Motion in a Straight Line

INTRODUCTION

The branch of physics which deals with the study of motion of material objects is called **mechanics**. Mechanics is divided into following branches.

(i) Statics :

Statics is the branch of mechanics which deals with the study of motion of objects under the effect of forces in equilibrium.

(ii) Kinematics :

It is that branch of mechanics which deals with the study of motion of object without taking into account the factors (i.e. nature of forces, nature of bodies etc.) which cause motion. Here time factor plays an essential role.

(iii) Dynamics :

It is that branch of mechanics which deals with the study of motion of objects taking into account the factors which cause motion.

Rest : An object is said to be at rest if it does not change its position with time, with respect to its surroudings. A book lying on a table, a person sitting in a chair are the examples of rest.

Motion : An object is said to be in motion if it changes its position with time, with respect to its moving on rails, a ship sailing on water, a man walking on road are some of the examples of motion, visible to the eye. Motion of gas molecules is an example of motion, invisible to the eye.

Rest & Motion are relative terms :

When we say that an object is at rest or in motion,then this statement is incomplete and meaningless. Basically, rest & motion are relative terms. An object which is at rest can also be in motion simultaneously. This can be illustrated as follows.

The passengers sitting in a moving bus are at rest with respect to each other but they are also in motion at the same time with respect to the objects like trees, buildings on the road side. So the motion and rest are relative terms.

Rectilinear motion :

If a particle moves in a fixed direction, the motion of this type is called rectilinear motion or one dimensional motion.

For example the motion of an ant on a wire is a rectilinear motion.

Two dimensional motion :

If the motion of a particle is in such a way that its position remains on a fixed plane, then the motion of a particle is called two dimensional motion.

For example the motion of a rolling ball on a horizontal plane or earth's surface, is a two dimensional motion.

Three dimensional motion :

If a line or plane is not fixed for the motion of a particle, then its motion is called three dimensional motion.

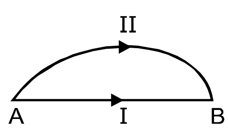
For example the motion of a flying bird is a three dimensional motion.

DISTANCE

The length of the actual path between initial & final positions of a particle in a given interval of time is called distance covered by the particle. Distance is the actual length of the path. It is the characteristic property of any path ie., path is always associated when we consider distance between two positions.

Distance beteen A & B while moving through path (1) may or may not be equal to the distance between A & B while moving through path (2)

Characteristics of Distance :



(i) It is a scalar quantity.

(ii) It depends on the path.

(iii) It never reduces with time.

(iv) Distance covered by a particle is always positive & can never be negative or zero.

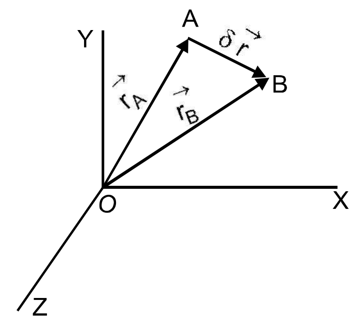
(v) Dimension : $\left[M^0 L^1 T^0\right]$

(vi) Unit : In C.G.S.system centimeter (cm), In S.I. system metre (m).

DISPLACEMENT

The shortest distance from the initial position to the final position of the particle is called displacement. The displacement of a particle is measured as the change in the position of the particle in a particular direction over a given time interval. It depends only on final & initial positions.

Displacement of a particle is a position vector of its final position w.r.t. initial position.



Position vector of A w.r.t. $O = \overrightarrow{OA}$

$\Rightarrow \quad \vec{r}_A = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$

Position vector of B w.r.t. $O = \overrightarrow{OB}$

$\Rightarrow \quad \vec{r}_B = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Displacement

$= \quad \overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Characteristics of Displacement :

(i) It is a vector quantity.

(ii) The displacement of a particle between any two points is equal to the shortest distance between them.

(iii) The displacement of an object in a given time interval can be positive, negative or zero.

(iv) Dimension : $\left[M^0 L^1 T^0\right]$

(v) Unit : In C.G.S. centimeter (cm), In S.I. system meter (m).

COMPARATIVE STUDY OF DISPLACEMENT AND DISTANCE

S.No.	Displacement	Distance
1.	It has single value bewteen two points	It may have more than one value between two points
2.	$0 \le \text{Displacement} \le 0$	Distance > 0
3.	Displacement can decrease with time	It can never decrease with time.
4.	It is a vector quantity	It is a scalar quantity

Special note:

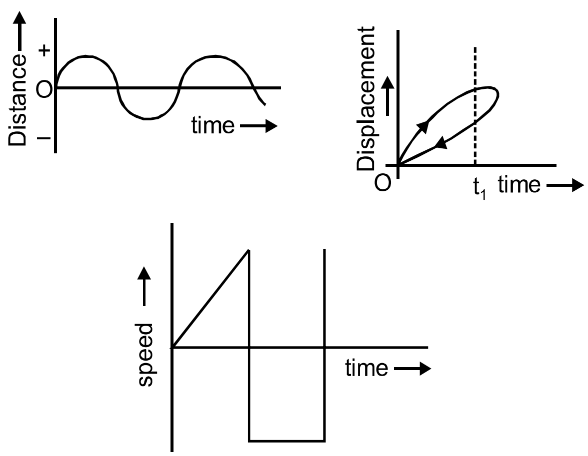
1. The actual disatnce travelled by a particle in the given interval of time is always equal to or greater than the magnitude of the displacement and in no case, it is less than the magnitude of the displacement, i.e., Distance $\geq$ | Displacement |

2. Displacement may be + ve, – ve or zero.

3. Distance, speed and time can never be negative.

4. At the same time particle cannot have two positions.

Some Impossible graphs :



Solved Examples

Ex.1 An old person moves on a semi circular track of radius 40 m during a morning walk. If he starts at one end of the track and reaches at the other end. Find the displacement of the person.

