COMPLEX NUMBERS AND QUADRATIC EQUATION

1. QUADRATIC EQUATION

A quadratic Polynomial f(x) when equated to zero is called Quadratic Equation.

e.g $3x^2 + 7x + 5 = 0$, $-9x^2 + 7x + 5 = 0$, $x^2 + 2x = 0$, $2x^2 = 0$

General form :

$ax^2 + bx + c = 0$

Where, a, b, c $\in$ C and a $\neq$ 0

3.1 Roots of a Quadratic Equation

The values of variable x which satisfy the quadratic equation is called as Roots (also called solutions or zeros) of a Quadratic Equation.

2. SOLUTION OF QUADRATIC EQUATION

4.1 Factorization Method :

Let $ax^2 + bx + c = a(x-\alpha)(x - \beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x - 1)(3x + 1) = 0$

$x = 1, -\frac{1}{3}$

Hindu Method (Sri Dharacharya Method) :

By completing the perfect square as

$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Adding and substracting $\left(\frac{b}{2a}\right)^2$

$$\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

Which gives, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence the Quadratic equation

$ax^2 + bx + c = 0$ ($a \neq 0$) has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note : Every quadratic equation has two and only two roots.

3. NATURE OF ROOTS

In Quadratic equation $ax^2 + bx + c = 0$, the term $b^2 - 4ac$ is called discriminant of the equation, which plays an important role in finding the nature of the roots. It is denoted by Δ or D.

(A) Suppose a, b, c $\in$ R and a $\neq$ 0 then

(i) If D > 0 $\Rightarrow$ Roots are Real and unequal

(ii) If D = 0 $\Rightarrow$ Roots are Real and equal and each equal t – b/2a

(iii) If D < 0 $\Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose a, b, c $\in$ Q, a $\neq$ 0 then

(i) If D > 0 and D is perfect square

$\Rightarrow$ Roots are unequal and Rational

(ii) If D > 0 and D is not perfect square

$\Rightarrow$ Roots are irrational and unequal

Conjugate Roots :

The Irrational and complex roots of a quadratic equation are always occurs in pairs. Therefore (a, b, c, $\in$Q)

If One Root	then Other Root
$\alpha + i\beta$	$\alpha - i\beta$
$\alpha + \sqrt{\beta}$	$\alpha - \sqrt{\beta}$

4. SUM AND PRODUCT OF ROOTS

If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, then,

(i) Sum of Roots

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of x}}{\text{Coefficient of } x^2}$$

(ii) Product of Roots

$$P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

e.g. In equation

$3x^2 + 4x - 5 = 0$

Sum of roots

$S = -\frac{4}{3}$,

Product of roots $P = -\frac{5}{3}$

Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are α and β then

(i) $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm\frac{\sqrt{b^2 - 4ac}}{a} = \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha+\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= -\frac{b\sqrt{b^2 - 4ac}}{a^2} = \frac{\pm\sqrt{D}}{a}$

(iv) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = -\frac{b(b^2-3ac)}{a^3}$

(v) $\alpha^3-\beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

$= \sqrt{(\alpha+\beta)^2-4\alpha\beta}\,\{(\alpha+\beta)^2-\alpha\beta\}$

$= \frac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$

(vi) $\alpha^4+\beta^4 = \{(\alpha+\beta)^2-2\alpha\beta\}^2 - 2\alpha^2\beta^2$

$= \left(\frac{b^2-2ac}{a^2}\right)^2 - 2\,\frac{c^2}{a^2}$

(vii) $\alpha^4-\beta^4 = (\alpha^2-\beta^2)(\alpha^2+\beta^2)$

$= \frac{-b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$

(viii) $\alpha^2+\alpha\beta+\beta^2 = (\alpha+\beta)^2-\alpha\beta$

(ix) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta+\beta^2\alpha = \alpha\beta(\alpha+\beta)$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4+\beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

5. FORMATION OF AN EQUATION WITH GIVEN ROOTS

A quadratic equation whose roots are α and β is given by

$(x-\alpha)(x-\beta) = 0$

$\therefore\ x^2 - \alpha x - \beta x + \alpha\beta = 0$

$\therefore\ x^2 - (\alpha+\beta)x + \alpha\beta = 0$

i.e x^2 – (sum of Roots)x + Product of Roots = 0

$\therefore\ x^2 - Sx + P = 0$

Equation in terms of the Roots of another Equation

If α,β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$

(Replace x by – x)

(ii) $1/\alpha,\ 1/\beta \Rightarrow cx^2 + bx + a = 0$

(Replace x by 1/x)

(iii) $\alpha^n,\ \beta^n;\ n \in N \Rightarrow a(x^{1/n})^2 + b(x^{1/n}) + c = 0$

(Replace x by $x^{1/n}$)

(iv) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$

(Replace x by x/k)

(v) $k+\alpha,\ k+\beta \Rightarrow a(x-k)^2 + b(x-k) + c = 0$

(Replace x by (x–k))

(vi) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$

(Replace x by kx)

(vii) $\alpha^{1/n},\ \beta^{1/n};\ n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$

(Replace x by x^n)

Symmetric Expressions

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, which do not change by interchanging α and β. To find the value of such an expression, we generally express that in terms of $\alpha+\beta$ and $\alpha\beta$.

