Quadratic Equation

POLYNOMIAL

Algebraic expression containing many terms is called Polynomial.

e.g $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots a_{n-1}x^{n-1} + a_nx^n$

where x is a variable, $a_0, a_1, a_2 \ldots\ldots\ldots a_n \in C$.

1. **Real Polynomial :** Let a_0, $a_1, a_2 \ldots\ldots a_n$ be real numbers and x is a real variable.

   Then $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ is called real polynomial of real variable x with real coefficients.

   eg. $-3x^3 - 4x^2 + 5x - 4$, $x^2 - 2x + 1$ etc. are real polynomials.

2. **Complex Polynomial:** If $a_0, a_1, a_2 \ldots a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ is called a complex polynomial of complex variable x with complex coefficients.

   eg.- $3x^2 - (2 + 4i)x + (5i - 4)$, $x^3 - 5ix^2 + (1+2i)x + 4$ etc. are complex polynomials.

3. **Degree of Polynomial :** Highest Power of variable x in a polynomial is called as a degree of polynomial.

   e.g. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots a_{n-1}x^{n-1} + a_nx^n$ is n degree polynomial.

   $f(x) = 4x^3 + 3x^2 - 7x + 5$ is 3 degree polynomial

   $f(x) = 3x - 4$ is single degree polynomial or Linear polynomial.

   $f(x) = bx$ is odd Linear polynomial

QUADRATIC EXPRESSION

A Polynomial of degree two of the form $ax^2 + bx + c$ $(a \neq 0)$ is called a quadratic expression in x.

e.g $3x^2 + 7x + 5$, $x^2 - 7x + 3$

General form :– $f(x) = ax^2 + bx + c$

where a, b, c $\in$ C and $a \neq 0$

QUADRATIC EQAUTION

An equation $ax^2 + bx + c = 0$ (where $a \neq 0$, and $a, b, c \in R$), is called a quadratic equation. Here a, b and c are called coefficient of the equation. This equation always has two roots. Let the roots be α and β.

1. **Roots of a Quadratic Equation**

   The values of variable x which satisfy the quadratic equation is called as Roots (also called solutions or zeros) of a Quadratic Equation.

SOLUTION OF QUADRATIC EQUATION

1. Factorization Method :

Let $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x-1)(3x+1) = 0$

$$x = 1, -\frac{1}{3}$$

2. Hindu Method {Sri Dharacharya Method} (Discriminant Formula) :

By completing the perfect square as

$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Adding and substracting $\left(\frac{b}{2a}\right)^2$

$$\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right] = 0$$

Which gives, $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Hence the Quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) has two roots, given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Note : Every quadratic equation has two and only two roots.

Solved Examples

Ex.1 The roots of the equation $x^2 - 2x - 8 = 0$ are -

(A) $-4, 2$ (B) $4, -2$

(C) $4, 2$ (D) $-4, -2$

Sol. Quadratic Equation $x^2 - 2x - 8 = 0$

After factorization $(x-4)(x+2) = 0$

$\Rightarrow x = 4, -2$

Ans.[B]

Ex.2 The roots of the equation $x^2 - 4x + 1 = 0$ are -

(A) $2 \pm \sqrt{3}$ (B) $2, 4$

(C) $-2 \pm \sqrt{3}$ (D) $\sqrt{3} \pm 2$

Sol. Here a = 1, b = 4, c = 1

Using Hindu Method

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Ans.[A]

NATURE OF ROOTS

In Quadratic equation $ax^2 + bx + c = 0$, the term $b^2 - 4ac$ is called discriminant of the equation, which plays an important role in finding the nature of the roots. It is denoted by Δ or D.

(A) Suppose a, b, c $\in$ R and a $\neq$ 0 then

(i) If $D > 0 \Rightarrow$ Roots are Real and unequal

(ii) If $D = 0 \Rightarrow$ Roots are Real and equal and each equal to $-b/2a$

(iii) If $D < 0 \Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose a, b, c $\in$ Q, a $\neq$ 0 then

(i) If $D > 0$ and D is perfect square

$\Rightarrow$ Roots are unequal and Rational

(ii) If $D > 0$ and D is not perfect square

$\Rightarrow$ Roots are irrational and unequal

CONJUGATE ROOTS

The Irrational and complex roots of a quadratic equation are always occurs in pairs. Therefore (a, b, c,$\in$Q)

If	One Root	then	Other Root
$\alpha + i\beta$	$\alpha - i\beta$	$\alpha + \sqrt{\beta}$	$\alpha - \sqrt{\beta}$

Solved Examples

Ex.3 The roots of the equation $x^2 - 2\sqrt{2}x + 1 = 0$ are

(A) Imaginary and different

(B) Real and different

(C) Real and equal

(D) Rational and different

Sol. The discriminant of the equation $\left(-2\sqrt{2}\right)^2 - 4(1)(1) = 8 - 4 = 4 > 0$ and a perfect square, so roots are real and different but we can't say that roots are rational because coefficients are not rational therefore.

$$\alpha, \beta = \frac{2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4}}{2} = \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1$$ this is irrational

$\therefore$ The roots are real and different.