Sol. Displacement = 2R = 2 x 40 = 80 meter.

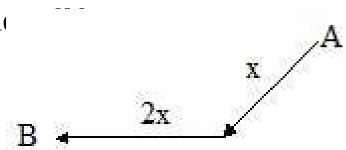
Ex.2 An athelete is running on a circular track of radius 50 meter. Calculate the displacement of the athlete after completing 5 rounds of the track.

Sol. Since final and initial positions are same .

Hence displacement of athlete will be $\Delta r = r - r = 0$

Ex.3 If a particle moves from point A to B then distance covered by particl

Sol. D = x + 2x = 3x



Ex.4 A monkey is moving on circular path of radius 80 m. Calcualte the distance covered by the monkey.

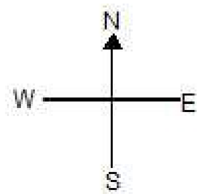
Sol. Distance = Circumference of the circle

$D = 2\pi R \quad \Rightarrow \quad D = 2\pi \times 80 \quad = 160 \times 3.14 \quad = 502.40\text{m}$

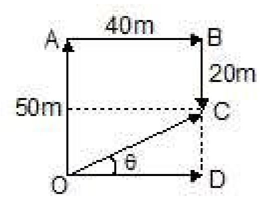
Ex.5 A man has to go 50 m due north, 40 m due east and 20 m due south to reach a field.

(a) What distance he has to walk to reach the field?

(b) What is his displacement from his house to the field?



Sol. Let origin be O then



(a) Distance covered

= OA + AB + BC

= 50 + 40 + 20

= 110 meter

(b) **First method :**

Displacement OC = $\sqrt{OD^2 + CO^2}$

= $\sqrt{40^2 + 30^2}$ = $10\sqrt{25} = 50$ meter

Second method :

Displacement $\vec{d} = 50\hat{j} + 40\hat{i} - 20\hat{j}$

= $30\hat{j} + 40\hat{i}$ $\quad |\vec{d}| = \sqrt{40^2 + 30^2}$ = 50 meter

Ex.6 A body covers $\frac{1}{4}$ th part of a circular path. Calulate the ratio of distance and displacement.

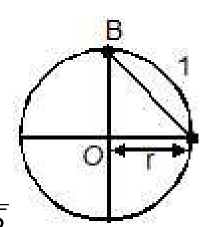
Sol. Distance = AB from path (1) = $\frac{2\pi r}{4} = \frac{\pi r}{2}$

Displacement = AB

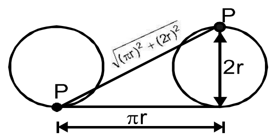
= $\sqrt{OA^2 + OB^2}$

= $\sqrt{r^2 + r^2} = r\sqrt{2}$

$\therefore \frac{\text{Distance}}{\text{Displacement}} = \frac{\pi r/2}{r\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$



Ex.7 A point P consider at contact point of a wheel on ground which rolls on ground without sliping then value of displacement of point P when wheel completes half of rotation - [If radius of wheel is 1 m]



Sol. Displacement

$= \sqrt{\pi^2 + 4}$ **Ans.**

SPEED

Speed of an object is defined as the time rate of change of position of the object in any direction. It is measured by the distance travelled by the object in unit time in any direction. i.e.,

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

(i) It is a scalar quantity.

(ii) It gives no idea about the direction of motion of the object.

(iii) It can be zero or positive but never negative.

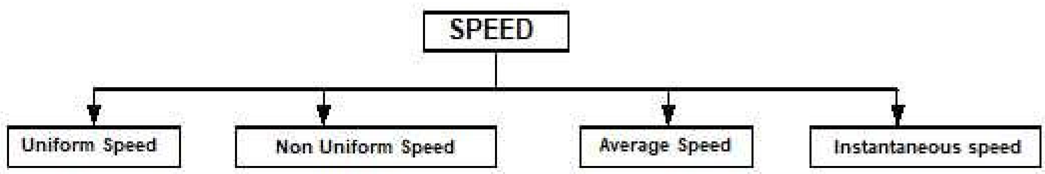
(iv) Unit: C.G.S. cm/sec, S.I. m/sec,

$$1\,\text{km/h} = \frac{100}{60 \times 60} = \frac{5}{18}\,\text{m/s}$$

$$1\,\text{km/h} = \frac{5}{18}\,\text{m/s}$$

(v) Dimension : $M^0 L^1 T^{-1}$

Types of speed :



(a) Uniform speed : An object is said to be moving with a uniform speed, if it covers equal distances in equal intervals of time, how soever small these intervals may be. The uiform speed is shown by straight line in distance time graph.

For example: suppose a train travels 1000 metre in 60 second. The train is said to be moving with uniform speed, if it travels 500 metre in 30 second, 250 metre in 15 second, 125 metre in 7.5 second and so on.

(b) Non Uniform Speed : An object is said to be moving with a variable speed if it covers equal distance in unequal intervals of time or unequal distances in equal intervals of time, howsoever small these intervals may be.

For example : suppose a train travels first 1000 metre in 60 second, next 1000 metre in 120 second and next 1000 metre in 50 second, then the train is moving with variable speed.

