Motion in a Plane

VECTOR

SCALAR AND VECTOR QUANTITIES

The physical quantities are two types : scalar and vector.

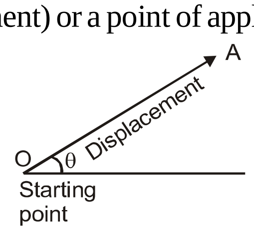
Scaler quantities : The quantities which have only magnitude, and no direction, are called 'scalar quantities', e.g. mass, distance, time, speed, volume, density, pressure, work, energy, power, charge, electric current, temperature, potential, specific heat, frequency etc.

Vector quantities : Certain quantities have both magnitude and direction, e.g. position, displacement, velocity, acceleration, force, weight, momentum, impulse, electric field, magnetic field, current density, etc. Such quantities are called 'vector quantities'

TYPES OF VECTOR

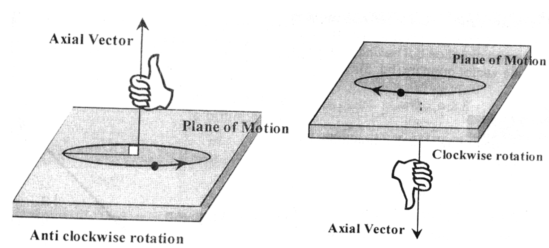
(a) Polar - Vectors : have starting point (like displacement) or a point of application (like force)

Ex. 0



(b) Axial - Vectors : Rotational effects are represented by axial vectors. They are along axis of rotation, direction denoted by right hand thumb rule or right hand screw rule.

Ex. Angular displacement, angular velocity, torque, angular momentum.



SOME OTHER TYPES OF VECTOR

[a] Negative of a vector : It has direction just opposite to given vector and have same magnitude fig.(a)

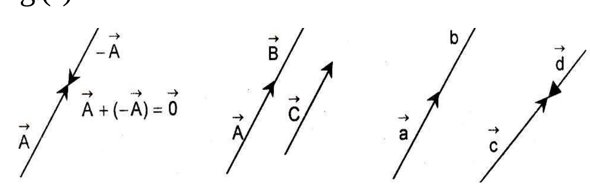


fig. (a) fig. (b) fig. (c) fig. (d)

[b] Zero vector or null vector : A vector will zero magnitude having no specific direction is called zero vector fig.(b)

(i) Multiplying a vector by zero. i.e. $0(\vec{A}) = \vec{0}$

(ii) By adding a negative vector to the given vector. $\vec{A} + (-\vec{A}) = \vec{0}$

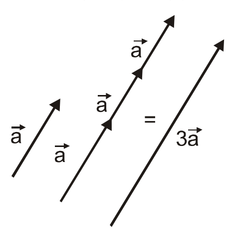
[c] Equal vector : Two vectors are called equal (or equivalent) vectors if they have equal magnitude, and same direction fig.(c) $\vec{A} = \vec{B} = \vec{C}$

[d] Collinear vectors : Two vectors acting along same straight line or along parallel straight line in same direction or in opposite direction fig.(d)

[e] Coplanar vectors : Three (or more) vectors are called coplanar vectors if they lie in the same plane or are parallel to the same plane. Two (free) vectors are always coplanar.

MULTIPLICATION OF A VECTOR WITH SCALAR

If a vector multiplied by a scalar quantity then we get a new vector in the same direction but having new magnitude = scalar quantity × magnitude of the vector quantity fig. $3 \times \vec{a} = 3a$



UNIT VECTOR

A vector having unit magnitude. It is used to denote the direction of a given vector.

$\vec{A} = \hat{a}.A$ $\hat{a}$ is unit vector along the direction of $\vec{A}$

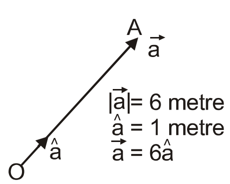
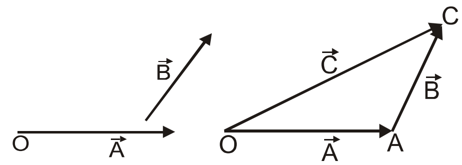


fig. (1)

Ex. As shown in fig.(1) if OA is displacement of vector in any orbiter direction then $\hat{a}$ is unit vector in the same direction.

VECTOR ADDITION

(i) Law of Triangle : If two sides of triangle are shown by two continuous vectors ($\vec{A}$ and $\vec{B}$) then third side of triangle in opposite direction shown resultant of two vectors ($\vec{C}$)



$\vec{C} = \vec{A} + \vec{B}$ $\vec{OC} = \vec{OA} + \vec{AC} \Rightarrow \vec{C} = \vec{A} + \vec{B}$

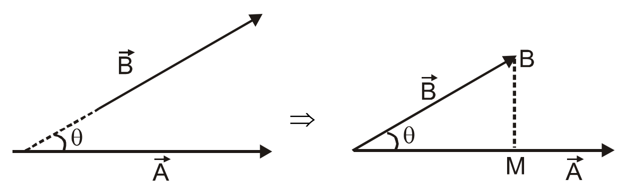
COMPONENT OF A VECTOR ALONG ANOTHER VECTOR

Two vector $\vec{A}$ and $\vec{B}$ there as shown below to get component of $\vec{B}$ along $\vec{A}$

shift $\vec{B}$ ‖ to it self and made it coinitial with $\vec{A}$, draw a perpendicular from B to OA let is meets $\vec{A}$ at Mθ is angle in between $\vec{A}$ and $\vec{B}$.

Component of $\vec{B}$, along $\vec{A} = OM$ = OB cos θ

= OB cosθ

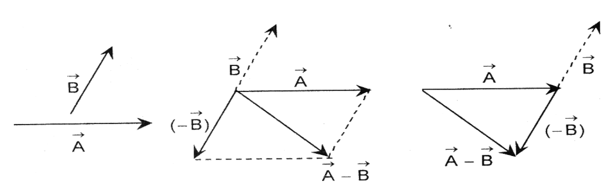


Note : (i) $\theta = 0$, OM maximum (ii) $\theta = 90°$, OM = zero

SUBTRACTION OF VECTORS

Vector which is want to subtracted just change direction of that vector and then add.

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



Solved Examples

Ex.1 Given $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + 3\hat{k}$

Determine (i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{b}$

Sol. (i) $\vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - \hat{k}) + (\hat{i} + \hat{j} + 3\hat{k})$

(ii) $\vec{a} - \vec{b} = (3\hat{i} + 2\hat{j} - \hat{k}) - (\hat{i} + \hat{j} + 3\hat{k})$

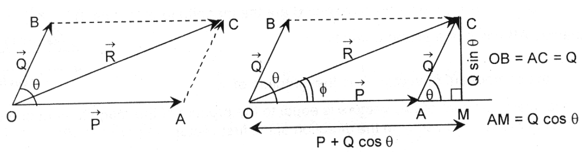
$= 3\hat{i} + 2\hat{j} - \hat{k} + \hat{i} + \hat{j} + 3\hat{k}$ $\quad$ $= 3\hat{i} + 2\hat{j} - \hat{k} - \hat{i} - \hat{j} - 3\hat{k}$

$= 4\hat{i} + 3\hat{j} + 2\hat{k}$ $\quad$ $= 2\hat{i} + \hat{j} - 4\hat{k}$

PARALLELOGRAM LAW OF VECTOR

To determine magnitude & direction of resultant vector, when two vectors act at an angle θ.

According to this law if two vectors $\vec{P}$ and $\vec{Q}$ are represented by two adjacent sides of a parallelogram both pointing outwards as shown in fig. The diagonal drawn through the intersection of the two vectors represents the resultant $\vec{R}$.



$\vec{R} = \vec{P} + \vec{Q}$

From triangle OCM

$OC^2 = OM^2 + CM^2$

$= (P + Q\cos\theta)^2 + (Q\sin\theta)^2$

$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$

Since $Q^2(\cos^2\theta + \sin^2\theta) = Q^2$

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

(Magnitude of resultant vector)

and $\tan\phi = \dfrac{CM}{OM} = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

or $\phi = \tan^{-1}\left[\dfrac{Q\sin\theta}{P + Q\cos\theta}\right]$

(Angle of resultant vector with P)

Important results

(a) $\theta = 0 \Rightarrow \vec{P} \parallel \vec{Q}$ then $R_{max} = P + Q$

(b) $\theta = 180^\circ \Rightarrow$ anti $\parallel$ then $R_{min} = P - Q$

(c) $\theta = 90^\circ \Rightarrow \vec{P} \perp \vec{Q}$ $R = \sqrt{a^2 + b^2}$ here $\tan\phi = \left(\dfrac{Q}{P}\right)$

(d) $|\vec{P}| = |\vec{Q}| = a$ $|\vec{R}|$ $2P\cos\dfrac{\theta}{2}$ and $\tan\phi = \dfrac{\theta}{2}$

$\therefore \phi = \dfrac{\theta}{2}$

(e) $|\vec{P}| = |\vec{Q}|$ and (i) $\theta = 60^\circ$ $R = \sqrt{3}\,P$

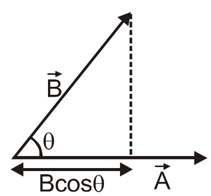
(ii) $\theta = 90^\circ$ $R = \sqrt{2}\,P$

(iii) $\theta = 120^\circ$ $R = P$

(f) If three vectors of equal magnitudes makes an angle of 120º with each other then the resultant vector will be zero. R = 0

(g) If n vectors of equal magnitudes makes the angle the of equal measure with each other then the resultant vector will be zero.

SCALAR PRODUCT OR DOT PRODUCT OF TWO VECTORS



If θ is the angle between $\vec{A}$ and $\vec{B}$.

Then $A(B\cos\theta) = \vec{A}.\vec{B}$, A and B are the magnitude of $\vec{A}$ and $\vec{B}$

The quantity AB cosθ is a scalar quantity.

B cosθ is the component of vector $\vec{B}$ in the direction of $\vec{A}$.

Hence, the scalar product of two vectors is equal to the product of the magnitude of one vector and the component of the second vector in the direction of the first vector.

Ex. [1] $W = \vec{F}.\vec{S}$ [2] $P = \vec{F}.\vec{V}$ [3] $\phi = \vec{B}.\vec{A}$

Propoerties of scalar product

(a) The scalar product is commutative.

i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ $\vec{A}.\vec{B}$ = A(B cosθ) = (A cosθ)

B = $\vec{B}.\vec{A}$

(b) The scalar product is distributive

$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$

(c) The scalar product of two mutually perpendiucular vectors is zero.

$\vec{A}.\vec{B} = AB\ \cos\ 90^o = 0$ $\vec{A}.\vec{B} = 0$ for $\vec{A} \perp \vec{B}$

$\boxed{\hat{i}.\hat{j} = 0, \quad \hat{j}\ \hat{k} = 0, \quad \hat{k}, \hat{i} = 0}$

(d) The scalar product of two parallel vectors is equal to the product of their magnitudes.

$\vec{A}.\vec{B} = AB\ \cos 0^o = AB$ cos 0° = 1

(e) The scalar product of a vector with itseft is equal to the square of the magnitude of the vector.

$\vec{A}.\vec{A} = A\,A\ \cos 0^o = A^2$

$\hat{i}.\hat{i} = 1, \quad \hat{j}.\hat{j} = 1, \quad \hat{k}.\hat{k} = 1$

(f) The scalar product of two vectors is equal to the sum of the products of their corresponding x, y, z components. Let $\vec{A}$ and $\vec{B}$ be two vectors.

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Their scalar product is given by

$\vec{A}.\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Solved Examples

Ex.2 Two vector $\vec{A} = \hat{i} + 2\hat{j} + 2\hat{k}$ $\vec{B} = \hat{i} + 3\hat{j} + 6\hat{k}$

Find (i) Dot product (ii) Angle between them

$A = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$

$B = \sqrt{(1)^2 + (3)^2 + (6)^2} = \sqrt{46}$

$\vec{A}.\vec{B} = (i + 2\hat{j} + 2\hat{k}).(i + 3\hat{j} + 6\hat{k})$

= 1 × 1 + 2 × 3 + 2 × 6 = 19

(ii) Angle between them

$\vec{A}.\vec{B} = AB\cos\theta \Rightarrow \cos\theta = \frac{\vec{A}.\vec{B}}{AB}$

$\cos\theta = \frac{19}{3\sqrt{46}} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{3\sqrt{46}}\right)$

Ex.3 A particle, under constant force $\hat{i} + \hat{j} - 2\hat{k}$ gets displaced from point A(2, –1, 3) to B(4, 3, 2). Find the work done by the force -

Sol. Force = $\hat{i} + \hat{j} - 2\hat{k}$

dispalcement

$= d = \vec{AB} = (4\hat{i} + 3\hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = (2\hat{i} + 4\hat{j} - \hat{k})$

work done = F.d = $(\hat{i} + \hat{j} - 2\hat{k}) . (2\hat{i} + 4\hat{j} - \hat{k})$

= 1 x 2 + 1 x 4 + (–2) x (–1) = 2 + 4 + 2

= 8 units

Ex.4 Show that the vectors **a** = $3\hat{i} - 2\hat{j} + \hat{k}$, $b = \hat{i} - 3\hat{j} + 5\hat{k}+$, $c = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

Sol. We have **b** + **c**

= $(\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k}) = 3\hat{i} - 2\hat{j} + \hat{k} = a$

⇒ a, b, c are coplanar

Hence no two of these vectors are parallel, therfore, the given vectors form a triangle.

a . c = $(3\hat{i} - 2\hat{j} + \hat{k}) . (2\hat{i} + \hat{j} - 4\hat{k})$

= 3 x 2 – 2 x 1 – 4 x 1 = 0

Hence the given vectors form a right angled triangle.

Ex.5 Find the value of λ so that the two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $-4\hat{i} - 6\hat{j} - \lambda\hat{k}$ are (i) parallel (ii) perpendicular to each other

Sol. Let $a = 2\hat{i} + 3\hat{j} - \hat{k}$ and $b = -4\hat{i} - 6\hat{j} + \lambda\hat{k}$

(i) a and b are parallel to each other

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ i.e. if $\frac{2}{-4} = \frac{3}{-6} = \frac{-1}{\lambda}$

(ii) a and b are perpendicular to each other if a.b = 0

i.e. if 2(–4) + 3(–6) + (–1) (λ) = 0

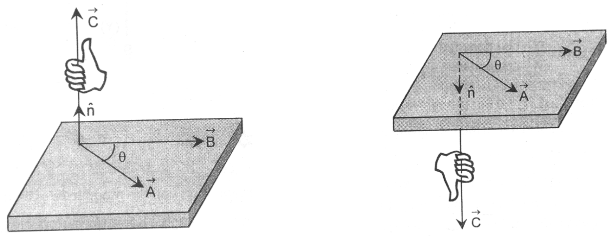
λ = –8 – 18 = –26

CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

Cross product of $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ is defined as :

$A\,B\sin\theta\ \ \hat{n} = \vec{A}\times\vec{B}\ \ \hat{n}\ \perp$ to plane of $\vec{A}$ and $\vec{B}$

Direction of $\hat{n}$ is given by right hand thumb rule. Curl the fingers of your right hand from $\vec{A}$ to $\vec{B}$. Then the direction of the erect thumb will point in the direction $\vec{A}\times\vec{B}$.