Ans. [B]

Ex.4 If the roots of the equation $x^2 + 2x + p = 0$ are real then the value of p is

(A) $p \le 1$ (B) $p \le 2$

(C) $p \le 3$ (D) None of these

Sol. Here a = 1, b = 2, c = p

$\therefore$ discriminant $= (2)^2 - 4(1)(p) \ge 0$ (since roots are real)

$= 4 - 4p \ge 0 \Rightarrow 4 \ge 4p$

$\Rightarrow p \le 1$

Ans. [A]

Ex.5 The roots of the quadratic equation $x^2 - 2(a + b)x + 2(a^2 + b^2) = 0$ are -

(A) Rational and different

(B) Rational and equal

(C) Irrational and different

(D) Imaginary and different

Sol. 1, B = –2 (a+b), C = 2 $(a^2 + b^2)$

$B^2 - 4AC = 1[2(a + b)]^2 - 4(1)(2a^2 + 2b^2)$

$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$

$= -4a^2 - 4b^2 + 8ab$

$= -4(a-b)^2 < 0$

So roots are imaginary and different.

Ans.[D]

Ex.6 The roots of the equation $(b+c)x^2 - (a+b+c)x + a = 0$ are $(a,b,c \in Q)$ -

(A) Real and different

(B) Rational and different

(C) Imaginary and different

(D) Real and equal

Sol. The discriminant of the equation is

$(a+b+c)^2 - 4(b+c)(a)$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4(b+c)a$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4ab - 4ac$

$= a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$

$(a-b-c)^2 > 0$

So roots are rational and different. **Ans.[B]**

SUM AND PRODUCT OF ROOTS

If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, then,

(i) Sum of Roots

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

(ii) Product of Roots

$$P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

e.g. In equation

$3x^2 + 4x - 5 = 0$

Sum of roots S $= -\frac{4}{3}$,

Product of roots P $= -\frac{5}{3}$

1. Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \ne 0)$ are α and β then :

(i) $(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\pm\dfrac{\sqrt{b^2-4ac}}{a}=\dfrac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\dfrac{b^2-2ac}{a^2}$

(iii) $\alpha^2-\beta^2=(\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$$=-\frac{b\sqrt{b^2-4ac}}{a^2}=\frac{\pm\sqrt{D}}{a}$$

(iv) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-\dfrac{b\left(b^2-3ac\right)}{a^3}$

(v) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$$=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\left\{(\alpha+\beta)^2-\alpha\beta\right\}$$

$$=\frac{\left(b^2-ac\right)\sqrt{b^2-4ac}}{a^3}$$

(vi) $\alpha^4+\beta^4=\left\{(\alpha+\beta)^2-2\alpha\beta\right\}^2-2\alpha^2\beta^2$

$$=\left(\frac{b^2-2ac}{a^2}\right)^2-2\frac{c^2}{a^2}$$

(vii) $\alpha^4-\beta^4=\left(\alpha^2-\beta^2\right)\left(\alpha^2+\beta^2\right)$

$$=\frac{-b\left(b^2-2ac\right)\sqrt{b^2-4ac}}{a^4}$$

(viii) $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta$

(ix) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)$

(xi) $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2=\dfrac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\dfrac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

Solved Examples

Ex.7 If the product of the roots of the quadratic equation $mx^2-2x+(2m-1)=0$ is 3 then the value of m is -

(A) 1 (B) 2

(C) –1 (D) 3

Sol. Product of the roots c/a = 3 = $\dfrac{2m-1}{m}$

$\therefore 3m-2m=-1 \quad \Rightarrow m=-1$

Ans. [C]

Ex.8 If the equation $(k-2)x^2-(k-4)x-2=0$ has difference of roots as 3 then the value of k is-

(A) 1,3 (B) 3,3/2

(C) 2, 3/2 (D) 3/2, 1

Sol. $(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

Now $\alpha+\beta=\dfrac{(k-4)}{(k-2)}$, $\alpha\beta=\dfrac{-2}{k-2}$

$$\therefore(\alpha-\beta)=\sqrt{\left(\frac{k-4}{k-2}\right)^2+\frac{8}{(k-2)}}$$

$$=\frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$$

$$\Rightarrow 3=\frac{\sqrt{k^2+16-8k+8k-16}}{(k-2)}$$

$$\Rightarrow 3k-6=\pm k$$

$$\therefore k=3,\ 3/2$$

Ans.[B]