(c) Average Speed : When an object is moving with a variable speed, then the average speed of the object is that constant speed with which the object covers the same distance in a given time as it does while moving with variable speed during the given time. Average speed for the given motion is defined as the ratio of the total distance travelled by the object to the total time take i.e.,

$$\text{Average speed } \bar{V} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Note : If any car covers distance x_1, x_2, ... in the time intervals t_1, t_2, ... then.

$$\bar{V} = \frac{x_1 + x_2 + x_3 + ... + x_n}{t_1 + t_2 + ... + t_n}$$

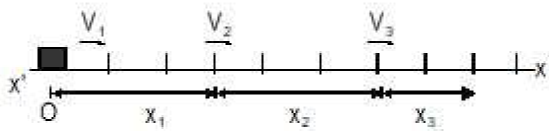
SOME IMPORTANT CASES RELATED TO AVERAGE SPEED :

Case : 1

If body covers distances x_1, x_2, and x_3 with speeds v_1, v_2, and v_3 respectively in same direction then average speed of body.

$$\Rightarrow \quad \bar{V} = \frac{x_1 + x_2 + x_3}{t_1 + t_2 + t_3}$$

here, $t_1 = \frac{x_1}{v_1}, t_2 = \frac{x_2}{v_2}, t_3 = \frac{x_3}{v_3}$ $\quad \bar{V} = \frac{x_1 + x_2 + x_3}{\frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{v_3}}$



If body covers equal distances with different speeds then, $x_1 = x_2 = x_3 = x$

$$\bar{V} = \frac{3x}{\frac{x}{v_1} + \frac{x}{v_2} + \frac{x}{v_3}} = \frac{3}{\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3}} = \frac{3v_1v_2v_3}{v_1v_2 + v_2v_3 + v_3v_1}$$

Case : 2

If any body travels with speeds v_1,v_2,v_3 during time intervals t_1,t_2,t_3 respectively then the average speed of the body wil be

Average speed

$$\bar{V} = \frac{x_1 + x_2 + x_3}{t_1 + t_2 + t_3} = \frac{v_1t_1 + v_2t_2 + v_3t_3}{t_1 + t_2 + t_3}$$

If $t_1 = t_2 = t_3 = t$

$$= \frac{(v_1 + v_2 + v_3) \times t}{3 \times t} = \frac{(v_1 + v_2 + v_3)}{3}$$

(d) Instantaneous speed :

The speed of the body at any instant of time or at a particular position is called instantaneous speed.

Let a body travel a distance Δx in the time interval Δt, then its average speed $= \frac{\Delta x}{\Delta t}$.

When $\Delta t \to 0$, then average speed of the body becomes the instantaneous speed.

$$\therefore \text{Instantaneous speed} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Note :

(a) Speedometer of the vehicle measures its instantaneous speed.

(b) In case of a uniform motion of an object, the instantaneous speed is equal to its uniform speed.

VELOCITY

It is defined as rate of change of displacement.

Characteristics of Velocity :

(i) It is a vector quantity.

(ii) Its direction is same as that of displacement.

(iii) Unit and dimension : Same as that of speed.

Types of Velcoity :

(a) Instantaneous Velocity

(b) Average Velocity

(c) Uniform Velocity

(d) Non-uniform Velocity

(a) Instantaneous Velocity : It is defined as the velocity at some particular instant.

$$\text{Instantaneous velocity} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

(b) Average Velocity :

$$\text{Average Velocity} = \frac{\text{Total Displacement}}{\text{Total time}}$$

(c) Uniform Velocity : A particle is said to have uniform velocity, if magnitudes as well as direction of its velocity remains same and this is possible only when the particles moves in same straight line reservering its direction.

(d) Non-uniform Velocity : A particle is said to have non-uniform velocity, if either of magnitude or direction of velocity changes (or both changes).

COMPARATIVE STUDY OF SPEED AND VELOCITY

S.No.	Speed	Velocity
1.	It is the time rate of change of distance of a body.	It is the time rate of charge of displacement of a particle
2.	It tells nothing about the direction of motion of the particle	It tells the direction of motion of the particle
3.	It can be positive or zero	It can be positive or negative or zero
4.	It is a scalar quantity	It is a vector quantity

Solved Examples

Ex.8 A man walks at a speed of 6 km/hr for 1 km and 8 km/hr for the next 1 km. What is his average speed for the walk of 2 km.

Sol. $\bar{V}=\frac{2v_1v_2}{v_1+v_2}=\frac{2\times6\times8}{6+8}=7\,km/h$

Ex.9 The distance travelled by a particle S = $10t^2$ (m). Find the value of instantaneous speed at t = 2 sec.

Sol. $V=\frac{dx}{dt}=\frac{d}{dt}\left(10t^2\right)=10(2t)=20t$

Put $t=2\,sec.\;V=20\times2=40\,m/s$

Ex.10 A car travels a distance A to B at a speed of 40 km/h and returns to A at a speed of 30 km/h.

(i) What is the average speed for the whole journey?

(ii) What is the average velocity ?

Sol. (i) Let AB = s, time taken to go from A to B, t_1 $=\frac{s}{40}h$ and time taken to go from B to A, $t_2=\frac{s}{30}h$

$\therefore$ total time taken = $t_1 + t_2$

$=\frac{s}{40}+\frac{s}{30}=\frac{(3+4)s}{120}=\frac{7s}{120}h$

Total distance travelled = s + s = 2s

$\therefore$ Average speed

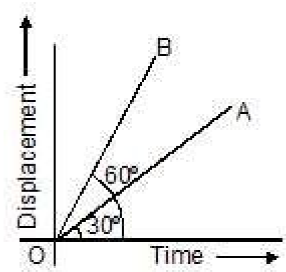
$V=\frac{dx}{dt}=\frac{d}{dt}\left(10t^2\right)=10(2t)=20t$

Put $t=2\,sec.\;V=20\times2=40\,m/s$

(ii) Total displacement = zero, since the car returns to the original position.

