 \simeq u$, $\qquad v_2 = \dfrac{m_1u + m_2u}{m_1 + m_2} = \dfrac{2m_1u}{m_1 + m_2}$; $v_2 = 2u$

2. If $m_1 = m_2 = m$ and e = 1 then

from eq. (iii) & (iv)

$$v_1 = \frac{m(u_1+u_2)+m(u_2-u_1)}{2m}$$

$v_1 = u_2$

In this way $v_2 = u_1$

i.e when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities.

14.3 Collision in two dimension (oblique) :

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

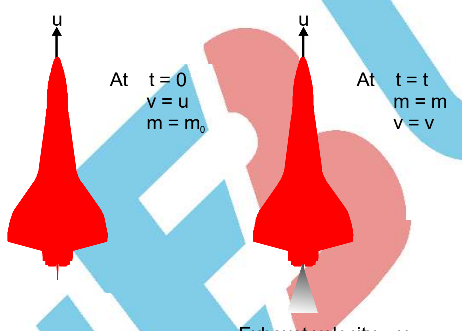
2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply Relative speed of separation = e (relative speed of approach)

14.4 Rocket Propulsion

Let m_0 be the mass of the rocket at time t = 0. m its mass at any time t and v its velocity at that moment. Initially let us suppose that the velocity of the rocket is u.



Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases.

Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time t = t

1. Thrust force on the rocket

$$F_t = v_r\left(-\frac{dm}{dt}\right) \quad \text{(upwards)}$$

2. Weight of the rocket

$$W = mg \quad \text{(downwards)}$$

3. Net force on the rocket

$$F_{net} = F_t - W \quad \text{(upwards)}$$

or $$F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or $$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$ or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

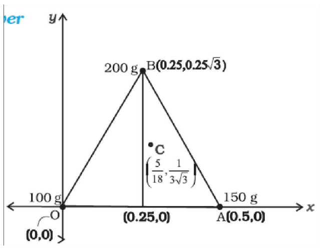
or $$\int_u^v dv = v_r\int_{m_0}^{m}\frac{-dm}{m} - g\int_0^t dt$$ or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$$

Thus, $$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right) \quad \text{...(i)}$$

SOLVED EXAMPLE

Ex.1 Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Ans.



With the X and Y axes chosen as shown in Fig., the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), $(0.25, 0.25\sqrt{3})$.

Let the masses 100 g, 150g and 200g be located at O, A and B be respectively. Then,

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)]\,gm}{(100 + 150 + 200)\ g}$$

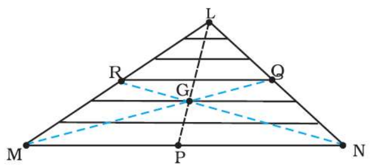
$$= \frac{75 + 50}{450}m = \frac{125}{450}m = \frac{5}{18}m$$

$$Y = \frac{|100(0) + 150(0) + 200(0.25\sqrt{3})|\ gm}{450\ g}$$

$$= \frac{50\sqrt{3}}{450}m = \frac{\sqrt{3}}{9}m = \frac{1}{3\sqrt{3}}m$$

Ex.2 Find the centre of mass of a triangular lamina.

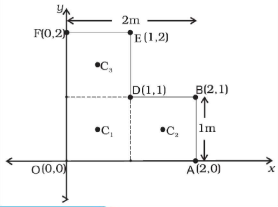
Ans. The lamina (ΔLMN) may be subdivided into narrow strips each parallel to the base (MN) as shown in Fig. 7.10



By symmetry each strip has its centre of mass at its midpoint. If we join the midpoint ofall the strips we get the median LP. The centre of mass of the triangle as a whole therefore, has to lie on the median LP. Similarly, we can argue that it lies on the median MQ and NR. This means the centre of mass lies on the point of concurrence of the medians, i.e. on the centroid G of the triangle.

Ex.3 Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.

Ans. Choosing the X and Y axes as shown in Fig. we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consist of 3 squares each of length 1m. The mass of each square is 1kg, since the lamina is uniform. The centres of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2,1/2), (3/2,1/2), (1/2,3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.



Hence

$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)]\,kg\,m}{(1 + 1 + 1)kg} = \frac{5}{6}m$$

$$Y = \frac{[1(1/2) + 1(1/2) + 1(1/2)]\,kg\,m}{(1 + 1 + 1)kg} = \frac{5}{6}m$$

The centre of mass of the L-shape lies on the line OD.