Ex.9 If α,β are roots of the equation $ax^2+bx+c=0$ then the value of $\dfrac{1}{(a\alpha+b)^2}+\dfrac{1}{(a\beta+b)^2}$ is -

(A) $\dfrac{b^2-2ac}{ac}$ (B) $\dfrac{2ac-b^2}{ac}$

(C) $\dfrac{b^2-2ac}{a^2c^2}$ (D) $\dfrac{b^2}{a^2c}$

Sol. Since α,β are the root of the ax^2+bx+c

then $a\alpha^2+b\alpha+c=0$

$\Rightarrow \alpha(a\alpha+b)+c=0$

$\Rightarrow (a\alpha+b)=-c/\alpha$ ...(1)

Similarly

$(a\beta+b)=-c/\beta$ ...(2)

$$\therefore\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}=\frac{1}{(-c/\alpha)^2}+\frac{1}{(-c/\beta)^2}$$

$$\Rightarrow\frac{\alpha^2}{c^2}+\frac{\beta^2}{c^2}=\frac{\alpha^2+\beta^2}{c^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{c^2}$$

$$=\frac{b^2/a^2-2c/a}{c^2}=\frac{b^2-2ac}{a^2c^2}$$

Ans.[C]

FORMATION OF AN EQUATION WITH GIVEN ROOTS

A quadratic equation whose roots are α and β is given by

$(x-\alpha)(x-\beta)=0$

$\Rightarrow x^2-\alpha x-\beta x+\alpha\beta=0$

$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$

i.e., x^2 – (Sum of roots)x + Product of roots = 0

1. Transformation of an Eqaution

If α,β are roots of the equation $ax^2+bx+c=0$ then the equation whose roots are

(i) $\frac{1}{\alpha},\frac{1}{\beta}$ is $cx^2+bx+a=0$ (Replace x by $\frac{1}{x}$)

(ii) $-\alpha,-\beta$ is $ax^2-bx+c=0$ (Replace x by $-x$)

(iii) $k+\alpha,k+\beta$ is $a(x-k)^2+b(x-k)+c=0$ {Replace x by $(x-k)$}

(iv) $\alpha^n,\beta^n\ (n\in N)$ is $a\left(x^{1/n}\right)^2+b\left(x^{1/n}\right)+c=0$ (Replace x by $x^{1/n}$)

(v) $\alpha^{1/n},\beta^{1/n}(n\in N)$ is $a\left(x^n\right)^2+b\left(x^n\right)+c=0$ (Replace x by x^n)

(vi) $k\alpha,k\beta$ is $ax^2+kbx+k^2c=0$ (Replace x by $\frac{x}{k}$)

(vii) $\frac{\alpha}{k},\frac{\beta}{k}$ is $k^2ax^2+kbx+c=0$ (Replace x by kx)

2. Symmetric Expressions

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, which do not change by interchanging α and β. To find the value of such an expression, we generally express that in terms of $\alpha+\beta$ and $\alpha\beta$.

Some examples of symmetric expressions are–

(i) $\alpha^2+\beta^2$ (ii) $\alpha^2+\alpha\beta+\beta^2$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}$ (iv) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

(v) $\alpha^2\beta+\beta^2\alpha$ (vi) $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2$

(vii) $\alpha^3+\beta^3$ (viii) $\alpha^4+\beta^4$

Solved Examples

Ex.10 If α,β are the root of a quadratic equation $x^2-3x+5=0$ then the equation whose roots are $\left(\alpha^2-3\alpha+7\right)$ and $\left(\beta^2-3\beta+7\right)$ is

(A) $x^2+4x+1=0$ (B) $x^2-4x-1=0$

(C) $x^2-4x+4=0$ (D) $x^2+2x+3=0$

Sol. Since α,β are the roots of equation $x^2-3x+5=0$

So $\alpha^2-3\alpha+5=0$; $\beta^2-3\beta+5=0$

$\therefore \alpha^2-3\alpha=-5$ ..... (i)

$\beta^2-3\beta=-5$ ..... (ii)

Putting the value from (i) and (ii) in $\left(\alpha^2-3\alpha+7\right)$ and $\left(\beta^2-3\beta+7\right)$

We get $(-5+7)$ and $(-5+7)$

$\therefore$ 2 and 2 are the roots

$\therefore$ The required equation is $x^2-4x+4=0$

Ans.[C]

Ex.11 The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be

(A) $x^2+4x-1=0$ (B) $x^2-4x-1=0$

(C) $x^2+4x+1=0$ (D) None of these

Sol. Given root $=\frac{1}{2+\sqrt{5}}=\sqrt{5}-2$

So the other root $=-\sqrt{5}-2$. Then sum of the roots $=-4$, product of the roots $=-1$

Hence the equation is $x^2+4x-1=0$

Ans.[A]