Therefore, average velocity $=\frac{\text{total displacement}}{\text{time taken}}=\frac{0}{2t}=0$

Ex.11 From the adjoining position time graph for two particles A and B the ratio of velocities $v_A : v_B$ will be



[1] 1 : 2 [2] $1:\sqrt{3}$ [3] $\sqrt{3}:1$ [4] $1:3$

Sol. $\frac{V_A}{V_B}=\frac{\tan\theta_a}{\tan\theta_b}=\frac{\tan30^0}{\tan60^0}=\frac{1/\sqrt{3}}{\sqrt{3}}=\frac{1}{3}$ **Ans. [4]**

Ex.12 The position of a particle moving on x-axis is given by x = $At^3 + Bt^2 + Ct + D$. The numerical value of A, B, C, D are 1, 4, –2 and 5 respectivley and S.I. units are used. Find velocity of the particle at t = 4 sec.

Sol. $V=\frac{dx}{dt}=\frac{d}{dt}$ $[At^3 + Bt^2 + Ct + D]$

or $V = 3At^2 + 2Bt + C$ at time t = 4 sec.

$V = 3A(4)^2 + 2B(4) + C$ $\quad V = 48A + 8B + C$

Considering A = 1, B = 4, C = –2

V = 48 (0) + 8 (4) + (–2) = 78 m/s

Ex.13 In a car race, car A takes a time of t sec. less than car B at the finish and passes the finishing point with a velocity υ m/s more than the car B. Assuming that the cars start from rest and travel with constant accelerations a_1 and a_2 respectively, show that v = $\left[\sqrt{a_1 a_2}\right]t$

Sol. Let the time taken by two cars to complete the journey be t_1 and t_2 and their velocities at the finish be υ_1 and υ_2 respectively.

Given that, $t_1 = t_2 - t$ and $\upsilon_1 = \upsilon_2 + \upsilon$ .... (1)

Now $S_1 = S = \frac{1}{2} a_1 t_1^2$ and $S_2 = S = \frac{1}{2} a_2 t_2^2$ ...... (2)

(At start, $v_1 = v_2 = 0$) .... (3)

Hence $a_1 t_1^2 = a_2 t_2^2 = 2s$

Also, $v_1 = a_1 t_1$ and $v_2 = a_2 t_2$

or $v_1 t_1 = a_1 t_1^2 = 2s$ and $v_2 t_2 = a_2 t_2^2 = 2s$

$\therefore$ $t_1 = \frac{2s}{v_1}$ and $t_2 = \frac{2s}{v_2}$

So, $t_2 - t_1 = 2s\left[\frac{1}{v_2} - \frac{1}{v_1}\right]$ .... (4)

From equations (1) and (4) we have

$2s\left[\frac{1}{v_2} - \frac{1}{v_1}\right] = t$ or $2s\left[\frac{v_1 - v_2}{v_1 v_2}\right] = t$

or $2s\left[\frac{v}{v_1 v_2}\right] = t$ or $v = \left[\frac{v_1 v_2}{2s}\right] t = \sqrt{\left\{\frac{v_1^2 v_2^2}{(2s)^2}\right\}} \times t$

$\sqrt{\left(\frac{v_1 v_2}{t_1 t_2}\right)} \times t = \sqrt{(a_1 a_2)} \times t$

ACCELERATION

It is defined as the rate of change of velocity.

(i) It is a vector quantity.

(ii) Its direction is same as that of change in velocity and not of the velcoity (That is why, acceleration in circular motion is towards the centre).

(iii) There are three ways possible in which change in velocity may occur.

When only direction	When only magnitude changes	When both the direction changes and magnitude change
To change the direction, net acceleration or net force should be perpendicular to direction of velocity. **Ex:** Uniform circular motion.	In this case, net force or net acceleration should be parallel or antiparallel to the direction of velocity. (straight line motion) **Ex:** When ball is thrown up under gravity.	In this case, net force or net acceleration has two components. One component is parallel or antiparallel to velocity and another one is perpendicular to velocity. **Ex:** Projectile motion

Types of acceleration -

(a) Instantaneous acceleration : It is defined as the acceleration of a body at some particular instant.

Instantaneous acceleration $= \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

(b) Average acceleration $= \vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$

(c) Uniform acceleration : A body is said to have uniform acceleration if magnitude and direction of the acceleration remains constant during particle motion.

Note : If a particle is moving with uniform acceleration, this does not necessarily imply that particle is moving in straight line.

Example : Parabolic motion

(d) Non-uniform acceleration : A body is said to have non-uniform acceleration, if magnitude or direction or both, change during motion.

Note :

(i) Acceleration is a vector with dimensions $[LT^{-2}]$ and SI units (m/s^2)

(ii) If acceleration is zero, velocity will be constant and motion will be uniform.

(iii) However if acceleration is constant acceleration is uniform but motion is non-uniform and if acceleration is not constt. both motion and acceleration are non-uniform.

(iv) If a force $\vec{F}$ acts on a particle of mass m then by Newton's law $\vec{a} = \vec{F}/m$

(v) As by definition, $\vec{v} = \frac{d\vec{s}}{dt}$ so,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{s}}{dt}\right) = \frac{d^2\vec{s}}{dt^2}$$

i.e. if $\vec{s}$ is given as a function of time, second time derivative of displacement gives acceleration.

(vi) If velocity is given as function of position then by the chain rule $a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$

$$\Rightarrow \quad a = v\frac{dv}{dx} \quad [\text{as } \frac{dx}{dt} = v]$$

(vii) As acceleration $\vec{a} = \frac{d\vec{v}}{dt}$, the slope of velocity time graph gives acceleration i.e. $\vec{a} = \frac{d\vec{v}}{dt} = \tan\theta$

(viii) The slope of $\vec{a}-t$ curve, i.e. $d\vec{a}/dt$ is a measure of rate of non-uniformity of acceleration. However we do not define this physical quantitiy as it in not involved in basic laws or equation of motion

(ix) Acceleration can be positive or negative. Positive acceleration means velocity is increasing with time while negative acceleration called retardation means velocity is decreasing with time.