Some examples of symmetric expressions are–

(i) $\alpha^2+\beta^2$ (ii) $\alpha^2+\alpha\beta+\beta^2$ (iii) $\frac{1}{\alpha}+\frac{1}{\beta}$

(iv) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (v) $\alpha^2\beta+\beta^2\alpha$ (vi) $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2$

(vii) $\alpha^3+\beta^3$ (viii) $\alpha^4+\beta^4$

6. THE REAL NUMBER SYSTEM

Natural Number (N) : The number which are used for counting are known as Natural Number (also known as set of Positive Integers) i.e. N = {1, 2, 3, ...}

Whole Number (W) : If ' 0 ' is included in the set of natural numbers then we get the set of Whole Numbers i.e.W

= {0, 1, 2, ...}

= {N} + {0}

Integers (Z or I) :If negative natural number is included in the set of whole number then we get set of Integers i.e.

Z or I = {............–3, –2, –1, 0, 1, 2, 3,............}

Rational Numbers (Q) : The numbers which are in the form of p/q (Where p, q $\in$ I, q $\neq$ 0) are called as Rational Number e.g. $\frac{2}{3}$, 3, $\frac{1}{3}$, 0.76, 1.2322 etc.

Irrational Numbers : The numbers which are not rational i.e. which can not be expressed in p/q form or whose decimal part is non terminating non repeating but which may represent magnitude of physical quantities. e.g. $\sqrt{2}$, $5^{1/3}$, π, e,.....etc.

Real Numbers (R) : The set of Rational and Irrational Number is called as set of Real Numbers i.e. N $\subset$ W $\subset$ Z $\subset$ Q $\subset$ R

Note :

(i) Number zero is neither positive nor negative but is an even number.
(ii) Square of a real number is always positive.
(iii) Between two real numbers there lie infinite real numbers.
(iv) The real number system is totally ordered, for any two numbers a, b $\in$R, we must say, either a < b or b < a or b = a.
(v) All real number can be represented by points on a straight line. This line is called as number line.
(vi) An Integer (Note) is said to be even, if it is divided by 2 other wise it is odd number.
(vii) The magnitude of a physical quantity may be expressed as a real number times, a standard unit.
(viii) Number ' 0 ' is an additive quantity
(ix) Number '1' is multiplicative quantity.
(x) Infinity (∞) is the concept of the number greater than greatest you can imagine. It is not a number, it is just a concept, so we do not associate equality with it.
(xi) Division by zero is meaning less.
(xii) A non zero integer p is called prime if p $\neq \pm$ 1 and its only divisors are $\pm$ 1 and $\pm$ p.

7. IMAGINARY NUMBER

Square root of a negative real number is an imaginary number, while solving equation $x^2 + 1 = 0$ we get $x = \pm\sqrt{-1}$ which is imaginary.

So the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus i = $\sqrt{-1}$

Further $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$ ....may be expressed as $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, $\pm$ 2i ....

Integral powers of iota

As we have seen $i = \sqrt{-1}$ so $i^2 = -1$

$i^3 = -i$ and $i^4 = 1$

Hence n $\in$ N, i^n = i, −1, −i, 1 attains four values according to the value of n,

so $i^{4n+1} = i$, $i^{4n+2} = -1$

$i^{4n+3} = -i$, i^{4n} or $i^{4n+4} = 1$

In other words $i^n = (-1)^{n/2}$ if n is even integer

$i^n = (-1)^{\frac{n-1}{2}}\, i$ if n is odd integer.

Note :–

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) $\sqrt{a.b} = \sqrt{a}.\sqrt{b}$ possible iff both a, b are non-negative. (incorrect). It is also true for one positive and one negative no.

e.g. $\sqrt{(-2)(3)} = \sqrt{-2}.\sqrt{3}$

only invalid when both are negative means

$\sqrt{a.b} \neq \sqrt{a}.\sqrt{b}$ iff a & b both are negative.

(iii) ' i ' is neither positive, zero nor negative, Due to this reason order relations are not defined for imaginary numbers.

8. COMPLEX NUMBER

A number of the form z = x+iy where x, y $\in$ R and $i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

Re (z) = x, Im (z) = y

Here if x = 0 the complex number is purely Imaginary and if y = 0 the complex number is purely Real.