Ex.12 The equation whose roots are 3 and 4 will be-

(A) $x^2 + 7x + 12 = 0$

(B) $x^2 - 7x + 12 = 0$

(C) $x^2 - x + 12 = 0$

(D) $x^2 + 7x - 12 = 0$

Sol. The quadratic equation is given by

x^2 – (sum of the roots) x + (product of roots) =0

$\therefore$ The required equation

$= x^2 - (3+4)\,x + 3.4 = 0$

$= x^2 - 7x + 12 = 0$ **Ans. [B]**

Ex.13 The quadratic equation with rational coefficients whose one root is $2 + \sqrt{3}$ is -

(A) $x^2 - 4x + 1 = 0$ (B) $x^2 + 4x + 1 = 0$

(C) $x^2 + 4x - 1 = 0$ (D) $x^2 + 2x + 1 = 0$

Sol. The required equation is

$x^2 - \{(2+\sqrt{3}) + (2-\sqrt{3})\}\,x$

$+ (2+\sqrt{3})(2-\sqrt{3}) = 0$

or $x^2 - 4x + 1 = 0$

Ans. [A]

Ex.14 If α,β are roots of the equation $x^2 - 5x + 6 = 0$ then the equation whose roots are $\alpha+3$ and $\beta+3$ is-

(A) $x^2 - 11x + 30 = 0$

(B) $(x-3)^2 - 5(x-3) + 6 = 0$

(C) Both (1) and (2)

(D) None

Sol. Let $\alpha + 3 = x$

$\therefore$ $\alpha = x - 3$ (Replace x by x – 3)

So the required equation is

$(x-3)^2 - 5(x-3) + 6 = 0$ ...(1)

$\Rightarrow x^2 - 6x + 9 - 5x + 15 + 6 = 0$

$\Rightarrow x^2 - 11x + 30 = 0$ ...(2)

Ans.[C]

Ex.15 If α,β are roots of the equation $2x^2 + x - 1 = 0$ then the equation whose roots are $1/\alpha$, $1/\beta$ will be -

(A) $x^2 + x - 2 = 0$ (B) $x^2 + 2x - 8 = 0$

(C) $x^2 - x - 2 = 0$ (D) None of these

Sol. From the given equation

$\alpha+\beta = -1/2$, $\alpha\beta = -1/2$

The required equation is-

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{-1/2}{-1/2}\right)x + \frac{1}{-1/2} = 0$$

$\Rightarrow x^2 - x - 2 = 0$ **Ans.[C]**

Short cut : Replace x by 1/x

$\Rightarrow 2(1/x)^2 + 1/x - 1 = 0 \Rightarrow x^2 - x - 2 = 0$

ROOTS UNDER PARTICULAR CASES

For the quadratic equation $ax^2 + bx + c = 0$

(i) If $b = 0$ $\Rightarrow$ roots are of equal magnitude but of opposite sign

(ii) If $c = 0$ $\Rightarrow$ one root is zero other is $-b/a$

(iii) If $b = c = 0$ $\Rightarrow$ both root are zero

(iv) If $a = c$ $\Rightarrow$ roots are reciprocal to each other

(v) If $\left.\begin{matrix} a>0 & c<0 \\ a<0 & c>0 \end{matrix}\right\}$ $\Rightarrow$ If Roots are of opposite signs

(vi) If $\left.\begin{matrix} a>0, b>0, c>0 \\ a<0, b<0, c<0 \end{matrix}\right\}$ $\Rightarrow$ both roots are negative.

(vii) If $\left.\begin{matrix} a>0, b<0, c>0 \\ a<0, b>0, c<0 \end{matrix}\right\}$ $\Rightarrow$ both roots are positive.

(viii) If sign of a = sign of b $\neq$ sign of c

$\Rightarrow$ Greater root in magnitude is negative.

(ix) If sign of b = sign of c $\neq$ sign of a

$\Rightarrow$ Greater root in magnitude is positive.

(x) If a+b+c = 0

$\Rightarrow$ one root is 1 and second root is c/a.

(xi) If a = b = c = 0 then equation will become an identity and will be satisfy by every value of x.