Example : (i) $\vec{\tau} = \vec{r}\times\vec{F}$ (ii) $\vec{J} = \vec{r}\times\vec{p}$

(iii) $\vec{v} = \vec{\omega}\times\vec{r}$ (iv) $\vec{a} = \vec{\alpha}\times\vec{r}$

Properties of vector product

(a) The vector product is 'not' commutative i.e

$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ $\boxed{\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}}$

(b) The vector product is distributive i.e.

$\boxed{\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}}$

(c) The magnitude of the vector product of two vectors mutually at right angles is equal to the product of the magnitudes of the vectors.

$\vec{A}\times\vec{B} = AB\ \sin 90^\circ\,\hat{n} = AB\ \hat{n},\ |\vec{A}\times\vec{B}| = AB$

(if $\theta = 90^\circ$)

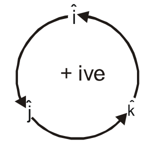
(d) The vector product of two parallel vectors is a null vector (or zero vector)

$\vec{A}\times\vec{B} = AB(\sin 0)\,\hat{n} = \vec{0}$ or 0

(e) The vector product of a vector by itself is a null vector (zero vector)

$\vec{A}\times\vec{B} = AA(\sin 0)\,\hat{n} = \vec{0}$ or 0

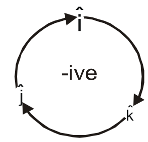
(f) The vector product of unit orthogonal vectors $\hat{i},\ \hat{j},\ \hat{k}$ have the following relations in the right handed coordinate system

(i) 

$\hat{i}\times\hat{j} = \hat{k}$ $\hat{j}\times\hat{i} = -\hat{k}$

$\hat{j}\times\hat{k} = \hat{i}$ $\hat{k}\times\hat{j} = -\hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$ $\hat{i}\times\hat{k} = -\hat{j}$



(ii) $\hat{i}\times\hat{i} = 0\quad \hat{j}\times\hat{j} = 0\quad \hat{k}\times\hat{k} = 0$

The magnitude of each of the vectors $\hat{i}, \hat{j}$ and $\hat{k}$ is 1 and the angle between any two of them is 90º. Therefore, we write $\hat{i}\times\hat{j} = (1)$ (1) sin 90º $\hat{n} = \hat{n}$, where $\hat{n}$ is a unit a vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$ i.e. it is just the third unit vector $\hat{k}$.

(g) The vector product of two vectors in terms of their x, y & z components can be expressed as a determinant. Let $\vec{A}$ and $\vec{B}$ be two vectors. Let us write their rectangular components :

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$= (A_yB_z - B_yA_z)\hat{i} - (A_xB_z - B_xA_z)\hat{j} + (A_xB_y - B_xA_z)\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Solved Examples

Ex.6 The vector from origin to the points A and B are $a = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $b = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of

(i) the triangle OAB

(ii) the parallelogram formed by **OA** and **OB** as adjacent sides.

Sol. Given **OA** = a = $3\hat{i} - 6\hat{j} + 2\hat{k}$ and **OB** = b $= 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore\ (a\times b) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (12 - 2)\ \hat{i} - (-6 - 4)\ \hat{j} + (3 + 12)\ \hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k}$

$\Rightarrow |a\times b| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$

(i) Area of ΔOAB $= \frac{1}{2}\ |a\times b| = \frac{5\sqrt{17}}{2}$ sq. units

(ii) Area of parallelogram formed by OA and OB as adjacent sides = $|a\times b| = 5\sqrt{17}$ sq. units.

Ex.7 Find a × b and b × a if

(i) $a = 3\hat{k} + 4\hat{j},\ b = \hat{i} + \hat{j} - \hat{k}$

(ii) a = (2, –1, 1) ; b = (3, 4, –1)

Sol. (i) $a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -3 \end{vmatrix} = -7\hat{i} + 3\hat{j} - 4\hat{k}$

$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 7\hat{i} - 3\hat{j} + 4\hat{k}$

(ii) $a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 11\hat{k}$

$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 11\hat{k}$

Tensor

If a physical quantity itself has no direction but has different values in different directions, then it is said to be TENSOR.

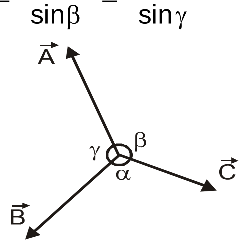
Ex. (a) Moment of inertia : It is not a vector as its direction is not be specified but has different values in different directions. Hence it is neither a scalar nor a vector but a Tensor.

(b) Stress : It is defined as force per unit area (acting at a point). It has no direction so it is not a vector but at a point it has different values for different areas so it is not a scalar also. It is actually a tensor.

LAMI'S THEOREM

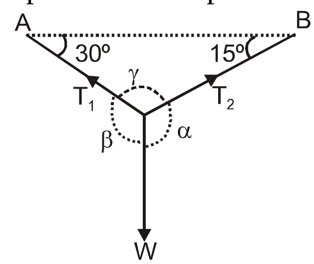
If three forces acting at a point are in equilibrium then each force is proportional to sine of the angle between the other two.

or $\dfrac{A}{\sin\alpha} = \dfrac{B}{\sin\beta} = \dfrac{C}{\sin\gamma}$



Solved Examples

Ex.8 A rope is stretched between two poles. A 50 N boy hangs from it, as shown in fig. Find the tensions in the two parts of the rope.



Sol. In fig. α = 90° + 15° = 105°

β = 90° + 30° = 120°

and γ = 180° – (30° + 15°) = 135°

Using Lami's Theorem, we have

$\dfrac{T_1}{\sin\alpha} = \dfrac{T_2}{\sin\beta} = \dfrac{W}{\sin\gamma}$

$\therefore\ T_1 = W \times \dfrac{\sin\alpha}{\sin\gamma} = 50 \times \dfrac{\sin 105^\circ}{\sin 135^\circ}$

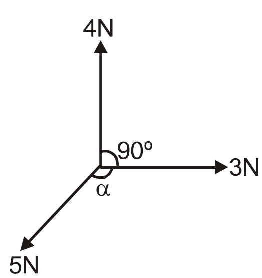
$= 50 \times \dfrac{\sin 75^\circ}{\sin 45^\circ} = \dfrac{50 \times 0.9659}{0.7071} = 68.3\ \text{N}$

$T_2 = \dfrac{W \sin\beta}{\sin\gamma} = \dfrac{50 \times \sin 120^\circ}{\sin 135^\circ} = \dfrac{50 \times \sin 60^\circ}{\sin 45^\circ}$

$= \dfrac{50 \times 0.8660}{0.7071} = 61.24\ \text{N}$

Ex.9 Three forces, 3N, 4N, 5N are acting on a body together and the body is in equilibrium . If the angle between 3N and 4N forces is 90°, then what is the angle between 3N and 5N forces?

Sol. The forces are shown in fig. Then according to Lami's theorem



$\dfrac{4}{\sin(\pi - \alpha)} = \dfrac{5}{\sin(\pi - 90^\circ)}$

Thus $\sin(\pi - \alpha) = \dfrac{4}{5}$

$\pi - \alpha = \sin^{-1}\left(\dfrac{4}{5}\right) = 53^\circ$

(remember the 3, 4, 5 triangle)

Thus α = 180 – 53° = 127°

Ex.10 The concurrent forces 8N, 6N and 15N are acting on an object. Is it possible to arrange the direction of forces in such a way that the object remain in equilibrium ?

Sol. Consider 8N and 6N as, say F_1 and F_2 forces. Then

$F_{max} = F_1 + F_2 = 14N$

$F_{min} = F_1 - F_2 = 2N$

For equilibrium, magnitude of the third force should satisfy, thus,

$14 \geq F_3 \geq 2$

which the F_3 = 15 newton **does not** satisfy, Therefore, equilibrium under 8N, 6N, 15 N forces is not possible

SPECIAL POINTS ABOUT VECTOR

* If a vector $\vec{A}$ is multiplied by zero, we get a vector whose magnitude is zero called null vector or Zero vector.
* The unit of $n\vec{A}$ is same as that of $\vec{A}$, if n is a pure real number.
* The unit of vector does not change on being multiplied by a dimensionless scalar.
* The unit of $n\vec{A}$ is different from that of $\vec{A}$, if n is a dimensional scalar.
* The multilication of velocity vector by time gives us displacement.
* Sum of non-coplanar forces can never be zero.
* Minimum number of equal forces required for a zero resultant is two.
* Minimum number of unequal forces required for a zero resultant is three.

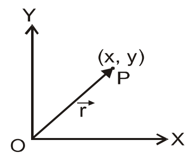
PROJECTILE MOTION

MOTION IN TWO DIMENSIONS

(i) In a two dimensional motion, the motion takes place in a plane.

(ii) If the plane in which motion takes place is taken as the (X-Y) plane, the coordinates of a point in this plane will be in this plane will be represented by (x, y) and the position vector $\vec{r}$ will be

$\vec{r} = x\hat{i} + y\hat{j}$

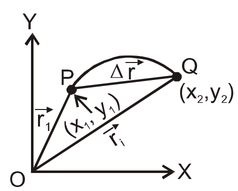


(iii) If (x_1, y_1) are the coordinates of the position of a particle at time t_1 and (x_2, y_2) are the coordinates of the particle at time t_2 then

$\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j}$

The displacement of the particle will be

$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\ \hat{i} + (y_2 - y_1)\hat{j}$



(iv) The magnitude of displacement is

$|\Delta\vec{r}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$

(v) At time t if the position is given by $\vec{r} = x\hat{i} + y\hat{j}$, the instantaneous velocity of the particle at this position will be

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} \quad = \quad v_x\hat{i} + v_y\hat{j}$$

Speed or magnitude of velocity will be $v = (v_x^2 + v_y^2)^{1/2}$

(vi) If the particle is displaced from position $\vec{r}_1$ at time t_1 to the position $\vec{r}_2$ at time t_2 then average velocity during the interval $(t_2 - t_1)$ is $\vec{v}_{av} = \dfrac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$

(vii) Motion with uniform velocity : Suppose the coordinates of a particle at t = 0 are (x_0, y_0) and after time t the coordinates become (x, y), then for uniform velocity

$$\vec{v} = \frac{\vec{r} - \vec{r}_0}{t - 0} = \frac{\vec{r} - \vec{r}_0}{t} \qquad \text{or} \qquad \vec{r} = \vec{r}_0 + \vec{v}t$$

$$\therefore\ x\hat{i} + y\hat{j} = (x_0\hat{i} + y_0\hat{j}) + (v_x\hat{i} + v_y\hat{j})t$$

$$= (x_0 + v_x t)\hat{i} + (y_0 + v_y t)\hat{j}$$

or $x = x_0 + v_x t$ and $y = y_0 + v_y\ t$

(viii) Motion with constant acceleration :

(a) If $\vec{a}$ is constant acceleration then $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

If the velocity is $\vec{v}_1$ at time t_1 and it becomes $\vec{v}_2$ at time t_2, then $d\vec{v} = \vec{a}\,dt$

Integrating $\displaystyle\int_{\vec{v}_1}^{\vec{v}_2} d\vec{v} = \vec{a}\int_{t_1}^{t_2} dt$ or $\vec{v}_2 - \vec{v}_1 = \vec{a}(t_2 - t_1)$

$$\therefore\ \vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

(b) Let the velocity at time t = 0 be $\vec{u}$ and the velocity at time t be $\vec{v}$.

$$\therefore\ \vec{a} = \frac{\vec{v} - \vec{u}}{t - 0} = \frac{\vec{v} - \vec{u}}{t} \qquad \text{or} \qquad \vec{v} = \vec{u} + \vec{a}t$$

The components of $\vec{v}$ will be

$v_x = u_x + a_x t$ and $v_y = u_y + a_y t$

(c) $\vec{v} = \dfrac{d\vec{r}}{dt} = \vec{u} + \vec{a}t \qquad \therefore \qquad d\vec{r} = (\vec{u} + \vec{a}t)dt$

Integrating from initial position $\vec{r} = \vec{r}_0$ at t = 0 to position $\vec{r}$ at time t

$$\int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_0^t (\vec{u} + \vec{a}t)\ dt \qquad \text{or} \qquad \vec{r} - \vec{r}_0 = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$\therefore\ \vec{r} = \vec{r}_0 + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

In terms of Cartesian components

$x = x_0 + u_x t + 1/2\ a_x\ t^2$

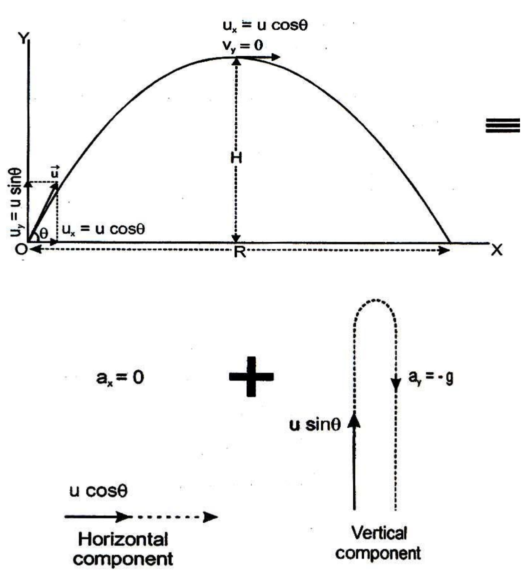
$y = y_0 + u_y t + 1/2\ a_y\ t^2$

PROJECTILE MOTION

Ground to ground projection : Consider the motion of a bullet which is fired from a gun so that its initial velocity $\vec{u}$ makes an angle θ with the horizontal. Take x-axis along ground and y-axis along vertical.

$\vec{u}$ can be resolved as $u_x = u\cos\theta$ (along horizontal) & $u_y = u\sin\theta$ (along vertical) motion of bullet can be resolved into horizontal and vertical motion.

* In horizontal direction there is no acceleration so it moves with constant velocity $v_x = u_x = u\cos\theta$ So distance traversed in time t is $x = u_x t$ or $x = (u\cos\theta)\,t$ or $t = \dfrac{x}{u\cos\theta}$. ....(i)



* The motion in the vertical direction is the same as that of a ball thrown upward with an initial velocity $u_y = u\sin\theta$ and acceleration = – g downward. So at time t vertical component of velocity

$v_y = u_y - gt = u\sin\theta - gt$ ....(ii)

Displacement along y direction

$y = (u\sin\theta)\,t - 1/2\,gt^2$ ....(iii)

Substituting the value of t from eq. (i) in eq. (iii)

we get $y = (u\sin\theta)\left(\dfrac{x}{u\cos\theta}\right) - \dfrac{1}{2}g\left(\dfrac{x}{u\cos\theta}\right)^2$

or $y = x\tan\theta - \dfrac{g}{2u^2\cos^2\theta}.x^2$. This is eq. of parabola.