A complex number may also be defined as an ordered pair of real numbers any may be denoted by the symbol (a, b). If we write z = (a, b) then a is called the real part and b the imaginary part of the complex number z.

Note :

(i) Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so 4 + 3i < 1 + 2i or i< 0 or i > 0 is meaning less.

(ii) If two complex numbers are equal, then their real and imaginary parts are separately equal. Thus if a + ib = c + id

$\Rightarrow$ a = c and b = d

so if z = 0 $\Rightarrow$ x + iy = 0 $\Rightarrow$ x = 0 and y = 0

The student must note that x, y $\in$ R and x, y $\neq$ 0. Then if

$x + y = 0 \Rightarrow x = y$ is correct

but $x + i\,y = 0 \Rightarrow x = -iy$ is incorrect

Hence a real number cannot be equal to the imaginary number, unless both are zero.

(iii) The complex number 0 is purely real and purely imaginary both.

Representation of a Complex Number :

(a) Cartesian Representation :

The complex number $z = x + iy = (x, y)$ is represented by a point P whose coordinates are refered to rectangular axis xox′ and yoy′, which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gussian plane.

Note :

(i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by $|z|$. Thus, $|z| = \sqrt{x^2 + y^2}$.

(ii) Angle of any complex number with positive direction of x-axis is called amplitude or argument of z.

Thus, amp (z) = arg (z) = $\theta = \tan^{-1}\frac{y}{x}$.

(b) Polar Representation : If $z = x + iy$ is a complex number then $z = r(\cos\theta + i\sin\theta)$ is a polar form of complex number z where

$x = r\cos\theta$, $y = r\sin\theta$ and

$r = \sqrt{x^2 + y^2} = |z|$.

(c) Exponential Form : If $z = x + iy$ is a complex number then its exponential form is $z = r\,e^{i\theta}$ where r is modulas and θ is amplitude of complex number.

(d) Vector Representation : If $z = x + iy$ is a complex number such that it represent point P(x, y) then its vector representation is $z = \vec{OP}$

Algebraic operations with Complex Number:

Addition $(a + ib) + (c + id)$

$= (a + c) + i(b + d)$

Subtraction $(a + ib) - (c + id)$

$= (a - c) + i(b - d)$

Multiplication $(a + ib)(c + id)$

$= ac + iad + ibc + i^2 bd$

$= (ac - bd) + i(ad + bc)$

Division $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

(when at least one of c and d is non zero)

$= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$

Properties of Algebraic operations with Complex Number

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties-

Commutativity : $z_1 + z_2 = z_2 + z_1$ & $z_1 z_2 = z_2 z_1$

Associativity : $(z_1 + z_2) + z_3$

$= z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3$

$= z_1(z_2 z_3)$

Identity element : If O = (0, 0) and 1 = (1, 0) then $z + 0 = 0 + z = z$ and $z.1 = 1.z = z$.

Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is $-z$ and multiplicative inverse of z is $\frac{1}{z}$.

Cancellation Law :

$\left.\begin{matrix} z_1 + z_2 = z_1 + z_3 \\ z_2 + z_1 = z_3 + z_1 \end{matrix}\right\} \Rightarrow z_2 = z_3$

and $z_1 \neq 0$ $\left.\begin{matrix} z_1 z_2 = z_1 z_3 \\ z_2 z_1 = z_3 z_1 \end{matrix}\right\} \Rightarrow z_2 = z_3$

Distributivity : $z_1(z_2 + z_3)$

$= z_1 z_2 + z_1 z_3$

and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$

Conjugate Complex Number :

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$ where $b \neq 0$ are said to be complex conjugate of each other (Here the complex conjugate is obtained by just changing the sign of i) e.g.conjugate of $z = -3 + 4i$ is $\bar{z} = -3 - 4i$.

Note : Image of any complex number in x-axis is called its conjugate.

Properties of Conjugate Complex Number

Let $z = a + ib$ and $\bar{z} = a - ib$ then

(i) $\overline{(\bar{z})} = z$ (ii) $z + \bar{z} = 2a = 2\,Re(z)$ = purely real

(iii) $z - \bar{z} = 2ib = 2i\,Im(z)$ = purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$ (viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(ix) $\overline{z^n} = (\bar{z})^n$ (x) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(xi) $z + \bar{z} = 0$ or $z = -\bar{z}$
$\Rightarrow$ z = 0 or z is purely imaginary

(xii) $z = \bar{z} \Rightarrow$ z is purely real

9. MODULUS OF A COMPLEX NUMBER

If $z = x + iy$ then modulus of z is equal to $\sqrt{x^2 + y^2}$ and it is denoted by $|z|$. Thus

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

Note :

Modulus of every complex number is a non negative real number.