Solved Examples

Ex.16 The roots of the equation $x^2 - 3x - 4 = 0$ are–

(A) Opposite and greater root in magnitude is positive

(B) Opposite and greater root in magnitude is negative

(C) Reciprocal to each other

(D) None of these

Sol. The roots of the equation $x^2 - 3x - 4 = 0$ are of opposite sign and greater root is positive

($\because$ a >0 , b< 0, c < 0) **Ans.[A]**

Ex.17 The roots of the equation $2x^2 - 3x + 2 = 0$ are -

(A) Negative of each other

(B) Reciprocal to each other

(C) Both roots are zero

(D) None of these

Sol. The roots of the equations $2x^2 - 3x + 2 = 0$ are reciprocal to each other because here a = c

Ans.[B]

Ex.18 If equation $\frac{x^2 - bx}{ax - c} = \frac{k-1}{k+1}$ has equal and opposite roots then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$

(C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Sol. Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k+1) = (k-1)(ax-c)$

$\Rightarrow x^2(k+1) - bx(k+1) = ax(k-1) - c(k-1)$

$\Rightarrow x^2(k+1) - bx(k-1) - ax(k-1) + c(k-1) = 0$

Now sum of roots = 0 ($\because \alpha - \alpha = 0$)

$\therefore$ b (k+1) + a (k–1) = 0

$\Rightarrow k = \frac{a-b}{a+b}$ **Ans.[B]**

Ex.19 The real values of a for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite signs are given by

(A) $a > 5$ (B) $0 < a < 4$

(C) $a > 0$ (D) $a > 7$

Sol. The roots of the given equation will be of opposite signs if they are real and their product is negative, i.e., Discriminant ≥ 0 and product of roots < 0.

$\Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0$ and $\frac{a^2 - 4a}{2} < 0$

$\Rightarrow a^2 - 4a < 0$

$\left[\because a^2 - 4a < 0 \Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0\right]$

$\Rightarrow 0 < a < 4$ **Ans.[B]**

CONDITION FOR COMMON ROOTS

1. Only One Root is Common : Let α be the common root of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ then

$\therefore a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

By Cramer's rule :

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$\therefore \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}, \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$, $\alpha \neq 0$.

$\therefore$ The condition for only one Root common is

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

2. Both roots are common : Then required conditions is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Note: Two different quadratic equation with rational coefficient cannot have single common root which is complex or irrational, as imaginary and surd roots always occur in pair.

Solved Examples

Ex.20 If one root of the equations $x^2+2x+3k=0$ and $2x^2+3x+5k=0$ is common then the values of k is -

(A) 1, 2 (B) 0, –1
(C) 1, 3 (D) None of these

Sol. Since one root is common, let the root is α.

$$\frac{\alpha^2}{10k-9k} = \frac{\alpha}{6k-5k} = \frac{1}{3-4}$$

$$\alpha^2 = -k \quad ...(1)$$

$$\alpha = -k \quad ...(2)$$

$\therefore$ $\alpha^2 = k^2$

$\Rightarrow$ $k^2 = -k \Rightarrow k^2 + k = 0$

$\Rightarrow$ $k(k+1) = 0$

$\Rightarrow$ $k = 0$ and $k = -1$ **Ans. [B]**

Ex.21 If the equations $2x^2 + x + k = 0$ and $x^2 + x/2 - 1 = 0$ have 2 common roots then the value of k is-

(A) 1 (B) 3
(C) –1 (D) –2

Sol. Since the given equation have two roots in common so from the condition

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{2}{1} = \frac{1}{1/2} = \frac{k}{-1}$$

$\therefore$ $k = -2$ **Ans.[D]**

Ex.22 If $x^2 + x - 1 = 0$ and $2x^2 - x + \lambda = 0$ have a common root then –

(A) $\lambda^2 - 7\lambda + 1 = 0$ (B) $\lambda^2 + 7\lambda - 1 = 0$
(C) $\lambda^2 + 7\lambda + 1 = 0$ (D) $\lambda^2 - 7\lambda - 1 = 0$

Sol. Let the common root is α then

$\alpha^2 + \alpha - 1 = 0$

$2\alpha^2 - \alpha + \lambda = 0$

By cross multiplication

$$\frac{\alpha^2}{\lambda-1} = \frac{\alpha}{-2-\lambda} = \frac{1}{-1-2}$$

$$\alpha^2 = \frac{\lambda-1}{-3} = \frac{1-\lambda}{3}, \quad \alpha = \frac{2+\lambda}{3}$$

$$\left(\frac{2+\lambda}{3}\right)^2 = \frac{1-\lambda}{3} \Rightarrow \lambda^2 + 7\lambda + 1 = 0$$ **Ans.[C]**

Ex. 23 If $x^2 + x - 1 = 0$ and $2x^2 - x + k = 0$ have a common root then

(A) $k^2 - 7k + 1 = 0$ (B) $k^2 + 7k + 1 = 0$
(C) $k^2 + 7k - 1 = 0$ (D) $k^2 - 7k - 1 = 0$

Sol. Let the common root is α then

$\alpha^2 + \alpha - 1 = 0$

$2\alpha^2 - \alpha + k = 0$

By cross multiplication

$$\frac{\alpha^2}{k-1} = \frac{\alpha}{-2-k} = \frac{1}{-1-2}$$

$$\alpha^2 = \frac{k-1}{-3} = \frac{1-k}{3}, \alpha = \frac{2+k}{3}$$

$$\left(\frac{2+k}{3}\right)^2 = \frac{1-k}{3} \Rightarrow k^2 + 7k + 1 = 0$$ **Ans. [B]**

QUADRATIC EXPRESSION

The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x. Where a, b and c are real $a \neq 0$. Let $y = ax^2 + bx + c$

$$\Rightarrow y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right\}$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 \quad .....(1)$$

where $D = b^2 - 4ac$

Equation (1) represents a parabola with vertex at $A\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, and axis of this parabola is parallel to y axis and it is $x = -b/2a$.