* The trajectory of projectile is parabolic

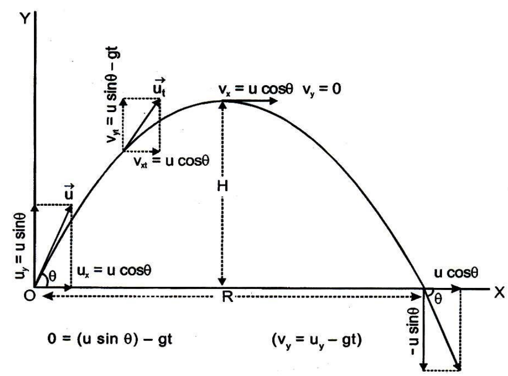
The projectile will rise to maximum height H (where $v_x = u\cos\theta$, $v_y = 0$) and the move down again to reach the ground at a distance R from origin.

Setting x = R and y = 0 (since projectile reaches ground again)

$$O = R\tan\theta - \frac{g}{2u^2\cos^2\theta}.R^2$$

We get $R = \dfrac{2u^2\cos^2\theta}{g}\times\dfrac{\sin\theta}{\cos\theta}$

or $R = \dfrac{2u^2}{g}.\sin\theta\cos\theta$ or Range $\boxed{\mathbf{R = \dfrac{u^2\sin 2\theta}{g}}}$



* If time for upward journey is t at highest point

$v_y = 0$ so $0 = (u\sin\theta) - gt$ $(v_y = u_y - gt)$

or $t = \dfrac{u\sin\theta}{g}$

$\therefore$ T = 2t

(it will take same time for downward journey)

$\therefore$ Time of flight $\boxed{\mathbf{T = \dfrac{2u\sin\theta}{g}}}$

* At the highest point y = H and $v_y = 0$

So that $H = \dfrac{u_y^2}{2g}$ $v_y^2 = u_y^2 - 2gy$

or Maximum Height $\boxed{\mathbf{H = \dfrac{u^2\sin^2\theta}{2g}}}$

Maximum height and Time of flight depends on Vertical component of Initial velocity.

* we can also determine R as follows $x = u_x\, t$

so $R = u_x \cdot T \quad = (u\cos\theta)\left(\dfrac{2u\sin\theta}{g}\right)$

or $R = \dfrac{u^2\sin 2\theta}{g}$ velocity at time t

$\vec{v}_t = v_{x\,t}\hat{i} + v_{y\,t}\hat{j} = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$

$v = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$

Note :

(i) Alternative equation of trajectory $y = x\tan\theta\left(1 - \dfrac{x}{R}\right)$ where $R = \dfrac{2u^2\sin\theta\cos\theta}{g}$

(ii) Vertical compound of velocity $v_y = 0$, when particle is at the highest point of trajectory.

(iii) Linear momentum at highest point = mu cos θ is in horizontal direction.

(iv) Vertical component of velocity is + ive when particle is moving up.

(v) Vertical component of velocity is –ive when particle is moving down.

(vi) Resultant velocity of particle at time t $v = \sqrt{v_x^2 + v_y^2}$ at an angle $\phi = \tan^{-1}\left(\dfrac{v_y}{v_x}\right)$.

(vii) Displacement from origin, $s = \sqrt{x^2 + y^2}$

Special Points :

(1) For projectile motion give :

$$T = \frac{2u\sin\theta}{g} \qquad R = \frac{u^2\sin 2\theta}{g} \qquad H = \frac{u^2\sin^2\theta}{2g}$$

(2) In case of projectile motion

The horizontal component of velocity (u cos θ), accleration (g) and mechanical energy remains constant.

Speed, velocity, vertical component of velocity (u sin θ), momentum, kinetic energy and potential energy all change. Velocity and K.E. are maximum at the point of projection, while minimum (but not zero) at the highest point.

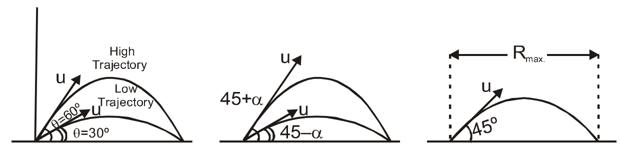
(3) If angle of projection is changed from

Same Range
$\alpha + \beta = 90^\circ$ α & β are two angles of projection with same velocity

$\theta \xrightarrow{\text{to}} \theta' = (90 - \theta)$,

then range

$$R' = \frac{u^2\sin 2\theta'}{g} = \frac{u^2\sin 2(90-\theta)}{g} = \frac{u^2\sin 2\theta}{g} = R$$



So a projectile has same range for angles of projection θ and (90 – θ)

But has different time of flight (T), Maximum height (H) & trajectories

Range is also same for $\theta_1 = 45^\circ - \alpha$ and $\theta_2 = 45^\circ + \alpha$. $\left[\text{equal } \dfrac{u^2\cos 2\alpha}{g}\right]$

(4) For maximum range

$R = R_{max} \Rightarrow 2\theta = 90^\circ$ for $\theta = 45^\circ$ $R_{max} = \dfrac{u^2}{g}$

[For sin 2θ = 1 = sin 90° or θ = 45°]

When range is maximum

Then maximum height reached

$H = \dfrac{u^2\sin^2 45^\circ}{2g}$ (When R_{max}) or $H = \dfrac{u^2}{4g}$

Hence maximum height teached (for R_{max}) $H = \dfrac{R_{max}}{4}$

(5) For height H to be maximum

$H = \dfrac{u^2\sin^2\theta}{2g} = \text{max}$

i.e. $\sin^2\theta = 1$ (max) or for θ = 90°

So that $H_{max} = \dfrac{u^2}{2g}$ When projected vertically (i.e. at θ = 90°) in this case

Range $R = \dfrac{u^2\sin(2\times 90^\circ)}{g} = \dfrac{u^2\sin 180^\circ}{g} = 0$

For vertical projection $H_{max} = \frac{u^2}{2g}$ and For oblique projection with same velocity $R_{max} = \frac{u^2}{g}$

so $H_{max} = \frac{R_{max}}{2}$

If a person can throw a projectile to a maximum distance (with $\theta = 45^o$) $R_{max} = \frac{u^2}{g}$

The maximum height to which he can throw the projectile (with $\theta = 90^o$) $H_{max} = \frac{R_{max}}{2}$

(6) At highest point

Potential energy will be max and equal to $(PE)_H$

$= mgH = mg . \frac{u^2 \sin^2\theta}{2g}$ or $(PE)_H = \frac{1}{2} mu^2 \sin^2\theta$.

While K.E. will be minimum (but not zero) and as at the highest point the vertical component of velocity is zero.

$(KE)_H = \frac{1}{2} mv_H{}^2 = \frac{1}{2} m(u \cos\theta)^2 = \frac{1}{2} mu^2 \cos^2\theta$

so $(PE)_H + (KE)_H = \frac{1}{2} mu^2 \sin^2\theta + \frac{1}{2} mu^2 \cos^2\theta$

$= \frac{1}{2} mu^2$ = Total Mechanical energy

So in projectile motion mechanical energy is conserved.

$$\left(\frac{PE}{KE}\right)_H = \frac{\frac{1}{2}mu^2\sin^2\theta}{\frac{1}{2}mu^2\cos^2\theta} = \tan^2\theta$$

So if $\theta = 45^o$ $\therefore$ $\tan^2\theta = 1$

PE = K.E. = $\frac{1}{2}$ M.E. at highest point i.e. if a body is projection at an angle $\theta = 45^o$ 10 the horizontal then at highest point, half of its M.E. is K.E. and half is P.E.

(7) In case of projectile motion if range R is n times the maximum height H, i.e. R = nH

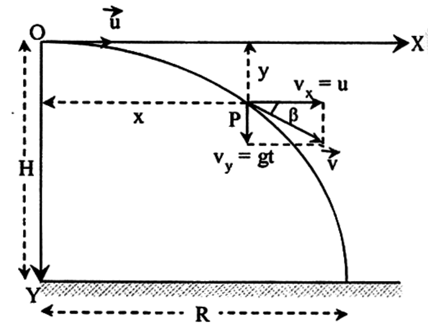
then $\frac{u^2 \sin 2\theta}{g} = n.\frac{u^2\sin^2\theta}{2g}$ or $2\cos\theta = \frac{n.\sin\theta}{2}$

or $\tan\theta = \frac{4}{n}$ $\Rightarrow$ $\theta = \tan^{-1}\left(\frac{4}{n}\right)$

(8) Weight of a body in projectile motion is zero as it is a freely falling body.

HORIZONTAL PROJECTILE MOTION TOP TO GROUND PROJECTION

Suppose a body is thrown horizontally from point O, with velocity u. Height of O from ground = H. Let x-axis be along horizontal and y-axis be vertically downwards origin O is at point of projection as shown is fig.



Let the particle be at P at a time t. The co-ordinates of P are (x, y).

Distance travelled along x-axis at time t with uniform velocity

i.e velocity of projection and without acceleration.

The horizontal component of velocity $v_x = u$

and horizontal displacement x = u . t .....(i)

displacement along vertical direction is y to calculate y, consider vertical motion of the projectile initial velocity in vertical direction $u_y = 0$. acceleration along y direction $a_y = g$ (acceleration due to gravity)

So $v_y = a_y t$ (y component of velocity at time t)

or $v_y = gt$ .....(ii) (as body were dropped from a height)

Resultant velocity at time t is $\vec{v} = v\,\hat{i} + (gt)\,\hat{j}$

$v = \sqrt{u^2 + (gt)^2}$

if β is the angle of velocity with x-axis (horizontal)

$\tan\beta = \frac{gt}{u}$ and $y = \frac{1}{2}gt^2$ ....(iii)

or $y = \frac{1}{2}g.\left(\frac{x}{u}\right)^2$ [from equation (i) $t = \frac{x}{u}$]

or $y = \frac{g}{2u^2}.x^2$ or $y = kx^2$

here $k = \frac{g}{2u^2}$ (k is constant)

This is equation of a parabola.

A body thrown horizontally from a certain height above the ground follows a parabolic trajectory till it hits the ground.

(i) Time of flight $T = \sqrt{\frac{2H}{g}}$ [as y = 1/2gt^2 $T = \sqrt{\frac{2H}{g}}$]

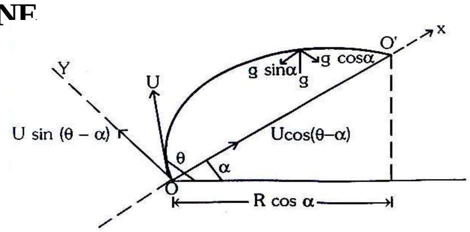
(ii) Range ⇒ horizontal distance covered = R.

R = u × time of flight

$R = u.\sqrt{\frac{2H}{g}}$ [$\because$ $H = \frac{g}{2u^2}R^2$]

(iii) Velocity when it hits the ground $v_g = \sqrt{u^2 + 2gH}$

PROJECTILE MOTION ON AN INCLINED PLANE



Time of Flight

Vertical displacements on inclined plane = 0

$S_y = U_y t + \frac{1}{2} a_y t^2$

$0 = U\sin(\theta - \alpha)\, t - 1/2\,(g\cos\alpha)t^2$

$t = \frac{2U\sin(\theta-\alpha)}{g\cos\alpha}$ check by $\alpha = 0$ $t = \frac{2U\sin\theta}{g}$

Which is time of flight on horizontal plane.

Range on inclined plane

$S_n = U_x t + 1/2\, a_x t^2$

$(O\,O') = U\cos(\theta - \alpha)t - 1/2\, g\sin\alpha\, t^2$

substituting the value of (t) $= U\cos(\theta - \alpha) \times$

$\frac{2U\sin(\theta-\alpha)}{g\cos\alpha} - \frac{1}{2}g\sin\alpha.\left\{\frac{2U\sin(\theta-\alpha)}{g\cos\alpha}\right\}^2$

$(OO') = R = \frac{2U^2\sin(\theta-\alpha)(\cos\theta)}{g\cos^2\alpha}$

condition for maximum range

$R = U^2 . \left[\frac{2\sin(\theta-\alpha)(\cos\theta)}{g\cos 2\alpha}\right] = U^2$

$\left[\frac{\sin(2\theta-\alpha)-\sin\alpha}{g\cos^2\alpha}\right]$ for R_{max} $\sin(2\theta - \alpha) = 1$

$2\theta - \alpha = \pi/2$ $\therefore$ $\theta = \alpha/2 + \pi/4$

$R_{max} = \frac{U^2[1-\sin\alpha]}{g[1-\sin^2\alpha]} => \frac{U^2}{g(1+\sin\alpha)}$ $R_{max} = \frac{U^2}{g(1+\sin\alpha)}$

Solved Examples

Ex.1 When the angle of elevation of a gun are 60° and 30° respectively. The height it shoots are h_1 and h_2 respectively. Then h_1/h_2 equals to.

(1) 3/1 (2) 1/3

(3) 1/2 (4) 2/1

Sol. For angle of elevation of 60°, we have maximum

height $h_1 = \frac{u^2\sin^2 60^o}{2g} = \frac{3u^2}{8g}$

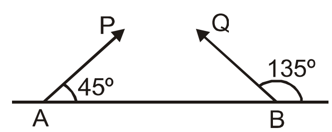
For angle of elevation of 30°, we have maximum

height $h_2 = \frac{u^2\sin^2 30^o}{2g} = \frac{u^2}{8g}$

$\frac{h_1}{h_2} = \frac{3}{1}$ Hence correct answer is (1)

(can only be solved after understanding conservation of linear momentum.)

Ex.2 Particle P and Q or mass 20 gm and 40 gm respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make 45° and 135° angles respectively with the horizontal AB. Each particle has an initial speed of 49 . The separation AB is 245 m. Both particles travel in the same vertical plane and undergo a collision. After collision P retraces its path. Determine the position of Q when it hits the ground.



How much time after the collision does the particle Q take to reach the ground.? {Take g = 9.8 m/s^2)

Sol. As the horizontal speed of two particles towards each other is same (u cos 45), they will meet at the middle of AB, i.e., at distance (245/2) = 122.5 from A toward B.

Now as

$R = \frac{u^2\sin 2\theta}{g} = R = \frac{u^2\sin 2\theta}{g} = \frac{49\times 49\times 1}{9.8} = 245$ m

so AB is the range and as the collision takes place at the middle of AB, so it is at the highest point of the trajectory.

Now applying conservation of linear momentum at the highest point along horizontal direction keeping in mind, $v_p = -u_p \cos 45^o$

20×10^{-3} u cos 45° – 40 × 10^{-3} u cos 45° = – 20 × 10^{-3} u cos 45° + v_Q

This give $v_Q = 0$ i.e., after collision, the velocity of Q at highest point is zero. So Q will fall freely under gravity and will hit the ground in the middle of AB, i.e., 122.5 m from A towards B.

Now as

$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{49 \times 49 \times 1}{2 \times 9.8 \times 2} = \frac{490}{8} = 61.25\text{m}$$

So time takes by Q to reach ground,

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\left(\frac{2 \times 490}{8 \times 9.8}\right)} = \frac{5}{\sqrt{2}}\text{s} = 3.536\text{ s}$$

(can only be solved after understanding conservation of linear momentum.)

Ex.3 A gun, kept on a straight horizontal road, is used to hit a car travelling along the same road away from the gun with a uniform speed of 72 km/hr. The car is at a distance of 500 m from the gun, when the gun is fired at an angle of 45° with the horizontal. Find

(a) the distance of the car from the gun when the shell hits it.