Ex.4 Find the scalar and vector products of two vectors. a = $\left(3\hat{i} - 4\hat{j} + 5\hat{k}\right)$ and b = $\left(-2\hat{i} + \hat{j} - 3\hat{k}\right)$

Ans. $a \cdot b = (3\hat{i} - 4\hat{j} + 5\hat{k})\cdot(-2\hat{i} + \hat{j} - 3\hat{k})$

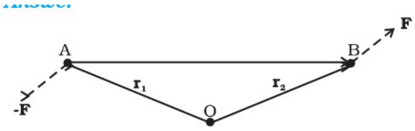
$= -6 - 4 - 15 = -25$

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{i} - \hat{j} - 5\hat{k}$$

Note $a \times b = -7\hat{i} + \hat{j} + 5\hat{k}$

Ex.5 Show that moment of a couple does not depend on the point about which you take the moments.

Ans.



Consider a couple as shown in Fig. acting on a rigid body. The forces **F** and **-F** act respectively at points B and A. These points have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to origin O. Let us take the moments of the forces about the origin. The moment of the couple = sum of the moments of the two forces making the couple

$= \mathbf{r}_1 \times (-\mathbf{F}) + \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F} = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}$

But $\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$, and hence $\mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1$. The moment of the couple, therefore, is **AB** × **F.** Clearly this is independent of the origin, the point about which we took the moments of the forces.

Ex.6 A metal bar 70 cm long and 4.00 kg in mass supported on two knife-edges placed 10 cm from each end. A 6.00 kg weight is suspended at 30 cm from one end. Find the reactions at the knife-edges. (Assume the bar to be of uniform cross section and homogeneous.)

Ans.

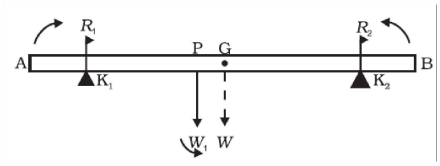


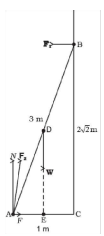
Figure shows the rod AB, the positions of the knife edges K1 and K2 , the centre of gravity of the rod at G and the suspended weight at P. The weight of the rod W acts at its centre of gravity G. The rod is uniform in cross section and homogeneous; hence G is at the centre of the rod; AB = 70 cm. AG = 35 cm, AP = 30 cm, PG = 5 cm, AK1= BK2 = 10 cm and K1G = K2G = 25 cm. Also, W= weight of the rod = 4.00 kg and W1= suspended weight = 6.00 kg; R_1 and R_2 are the normal reactions of the support at the knife edges. For translational equilibrium of the rod,

$R_1 + R_2 - W_1 - W = 0$ (i)

W_1 and W act vertically down and R_1 and R_2 act vertically up. For considering rotational equilibrium, we take moments of the forces. A convenient point to take moments about is G. The moments of R_2 and W_1 are anticlockwise (+ve), whereas the moment of R_1 is clockwise (-ve). For rotational equilibrium, $-R_1 (K_1G) + W_1 (PG) + R_2 (K_2G) = 0$ (ii) It is given that W = 4.00g N and W_1 = 6.00g N, where g = acceleration due to gravity. We take g = 9.8 m/s². With numerical values inserted, from (i) $R_1 + R_2 - 4.00g - 6.00g = 0$ or $R_1 + R_2 = 10.00g$ N (iii) = 98.00 N From (ii) $-0.25 R_1 + 0.05 W_1 + 0.25 R_2 = 0$ or $R_2 - R_1 = 1.2g$ N = 11.76 N (iv) From (iii) and (iv), R_1 = 54.88 N, R_2 = 43.12 N

Thus the reactions of the support are about 55 N at K_1 and 43 N at K_2.

Ex.7 A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in Fig. Find the reaction forces of the wall and the floor.



Ans. The ladder AB is 3 m long, its foot A is at distance AC = 1 m from the wall. From Pythagoras theorem, BC = $2\sqrt{2}$ m. The forces on the ladder are its weight W acting at its centre of gravity D, reaction forces F_1 and F_2 of the wall and the floor respectively. Force F_1 is perpendicular to the wall, since the wall is frictionless. Force F_2 is resolved into two components, the normal reaction N and the force of friction F. that F prevents the ladder from sliding away from the wall and is therefore directed toward the wall. For translational equilibrium, taking the forces in the vertical direction, N – W = 0 (i) Taking the forces in the horizontal direction, $F - F_1 = 0$ (ii) For rotational equilibrium, taking the moments of the forces about

A, $2\sqrt{2}\ F_1 - (1/2)\ W = 0$ (iii)

Now W = 20 g = 20 × 9.8 N = 196.0 N

From (i) N = 196.0

From (iii) $F_1 = W 4\sqrt{2} = 196.0/4\sqrt{2} = 34.6$N

From (ii) 1 F = F = 34.6 N

$F_2 = \sqrt{F^2 + N^2} = 199.0$ N

The force F_2 makes an angle ? with the horizontal,

$\tan\alpha = N F = 4\sqrt{2}\ \ \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^o$

Ex.8 What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end?