Properties of modulus of a Complex Number

(i) $|z| \geq 0$
(ii) $-|z| \leq \text{Re}(z) \leq |z|$
(iii) $-|z| \leq \text{Im}(z) \leq |z|$
(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
(v) $z\bar{z} = |z|^2$
(vi) $|z_1 z_2| = |z_1|\ |z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

(viii) $|z|^n = |z^n|$, $n \in N$

(ix) $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$

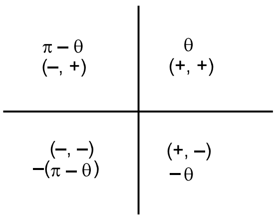
(x) $z^{-1} = \frac{\bar{z}}{|z|^2}$

(xi) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1 \bar{z}_2)$
(xii) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$
(xiii) $|re^{i\theta}| = r$

10. AMPLITUDE OR ARGUMENT OF A COMPLEX NUMBER :

The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis

For finding the argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.



Note :

(i) Principle value of any complex number lies between $-\pi < \theta \leq \pi$.
(ii) Amplitude of a complex number is a many valued function. If θ is the argument of a complex number then $(2n\pi+\theta)$ is also argument of complex number.
(iii) Argument of zero is not defined.
(iv) If a complex number is multiplied by iota (i) its amplitude will be increased by $\pi/2$ and will be decreased by $\pi/2$, if is multiplied by $-i$.
(v) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

Properties of argument of a Complex Number

(i) amp (any real positive number) = 0
(ii) amp (any real negative number) = π
(iii) amp $(z - \bar{z}) = \pm\,\pi/2$
(iv) amp $(z_1 . z_2)$ = amp (z_1) + amp (z_2)

(v) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2)

(vi) amp $(\bar{z})$ = – amp (z) = amp (1/z)
(vii) amp (– z) = amp (z) $\pm\,\pi$ (viii) amp (z^n) = n amp (z)
(ix) amp (iy) $= \pi/2$ if $y > 0$
$= -\pi/2$, if $y < 0$
(x) amp (z) + amp $(\bar{z})$ = 0

11. SQUARE ROOT OF A COMPLEX NUMBER

The square root of $z = a + ib$ is -

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \quad \text{for } b > 0$$

and $$\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right] \quad \text{for } b < 0$$

Note :

(i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$

(Here b = 1)

(ii) The square root of – i is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$

(Here b = –1)
(iii) The square root of ω is $\pm\,\omega^2$
(iv) The square root of ω^2 is $\pm\,\omega$

12. TRIANGLE INEQUALITIES

(i) $|z_1 \pm z_2| \leq |z_1| + |z_2|$
(ii) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

SOLVED PROBLEMS

Ex.1 Find the number of real roots of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$

Sol. Let $e^{\sin x} = y$ then given equation reduces to

$$y - \frac{1}{y} - 4 = 0$$

$\Rightarrow y^2 - 4y - 1 = 0$

$\Rightarrow y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$

$= 4.23, - 0.23$

But $y = e^{\sin x}$ is never negative. So

$y = e^{\sin x} = 4.23$

$\Rightarrow \sin x = \log 4.23 > 1$

which is not possible. Hence the equation has no real root.

Ex.2 Show that the Both roots of the equation $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are real

Sol. The given equation can be written in the following form :

$3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$

Here discriminant

$= 4(a + b + c)^2 - 12(ab + bc + ca)$

$= 4[(a^2 + b^2 + c^2) - (ab + bc + ca)] > 0$

$[\because a^2 + b^2 + c^2 > ab + bc + ca]$

$\therefore$ Both roots are real.

Ex.3 If a < b < c < d, then show that roots of $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and unequal

Sol. Here

$3x^2 - (a+c+2b+2d)x + (ac+2bd) = 0$

$\therefore$ Discriminant

$= (a+c+2b+2d)^2 - 12(ac+2bd)$

$= [(a+2d)-(c+2b)]^2 + 4(a+2d)(c+2b) - 12(ac+2bd)$

$= [(a+2d)-(c+2b)]^2 + 8(c-b)(d-a) > 0.$

Ex.4 For the equation $\frac{1}{x+a} - \frac{1}{x+b} = \frac{1}{x+c}$, if the product of roots is zero. find the sum of roots

Sol. $\frac{1}{x+a} - \frac{1}{x+b} = \frac{1}{x+c}$

$$\frac{b-a}{x^2 + (b+a)x + ab} = \frac{1}{x+c}$$

or $x^2 + (a + b)x + ab = (b - a)x + (b - a)c$

or $x^2 + 2ax + ab + ca - bc = 0$

Since product of the roots $= 0$

$ab + ca - bc = 0 \Rightarrow a = \frac{bc}{b+c}$

Thus sum of roots $= -2a = \frac{-2bc}{b+c}$

Ex.5 If α, β are the roots of $x^2 - p(x+1) - c = 0$ then find $\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}$