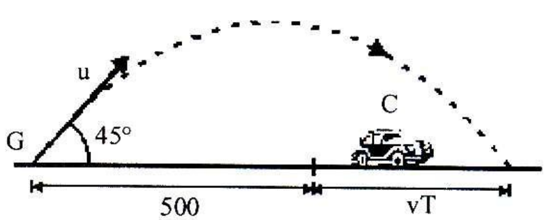
(b) The speed of projection of the shell from the gun. ($g = 9.8$ m/s^2)

Sol. The speed of the car v = 72 × (5/18) = 20 m/s

The time of flight of projectile $T = \frac{2u\sin\theta}{g} = \frac{u\sqrt{2}}{g}$

[as θ = 45°] ......(1) and range of projection

$R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2}{g}$ ....(2)



According to given problem, R = 500 + vT

Substituting the values of T and R from eq. (1) and (2) in the above,

$$\frac{u^2}{g} = 500 + \frac{u\sqrt{2}}{g} \times 20$$

or $u^2 - 20\sqrt{2}\,u - 4900 = 0$

or u = (1/2) $[20\sqrt{2} \pm \sqrt{(800 + 4 \times 4900)}]$

or u = $10[\sqrt{2} \pm \sqrt{51}]$

As negative sign of u is physically unacceptable, u = 10 [1.414 + 7.141] = 85.56 m/s

Substituting the above value of u in eq. (2)

$$R = \frac{u^2}{g} = \frac{(85.56)^2}{9.8} = 746.9\text{m}$$

Ex.4 A person observes a bird on a tree 39.6 meter high and at a distance of 59.2 meter. With what velocity the person should throw an arrow at an angle of 45° so that it may hit the bird ?

Sol. Let u m/s be the initial velocity of the arrow and let it hit the bird after t second.

The horizontal component of velocity u is u cos 45° = u/$\sqrt{2}$.

Hence the horizontal distance travelled in t second

is $\frac{u}{\sqrt{2}} \times t = 59.2$ ....(i)

The vertical distance travelled in t second is 39.6 m, acceleration g = – 9.8 m/s^2 and initial vertical velocity = u sin 45° = u/$\sqrt{2}$. Hence by the formula

h = ut + $\frac{1}{2}$ gt^2,

we have $39.6 = \frac{u}{\sqrt{2}}t - \frac{1}{2}(9.8)t^2$ ....(ii)

by eq. (i) and (ii), we get

39.6 = 59.2 – 1/2(9.8) t^{2}o

or $t^2 = \frac{59.2 - 39.6}{4.9} = \frac{19.6}{4.9} = 4$

∴ t = 2 seconds

Substituting this value of t in eq. (i), we get

$$\frac{u}{\sqrt{2}} \times 2 = 59.2$$

∴ **u = 29.6$\sqrt{2}$ m/s** Ans.

Ex.5 An enemy aircraft is flying at a height of 875 m from the surface of earth with a uniform velocity of 150 m/s in horizontal direction. When this aircraft is just over an antiaircraft gun, a shell is fired from the antiaircraft gun with velocity u at an angle θ above the horizontal direction. This shell hits the aircraft 35 second after being fired. Determine u and θ. g = 10 m/s^2 and tan 53º = 4/3.

Sol. The shell will hit the aircraft provided the horizontal velocity of the shell u cosθ, is equal to the horizontal velocity of the aircraft.

That is $u \cos\theta = 150\ ms^{-1}$ .....(i)

The vertical distance covered by the shell is h = 875 m, acceleration is $g = -10\ ms^{-2}$ and the initial vertical velocity of the shell is $u \sin\theta$. The shell hits the aircraft in t = 35 s. Putting these values in the relation $h = ut + 1/2\ gt^2$, we have 875

$= (u \sin\theta) \times 35 + 1/2\,(-10) \times (35)^2$

Solving : $u \sin\theta = 200\ ms^{-1}$ ....(ii)

From eq. (i) and (ii) $u^2 = (150)^2 + (200)^2$

= 22,500 + 40,000 = 62,500. $\therefore$ **$u = 250\ ms^{-1}$**

Dividing eq. (ii) by eq. (i), we get $\tan\theta = \frac{200}{150} = \frac{4}{3}$

$\therefore\ \theta = \tan^{-1}(4/3) = \mathbf{53^o}$

Ex.6 A bomb is dropped on an enemy post by an aeroplane flying with a horizontal velocity of 60 km/hr and at a height of 490 m. How far the aeroplane must be from the enemy post at the time of dropping the bomb, so that it may directly hit the target ? (g = 9.8 m/s^2)

What is the trajectory of the bomb as seen by an observer on the earth ? What as seen by a person sitting inside the aeroplane ?

Sol. The horizontal and vertical velocities of the bomb are independent to each other. The time taken by the bomb to hit the target can be calculated by its vertical motion. Let this time be t. Putting h = 490 m and g = 9.8 m/s^2 in the formula $h = 1/2\ gt^2$, we have $490 = 1/2 \times 9.8 \times t^2$

$$\therefore\ t = \sqrt{\frac{2 \times 490}{9.8}} = 10\ \ s$$

The bomb will hit the target after 10 second of its dropping, The horizontals velocity of the bomb is 60 km/h which is constant. Hence the horizontal distance travelled by the bomb in 10 second (horizontal velocity × time) is 60 km/h × 10 s = 60 km/h × $\frac{10}{60 \times 60}$ h = 1/6 km.

Hence the distance of aeroplane from the enemy post is 1/6 km = 1000/6 = 500/3 meter.

The trajectory of the bomb as seen by an observer on the ground is parabola. Since the horizontal velocity of the bomb is the same as that of the aeroplane, then falling bomb will always remain below the aeroplane. Hence the person sitting inside the aeroplane will observe the bomb falling **vertically downwards**.

Ex.7 A bomb is fired from a cannon with a velocity of 1000 m/s making an angle of 30º with the horizontal (g = 9.8 m/s^2). (i) What is the time taken by the bomb to reach the highest point ? (ii) What is the total time of its motion ? (iii) With what speed the bomb will hit the ground and what will be its direction of motion while hitting ? (iv) What is the maximum height attained by the bomb ? (v) At what distance from the cannon the bomb will hit the ground ?

Sol. (i) Let u be the initial velocity of the bomb and θ_0 its angle of projection. The time taken by the bomb to reach the highest point is given by

$$t = \frac{u \sin\theta_0}{g} = \frac{1000 \times \sin 30^o}{9.8} = \frac{1000 \times 1/2}{9.8} = 51 sec.$$

(ii) The total time of its motion is, T = 2t = 2 × 51 = 102 seconds.

(iii) The bomb will hit the ground with the same speed with which it was fired. Hence its speed of hitting = 100 m/s. Also, the angle of hitting with respect to the horizontal is 30º

(iv) The maximum height attained by the bomb is

$$h = \frac{u^2 \sin^2\theta_0}{2g} = \frac{(1000)^2 \times \sin^2 30^o}{2 \times 9.8}$$

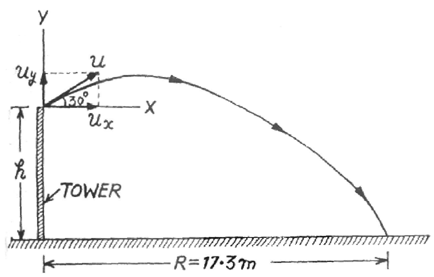
$$= \frac{(1000)^2 \times (0.5)^2}{2 \times 9.8} = 1.27 \times 10^4 \text{meter}$$

(v) Horizontal range is

$$R = \frac{u^2 \sin 2\theta_0}{g} = \frac{(1000)^2 \times \sin 60^o}{9.8} = 8.83 \times 10^4 \text{meter}$$

Ex.8 A ball is thrown from the top of a tower with an initial velocity of 10 m/s at an angle of 30º above the horizontal. It hits the ground at a distance of 17.3 m from the base of tower. Calculate the height of the tower. ($g = 10\ m/s^2$)

Sol.



The angle of projection of the ball is θ_0 (= 30º) and the velocity of projection is u (= 10 m/s). Resolving u in horizontal and vertical components, we have horizontal component, $u_x = u \cos \theta_0 = 10 \cos 30^\circ = 8.65$ m/s and vertical component (upwards),

$u_y = u \sin 30^\circ = 5.0$ m/s

If the ball hits the ground after t seconds of projection, then the horizontal range is $R = u_x \times t = 8.65\,t$ meter

$$\therefore \quad t = \frac{R}{8.65} = \frac{17.3m}{8.65m/s} = 2.0s$$

If h be the height of the tower, then

$h = u'_y\, t + 1/2\, gt^2$,

where u_y' is the vertical component (downward) of the velocity of the ball

Here $u_y' = -u_y = -5.0$ m/s and t = 2.0 s

$\therefore \quad h = (-5.0) \times 2.0 + 1/2 \times 10 \times (2.0)^2$

$= -10 + 20 = 10$ meter.

CIRCULAR MOTION

FOUNDAMENTAL PARAMETRE OF CIRCULAR MOTION

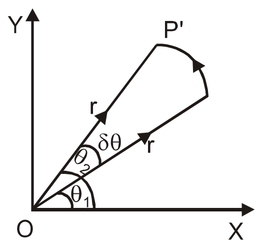
Radius vector : The vector joining the centre of the circle and the center of the particle performing circular motion is called radius vector.

It has constant magnitude and variable direction

Angular displacement ($\delta\theta$ or θ)

The angle described by radius vector is called angular displacement.

Infinitesimal angular displacement is a vector quantity. However, finite angular displacement is a scalar quantity.



* S.I Unit Radian
* Dimension : $M^0L^0T^0$
* 1 radian $= \frac{360}{2\pi}$
* No. Of revolution $= \frac{\text{angular dispalcement}}{2\pi}$
* In 1 revolution $\Delta\theta = 360^\circ = 2\pi$ radian
* In N revolution $\Delta\theta = 360^\circ \times N = 2\pi N$ radian
* Clockwise rotation is taken as negative
* Anticlockwise rotation is taken as positive

Solved Examples

Ex.1 If a particle complete one and half revolution along the circumference of a circle then its angular displacement is -

(1) 0 (2) π
(3) 2π (4) 2π

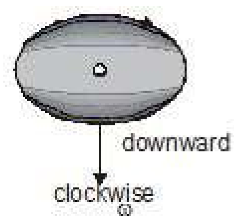
Sol. (4) = 3π

Angular velocity (ω)

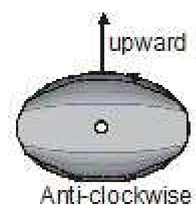
* The rate of change of angular displacement with time is called angular velocity.
* It is a vector quantity.
* The angle traced per unit time by the radius vector is called angular speed.
* Instantaneous angular velocity

$$= \omega = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} \quad \text{or} \quad \omega = \frac{d\theta}{dt}$$

* Average angular velocity $\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$
* **S.I. Unit** : rad/sec
* **Dimension** : $M^0L^0T^{-1}$
* **Direction :** Infinitesimal angular displacement, angular velocity and angular acceleration are vector quantities whose direction is given by right hand rule.



* **Right hand Rule :** Imagine the axis of rotation to be held in the right hand with fingers curled round the axis and the thumb stretched along the axis.



If the curled fingers denote the sense of rotation, then the thumb denoted the direction of the angular velocity (or angular acceleration of infinitesimal angular displacement .

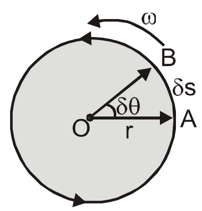
Angular acceleration (a)

* The rate of change of angular velocity with time is called angular acceleration.
* Average angular acceleration $\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$
* Instantaneous angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
* It is a vector quantity, whose direction is along the change in direction of angular velocity.

* **S.I. Unit :** radian/sec^2

* **Dimension :** $M^0L^0T^{-2}$

Relation between angular velocity and linear velocity



* Suppose the particle moves along a circular path from point A to point B in infinitesimally small time δt.

As, $\delta t \to 0$, $\delta\theta \to 0$

$\therefore$ arc AB = chord AB i.e. displacement of the particle is along a straight line.

$\therefore$ Linear velocity, $v = \lim_{\delta t\to 0}\frac{\delta s}{\delta t}$ But, $\delta s = r.\delta\theta$

$\therefore$ $v = \lim_{\delta t\to 0}\frac{r.\delta s}{\delta t} = r \lim_{\delta t\to 0}\frac{\delta\theta}{\delta t}$

But, $\lim_{\delta t\to 0}\frac{\delta\theta}{\delta t} = \omega$ = angular velocity

$\therefore$ $v = r\,.\,\omega$ [For circular motion only]

i.e. (linear velocity) = (Radian) × (angular velocity)

In vector notation, $\vec{v} = \vec{\omega} \times \vec{r}$ [in general]

* Linear velocity of a particle performing circular motion is the vector product of its angular velocity and radius vector.

Relation between angular acceleration & linear acceleration

For perfect circular motion we know $v = \omega r$

on differentiating with respect to time

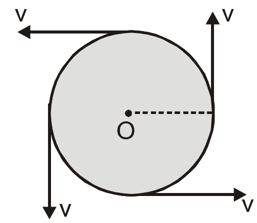
we get $\frac{dv}{dt} = r\frac{d\omega}{dt}$ $a = r\,\alpha$

In vector form $\vec{a} = \vec{\alpha} \times \vec{r}$

(linear acc.) = (angular acc) × (radius)

TYPES OF CIRCULAR MOTION

uniform circular motion



* Uniform circular motion : Motion of a particle along the circumference of a circle with a constant speed is called uniform circular motion. Uniform circular motion is an accelerated motion.

In case of uniform circular motion :

* speed remains constant. v = constant and $v = \omega r$

angular velocity ω = constant

motion will be periodic with time period $= T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$

* Frequency of uniform circular motion : The number of revolutions performed per unit time by the particle performing uniform circular motion is called the frequency (n)

$\therefore$ $n = \frac{1}{T} = \frac{v}{2\pi r} = \frac{\omega}{2\pi}$

S.I. unit of frequency is Hz.

* As ω = constant, from $\omega = \omega_0 + \alpha t$

angular acceleration $\alpha = 0$

As $a_t = \alpha r$, tangential acc. $a_t = 0$

* As $a_t = 0$, $a = (a_r^2 + a_t^2)^{1/2}$ yields $a = a_r$, i.e. acceleration is not zero but along radius towards centre and has magnitude $a = a_r = (v^2/r) = r\omega^2$

(v) Speed and magnitude of acceleration are constant. but their directions are always changing so velocity and acceleration are not constant. Direction of $\vec{v}$ is always along the tangent while that of $\vec{a_r}$ along the radius $\vec{v} \perp \vec{a_r}$

* If the moving body comes to rest, i.e. $\vec{v} \to 0$, the body will move along the radius towards the centre and if radial acceleration a_r vanishes, the body will fly off along the tangent. So a tangential velocity and a radial acceleration (hence force) is a must for uniform circular motion.