Special note - I

If the motion of a particle is accelerated translatory (without change in direction) $\vec{v} = |\vec{v}|\hat{n}$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{d}{dt}[|\vec{v}|\hat{n}] = \hat{n}\frac{d|\vec{v}|}{dt} \quad [\text{as } \vec{n} \text{ is constt.}]$$

$$\Rightarrow \left|\frac{d\vec{v}}{dt}\right| = \frac{d\vec{v}}{dt} (\neq 0)$$

Howeverm, if motion is uniform translatory, both these will still be equal but zero.

Special note - II

(i) $\frac{d|\vec{v}|}{dt} = 0$ while $\left|\frac{d\vec{v}}{dt}\right| \neq 0$ (it is possible.)

(ii) $\frac{d|\vec{v}|}{dt} \neq 0$ while $\left|\frac{d\vec{v}}{dt}\right| = 0$ (it is not possible.)

EFFECTIVE USE OF MATHEMATICAL TOOLS IN SOLVING PROBLEMS OF ONE-DIMENSIONAL MOTION

If displacement-time equation is given, we can get velocity-time equation with the help of differentiation. Again, we can get acceleration-time equation with the help of differentiation.

If acceleration-time equation is given, we can get velocity-time equation by integration. From velocity equation, we can get displacement-time equation by integration.

$$\begin{tikzcd} \boxed{\text{Displacement}} \arrow[r, shift left, "\text{Differentiation}"] & \boxed{\text{Velocity}} \arrow[l, shift left, "\text{Integration}"] \arrow[r, shift left, "\text{Differentiation}"] & \boxed{\text{Acceleration}} \arrow[l, shift left, "\text{Integration}"] \end{tikzcd}$$

Solved Examples

Ex.14 The displacement of a particle is given by $y = a + bt + ct^2 + dt^4$. Find the acceleration of a particle.

Sol. $v = \frac{dy}{dt} = \frac{d}{dt}\left(a + bt + ct^2 + dt^4\right) = b + 2ct + 4dt^3$

$a = \frac{dv}{dt} = 2c + 12dt^2$

Ex.15 If the displacement of a particle is $(2t^2 + t + 5)$ meter then, what will be acceleration at t = 5 sec.

Sol. $v = \frac{dx}{dt} = \frac{d}{dt}\left(2t^2 + t + 5\right) = 4t + 1\ m/s$ and

$a = \frac{dv}{dt} = \frac{d}{dt}(4t+1)$ $a = 4\ m/s^2$

Ex.16 The velocity of a particle moving in the x direction varies as $V = \alpha\sqrt{x}$ where α is a constant. Assuming that at the moment t = 0 the particle was located at the point x = 0. Find the acceleration.

Sol. $a = \frac{dv}{dt} = \alpha\frac{d}{dt}\sqrt{x} = \alpha.\frac{1}{2}x^{-1/2}.\frac{dx}{dt}$

$= \alpha.\frac{1}{2\sqrt{x}}.\alpha\sqrt{x} \Rightarrow a = \frac{\alpha^2}{2}$

Ex.17 The velocity of any particle is related with its displacement As; $x = \sqrt{v+1}$, Calculate acceleration at x = 5 m.

Sol. $x = \sqrt{v+1}$ $\quad x^2 = v + 1$ $\quad v = \left(x^2 - 1\right)$

Therefore

$a = \frac{dv}{dt} = \frac{d}{dt}\left(x^2 - 1\right) = 2x\frac{dx}{dt} = 2x\ v = 2x\left(x^2 - 1\right)$

at x = 5 m $\quad a = 2 \times 5(25 - 1) = 240\,m/s^2$

Ex.18 The velocity of a particle moving in the positive direction of x-axis varies as $v = \alpha\sqrt{x}$ where α is positive constant. Assuming that at the moment t = 0, the particle was located at x = 0 find, (i) the time dependance of the velocity and the acceleration of the particle and (ii) the mean velocity of the particle averaged over the time that the particle takes to cover first s metres of the path.

Sol. (i) Given that $v = \alpha\sqrt{x}$ or $\frac{dx}{dt} = \alpha\sqrt{x}$

$\therefore \quad \frac{dx}{\sqrt{x}} = \alpha\,dt$ or $\int_0^x \frac{dx}{\sqrt{x}} = \int_0^t \alpha\,dt$

Hence $\quad 2\sqrt{x} = \alpha\,t$ or $x = \left(\alpha^2 t^2/4\right)$

Velocity $\quad \frac{dx}{dt} = \frac{1}{2}\alpha^2 t$ and

Acceleration $\quad \frac{d^2x}{dt^2} = \frac{1}{2}\alpha^2$

(ii) Time taken to cover first s metres

$s = \frac{\alpha^2 t^2}{4}$ or $t^2 = \frac{4s}{\alpha^2}$ or $t = \frac{2\sqrt{s}}{\alpha}$

$\bar{v} = \frac{\text{total distance}}{\text{total time}} = \frac{s\alpha}{2\sqrt{s}}$ or $\bar{v} = \frac{1}{2}\sqrt{s}\,\alpha$

Ex.19 A particle moves in the plane xy with constant acceleration a directed along the negative y-axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the partcle at the origin of coordinates.