Sol. Here the equation is

$x^2 - p(x + 1) - c = 0$

$\therefore \alpha + \beta = p, \ \alpha\beta = -(p + c)$

$\Rightarrow (\alpha + 1)(\beta + 1) = 1 - c$

Now given expression

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (1-c)},$$

Putting value of $1 - c = (\alpha + 1)(\beta + 1)$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha}$$

$$= \frac{\alpha+1-\beta-1}{\alpha-\beta} = 1$$

Ex.6 Find the quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$

Sol. Given root $= \frac{1}{2+\sqrt{5}} = \sqrt{5} - 2$

So the other root $= -\sqrt{5} - 2$. Then sum of the roots $= -4$, product of the roots $= -1$

Hence the equation is $x^2 + 4x - 1 = 0$

Ex.7 Find the value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots of $x^2 - x + m = 0$

Sol. Let α be the root of $x^2 - x + m = 0$ and 2α be the root of $x^2 - 3x + 2m = 0$. Then,

$\alpha^2 - \alpha + m = 0$ and

$4\alpha^2 - 6\alpha + 2m = 0$

$\Rightarrow \frac{\alpha^2}{-4m} = \frac{\alpha}{-2m} = \frac{1}{2} \Rightarrow m^2 = -2m$

$\Rightarrow m = 0, \ m = -2$

Ex.8 If roots of the equation $x^2 + ax + 25 = 0$ are in the ratio of 2 : 3 then find the value of a

Sol. Here $k = 2/3$

so from the condition

$$\frac{(k+1)^2}{k} = \frac{b^2}{ac}$$

$$\frac{(2/3+1)^2}{2/3} = \frac{a^2}{25}$$

$\Rightarrow \frac{25}{9} \times \frac{3}{2} = \frac{a^2}{25} \Rightarrow \frac{25}{6} = \frac{a^2}{25}$

$\therefore\ a^2 = \frac{25 \times 25}{6} \Rightarrow a = \frac{\pm 25}{\sqrt{6}}$

Ex.9 If $x = 2 + \sqrt{3}$ then find the value of $x^3 - 7x^2 + 13x - 12$

Sol. $x = 2 + \sqrt{3}$

$\Rightarrow x - 2 = \sqrt{3}$

$\Rightarrow x^2 + 4 - 4x = 3$

$\Rightarrow x^2 - 4x = -1$

$\Rightarrow x^2 - 4x + 1 = 0$

Now we can write the given equation as

$x^3 - 7x^2 + 13x - 12$

$= x(x^2 - 4x + 1) - 3x^2 + 12x - 12$

$= x(x^2 - 4x + 1) - 3(x^2 - 4x + 1) - 9$

Now putting the value of

$x^2 - 4x + 1 = 0$

$= x(0) - 3(0) - 9 = -9$

Ex.10 Find $\sqrt{-2}\ \sqrt{-3}$

Sol. $\sqrt{-2} \times \sqrt{-3} = \sqrt{2}i \times \sqrt{3}i$

$= \sqrt{6}(i)^2 = -\sqrt{6}$

Ex.11 If x be real, find the relation between a and b, when $\frac{1-ix}{1+ix} = a - ib$

Sol. $\because \frac{1-ix}{1+ix} = a - ib$

on taking modulus; we get

$|a - ib| = \left|\frac{1-ix}{1+ix}\right|$

$\Rightarrow \sqrt{a^2+b^2} = \left|\frac{1-ix}{1+ix}\right| = \frac{|1-ix|}{|1+ix|} = 1$

$\therefore\ a^2 + b^2 = 1$

Ex.12 If the vertices of any quadrilateral are A = 1 + 2i, B = –3 + i, C = –2 – 3i, and D = 2 – 2i, show that it is a square

Sol. A = (1, 2), B = (–3, 1)

C = (–2, –3), D = (2, –2)

$\therefore\ AB = \sqrt{(-3-1)^2 + (1-2)^2} = \sqrt{17}$

$BC = \sqrt{(-2+3)^2 + (-3-1)^2} = \sqrt{17}$

$CD = \sqrt{(2+2)^2 + (-2+3)^2} = \sqrt{17}$

$DA = \sqrt{(1-2)^2 + (2+2)^2} = \sqrt{17}$

Diagonal

$AC = \sqrt{(-2-1)^2 + (-3-2)^2} = \sqrt{34}$

and $BD = \sqrt{(2+3)^2 + (-2-1)^2} = \sqrt{34}$

$\because$ AB = BC = CD = DA and AC = BD

$\therefore$ ABCD is a square

Ex.13 If $z = \left(\frac{1}{2}, 1\right)$, then find the value of z^{-1}

Sol. $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(1/2) - i}{(1/2)^2 + 1} = \frac{2}{5} - \frac{4}{5}i = \left(\frac{2}{5}, -\frac{4}{5}\right)$

Ex.14 If $\frac{\tan\theta - i\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{1 + 2i\sin\frac{\theta}{2}}$ is purely imaginary then find the general value of θ

Sol. Multiply above and below by conjugate of denominator and put real part equal to zero.