* As $\vec{F} = \frac{mv^2}{r} \neq 0$, so the body is not in equilibrium and linear momentum of the particle moving on the circle is not conserved. However, as the force is contral, i.e.,

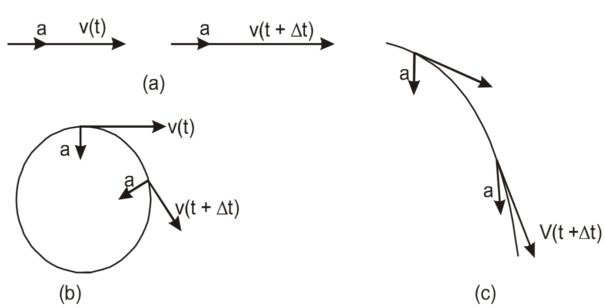
$\vec{\tau} = 0$, so angular momentum is conserved, i.e., $\vec{p} \neq$ constant but $\vec{L}$ = constant

* The work done by centripetal force is always zero as it is perpendicular to velocity and hence displacement. By work-energy theorem as work done = change in kinetic energy $\Delta K = 0$

So K (kinetic energy) remains constant

e.g. Planets revolving around the sun, motion of an electron around the nucleus in an atom

SPECIAL POINTS

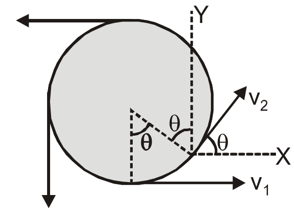


* In one dimensional motion, acceleration is always parallel to velocity and changes only the magnitude of the velocity vector.

* In uniform circular motion, acceleration is always perpendicular to velocity and changes only the direction of the velocity vector.

* IN the more general case, like projectile motion, acceleration is neither parallel nor perpendicular to figure summarizes these three cases.

* If a particle moving with uniform speed v on a circle of radius r suffers angular displacement θ in time Δt then change in its velocity.



$\Delta\vec{v} = \Delta\vec{v}_2 - \Delta\vec{v}_1$ $\qquad \vec{v}_1 = \vec{v}_1\,\hat{i}$

$\vec{v}_2 = \vec{v}_2\cos\theta\,\hat{i} + \vec{v}_2\sin\theta\,\hat{j}$

$\Delta\vec{v} = (\vec{v}_2\cos\theta - \vec{v}_1)\,\hat{i} + \vec{v}_2\sin^2\,\hat{j}$

$|\Delta\vec{v}| = \sqrt{(\vec{v}_2\cos\theta - \vec{v}_1)^2 + \vec{v}_2\sin^2}$

$|\Delta\vec{v}| = \sqrt{2v^2 - 2v^2\cos\theta} \quad = \sqrt{2v^2(1-\cos\theta)}$

$= \sqrt{2v^2\left(2\sin^2\frac{\theta}{2}\right)} \quad (\because\ v_1 = v_2 = v)$

$|\Delta\vec{v}| = 2v\sin\frac{\theta}{2}$

Solved Examples

Ex.2 A particle is moving in a circle of radius r centrad at O with constant speed v. What is the change in velocity in moving from A to B ? Given $\angle AOB = 40^o$.

Sol. $|\Delta\vec{v}|$ = 2v sin40º/2 = 2 v sin 20º

Non uniform circular motion

A circular motion in which both the direction and magnitude of the velocity changes is called nonuniform circular motion.

* A merry-go-round spinning up from rest to full speed, or a ball whirling around in a vertical circle.

* The acceleration is neither parallel nor perpendicular to the velocity.

* We can resolve the acceleration vector into two components :

* **Radial Acceleration :** a_r perpendicular to the velocity ⇒ changes only the directions of velocity Acts just like the acceleration in uniform circular motion.

a_c = or $a_r = \frac{v^2}{r}$

* **Centripetal force** : $F_c = \frac{mv^2}{r} = m\omega^2 r$

* **Tagential acceleration** : a_r parallel to the velocity (since it is tangent to the path)

$\Rightarrow$ changes magnitude of the velocity acts just like one-dimensional acceleration $\Rightarrow$ $a_t = \frac{dv}{dt}$

Tangential acceleration : $a_t = \frac{dv}{dt}$, where $v = \frac{ds}{dt}$ and s = length of arc

* **Tangential force** : $F_t = ma_t$

The net acceleration vector is obtained by vector addition of these two components.

$$\boxed{a = \sqrt{a_r^2 + a_t^2}}$$

(a) In non-uniform circular motion :

speed $|\vec{v}| \neq$ constant angular velocity $\omega \neq$ constant

i.e. speed $\neq$ constant i.e. angular velocity $\neq$ constant

(b) In at any instant

$\Rightarrow$ v = magnitude of velocity of particle

$\Rightarrow$ r = radius of circular path

$\Rightarrow$ ω = angular velocity of a particle

then, at that instant $v = r\omega$

* **Net force on the particle**

$$\vec{F} = \vec{F}_c + \vec{F}_t \quad \Rightarrow \quad F = \sqrt{F_c^2 + F_t^2}$$

If θ is the angle made by $F = F_c$,

then $\tan\theta = \frac{F_t}{F_c} \Rightarrow \theta = \tan^{-1}\left[\frac{F_t}{F_c}\right]$ [Note angle between F_c and F_t is 90°]

Angle between F and F_t is $(90° - \theta)$

* **Net acceleration :** $a = \sqrt{a_c^2 + a_t^2} = \frac{F_{net}}{m}$

The angle made by 'a' with a_c, $\tan\theta = \frac{a_t}{a_c} = \frac{F_t}{F_c}$

Special note :

* In both uniform and non-uniform circular motion F_c is perpendicular to velocity.

So work done by centripetal force will be zero in both the cases.

* In uniform circular motion $F_t = 0$, as $= a_t = 0$, so work done will be zero by tangential force.

But in non-uniform circular motion $F_t \neq 0$, so work done by tangential force is non zero.

Rate of work done by net force in non-uniform circular motion = rate of work done by tangential force

$$\Rightarrow P = \frac{dW}{dt} = \vec{F}_t . \vec{v} = \vec{F}_t . \frac{d\vec{x}}{dt}$$

* In a circle as tangent and radius are always normal to each other, so $\vec{a}_t \perp \vec{a}_r$.

Net acceleration in case of circular motion $a = a_r^2 = a_t^2$

Here is must be noted that a_t governs the magnitude of $\vec{v}$ while a_r its direction of motion so that

if $a_r = 0$ and $a_t = 0$ $a \to 0$

$\Rightarrow$ motion is uniform translatory

if $a_r = 0$ and $a_t \neq 0$ $a \to a_t$

$\Rightarrow$ motion is accelerated translatory

if $a_r \neq 0$ and $a_t = 0$ $a \to a_r$

$\Rightarrow$ motion is uniform circular

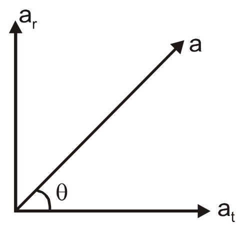
if $a_r \neq 0$ and $a_t \neq 0$ $a \to \sqrt{a_r^2 + a_t^2}$

$\Rightarrow$ motion is non-uniform circular

Solved Examples

Ex.3 A road makes a 90° bend with a radius of 190 m. A car enters the bend moving at 20 m/s. Finding this too fast, the driver decelerates at 0.92 m/s^2. Determine the acceleration of the car when its speed rounding the bend has dropped to 15 m/s.

Sol. Since it is rounding a curve, the car has a radial acceleration associated with its changing direction, in addition to the tangential deceleration that changes its speed. We are given that $a_t = 0.92$ m/s^2; since the car is slowing down, the tangential acceleration is directed opposite the velocity.



The radial acceleration is $a_r = \frac{v^2}{r} = \frac{(15 m/s)^2}{190 m} = 1.2$ m/s^2

Magnitude of net acceleration,

$a = \sqrt{a_r^2 + a_t^2}$ = [(1.2 m/s)^2 + (0.92 m/s)^2]$^{1/2}$

= 1.5 m/s^2 and points at an angle

$$\theta = \tan^{-1}\left(\frac{a_r}{a_t}\right) = \tan^{-1}\left(\frac{1.2 m/s^2}{0.92 m/s^2}\right) = 53^o$$

relative to the tangent line to the circle.

Ex.4 A particle lis constrained to move in a circular path of radius r = 6m. Its velocity varies with time according to the relation v = 2t (m/s). Determine its (i) centripetal acceleration, (ii) tangential acceleration, (iii) instantaneous acceleration at (a) t = 0 sec. and (b) t = 3 sec.

Sol. (a) At = 0, v = 0, Thus $a_r = 0$

but $\frac{dv}{dt}$ =2 thus a_t = 2 m/s^2 and a

= $\sqrt{a_t^2 + a_r^2}$ = 2 m/s^2

(b) At t = 3 sec. v = 6 m/s so $a_r = \frac{v^2}{r} = \frac{(6)^2}{6} = 6$

m/s^2 and $a_t = \frac{dv}{dt} = 2$ m/s^2

Therefore, a = $a = \sqrt{2^2 + 6^2}$ = $\sqrt{40}$ m/s^2

Ex.5 The kinetic energy of a particle moving along a circle of radius r depends on distance covered s as K = As^2 where A is a const. Find the force acting on the particle as a function of s.

Sol. According to given problem

$\frac{1}{2}$ mv^2 = As^2 or $v = s\sqrt{\frac{2A}{m}}$ ....(1)

So $a_r = \frac{v^2}{r} = \frac{2As^2}{mr}$ ....(2)

Further more as $a_t = \frac{dv}{dt} = \frac{dv}{ds}.\frac{ds}{dt} = v\frac{dv}{ds}$ ....(3)

from eqn. (1), $v = s\sqrt{(2A/m)}$

$\Rightarrow \frac{dv}{ds} = \sqrt{\frac{2A}{m}}$ ....(4)

Substitute values from eqn. (1) & eqn. (4) in eqn. (3)

$$a_t = \left[s\sqrt{\frac{2A}{m}}\right]\left[\sqrt{\frac{2A}{m}}\right] = \frac{2As}{m}$$

so $a = \sqrt{a_r^2 + a_t^2}$ = $\sqrt{\left[\frac{2As^2}{mr}\right]^2 + \left[\frac{2As}{m}\right]^2}$

i.e. $a = \frac{2As}{m}\sqrt{1+[s/r]^2}$

so F = ma = 2As $\sqrt{1+[s/r]^2}$

Ex.6 A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration as is varying with time t as $a_c = k^2rt^2$, where k is a constant . Determine the power delivered to particle by the forces acting on it.

Sol. If v is instantaneous velocity, centripetal acceleration $a_c = \frac{v^2}{r} \Rightarrow \frac{v^2}{r} = k^2rt^2 \Rightarrow v = krt$

In circular motion work done by centripetally force is always zero & work is done only by tangential force.

$\because$ Tangent acceleration $a_t = \frac{dv}{dt} = \frac{d}{dt}$ (krt) = kr

$\therefore$ Tangential force $F_t = ma_t$ = mkr

Power P = $F_t v$ = (mkr) (krt) = mk^{2}r^{2}t

CENTRIPETAL AND CENTRIFUGAL FORCE

Centripetal force : In uniform circular motion the force acting on the particle along the radius and towards the centre keeps the body moving along the circular path . This force is called centripetal force.

Explanation :

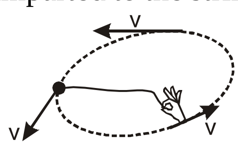
(i) Centripetal force in necessary for uniform circular motion.

(ii) It is along the radius and towards the centre.

(iii) Centripetal force = [mass] × [centripetal acceleration] = $\frac{mv^2}{r}$ = $mr\omega^2$

(iv) Centripetal force is due to known interaction. Therefore it is a real force.

* If an object tied to a string its revolved uniformly in a horizontal circle, the centripetal force is due to the tension imparted to the string by the hand.



* When a satellite is revolving in circular orbit round the earth, the centripetal force is due to the gravitational force of attraction between the satellite and the earth.

* In an atom, an electron revolves in a circular orbit round the nucleus. The centripetal force is due to the electrostatic force of attraction between the positively charged nucleus and negatively charged

Solved Examples

Ex.7 Stone of mass 1kg is whirled in a circular path of radius 1 m. Find out the tension in the string if the linear velocity is 10 m/s ?

Sol. Tension $\frac{mv^2}{R} = \frac{1\times(10)^2}{1} = 100$ N

Ex.8 A satellite of mass 10^7 kg is revolving around the earth with a time period of 30 days at a height of 1600 km. Find out the force of attraction on satellite by earth?

Sol. Force = $m\omega^2R$ and $\frac{2\pi}{T} = \frac{2\times3.4}{30\times86400} = \frac{6.28}{2.59\times10^6}$

Force = $m\omega^2r$ = $\left(\frac{6.28}{2.59\times10^6}\right)^2 \times 10^7 \times (6400 + 1600) \times 10^3 = 2.34 \times 10^6$ N

Centrifugal force

The pseudo force experienced by a particle performing uniform circular motion due to accelerated frame of reference which is along the radius and directed way from the centre is called centrifugal force.

Explanation :

(i) Centrifugal force is a pseudo force as it is experienced due to accelerated frame of reference. The interaction of origin and away from the centre.

(ii) It is along the radius and away from the centre.

(iii) The centrifugal force in having the same magnitude as that of centripetal force. But, its direction is opposite to that of centripetal force . It is not due to reaction of centripetal force because without action, reaction not possible, but centrifugal force can exists without centripetal force.

(iv) Magnitude of the centrifugal force is $\frac{mv^2}{r}$ or $mr\omega^2$

Note : Pseudo force acts in non inertial frame i.e. accelerated frame of reference in which Neutron's law's of motion do not hold good.

* When a car moving along a horizontal curve takes a turn, the person in the car experiences a push in the outward direction.

* The coin placed slightly away from the centre of a rotating gramophone disc slips towards the edge of the disc.

* A cyclist moving fast along a curved road has to lean inwards to keep his balance.

Difference between centripetal force and centrifugal force

Centripetal force	Centrifugal force
☞ Centripetal force is directed along the radius, towards the centre of the circle ☞ It is a real force. ☞ This force produces uniform motion. ☞ It arises in both inertial and non-inertial frames of reference. ☞ e.g. when a satellite is revolving in circular orbit round the earth, the centripetal force is due to the gravitational force of attraction	☞ Centrifugal force is directed along the radius, away from the centre of the circle. ☞ It is a pseudo force. ☞ This force is the effect of uniform circular motion. ☞ It arises only in the non-inertial frame of reference of in a rotating frame of reference. ☞ e.g. along a curved road the passenger in the vehicle has a feeling of push in the outward direction. This push is due to centrifugal force

Applications of centrifugal force

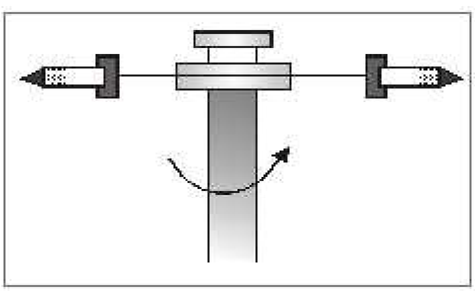
* The centrifugal pump used to lift the water works on the principle of centrifugal force.

* A cream-separator used in the diary work, works on the principle of centrifugal force.

* Centrifuge used for the separation of suspended particle from the liquid, works on the principle of centrifugal force.

* Centrifugal drier.