Sol. Given that $y = px - qx^2$ and

$$\frac{d^2y}{dt^2} = p\frac{d^2x}{dt^2} - 2qx\frac{d^2x}{dt^2} - 2q\left(\frac{dx}{dt}\right)^2$$

or $\quad -a = -2q\left(\frac{dx}{dt}\right)^2 = -2q\upsilon_x^2$

$\because \quad \frac{d^2x}{dt^2} = 0$

(no acceleration along x-axis) and $\frac{d^2y}{dt^2} = -a$

$\therefore \quad \upsilon_x^2 = \frac{a}{2q}$ or $\upsilon_x = \sqrt{\frac{a}{2q}}$

Further, $\left(\frac{dy}{dt}\right)_{x=0} = p\frac{dx}{dt}$ or $\upsilon_y = p\,\upsilon_x$

$\therefore \quad \upsilon_y = p\sqrt{\left(\frac{a}{2q}\right)}$

Now $\quad \upsilon = \sqrt{\left(\upsilon_x^2 + \upsilon_y^2\right)} = \sqrt{\left(\frac{a}{2q} + \frac{ap^2}{2q}\right)}$

or $\upsilon = \sqrt{\left[\frac{a\left(p^2+1\right)}{2q}\right]}$ **Ans.**

Ex.20 A particle is moving in a plane with velocity given by $u = u_0 i + (a\omega\cos\omega t) j$, where i and j are unit vectors along x and y axes respectively. If particle is at the origin at t = 0.

(a) Calculate the trajectory of the particle.

(b) Find the distance from the origin at time $3\pi/2\omega$.

Sol. (a) Given that $u = u_0 i + (a\omega\cos\omega t) j$,

Hence, velocity along x-axis $u_x = u_0$ .... (1)

Velocity along y-axis $u_y = a\omega\cos\omega t$ .... (2)

We know that $\upsilon = \dfrac{ds}{dt}$ or $s = \int \upsilon.dt$

So from equations (1) and (2), we have

Displacement at time t in horizontal direction

$$x = \int u_0\, dt = u_0 .t \qquad .... (3)$$

Displacement in vertical direction

$$y = \int a\omega\cos\omega t\, dt = a\sin\omega t \qquad .... (4)$$

Eliminating t from equations (3) and (4) we get

$$y = a\sin(\omega x/u_0) \qquad .... (5)$$

Equation (5) gives the trajectory of the particle.

(b) At time, $t = 3\pi/2\omega$

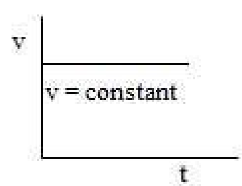
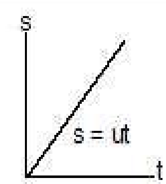
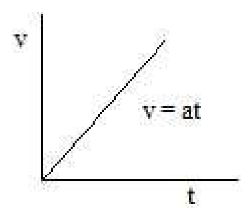
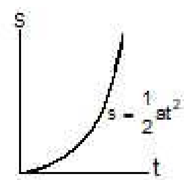
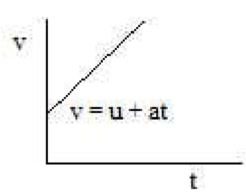
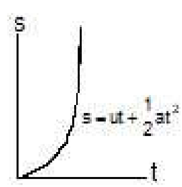
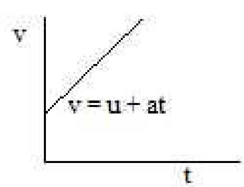
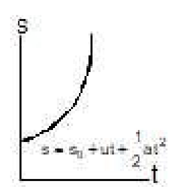
$x = u_0(3\pi/2\omega)$ and $y = a\sin 3\pi/2 = -a$

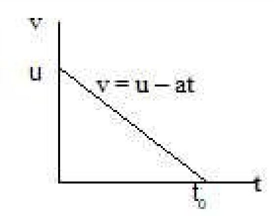
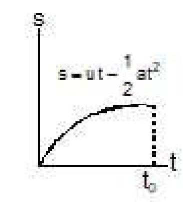
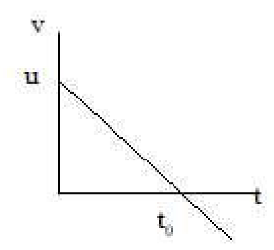
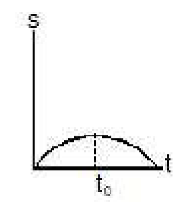
$\therefore$ Distance of the particle from the origin

$$= \sqrt{(x^2+y^2)} = \sqrt{\left[\left(\frac{3\pi\, u_0}{2\omega}\right)^2 + a^2\right]} \quad \textbf{Ans.}$$

SOME IMPORTANT GRAPHS RELATED TO MOTION

All the following graphs are drawn for one-dimensional motion with uniform velocity or with constant acceleration.

Different case	v-t graph	s-t graph	Important Point
1. Uniform motion			(i) Slope of s-t graph = v = constant (ii) In s-t graph s = 0 at t = 0
2. Uniformly accelerated motion with u = 0 and s = 0 at t = 0			(i) u = 0, ie., v = 0 at t = 0 (ii) u = 0, i.e, slope of s-t graph at t = 0, should be zero (iii) a or slope of v-t graph is constant.
3. Uniformly accelerated motion with $u \neq 0$ but s = 0 at t = 0			(i) $u \neq 0$, i.e., v or slope of s-t graph at t = 0 is not zero. (ii) v or slope of s-t graph gradually goes on increasing.
4. Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at t = 0			(i) $s = s_0$ at t = 0

5. Uniformly retarded motion till velocity becomes zero.			(i) slope of s-t graph at $t = 0$ gives u (ii) slope of s-t graph at $t = t_0$ becomes zero. (iii) In this case u can't be zero.
6. Uniformly retarded then accelerated in opposite direction			(i) At time $t = t_0$, $v = 0$ or slope of s-t graph is zero. (ii) In s-t graph slope or velocity first decreases then increases with opposite sign.

MOTION WITH UNIFORM ACCELERATION

EQUATIONS OF MOTION :

Motion **under uniform** acceleration is described by the following equations.