$= \frac{\tan\theta - i\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{1 + 2i\sin\frac{\theta}{2}} \times \frac{1 - 2i\sin\frac{\theta}{2}}{1 - 2i\sin\frac{\theta}{2}}$

$= \frac{\tan\theta - 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) - i\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2} + 2\tan\theta\sin\frac{\theta}{2}\right)}{1 + 4\sin^2\frac{\theta}{2}}$

$\therefore\ \tan\theta - 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) = 0$

$\Rightarrow \frac{\sin\theta}{\cos\theta} - (1 - \cos\theta) - \sin\theta = 0$

$\Rightarrow \sin\theta\left(\frac{1-\cos\theta}{\cos\theta}\right) - (1 - \cos\theta) = 0$

$\Rightarrow (1 - \cos\theta)(\tan\theta - 1) = 0$

$\cos\theta = 1 \Rightarrow \theta = 2n\pi$ and

$\tan\theta = 1 \Rightarrow \theta = n\pi + \frac{\pi}{4}$

Ex.15 The complex numbers sin x + i cos 2x and cos x – i sin 2x are conjugate to each other then find x

Sol. $\sin x + i\cos 2x = \cos x + i\sin 2x$

$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow x = n\pi + \frac{\pi}{4}$ and $x = \frac{n\pi}{2} + \frac{\pi}{8}$

$\Rightarrow x \in \left\{\ldots, \frac{-7\pi}{4}, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots\right\}$

$\cap \left\{\ldots, \frac{-7\pi}{8}, \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \ldots\right\}$

$\Rightarrow$ there is no common value of x.

Ex.16 If z_1, z_2 are any two complex numbers and a, b are any two real numbers, then prove that $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2+b^2)(|z_1|^2 + |z_2|^2)$

Sol. Expression

$= (az_1 - bz_2)\overline{(az_1 - bz_2)} + (bz_1 + az_2)\overline{(bz_1 + az_2)}$

$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2)$

$+ (bz_1 + az_2)(b\bar{z}_1 + a\bar{z}_2)$

$= a^2 |z_1|^2 + b^2 |z_2|^2 + b^2 |z_1|^2 + a^2 |z_2|^2$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

Ex.17 Find the amplitude of $1-\cos\theta - i\sin\theta$

Sol. Let

$z = 1 - \cos\theta - i\sin\theta = r(\cos\phi + i\sin\phi)$

$\therefore \tan\phi = -\frac{\sin\theta}{1-\cos\theta}$

$= \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2)$

$= -\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

or $\tan\phi = \tan\left(\frac{\theta}{2} - \frac{\pi}{2}\right)$

$\therefore \text{amp}(z) = \frac{\theta}{2} - \frac{\pi}{2}$

Ex.18 If $x_n = \cos(\pi/2^n) + i\sin(\pi/2^n)$, then find $x_1 x_2 x_3 \ldots\ldots\ldots\infty$

Sol. $x_1 x_2 x_3 \ldots\ldots\infty$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots\right)$

$+ i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots\right)$

Ex.19 If $z_1 = 10+6i$, $z_2 = 4+6i$ and z is a complex number such that $\text{amp}\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$, then find $|z-7-9i|$

Sol. If $z = x + iy$, then $\text{amp}\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0$ ...(1)

Now $|z - 7 - 9i|$

$= \sqrt{x^2 + y^2 - 14x - 18y + 130}$

$= 3\sqrt{2}$. (from 1)

Ex.20 Find the polar form of complex number

$z = \frac{\{\cos(\pi/3) - i\sin(\pi/3)\}(\sqrt{3}+i)}{i-1}$

Sol. Here $|z|$

$= \frac{|\cos(\pi/3) - i\sin(\pi/3)||\sqrt{3}+i|}{|i-1|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Again $\text{amp}(z) = \text{amp}\{\cos(\pi/3) - i\sin(\pi/3)\} + \text{amp}(\sqrt{3} + i) - \text{amp}(-1 + i)$

$= -\frac{\pi}{3} + \frac{\pi}{6} - \left(\pi - \frac{\pi}{4}\right) = -\frac{11\pi}{12}$

Therefore

$z = \sqrt{2}\left\{\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right\}$

$= \sqrt{2}\left\{\cos\left(-\frac{11\pi}{12} + 2\pi\right) + i\sin\left(-\frac{11\pi}{12} + 2\pi\right)\right\}$

$= \sqrt{2}\left\{\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right\}$

Ex.21 Find the Square root of $-8 - 6i$

Sol. Let $\sqrt{-8-6i} = \pm(a + ib)$

$\Rightarrow -8 - 6i = a^2 - b^2 + 2iab$

$\Rightarrow a^2 - b^2 = -8$ ...[1]