A Centrifuge



* A centrifuge works on the principle of centrifugal force.

* The centrifuge consists of two steel tubes suspended from the ends of a horizontal bar which can be rotated at high speed in a horizontal plane by an electric motor.

* The tubes are filled with the liquid and the bar is set into rotation.

* Due to rotational motion, the tubes get tied and finally become horizontal.

* Due to heavy mass, the heavier particles experience more centrifugal force than that of the liquid particles. Therefore, is then stopped so that the tubes becomes vertical.

Solved Examples

Ex.9 Two balls of equal masses are attached to a string at distance 1 m and 2 m from one end as shown in fig. The string with masses is then moved in a horizontal circle with constant speed. Find the ratio of the tension T_1 and T_2 ?

Sol. Let the balls of the two circles are r_1 and r_2. The linear speed of the two masses are $v_1 = \omega r_1$, $v_2 = \omega r_2$

where ω is the angular speed of the circular motion.

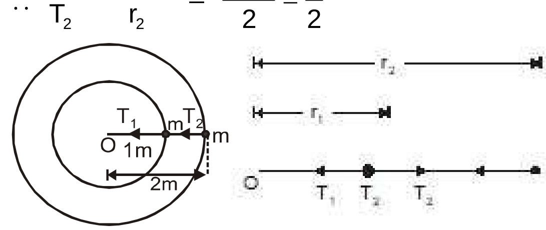
The tension in the strings are such that

$$T_2 = \frac{mv_2^2}{r_2} = m\omega^2 r_2$$

$$T_1 - T_2 = \frac{mv_1^2}{r_1} = m\omega^2 r_1$$

$$\therefore T_1 = m\omega^2 r_1 + T_2 = m\omega^2 (r_1 + r_2)$$

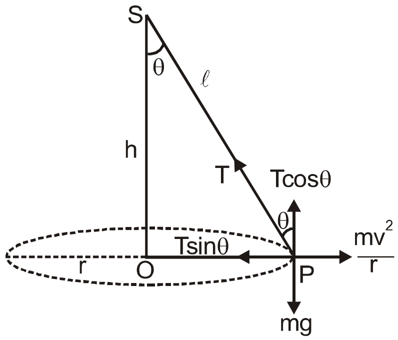
$$\therefore \frac{T_1}{T_2} = \frac{r_1 + r_2}{r_2} = \frac{1+2}{2} = \frac{3}{2}$$



CONICAL PENDULUM

(This the best example of uniform circular motion)

A conical pendulum consists of a body attached to a string, such that it can revolve in a horizontal circle with uniform speed. The string traces out a cone in the space.



The force acting on the bob are (a) Tension T (b) weight mg

The horizontal component T sinθ of the tension T provides the centripetal force and the vertical component Tcos θ balances the weight to bob

$\therefore$ $T \sin\theta = \frac{mv^2}{r}$ and $T\cos\theta = mg$

From these equations

$T = mg\sqrt{1+\frac{v^4}{r^2g^2}}$ ....(i) and $\tan\theta = \frac{v^2}{rg}$ ....(ii)

If h = height of conical pendulum

$\tan\theta = \frac{OP}{OS} = \frac{r}{h}$ ....(iii)

From (ii) & (iii) $\frac{v^2}{rg} = \frac{r}{h} \Rightarrow \omega^2 = \frac{v^2}{r^2} = \frac{g}{h}$

The time period of revolution

$$T = 2\pi = \sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$$

Hints to solve numerical problem (UCM)

(i) First show all force acting on a particle

(ii) Resolve these forces along radius and tangent.

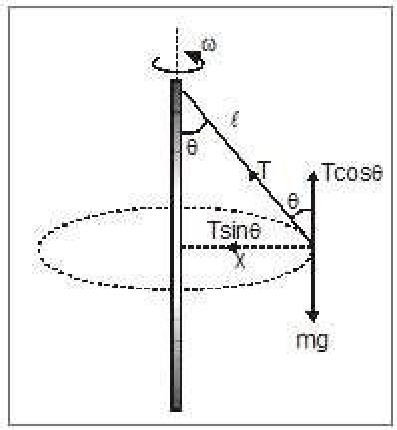
(iii) Resultant force along radial direction provides necessary centripetal force.

(iv) Resultant force along tangent = $Ma_T = 0$ (a_T = tangential acceleration)

Solved Examples

Ex.10 A vertical rod is rotating about its axis with a uniform angular speed ω. A simple pendulum of length ℓ is attached to its upper end what is its inclination with the rod ?

Sol. Let the radius of the circle in which the bob is rotating is, the tension in the string is T, weight of the bob mg, and inclination of the string θ. Then T cos θ balances the weight mg and T sin θ provides the centripetal force necessary for circular motion.



That is -

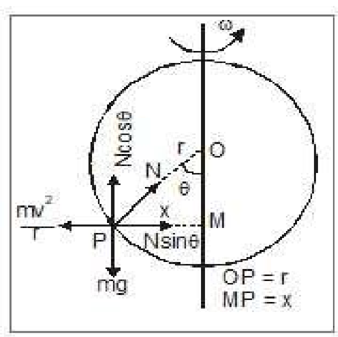
$T\cos\theta = mg$ and $T\sin\theta = m\omega^2 x$

but $x = \ell\sin\theta$ $\therefore T = m\omega^2\ell$

and $\cos\theta = \frac{mg}{T} = \frac{mg}{m\omega^2\ell}$ or $\theta = \cos^{-1}\left(\frac{g}{\omega^2\ell}\right)$

Ex.11 A circular loop has a small bead which can slide on it without friction. The radius of the loop is r. Keeping the loop vertically it is rotated about a vertical diameter at a constant angular speed ω. What is the value of angle θ, when the bead is in dynamic equilibrium ?

Sol. Centripetal force is provided by the horizontal component of the normal reaction N.



The vertical component balances the weight. Thus

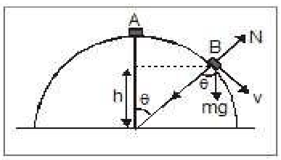
$N \sin\theta = m\omega^2 x$ and $N\cos\theta = mg$

Also $x = r\sin\theta \Rightarrow$ $N = m\omega^2 r$

$\cos\theta = \dfrac{g}{\omega^2 r}$ or $\theta = \cos^{-1}\left(\dfrac{g}{r\omega^2}\right)$

Ex.12 A particle of mass m slides down from the vertex of semihemisphere, without any initial velocity. At what height from horizontal will the particle leave the sphere.

Sol. Let the particle leave the sphere at height h, $\dfrac{mv^2}{R} = mg\cos\theta - N$



When the particle leaves the sphere N = 0,

$\dfrac{mv^2}{R} = mg\cos\theta \Rightarrow v^2 = gR\cos\theta$

According to law of conservation of energy

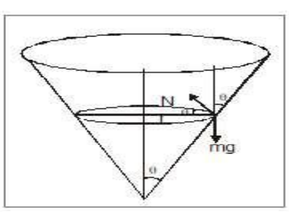
(K . E.+ P. E.) at A =(K . E.+ P. E.) at B

$\Rightarrow o + mgR = \dfrac{1}{2}mv^2 + mgh \Rightarrow v^2 = 2g(R-h)$.......(2)

From (1) & (2) $h = \dfrac{1}{2}R$, Also $\cos\theta = 2/3$

Ex.13 A particle describes a horizontal circle of radius r in a funnel type vessel of frictionless surface with half one angle θ (as shown in figure). If mass of the particle is m, then in dynamical equilibrium the speed of the particle must be-

Sol. The normal reaction N and weight mg are the only forces acting on the particle (inertial frame view),The N is making an angle $\left(\dfrac{\pi}{2}-\theta\right)$ with the vertical. The vertical component of N balances the weight mg and the horizontal component provides the centripetal force required for circular motion.



Thus

$N\cos\left(\dfrac{\pi}{2}-\theta\right) = mg$ $\quad N\sin\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{mv^2}{r}$

or $N\sin\theta = mg.$ $\quad N\cos\theta = \dfrac{mv^2}{r}$,

on dividing we get $\tan\theta = \dfrac{rg}{v^2}$, so $v = \sqrt{\dfrac{rg}{\tan\theta}}$

Ex.14 Prove that a motor car moving over a

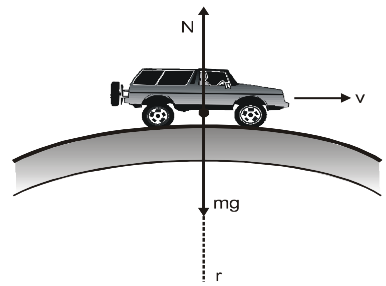
(i) Convex bridge is lighter than the same car resting on the same bridge.

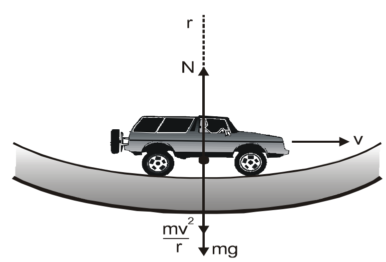
(ii) Concave bridge is heavier than the same car resting on the same bridge.

Sol. Apparent weight of car = N (normal reaction)

(i) Convex bridge

The motion of the motor car over a convex bridge is the motion along the segment of a circle. The centripetal force is provided by the difference of weight mg of the car and the normal reaction N of the bridge.





$\therefore \quad mg - N = \dfrac{mv^2}{r}$ $\quad$ or $N = mg - \dfrac{mv^2}{r}$

Clearly N < mg, i.e., the apparent weight of the moving car is less than the weight of the stationary car.

(ii) Concave bridge $N - mg = \dfrac{mv^2}{r}$

Apparent weight $N = mg + \dfrac{mv^2}{r}$

MOTION IN VERTICAL CIRCLE

Motion of a body suspended by string :

This is the best example of non-uniform circular motion.

Suppose a particle of mass m is attached to an inexcusable light string of length r. The particle is moving in a vertical circle of radius r, about a fixed point O.

At lost point A velocity of particle = u

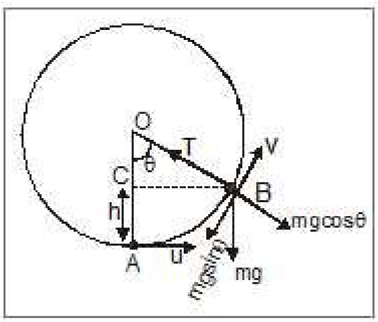
(in horizontal direction)

After covering $\angle\theta$ velocity of particle = v

(at point B)

Resolve weight (mg) into two components (i) mg cos θ (along radial direction) (ii) mg sin θ (tangential direction)

Then force $T - mg\cos\theta$ provides necessary centripetal force



$$T - mg\cos\theta = \frac{mv^2}{r} \qquad \text{......(i)}$$

$$\Delta\ OCB \quad \cos\theta = \frac{r-h}{r} \qquad \text{......(ii)}$$

or $h = r(1-\cos\theta)$

By conservation of energy at point A & B

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh \quad \text{or} \quad u^2 = v^2 + 2gh$$

or $\boxed{v^2 = u^2 - 2gh}$ ....(iii)

Substitute value of cos θ and v^2 in equn. (i)

$$T - mg\left[\frac{r-h}{r}\right] = \frac{m}{r}(u^2 - 2gh) \quad \text{or}$$

$$T = \frac{m}{r}[u^2 - 2gh + gr - gh) \quad \text{or}$$

$$T = \frac{m}{r}[u^2 + gr - 3gh] \quad \text{............(iv)}$$

(i) If velocity becomes zero at height h_1

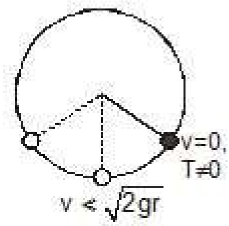
$$O = u^2 - 2gh, \quad \text{or } h_1 = \frac{u^2}{2g} \quad \text{........ (v)}$$

(ii) If tension becomes zero at height h_2

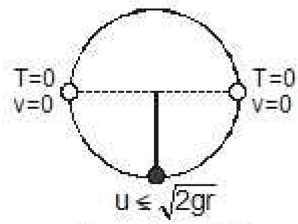
$$O = \frac{m}{r}[u^2 + gr - 3gh_2]$$

or $u^2 + gr - 3gh_2 = 0$ or $h_2 = \dfrac{u^2 + gr}{3g}$ ...(vi)

(iii) Case of oscillation



Case of oscillation

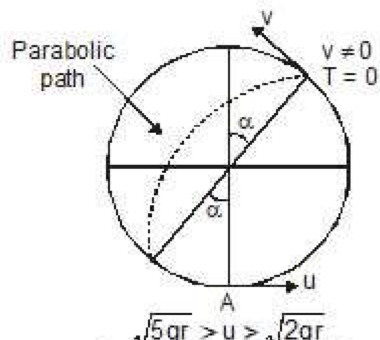


Case of oscillation semi-circular path

It $v = 0,\ T \neq 0$ then $h_1 < h_2$ $\quad \dfrac{u^2}{2g} < \dfrac{u^2 + gr}{3g}$

$3u^2 < 2u^2 + 2gr \quad u^2 < 2gr \quad u < \sqrt{2gr}$

(iv) Case of leaving the circle



Case of leaving the circle

If $v \neq 0,\ T = 0$ then $h_1 > h_2$ $\quad \dfrac{u^2}{2g} > \dfrac{u^2 + gr}{3g}$

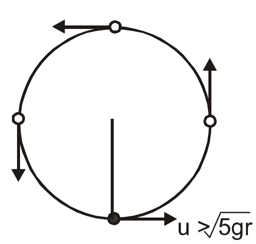
$3u^2 > 2u^2 + 2gr \quad u^2 > 2gr \qquad u > \sqrt{2gr}$

$\sqrt{5gr} > u > \sqrt{2gr}$

(v) Case of complete the circle

$u \geq \sqrt{5gr} \quad T \geq 0 \qquad v \neq 0$

Case of complete the circle or looping the loop



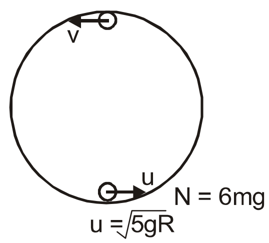
Special Note :

The same conditions apply if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.

This is shown in the figure given belw

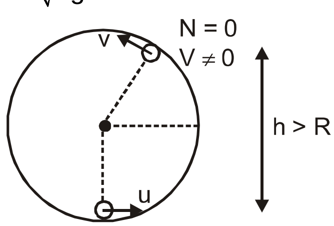
$v = \sqrt{gR}$ N = 0

(i) Condition of looping the loop is u $\geq \sqrt{5gR}$

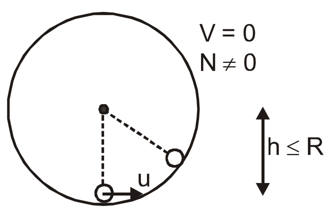


(ii) Condition of leaving the circle

$\sqrt{2gR} < u < \sqrt{5gR}$



(iii) Condition of oscillation is 0 < u $\geq \sqrt{2gR}$



Solved Examples

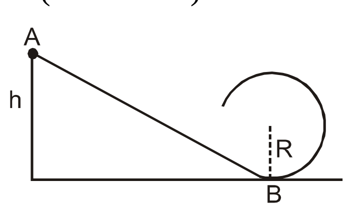
Ex.15 A ball is released from height h as shown in fig. Find the condition for the particle to complete the circular path.