Ans. For the rod of mass M and length l, I = $M\ell^2/12$. Using the parallel axes theorem,

$I' = I + Ma^2$ with $a = \ell/2$ we get,

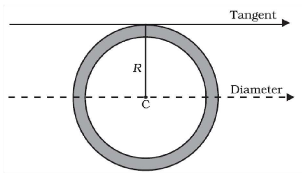
$$I' = M\frac{\ell^2}{12} + M\left(\frac{\ell}{2}\right)^2 = \frac{M\ell^2}{3}$$

We can check this independently since I is half the moment of inertia of a rod of mass 2M and length 2l about its midpoint, $I' = 2M.\frac{4\ell^2}{12} \times \frac{1}{2} = \frac{M\ell^2}{3}$

Ex.9 What is the moment of inertia of a ring about a tangent to the circle of the ring?

Ans.

The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring. The distance between these two parallel axes is R, the radius of the ring. Using the parallel axes theorem,



$$I_{tangent} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

Exercise - I | UNSOLVED PROBLEMS

Q.1 Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body ?

Q.2 In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Q.3 A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system ?

Q.4 Three mass points m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1 ?

Q.5 What is the moment of inertia of a uniform circular disc of radius R and mass M about an axis (i) passing through is centre and normal to the disc ; (ii) passing through a point on its edge and normal to the disc ? The moment of inertia of the disc about any of its diameters is given to be (1/4) MR^2.

Q.6 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder ? What is the magnitude of angular momentum of the cylinder about its axis ?

Q.7 (A) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value ? Assume that the turntable rotates without friction. (b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy ?

Q.8 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N ? What is the linear acceleration of the rope ? Assume that there is no slipping.

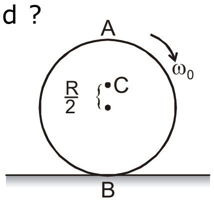
Q.9 A uniform solid cylinder of mass 5 kg and radius 30 cm, and free to rotate about its axis, receives an angular impulse of 3 kg m s^{-1} initially followed by a similar impulse after every 4 s. What is the angular speed of the cylinder 30 s after the initial impulse ? The cylinder is at rest initially.

Q.10 To maintain a rotor at a uniform angular speed or 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine ?
(Note : uniform angular velocity in the absence of friction implies zero torque. In particle, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

Q.11 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time.

Q.12 Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system ? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy ? Take $\omega_1 \neq \omega_2$.

Q.13 A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. Figure ? Will the disc roll in the direction indicated ?



Q.14 Explain why friction is necessary to make the disc in figure roll in the direction indicated.

(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

(b) What is the force of friction after perfect rolling begins ?

Q.15 A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10 π rad s^{-1}. Which of the two will start to roll earlier ? The co-efficient of kinetic friction is $\mu_K = 0.2$.

Q.16 A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The co-efficient of static friction $\mu_s = 0.25$.

(a) How much is the force of friction acting on the cylinder ?

(b) What is the work done against friction during rolling ?

(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly ?

Q.17 A ring, a disc and a sphere, all of the same radius and mass, roll down on an inclined plane from the same height h. Which of the three reaches the bottom (i) first, (ii) last ?

Q.18 A solid cylinder of mass 20 kg and radius 0.12 m rotating with initial angular speed of 125 rad s^{-1} is placed lightly (i.e. without any translational push) on a horizontal table with co-efficient of kinetic friction $\mu_K = 0.15$, between the cylinder and the table.

(a) After how long does the cylinder start rolling ?

(b) What is the initial (i) translational energy, (ii) rotational energy, and (iii) total energy of the cylinder?

(c) Is the final total energy equal to the initial total energy, (b) rotational energy, and (c) total energy of the cylinder?

(d) Is the final total energy equal to the initial total energy of motion of the cylinder? If not, where does the difference of energy disappear?

(e) Account for the loss of total energy of motion in the following way; find the work done by friction on the body for its translational motion, and the work done against friction by the body as regards its rotational motion. Show that the net work done by friction on the body is negative, equal in magnitude of the loss of total energy computed in (d) above.

Q.19 Read each statement below carefully, and state, with reasons, if it is true or false :

(a) During rolling without slipping on a fixed inclined surface, the force of friction acts in the same direction as the direction of motion of the CM of the body (only weight and contact force act).

(b) The instantaneous speed of the point of contact during rolling without slipping on a fixed surface is zero.

(c) The instantaneous acceleration of the point of contact during rolling without slipping on a fixed surface is zero.

(d) For rolling without slipping on a fixed surface motion, total work done by friction is zero.

(e) A wheel moving down a perfectly frictionless fixed inclined plane will undergo rolling with slipping (only weight and contact force act).