$2ab = -6 \Rightarrow ab = -3$ ...[2]

$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$

$= (-8)^2 + (-6)^2 = 64 + 36 = 100$

$\Rightarrow a^2 + b^2 = 10$ ...[3]

from equation (2) and (3)

$a = 1, b = -3$

So, $\sqrt{-8-6i} = \pm(1 - 3i)$

Ex.22 If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then find k

Sol. Here $x + iy = (a - ib)^3$

$= (a^3 - 3ab^2) + i(-3a^2b + b^3)$

$\Rightarrow x = a^3 - 3ab^2, \; y = b^3 - 3a^2b$

$\Rightarrow \frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (b^2 - 3a^2)$

$= 4(a^2 - b^2) \Rightarrow k = 4$

Ex.23 Find the value of

$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

Sol. $\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$= -i\left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right) = -i\,e^{\frac{2\pi k i}{7}}$

$\therefore \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$= -i\left[e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + \ldots\ldots 6 \text{ terms}\right]$

$= -i\,e^{\frac{2\pi i}{7}}\left\{\frac{1 - e^{\frac{12\pi i}{7}}}{1 - e^{\frac{2\pi i}{7}}}\right\}$ $(\because e^{2\pi i} = 1)$

$= -i\left\{\frac{e^{\frac{2\pi i}{7}} - 1}{1 - e^{\frac{2\pi i}{7}}}\right\} = i$

EXERCISE - I UNSOLVED PROBLEMS

Q.1 Evaluate :
(i) i^9 (ii) i^{51} (iii) i^{342}

Q.2 Evaluate :
(i) i^{-63} (ii) i^{-38} (iii) i^{-130}

Q.3 Evaluate :
(i) $(i^9 + i^{19})$ (ii) $(i^{131} + i^{49})$

Q.4 Evaluate: (i) $\left(i^{29}+\frac{1}{i^{29}}\right)$ (ii) $\left(i^{37}+\frac{1}{i^{67}}\right)$

Show that :

Q.5 $1 + i^2 + i^4 + i^6 = 0$

Q.6 $1 + i^{10} + i^{100} - i^{1000} = 0$

Q.7 $i^{104} + i^{109} + i^{114} + i^{119} = 0$

Q.8 $6. i^{54} + 5. i^{37} - 2. i^{11} + 6. i^{68} = 7. i$

Q.9 $\frac{1}{i}-\frac{1}{i^2}+\frac{1}{i^3}=\frac{1}{i^4}=0$

Q.10 $(1 + i^{14} + i^{18} + i^{22})$ is a real number.

Q.11 Evaluate: (i) $\sqrt{-81}$ (ii) $\sqrt{-25}\times\sqrt{-81}$

Q.12 Evaluate $4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$

Q.13 Simplify and express the result in the form (a + ib) :
(i) $(3+\sqrt{-16})-(4-\sqrt{-9})$ (ii) (7 – 5i) (3 + i)
(iii) $(5+\sqrt{-7})(5-\sqrt{-7})$ (iv) $(4+i)^2$

Q.14 Write each of the following in the form (a + ib):
(i) $(3-2i)^2$ (ii) $(2+\sqrt{-3})^2$ (iii) $(2+5i)^3$
(iv) $(1-4i)^3$ (v) $(3-2i)^{-1}$

Q.15 Express each of the following in the form (a+ib):
(i) $\frac{1}{(3+4i)}$ (ii) $\frac{1-i}{1+i}$ (iii) $\frac{(2+3i)^2}{(2-i)}$

Q.16 Show that $\left\{\frac{(\sqrt{7}+i\sqrt{3})}{(\sqrt{7}-i\sqrt{3})}+\frac{(\sqrt{7}-i\sqrt{3})}{(\sqrt{7}+i\sqrt{3})}\right\}$ is real.

Q.17 Find real values of θ for which $\left(\frac{3+2i\sin\theta}{1-2i\sin\theta}\right)$ is purely real.

Q.18 Find the real values of x and y for which :
(i) (1 – i) x + (1 + i) y = 1 – 3i
(ii) (x + iy) (2 – 3i) = (4 + i)

Q.19 Find the conjugate of each of the following
(i) (–5 – 2i) (ii) $\frac{1}{(4+3i)}$ (iii) $\frac{(1+i)^2}{(3-i)}$

Q.20 Separate $\frac{3+\sqrt{-1}}{2-\sqrt{-1}}$ into real and imaginary parts, and hence find its modulus

Q.21 If z is a complex number such that |z| = 1, prove that $\left(\frac{z-1}{z+1}\right)$, where z ≠ -1.