GREATEST AND LEAST VALUE OF QUADRATIC EXPRESSION

(i) If $a > 0$, then the quadratic expression $y = ax^2 + bx + c$ has no greatest value but it has least value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

(ii) If $a < 0$, then the quadratic expression $y = ax^2 + bx + c$ has no least value but it has greatest value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

Solved Examples

Ex.24 The range of the values of $\frac{x}{x^2+4}$ for all real value of x is

(A) $\frac{-1}{2} \le y \le \frac{1}{2}$ (B) $\frac{-1}{4} \le y \le \frac{1}{4}$

(C) $\frac{-1}{6} \le y \le \frac{1}{6}$ (D) None of these

Sol. Let $y = \frac{x}{x^2+4}$

$\Rightarrow x^2y - x + 4y = 0$

Now, $x \in R \Rightarrow B^2 - 4AC \ge 0 \Rightarrow 1 - 4y.4y \ge 0$

$= (4y-1)(4y+1) \le 0$

$\therefore \frac{-1}{4} \le y \le \frac{1}{4}$ **Ans.[B]**

Ex.25 If the roots of the quadratic equation $x^2 - 4x - \log_3 a = 0$ are real, then the least value of a is

(A) 81 (B) $\frac{1}{81}$

(C) $\frac{1}{64}$ (D) None of these

Sol. Since the roots of the given equation are real.

$\therefore$ Discriminant $\ge 0 \Rightarrow 16 + 4\log_3 a \ge 0$

$\Rightarrow \log_3 a \ge -4 \Rightarrow a \ge 3^{-4} \Rightarrow a \ge \frac{1}{81}$

Hence, the least value of a is $\frac{1}{81}$ **Ans.[B]**

Ex.26 The minimum value of the expression $4x^2 + 2x + 1$ is-

(A) 1/4 (B) 1/2

(C) 3/4 (D) 1

Sol. Since a = 4 > 0 therefore its minimum value is

$$= \frac{4(4)(1)-(2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$$

Ans.[C]

Ex.27 The maximum value of $5 + 20x - 4x^2$ for all real value of x is-

(A) 10 (B) 20

(C) 25 (D) 30

Sol. Since a = –4 < 0 therefore its maximum value is

$$= \frac{4(-4)(5)-(20)^2}{4(-4)} = \frac{-80-400}{-16} = \frac{-480}{-16} = 30$$

Ans.[D]

FEW GRAPHS OF QUADRATIC EXPRESSION

(i) $y = x^2$ (a = 1, b = 0, c = 0)

Vertex (0, 0)

Axis of the parabola x = 0

This parabola opens in upward direction.

(ii) $y = -x^2 + 2x + 1$ (a = – 1, b = 2, c = 1)

vertex (1,2)

Axis of the parabola x = 1

This parabola opens downward.

Note: If a > 0, shape of parabola is upward.

And if a < 0, shape of parabola is downward.

Values of x where curve crosses x-axis will be roots of the equation $ax^2 + bx + c = 0$ as y = 0 on these points. If a curve is above x-axis for all x, then it means that it does not intersect x-axis or we can say equation $ax^2 + bx + c = 0$ have imaginary roots. This gives rise to following cases:

Cases:

Let $f(x) = ax^2 + bx + c$ where $a, b, c, \in R$ $(a \neq 0)$.

For some value of x, f(x) may be positive, negative or zero. This gives rise to the following cases:

(i) $a > 0$ and $b^2 - 4ac < 0$

$\Leftrightarrow f(x) > 0, \ \forall \ x \in R$

In this case whole of the parabola lies above the x - axis.

(ii) $a > 0$ and $b^2 - 4ac = 0 \Leftrightarrow f(x) \geq 0, \ \forall \ x \in R$

In this case whole of the parabola lies above the x-axis except at one point where it touches the x – axis. The point is $\left(-\frac{b}{2a}, 0\right)$.

(iii) $a > 0$ and $b^2 - 4ac > 0$

Let $f(x) = 0$ have two real roots α and β

(say $\alpha < \beta$)

then $f(x) > 0, \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

and $f(x) < 0, \ \forall \ x \in (\alpha, \beta)$

(iv) $a < 0$ and $b^2 - 4ac < 0$

$\Leftrightarrow f(x) < 0, \ \forall \ x \in R$

In this case whole of the parabola lies below the x–axis.