$v = u + at$ $\quad s = ut + \frac{1}{2}$ $\quad at^2v^2 = u^2 + 2as$

1. The velocity time relation

$a = \frac{dv}{dt}$ $\qquad dv = adt$

If the velocity of particle at time t_1 is v_1 and at time t_2 is v_2

$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$ $\qquad [v]_{v_1}^{v_2} = a[t]_{t_1}^{t_2}$

$v_2 - v_1 = a[t_2 - t_1]$ $\qquad v_2 = v_1 + a[t_2 - t_1]$

If $t_1 = 0, v_1 = u$ and $t_2 = t, v_2 = v$ $\quad v = u + a\,t$

2. Position-time relation

By $v = \frac{dx}{dt}$ $\quad$ or $\quad dx = vdt$

$\Rightarrow \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt \Rightarrow [x]_{x_1}^{x_2} = \left[ut + \frac{1}{2}at^2\right]_0^t$

$x_2 - x_1 = ut + \frac{1}{2}at^2$ $\qquad s = ut + \frac{1}{2}at^2$

3. Velocity-position relation

$a = \frac{dv}{dt} = \frac{dv}{dt}\cdot\frac{dx}{dx} = v\frac{dv}{dx}$ $\quad$ or $\quad a\,dx = v\,dv$

$\int_{x_1}^{x_2} a\,dx = \int_{v_1}^{v_2} v\,dv$ $\qquad a[x]_{x_1}^{x_2} = \left[\frac{v^2}{2}\right]_{v_1}^{v_2}$

$2a(x_2 - x_1) = v_2^2 - v_1^2$ $\qquad v^2 - u^2 = 2as$

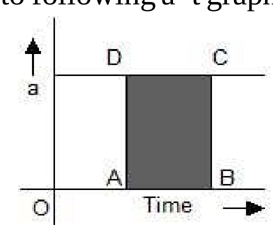
$v^2 = u^2 + 2as$

4. Distance travelled in n^{th} second of uniformly accelerated motion :

$S_{n^{th}} = u + \frac{a}{2}(2n - 1)$

The calculation of speed and distance by acceleration-time graph:

Let a particle be moving with uniform acceleration according to following a–t graph –



$dv = a\,dt$ or $\int_u^v dv = \int_{t_1}^{t_2} a\,dt$

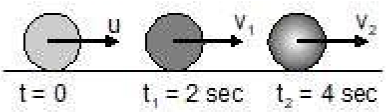
$(v)_u^v = a(t)_{t_1}^{t_2}$ $\qquad v - u = a(t_2 - t_1)$

Therefore difference in magnitude of velocity $(v - u) = AB \times AD$

$v - u$ = Area of rectangle ABCD = area under $a - t$ graph

Solved Examples

Ex.21 The velocity acquired by a body moving with uniform acceleration is 20 m/s in first 2 sec and 40 m/s in first 4 sec. Calculate initial velocity.



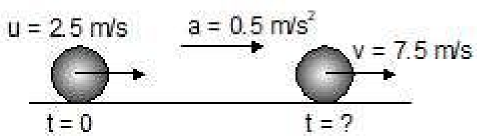
Sol. $a = \frac{v_2 - v_1}{t_2 - t_1}$ $\quad a = \frac{40-20}{4-2} = \frac{20}{2} = 10\,m/s^2$

Now, $v = u + at \quad v_1 = u + at_1$

$\Rightarrow \quad 20 = u + 10 \times 2$

$\Rightarrow \quad 20 = u + 20 \quad \Rightarrow \quad u = 0\,m/s$ **Ans**

Ex.22 A particle starts with initial velocity 2.5 m/s along the x direction and accelerates uniformly at the rate 50 cm/s^2. Find time taken to increase the velocity to 7.5 m/s.



Sol. $v = u + at \qquad 7.5 = 2.5 + 0.5\,t$

$5.0 = 0.5\,t \qquad t = \frac{50}{5} = 10\,sec$

Ex.23 A particle starts with a constant acceleration. At a time t second speed is found to be 100 m/s and one second later speed becomes 150 m/s. Find acceleration of the particle.

Sol. From equation (1) of motion v = u + at

$\Rightarrow$ 100 = 0 + at $\qquad$ 100 = at $\qquad$ ... (1)

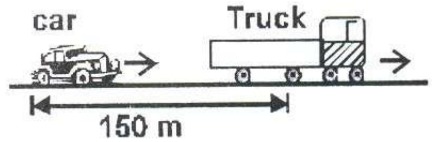
Now consider velocity one second later -

$v' = 0 + a(t+1) \quad \Rightarrow \quad 150 = a(t+1)$ ... (2)

On subtracting equation (1) from equation (2)

$a = 50\,m/s^2$

Ex.24 A truck starts from rest with an acceleration of 1.5 ms^{-2} while a car 150 metre behind starts from rest with an acceleration of 2 ms^{-2}. (a) How long will it take before both the truck and car are side by side and (b) How much distance is travelled by each.



Sol. (a) $s_T = \frac{1}{2}at^2 \qquad s_T = \frac{1}{2}(1.5)t^2$ .... (1)

Distance covered by car when car one overtakes the truck

$s_c = \frac{1}{2}(2)t^2 \qquad (s_T + 150) = \frac{1}{2}(2)t^2$ .... (2)

Divide equation (2) by equation (1) $\frac{s_T + 150}{s_T} = \frac{2}{1.5}$

$\Rightarrow \quad 1 + \frac{150}{s_T} = \frac{20}{15} = \frac{4}{3}$

$\Rightarrow \quad \frac{150}{s_T} = \frac{4}{3} - 1 = \frac{1}{3} \quad$ or $\quad s_T = 450$

Distance travelled by car = 450 + 150 = 600 metre

(b) Now by equation (1) $s_T = \frac{1}{2}at^2$

$450 = \frac{1}{2} \times 1.5 \times t^2$

$t^2 = \frac{450 \times 2}{1.5} \quad \Rightarrow \quad t = \sqrt{300 \times 2} = 24.5\,sec$

Therefore car will overtake the truck after 24.5 second.