Sol. According to law of conservation of energy (K.E. + P.E) at A = (K.E. + P.E) at B

$\Rightarrow$ 0 + mgh

$= \frac{1}{2}$ mv^2 + 0

$\Rightarrow$ $v = \sqrt{2gh}$



But velocity at the lowest point of circle,

$v \geq \sqrt{5gR} \Rightarrow \sqrt{2gh} \geq \sqrt{5gR} \Rightarrow h \geq \frac{5R}{2}$

Ex.16 A body weighing 0.4 kg is whirled in a vertical circle making 2 revolutions per second. If the radius of the circle is 1.2 m, find the tension in the string, when the body is (a) at the top of the circle (b) at the bottom of the circle. Given : g = 9.8 ms^{-2} and π = 1.2 m

Sol. Mass m = 0.4 kg time period = $\frac{1}{2}$ second and radius, r =1.2 m

Angular velocity, $\omega = \frac{2\pi}{1/2} = 4\pi$ rad s^{-1} = 12.56 rad s^{-1}

(a) At the top of the circle,

$T = \frac{mv^2}{r} - mg = mr\omega^2 - mg = m\,(r\omega^2 - g)$

= 0.4 (1.2 × 12.56 × 12.56 – 9.8) N = 71.8 N

(b) At the lowest point, T = m(rω^2 + g) = 79.64 m

Ex.17 In a circus a motorcyclist moves in vertical loop inside a 'death well' (a hollow spherical chamber with holes, so that the spectators can watch from outside). Explain clearly why the motorcyclist does not drop down when the is at the uppermost point, with no support from below. What is the minimum speed required to perform a vertical loop is the radius of the chamber is 25 m.

Sol. When the motorcyclist is at the highest point of te death-well, the normal reaction R on the motorcyclist by the ceiling of the chamber acts downwards. His weight mg also acts downwards. These two forces are balanced by the outward centrifugal force acting on him.

$\therefore\ R + mg = \frac{mv^2}{r}$ ....(i) r = radius of the circle

Here v is the speed of the motorcyclist and m is the mass of the motorcyclist (including the mass the motor cycle). Because of the balancing of the forces, the motorcyclist does not fall down.

The minimum speed required to perform a vertical loop is given by equation (i), when R = 0

$\therefore\ mg = \frac{mv^2_{min}}{r}$ or $v^2_{min} = gr$

or $v_{min} = \sqrt{gr} = \sqrt{9.8 \times 25}$ ms^{-1} = 15.65 ms^{-1}

So, the minimum speed at the top required to perform a vertical loop is 15.65 ms^{-1}.

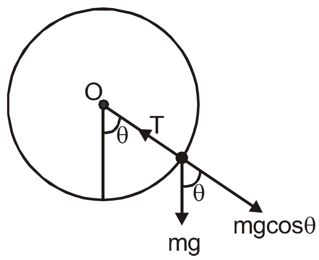
Ex.18 A 4kg ball in swung in a vertical circle at the end of a cord 1 m long. What is the maximum speed which is can swing if the cord can sustain maximum tension of 163.6 N ?

Sol. Maximum tension = $T = \frac{mv^2}{r} + mg$

$\therefore \quad \frac{mv^2}{r} = T - mg$ or $\frac{4v^2}{1} = 163.6 - 4 \times 9.8$

solving we get $v = 6$ m/sec

Ex.19 A small body of mass m = 0.1 kg swings in a vertical circle at the end of a chord of length 1 m. Its speed is 2 m/s when the chord makes an angle $\theta = 30^\circ$ with the vertical. Find the tension in the chord.



Sol. The equation of motion is

$$T - mg\cos\theta = \frac{mv^2}{r} \text{ or } T = mg\cos\theta + \frac{mv^2}{r}$$

Substituting the given values, we get

$$T = 0.1 \times 9.8 \times \cos 30 + \frac{0.1 \times (2)^2}{1} = 0.98 \times \left(\frac{\sqrt{3}}{2}\right) + 0.4$$

$= 0.85 + 0.4 = 1.25$ N

Important Point :

* If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottom most points in not $\sqrt{5gR}$. Because in this case, velocity of the particle at the topmost point can be zero also. Using conservation of mechanical energy between points A and B as shown in fig. (a) we get

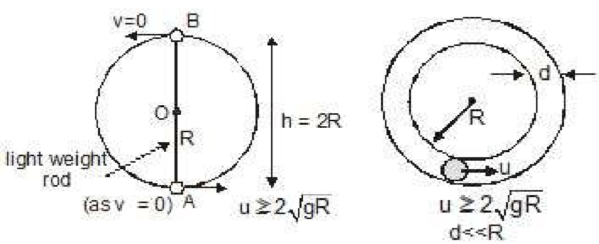


fig (a) fig (b)

$\frac{1}{2} m (u^2 - v^2) = mgh$ or $\frac{1}{2} mu^2 = mg(2R)$

$\therefore u = 2\sqrt{gR}$

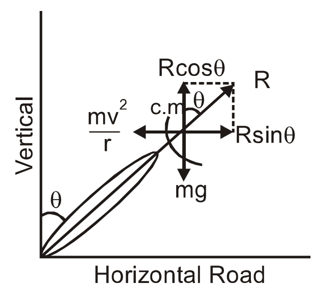
Therefore, the minimum values of u in this case is $2\sqrt{gR}$

Same in the case when a particle is compelled to move inside a smooth vertical tube as shown in fig. (b)

PRATICAL APPLICATION OF CIRCULAR MOTION

A Cyclist making A turn

Let a cyclist moving on a circular path of radius r bend away from the vertical by an angle θ. If R is the reaction of the ground, then R may be resolved into two components horizontal and vertical. The vertical component R cos θ balances the weight mg of the cyclist and the horizontal component R sin θ provides the necessary centripetal force for circular motion.



$R\sin\theta = \frac{mv^2}{r}$ ....(1)

and $R\cos\theta = mg$ ....(2)

Dividing (1) by (2),

we get $\tan\theta \; \frac{v^2}{rg}$ ....(3)

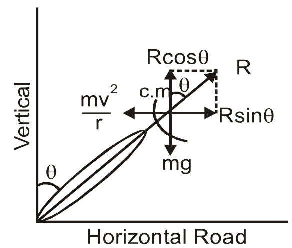
For less beding of cyclist, his speed v should be smaller and radius r of circular path should be greater.

If μ is coefficient of friction, then for no skidding of cycle (or overturning of cyclist)

$\mu \geq \tan\theta$ ......(4) $\quad \mu \geq \frac{v^2}{rg}$

An aeroplane making a turn

In order to makes a circular turn, a plane must roll at some angle θ in such a manner that the horizontal component of the lift force L provides the necessary centripetal force for circular motion. The vertical component of the lift force balances the weight of the plane.



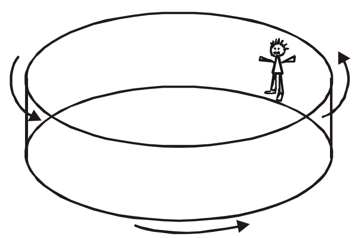
$L \sin\theta = \dfrac{mv^2}{r}$ and $L \cos\theta = mg$

or the angle θ should be such that $\tan\theta = \dfrac{v^2}{rg}$

Death well and rotor

Example of uniform circular motion

In 'death well' a person drives a bicycle on a vertical surface of a large wooden well.



(a) A passenger on a 'rotor ride'

In 'death well' walls are at rest while person revolves.

In a rotor at a certain angular speed of rotor a person hangs resting against the wall without any floor.

In rotor person is at rest and the walls rotate.

In both these cases friction balances the weight of person while reaction provides the centripetal force necessary for circular motion i.e.

Force of fiction $F_S = mg$ and Normal reaction F_N

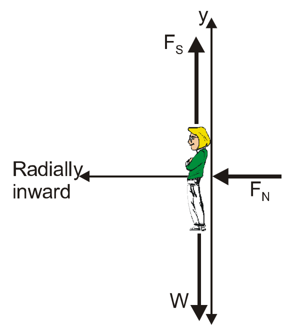
$= \dfrac{mv^2}{r}$ so $\dfrac{F_N}{F_S} = \dfrac{v^2}{rg}$, i.e., $v = \sqrt{\dfrac{rgF_N}{F_S}}$

Now for v to be minimum F_S must be maximum,

i.e., $v_{min} = \sqrt{\dfrac{rg}{\mu}}$ [as $F_{S\,max} = \mu F_N$]

Solved Examples

Ex.20 A 62 kg woman is a passenger in a "rotor ride" at an amusement park. A drum of radius 5.0 m is spun with an angular velocity of 25 rpm. The woman is pressed against the wall of the rotating drum as shown in fig. (a) Calculate the normal force of the drum of the woman (the centripetal force that prevents her from leaving her circular path). (b) While the drum rotates, the floor is lowered. A vertical static friction force supports the woman's weight. What must the coefficient of friction be to support her weight ?



(b) A force diagram for the person

Sol. Normal force exerted by the drum on woman towards the centre $F_N = ma_c = m\omega^2 r$

$$= 62\ \text{kg} \times \left(25\frac{\text{rev}}{\text{min}} \times \frac{2\pi\text{rad}}{1\text{rev}} \times \frac{1\text{min}}{60\text{s}}\right)^2 \times 5\text{m} = 2100\ \text{N}$$

(b) $\mu F_N = F = mg$ ...(2) dividing eqn. (2) be eq. (1)

$$\mu = \frac{g}{\omega^2 r} = \left(\frac{60}{2\pi \times 25}\right)^2 \times \frac{10}{5} = 0.292$$

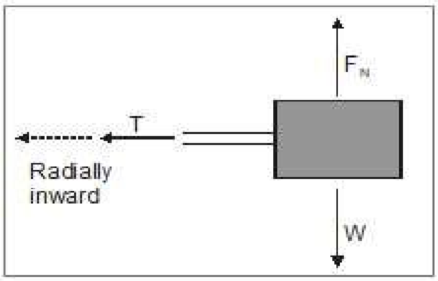
Ex.21 A 1.1 kg block slides on a horizontal frictionless surface in a circular path at the end of a 0.50 m long string. (a) Calculate the block's speed if the tension in the string in 86 N. (b) By what percent does the tension change if the block speed decreases by 10 percent?

Sol. (a) Force diagram for the block is shown in fig. The upward normal force balances the block's weight. The tension force of the string on the block provides the centripetal force that keeps the block moving in a circle. Newton's second law for forces along the radial direction is

$$\Sigma F \text{ (in radial direction)} = T = \frac{mv^2}{r},$$

or $v = \sqrt{\frac{Tr}{m}} = \sqrt{\frac{(80N)(0.50m)}{1.2\text{ kg}}} = 5.0$ m/s

(b) A 10 percent reduction in the speed results in a speed v' = 5.4 m/s. The new tension is



$$T' = \frac{mv'^2}{r} - \frac{(1.2kg)\,(5.4m/s)^2}{0.50m} = 70N$$

Thus, $\frac{T'}{T} = \frac{70N}{86N} = 0.81$

percentage reduction in the tension is about 19%.

The same result is obtained using a proportionality method.

$$\frac{T'}{T} = \frac{(mv'^2/r)}{(mv^2/r)} = \left(\frac{v'}{v}\right)^2 = \left(\frac{0.90}{v}\right)^2 = 0.81$$

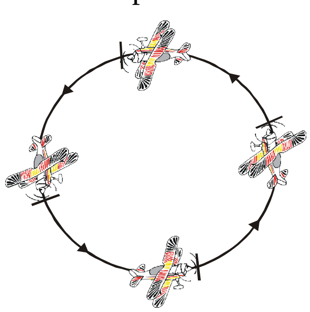
Looping the loop

This is the best example of non uniform circular motion in vertical plane.

For looping the pilot of the plane puts off the engine at lowest point and traverses a vertical loop. (with variable velocity)

Solved Examples

Ex.22 An aeroplane moves at 64 m/s in a vertical loop of radius 120 m, as shown in fig. Calculate the force of the plane's seat on 1 72 kg pilot while passing through the bottom part of the loop.

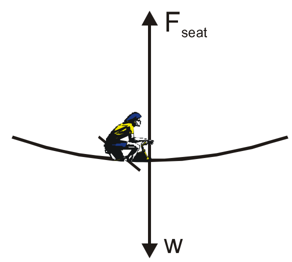


Sol. Two forces acts on the pilot his downward weigth force w and the upward force of the aeroplane's seat F_{seat}. Because the pilot moves in a circular path, these forces along the radial direction must, according to Newton's seconds law ($\Sigma F = ma$), equal the pilot's mass times his centripetal acceleration, where

$a_c = v^2/r$. We find that ΣF (in radial direction)

$= F_{seat} - w = \frac{mv^2}{r}$

Remember that force pointing towards the center of the circle (F_{seat}) are positive & those pointing away from the center (w) are negative.



Substituting $\omega = mg$ and rearranging, we find that the force of the aeroplane seat on the pilot is

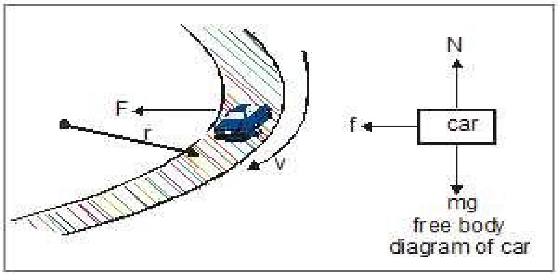
$$F_{seat} = m\left(\frac{v^2}{r} + g\right) = 72\text{ kg}\left[\frac{64(m/s)^2}{120m} + 9.8m/s^2\right]$$

$= 72$ kg $(34.1\text{ m/s}^2 + 9.8\text{m/s}^2) = 3160.8$ N

The pilot in this example feels very heavy. To keep him in the circular path, the seat must push the pilot upwards with a force of 3160 N, 4.5 times his normal weight. He experiences an acceleration of 4.5 g, that is, 4.5 times the acceleration of gravity.

A car taking a turn on a level road

When a car takes a turn on a level road, the portion of the turn can be approximated by an arc of a circle of radius r (see fig.) If the car makes the turn at a constant speed v, then there must be some centripetal force acting on the car. This force is generated by the friction between the tyre and the road. (car has a tendency to slip radially outward, so frictional force acts inwards)



μ_s is coefficient of static friction

N = mg is the normal reaction of the surface

The maximum safe velocity v is -

$$\frac{mv^2}{r} - \mu_S N = \mu_S mg \quad \text{or} \quad \mu_S = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{\mu_S rg}$$

It is independent of the mass of the car. The safe velocity is same for all vehicles of larger and smaller mass.

Solved Examples

Ex.23 A car is travelling at 30 km/h in a circle of radius 60 m. What is the minimum value of μ_s for the car to make the turn without skidding ?