(v) $a < 0$ and $b^2 - 4ac = 0$

$\Leftrightarrow f(x) \leq 0, \ \forall \ x \in R$

In this case whole of the parabola lies below the x–axis, except at one point

where it touches the x–axis.

The point is $\left(-\frac{b}{2a}, 0\right)$.

(vi) $a < 0$ and $b^2 - 4ac > 0$

Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$.

Then $f(x) < 0, \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0, \ \forall \ x \in (\alpha, \beta)$

LOCATION OF ROOTS
(Interval in which roots lie)

In some problems we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c. Since $a \neq 0$, we can take $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$.

(i) If both the roots are positive i.e., they lie in $(0, \infty)$, then the sum of the roots as well as the product of the roots must be positive.

$\Rightarrow \alpha + \beta = -\frac{b}{a} > 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$.

Similarly, if both the roots are negative i.e. they lie in $(-\infty, 0)$ then the sum of the roots must be negative and the product of the roots must be positive.

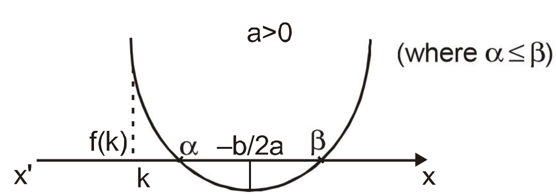
i.e. $\alpha + \beta = -\frac{b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$

Both the roots are of the same sign if a and c are of same sign. Now if b has the same sign that of a, both roots are negative or else both roots are positive.

If a and c are of opposite sign both roots are of opposite sign.

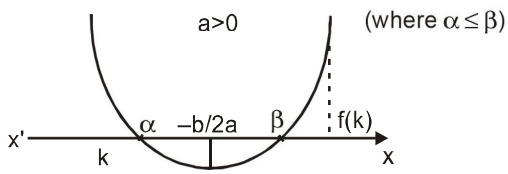
(ii) Both the roots are greater than given number k if the following three conditions are satisfied

$D \geq 0, \quad -\frac{b}{2a} > k$ and $f(k) > 0$.



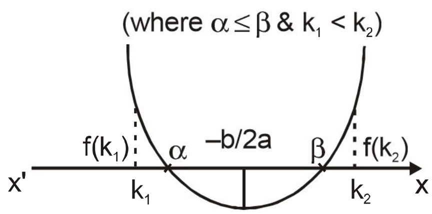
(iii) Both the roots will be less than a given number k if the following conditions are satisfied:

$D \geq 0,\ \ -\frac{b}{2a} < k$ and $f(k) > 0$

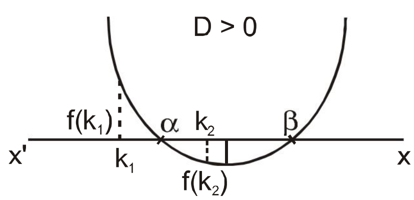


(iv) Both the roots will lie in the given interval (k_1, k_2) if the following conditions are satisfied:

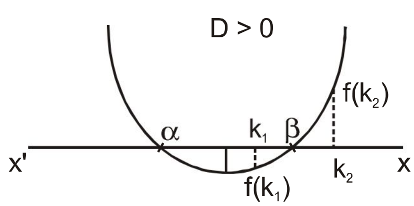
$D \geq 0,\ \ k_1 < -\frac{b}{2a} < k_2$ and $f(k_1) > 0,\ \ f(k_2) > 0$



(v) Exactly one of the root lies in the given interval (k_1, k_2) if $f(k_1).f(k_2) < 0$

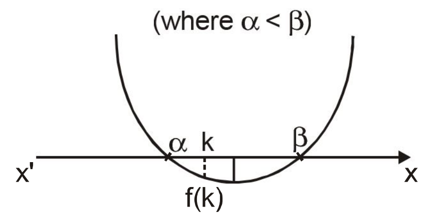


$f(k_1) > 0$ & $f(k_2) < 0$



$f(k_1) < 0$ & $f(k_2) > 0$

(vi) A given number k will lie between the roots if $f(k) < 0$, $D > 0$.



In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow f(0) < 0$.

Solved Examples

Ex.28 If f(x) is a quadratic expression which is positive for all real values of x and $g(x) = f(x) + f'(x) + f''(x)$, then for any real value of x-

(A) $g(x) < 0$ (B) $g(x) > 0$

(C) $g(x) = 0$ (D) $g(x) \geq 0$

Sol. Let $f(x) = ax^2 + bx + c$, then

$g(x) = (ax^2 + bx + c) + 2ax + b + 2a$

$= ax^2 + (b + 2a)x + (c + b + 2a)$

$\because f(x) > 0$, therefore $b^2 - 4ac < 0$ and $a > 0$.