Q.23 If $(x+iy)^{1/3}=(a+ib)$, prove that $\left(\frac{x}{a}+\frac{y}{b}\right)=(4a^2-b^2)$.

Q.24 If (x + iy) = $\frac{(a+ib)}{(a-ib)}$, prove that $(x^2 + y^2) = 1$.

Q.25 If z = (x + iy) and w = $\frac{1-iz}{z-i}$ such that |w| = 1, then show that z is purely real.

Represent each of the following in the polar form (Question No. 26 to Q. 30)

Q.26 –1+i **Q.27** $-4+4\sqrt{3}\,i$ **Q.28** $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

Q.29 $1-\sqrt{3}\,i$ **Q.30** $\frac{(5-i)}{(2-3i)}$

Solve : (Question No. 31 to Q. 45)

Q.31 $x^2 + 5 = 0$
Q.32 $2x^2 + 1 = 0$
Q.33 $x^2 - x + 2 = 0$
Q.34 $x^2 + 2x + 2 = 0$
Q.35 $2x^2 - 4x + 3 = 0$
Q.36 $x^2 + 3x + 5 = 0$
Q.37 $25x^2 - 30x + 11 = 0$
Q.38 $8x^2 + 2x + 1 = 0$
Q.39 $27x^2 + 10x + 1 = 0$
Q.40 $2x^2 - \sqrt{3}\,x + 1 = 0$
Q.41 $x^2 + 13 = 4x$
Q.42 $3x^2 + 5 = 7x$
Q.43 $17x^2 + 1 = 8x$
Q.44 $x^2 + 3ix + 10 = 0$
Q.45 $2x^2 + 3ix + 2 = 0$

ANSWER KEY

1. (i) i (ii) –i (iii) –1 **2.** (i) i (ii) –1 (iii) –1 **3.** (i) 0 (ii) 0

4. (i) 0 (ii) 2i **11.** (i) 9 i (ii) –45 **12.** 11 i

13. (i) (–1 + 7 i) (ii) (26 – 8 i) (iii) 32 (iv) 15 + 8 i

14. (i) (5 – 12i) (ii) $(1 + 4\sqrt{3}\,i)$ (iii) (–142 – 65 i) (iv) (–47 + 52i) (v) $\left(\frac{3}{13}+\frac{2}{13}i\right)$

15. (i) $\left(\frac{3}{25}-\frac{4}{25}i\right)$ (ii) 0 – i (iii) $\left(\frac{-22}{5}+\frac{19}{5}i\right)$ **17.** $\theta = n\pi,\ n \in Z$ **18.** (i) x = 2, y –1 (ii) $x=\frac{5}{13}, y=\frac{14}{13}$

19. (i) (–5 + 2i) (ii) $\frac{1}{25}(4+3i)$ (iii) $\left(-\frac{1}{5}-\frac{3}{5}i\right)$ **20.** $(1+i), \sqrt{2}$

26. $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right), \sqrt{2}, \frac{3\pi}{4}$ **27.** $8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right), 8, \frac{2\pi}{3}$ **28.** $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$, 2, $\frac{\pi}{3}$

29. $2\left\{\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right\}, 2, \frac{-\pi}{3}$ **30.** $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right), \sqrt{2}, \frac{\pi}{4}$ **31.** $(\sqrt{5}\,i - \sqrt{5}\,i)$

32. $\left\{\frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}}i\right\}$ **33.** $\left\{\frac{1+\sqrt{7}i}{2}, \frac{1-\sqrt{7}i}{2}\right\}$ **34.** {–1 + i, –1 – i} **35.** $\left\{1+\frac{1}{\sqrt{2}}i, 1-\frac{1}{\sqrt{2}}i\right\}$

36. $\left\{\frac{-3}{2}+\frac{\sqrt{11}}{2}i, \frac{-3}{2}-\frac{\sqrt{11}}{2}i\right\}$ **37.** $\left\{\frac{3+\sqrt{2}i}{5}, 1-\frac{3-\sqrt{2}i}{5}\right\}$ **38.** $\left\{\frac{-1+\sqrt{7}i}{8}, \frac{-1-\sqrt{7}i}{8}\right\}$

39. $\left\{\frac{-5+\sqrt{2}i}{27}, \frac{-5-\sqrt{2}i}{27}\right\}$ **40.** $\left\{\frac{\sqrt{3}+\sqrt{5}i}{4}, \frac{\sqrt{3}-\sqrt{5}i}{4}\right\}$ **41.** {2 + 3i, 2 – 3i}

42. $\left\{\frac{7+\sqrt{11}i}{6}, \frac{7-\sqrt{11}i}{6}\right\}$ **43.** $\left\{\frac{4+i}{17}, \frac{4-i}{17}\right\}$ **44.** {2i, – 5i}

45. $\left\{\frac{1}{2}i, -2i\right\}$