Sol. The minimum μ_S should be that

$$\mu_S\, mg = \frac{mv^2}{r} \quad \text{or} \quad \mu_S = \frac{v^2}{rg}$$

Hare $v = 30\ \frac{km}{h} = \frac{30 \times 1000}{3600} = \frac{25}{3}$ m/s

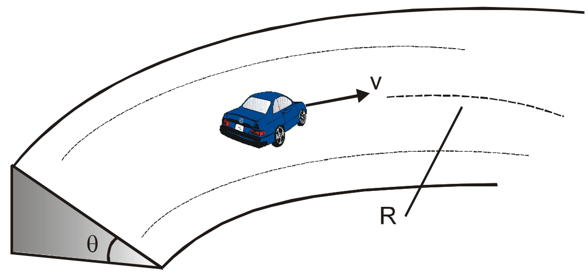
$$\mu_S = \frac{25}{3} \times \frac{25}{3} \times \frac{1}{60 \times 10} = 0.115$$

For all values of μ_S greater than or equal to the above value, the car can make the turn without skidding. If the speed of the car is high so that minimum μ_S is greater than the standard values (rubber tyre on dry concrete μ_S = 1 and on wet concrete μ_S = 0.7), then the car will skid.

Banking of road

If a cyclist takes a turn, he can bend from his vertical position.

This is not possible in the case of car, truck of train. The tilting of the vehicle is achieved by raising the outer edge of the circular track, slightly above the inner edge.



This is known as banking of curved track.

The angle of inclination with the horizontal is called the angle of banking.

* If driver moves with slow velocity that friction does not play any role in negotiating the turn.

The various forces acting on the vehicle are :

(i) Weight of the vehicle (mg) in the downward direction.

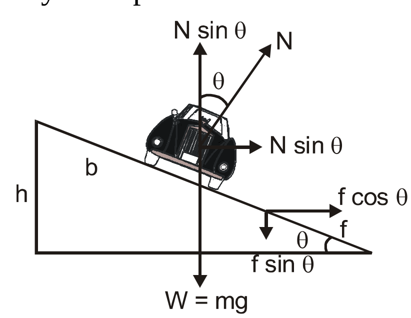
(ii) Normal reaction (N) perpendicular to the inclined surface of the road.

Resolve N in two components.

* N cosθ, vertically upwards which balances weight of the vehicle.

$\therefore$ N cosθ = mg ....(i)

* N sin θ, in horizontal direction which provides necessary centripetal force.



$$\therefore N \sin\theta = \frac{mv^2}{r} \quad ....(ii)$$

on dividing eqn. (ii) by eqn. (i)

$$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{r}}{mg} \quad \text{or} \quad \tan\theta = \frac{v^2}{rg} \quad \therefore \quad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

Where m is the mass of the vehicle, r is radius of curvature of the road, v is speed of the vehicle and θ is the banking angle (sinθ =h/b) .

* Factors that decide the value of angle of banking are as follows :

Thus, there is no need of mass of the vehicle to express the value of angle of banking

i.e. angle of banking $\Rightarrow$ does not dependent on the mass of the vehicle.

$\because$ $v^2 = gr \tan\theta$

$\therefore$ $v = \sqrt{gr \tan\theta}$ (maximum safe speed)

This gives the maximum safe speed of the vehicle. In actual practice, some frictional forces are always present. So, the maximum safe velocity is always much greater than that given by the above equation. While construction the curved track, the value of θ is calculated for fixed values of v_{max} and r. This explains why along the curved roads, the speed limit at which the curve is to be negotiated is clearly incited on sign boards.

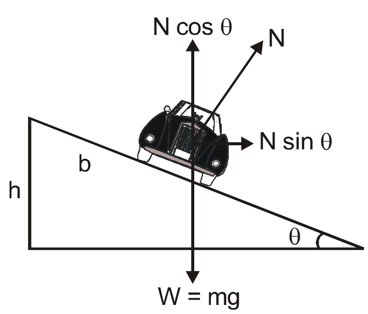
The outer side of the road is raised by $h = b \times \theta$.

When θ i small, then $\tan\theta \approx \sin\theta = \frac{h}{b}$;

Also $\tan\theta = \frac{v^2}{rg}$ $\therefore$ $\frac{v^2}{rg} = \frac{h}{b}$ or $h = \frac{v^2}{rg} \times b$

This gives us the height through which outer edge is raised above the inner edge.

* If the driver moves faster than the safe speed mentioned above, a friction force must act parallel to the road, inwards towards centre of the turn.



In this case forces acting on the vehicle are :

(i) Weight of the vehicle (mg) in the downward direction.

(ii) Normal reaction perpendicular to the inclined plane of the road.

N cos θ and N sin θ are the two rectangular components of N.

f cos θ and f sinθ are the two rectangular components of f.

The car does not have any vertical motion.

$\therefore$ $mg + f\sin\theta = N\cos\theta$

or $mg = N\cos\theta - f\sin\theta$

But $f = \mu N$ where $\mu \le \mu_s$.

$\therefore$ $mg = N\cos\theta - \mu N \sin\theta$

The foces N sin θ and f cos θ together provide the necessary centripetal force.

$\therefore$ $\frac{mv^2}{r} = N\sin\theta + f\cos\theta$

or $\frac{mv^2}{r} = N\sin\theta + \mu N\cos\theta$

Dividing equation (ii) by eqn. (i) we get

$$\frac{\frac{mv^2}{r}}{mg} = \frac{N\sin\theta + \mu N\cos\theta}{N\cos\theta - \mu N\sin\theta} \quad \text{or} \quad \frac{v^2}{rg} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}$$

$$\text{or} \quad \frac{v^2}{rg} = \frac{\cos\theta(\tan\theta + \mu)}{\cos\theta(1 - \mu\tan\theta)} \quad \text{or} \quad v^2 = \frac{\tan\theta + \mu}{1 - \mu\tan\theta} rg$$

$$\text{or} \quad v = \sqrt{\left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right) rg}$$

The best speed to negotiate a curve is obtained by putting $\mu = 0$

$\therefore$ $v = \sqrt{rg\tan\theta}$

with this speed, there will be minimum wear and tear of the types.

Solved Examples

Ex.24 At what should a highway be banked for cars travelling at a speed of 100 km/h if the radius of the road is 400 m and no frictional forces are involved?

Sol. The banking should be done at an angle θ such that

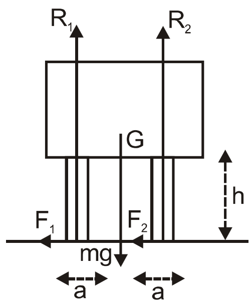
$$\tan\theta = \frac{v^2}{rg} = \frac{\frac{250}{9} \times \frac{250}{9}}{400 \times 10} \quad \text{or } \tan\theta = \frac{625}{81 \times 40} = 0.19$$

or $\theta = \tan^{-1} 0.19 \approx 0.19$ radian

$\approx 0.19 \times 57.3^{\circ}$ $\approx 11^{\circ}$

Condition of overturning

Here, we shell find the condition for the car to overturn. Let the distance between the centres of wheels of the car be 2a and the centre of gravity be h meters above the ground (road). The different forces acting on the car are shown in the fig.

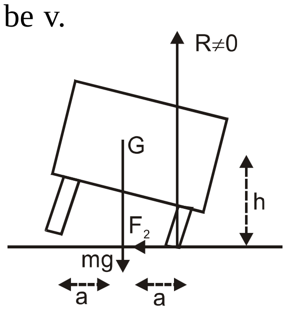


* The weigh mg of the car acts downwards through centre of gravity G.

* The normal reactions of the ground R_1 and R_2 on the inner and outer wheels respectively. These act vertically upwards.

* Let force of friction $F_1 + F_2$ between wheels and ground towards the centre of the turn.

Let the radius of circular path be r and the speed of the car be v.



Since there is no vertical motion, equating the vertical forces, we have

$R_1 + R_2 = mg$ ....(1)

The horizontal force = centripetal force for motion in a circle

So, $F = F_1 + F_2 = \frac{mv^2}{r}$ ....(2)

Taking moments about the centre of mass G.

$(F_1 + F_2)\,h = R_1a = R_2a$

$\therefore F_1 + F_2 = (R_2 - R_1) = \frac{a}{h}$ ....(3)

Combining this with equation (2) to eliminate $F_1 + F_2$ gives $R_2 - R_1 = \frac{hmv^2}{ar}$ ....(4)

We now have two simultaneous equations, (1) and (4), for R_1 and R_2. Solving these by adding and subtracting, we find that

$2R_1 = mg - \frac{hmv^2}{ar}$ and $2R_2 = mg + \frac{hmv^2}{ar}$

From these expressions it is clear.

Inner wheels will leave the ground when R_1 will become zero and the car begins to overturn,

i..e, $mg = \frac{hmv^2}{ar}$

So the limiting speed is given by $v^2 = \frac{gra}{h}$ as required.

Solved Examples

Ex.25 The radius of curvature of a railway line at a place when the train is moving with a speed of 36 kmh $^{-1}$ is 1000 m, the distance between the two rails being 1.5 metre. Calculate the elevation of the outer rail above the inner rail so that there may be no side pressure on the rails.

Sol. Velocity, $v = 36\ \text{km h}^{-1} = \frac{36 \times 1000}{3600}\ \text{ms}^{-1}$

$= 10\ \text{ms}^{-1}$ radius, $r = 1000$ m; $\tan\theta = \frac{v^2}{rg}$

$= 1000 \times 9.8 = \frac{1}{9.8}$

Let h be the height through which outer rail is raised. Let ℓ be the distance between the two rails.

Then, $\tan\theta = \frac{h}{\ell}$ [$\because \theta$ is very small] or $h = \ell \tan\theta$

$h = 1.5 \times \frac{1}{98}$ m $= 0.0153$ m [$\because \ell = 1.5$ m]

Ex.26 An aircraft executes a horizontal loop oat a speed of 720 km h^{-1} with its wing banked at 15º. Calculate the radius of the loop.

Sol. Speed , $v = 720\ \text{km h}^{-1} = \frac{720 \times 1000}{3600}\ \text{ms}^{-1} = 200\ \text{ms}^{-1}$ and $\tan\theta = \tan 15^\circ = 0.2679$

$\tan\theta = \frac{v^2}{rg}$ or $r = \frac{v^2}{g\tan\theta} = \frac{200 \times 200}{9.8 \times 0.2679}$ m

$= 1523.7\text{m} = 15.24$ km,

Ex.27 A train rounds an unbanked circular bend of radius 30 m at a speed of 54 km h^{-1}. The mass of the train is 10^6 kg. What provides the centripetal force required for this propose? The engine or the rails? The outer of inner rails ? Which rail will wear out faster, the outer or the inner rail ? What is the angle of banking required to prevent wearing out of the rails?

Sol. $r = 30$ m, $v = 54$ km $h^{-1} = \frac{54 \times 5}{18}$ ms^{-1}

$= 15$ ms^{-1} $m = 10^6$ kg, $\theta = ?$

(i) The centripetal force is provided by the lateral thrust by the outer rail on the flanges of the wheel of the train. The train causes an equal and opposite thrust on the outer rail (Newton's third law of motion).

Thus, the outer rails wears out faster.

(ii) $\tan\theta = \frac{v^2}{rg} = \frac{15 \times 15}{60 \times 9.8} = 0.7653$

$\therefore$ $\theta = \tan^{-1}(0.7653) = 37.43^\circ$

Ex.28 A circular race tack of radius 300 m is banked at an angle of 150º. If the coefficient of friction between the wheels of a race car and the road is 0.2, what is the (a) optimum speed of the race car to avoid wear and tear of tyres, and the (b) maximum permissible speed to avoid slipping?

Sol. (a) on a banked road, the horizontal component of the normal reaction and the friction force contribute to provide centripetal force to keep the car moving o a circular turn without slipping. At the optimum speed, the component of the normal reaction is enough to provide the required centripetal force. In this case, the frictional force is not required. The optimum speed is given by

$v_0 = (rg \tan\theta)^{1/2} = (300 \times 9.8 \tan 15^\circ)^{1/2}$ ms^{-1} $= 28.1$ ms^{-1}

(b) The maximum permissible speed if given by

$v_{max} = \left(\frac{\mu_s + \tan\theta}{1 - \mu_s \tan\theta}\right)^{1/2}$ Substituting values and simplifying, we get $v_{max} = 38.1$ ms^{-1}.

SPECIAL POINTS ABOUT CIRCULAR MOTION

* Centripetal force does not increase the kinetic energy of the particle moving in circular path, hence the work done by the force is zero.
* Centrifuges are the apparatuses used to separate small and big particles from a liquid.
* The physical quantities which remain constant for a particle moving in circular path are speed, kinetic energy and angular momentum.
* If a body is moving on a curved road with speed greater than the speed limit, the reaction at the inner wheel disappears and it will leave the ground first.
* On unbanked curved roads the minimum radius of curvature of the curve for safe driving is $r = v^2/\mu g$, where v is the speed of the vehicle and μ is the coefficient of friction.
* The skidding of a vehicle will occur if $v^2/r > \mu g$ i.e., skidding will take place if the speed is large, the curve is sharp and μ is small.
* If r is the radius of curvature of the speed breaker, then the maximum speed with which the vehicle can run on it swithout leaving contact with the ground is $v = \sqrt{(gr)}$,
* While taking a turn on the level road sometimes vehicles overturn due to centrifugal force.

POINTS TO BE REMEMBER

* **Uniform motion in a circle -**

Angular velocity $\omega = \frac{d\theta}{dt} = 2\pi n = \frac{2\pi}{T}$

Linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$

$v = \omega r$ when $\vec{\omega}$ and $\vec{r}$ are perpendicular to each other.

Centripetal acceleration $a = \frac{v^2}{r} = \omega^2 r = \omega v = 4\pi^2 n^2 r$

* **Equations of motion -**

For constant angular acceleration -

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

*** Motion of a car on a plane circular road -**

For motion without skidding

$$\frac{Mv_{max}^2}{r} = \mu M_g, v_{max}\sqrt{\mu rg}$$

*** Motion on a banked road -**

Angle of banking = θ $\tan\theta = \frac{h}{b}$

Maximum safe speed at the bend

$$v_{max} = \left[\frac{rg(\mu + \tan\theta)}{1-(\mu\tan\theta)}\right]^{1/2}$$

If friction is negligible $v_{max} = \sqrt{rg\tan\theta} = \sqrt{\frac{rhg}{b}}$

and $\tan\theta = \frac{v^2 max}{rg}$

*** Motion of cyclist on a curve -**

In equilibrium angle with vertical is θ then $\tan\theta = \frac{v^2}{rg}$

Maximum safe speed = $v_{max} = \sqrt{\mu rg}$

*** Motion in a vertical circle (particle tied to string) -**

At the top position - Tension $T_A = m\left(\frac{v_A^2}{r} - g\right)$

For $T_A = 0$, critical speed $= \sqrt{gr}$

At the bottom - Tension $T_B = m\left(\frac{v_B^2}{r} + g\right)$

For completing the circular motion minimum speed at the bottom $v_B = \sqrt{5gr},$

Tension $T_B = 6mg$

*** Conical pendulum (Motion in a horizontal circle)**

Tension is string $= \frac{mg\ell}{(\ell^2 - r^2)^{1/2}}$

Angular velocity $= \sqrt{\frac{g}{\ell\cos\theta}}$

Periodic time $= 2\pi\sqrt{\frac{\ell\cos\theta}{g}} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$