Now for g(x),

Discriminant $= (b+2a)^2 - 4a(c+b+2a)$

$= b^2 - 4ac - 4a^2 < 0$

$(\because b^2 - 4ac < 0,\ -4a^2 < 0)$

Therefore signs of g(x) and a are same i.e. $g(x) > 0$. **Ans.[B]**

Ex.29 For real values of x, $2x^2 + 5x - 3 > 0$, if-

(A) $x < -2$ (B) $x > 0$

(C) $x > 1$ (D) None of these

Sol. Discriminant $b^2 - 4ac = 25 + 24 = 49 > 0$

$\Rightarrow$ Roots are real.

$\Rightarrow$ The given expression is positive for those real values of x for which $x \notin (-3, 1/2)$, because $a = 2 > 0$.

$\Rightarrow x > 1$ is true. **Ans.[C]**

THEORY OF EQUATIONS

(i) If $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, $(a_0, a_1, a_2, ..., a_n \in R$ and $a_n \neq 0)$ then $p(x) = 0$ has exactly n roots. (real/complex)

(ii) Imaginary roots always occur in conjugate pairs. If $\beta \neq 0$ and $\alpha + i\beta$ is a root of p(x), then $\alpha - i\beta$ is also a root.

(iii) A polynomial equation in x of odd degree has at least one real root (moreover it has odd no. of real roots).

(iv) If $x_1, ..., x_n$ are the roots of $p(x) = 0$, then p(x) can be written in the form $p(x) \equiv a_n(x - x_1)...(x - x_n)$.

(v) If α is a root of $p(x) = 0$, then $(x - \alpha)$ is a factor of p(x) and vice - versa.

(vi) If $x_1, ..., x_n$ are the roots of $p(x) \equiv a_n x^n + ... + a_0 = 0$, $a_n \neq 0$.

Then $\sum_{i=1}^{n} x_i = -\dfrac{a_{n-1}}{a_n}$,

$$\sum_{1\le i<j\le n} x_i x_j = \frac{a_{n-2}}{a_n},$$

$$x_1 x_2 ... x_n = (-1)^n \frac{a_o}{a_n}.$$

(vii) If equation $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$ has more than n distinct roots then f(x) is identically zero.

(viii) If p(a) and p(b) (a < b) are of opposite sign, then $p(x) = 0$ has odd number of real roots in (a, b), i.e. it has at least one real root in (a, b) and if p(a) and p(b) are of same sign then $p(x) = 0$ has even number of real roots in (a, b).

(ix) If coefficients of p(x) (polynomial in x written in descending order) have 'm' changes in signs, then $p(x) = 0$ have at the most 'm' positive real roots and if $p(-x)$ have 't' changes in sign, then $p(x) = 0$ have at most 't' negative real roots. By this we can find maximum number of real roots.

Some Important Points

(i) Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots and if the equation has more than n roots, it is an identity.

(ii) If roots of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio $\left(\text{i.e. } \dfrac{\alpha_1}{\beta_1} = \dfrac{\alpha_2}{\beta_2}\right)$ then $\boxed{\dfrac{b_1^2}{b_2^2} = \dfrac{a_1c_1}{a_2c_2}}$

(iii) If one root is k times the other root of quadratic equation $a_1x^2 + b_1x + c_1 = 0$ then $\boxed{\dfrac{(k+1)^2}{k} = \dfrac{b^2}{ac}}$

QUADRATIC EXPRESSION IN TWO VARIABLE

The general form of a quadratic expression in two variables x, y is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

The condition that this expression may be resolved into two linear rational factors is

$$\boxed{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$$

or

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example : For what value of m the expression $y^2 + 2xy + 2x + my - 3$ can be resolved into two rational factors?

Sol. Here a = 0, b = 1, c = – 3

h = 1, g = 1, g = m/2

So $\Delta = 0 \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$

$\Rightarrow$ (m/2 + 3) + (m/2 – 1) = 0

$\Rightarrow$ m + 2 = 0

$\Rightarrow$ m = – 2

Solved Examples

Ex.30 For what value of m the expression $y^2 + 4xy + 4x + my - 2$ can be resolved into two rational factors-

(A) 1 (B) –1
(C) 2 (D) –2

Sol. Here a = 0 , b = 1, c = –2

h = 2 , g = 2, f = m/2

So $\Delta = 0 \Rightarrow \begin{vmatrix} 0 & 2 & 2 \\ 2 & 1 & m/2 \\ 2 & m/2 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad 2(4+m) + 2(m-2) = 0$

$\Rightarrow \quad 4m + 4 = 0$

$\Rightarrow$ m = –1 **Ans. [B]**