Ex.25 A body travels a distance of 2 m in 2 seconds and 2.2 m in next 4 seconds. What will be the velocity of the body at the end of 7th second from the start.

Sol. Here, case (i) s = 2m, t = 2s

case (ii) t = 2 + 4 = 6s

Let u and a be the initial velocity and uniform acceleration of the body.

we know that, $\quad s = ut + \frac{1}{2}at^2$

Case (i) $2 = u \times 2 + \frac{1}{2}a \times 2^2$

or 1 = u + a $\qquad$ .... (1)

Case (ii) $4.2 = u \times 6 + \frac{1}{2}a \times 6^2$

or $0.7 = u + 3a$ ... (2)

Subtracting (2) from (1), we get

$0.3 = 0 - 2a = -2a$

or $a = -0.3/2 = -0.15\,ms^{-2}$

From (i), $u = 1 - a = 1 + 0.15$ $\quad u = 1.15\,ms^{-1}$

For the velocity of body at the end of 7th second, we have

$u = 1.15\,ms^{-1}; a = -0.15\,ms^{-2}$,

v = ?, t = 7s $\qquad$ As, v = u + at

$\therefore$ v = 1.15 + (–0.15) x 7 $\qquad$ v = 0.1 m/s

Ex.26 A body travels a distance of 20 m in the 7th second and 24 m in 9th second. How much distance shall it travel in the 15th second?

Sol. Here, s_7 = 20 m ; s_9 = 24 m, s_{15} = ?

Let u and a be the initial velocity and uniform acceleration of the body.

We know that, $s_n = u + \frac{a}{2}(2n - 1)$

$\therefore$ $s_7 = u + \frac{a}{2}(2 \times 7 - 1)$ or $20 = u + \frac{13a}{2}$ .... (i)

and $s_9 + u + \frac{a}{2}(2 \times 9 - 1)$ or $24 = u + \frac{17}{2}a$ ... (ii)

Subtracting (ii) form (i), we get $\quad 4 = 2a$

or a = 2 ms^{-2}

Putting this value in (i), we get

$20 = u + \frac{13}{2} \times 2$ or 20 = u + 13

or u = 20 – 13 = 7 ms^{-1}

Hence, $s_{15} = u + \frac{a}{2}$ (2 × 15 – 1) = 7 + $\frac{2}{2}$ × 29

s_{15} = 36 m $\qquad$ **Ans.**

Ex.27 A person travelling at 43.2 km/h applies the brakes giving a deceleration of 6 m/s^2 to his scooter. How far will it travel before stopping?

Sol. Here, $u = 43.2\,km/h = 43.2 \times \frac{5}{18}\,m/s$

Deceleration; a = 6 m/s^2 v = 0 $\quad$ s = ?

Using $v^2 = u^2 - 2as$ $\quad 0 = (12)^2 - 2 \times 6\,s$

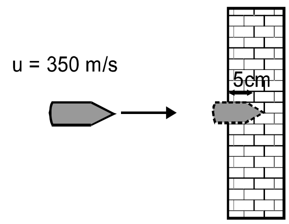
144 = 2 x 6s

$s = \frac{144}{12} = 12\,m$ $\qquad$ **Ans.**

Ex.28 A bullet going with speed 350 m/s enters in a concrete wall and penetrates a distance of 5 cm before coming to rest. Find deceleration.

Sol. Here, u = 350 m/s, $\quad$ s = 5 cm,

v = 0 m/s $\quad$ and $\quad$ a = ?



By using $v^2 = u^2 - 2as$ $\quad 0 = u^2 - 2as$

$u^2 = 2as$ or $a = \frac{u^2}{2s}$

$a = \frac{350 \times 350}{2 \times 0.05} = 12.25 \times 10^5$ m/sec^2

Ex.29 A driver takes 0.20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving a car at a speed of 54 km/h and the brakes cause a deceleration of 6.0 m/s^2, find the distance travelled by the car after he seeds the need to put the brakes on

Sol. Distance covered by the car during the application of brakes by driver -

u = 54 km/h = $54 \times \frac{5}{18}\,m/s$ = 15 m/s

s_1 = ut $\quad$ or $\quad$ s_1 = 15 x 0.2 = 3.0 meter

After applying the brakes;

v = 0 $\quad$ u = 15 m/s, $\quad$ a = 6 m/s^2 $\quad$ s_2 = ?

Using $\quad v^2 = u^2 - 2as$

$0 = (15)^2 - 2 \times 6 \times s_2$

12 s_2 = –225

$\Rightarrow s_2 = \frac{225}{12} = 18.75\,metre$

Distance travelled by the car after driver sees the need for it

$s = s_1 + s_2$

s = 3 + 18.75 = 21.75 metre. $\quad$ **Ans.**

MOTION UNDER GRAVITY

The most important example of motion in a straight line with constant acceleration is motion under gravity. In case of motion under gravity.

(1) The acceleration is constant, i.e., $a = g = 9.8$ m/s^2 and directed vertically downwards.

(2) The motion is in vacuum, i.e., viscous force or thrust of the medium has no effect on the motion.

[1] Body falling freely under gravity :

Taking initial position as origin and downward direction of motion as positive, we have

$u = 0$ [as body starts from rest]

$a = +g$

[as acceleration is in the direction of motion]

So, if the body acquires velocity v after falling a distance h in time t, equations of motion, viz.

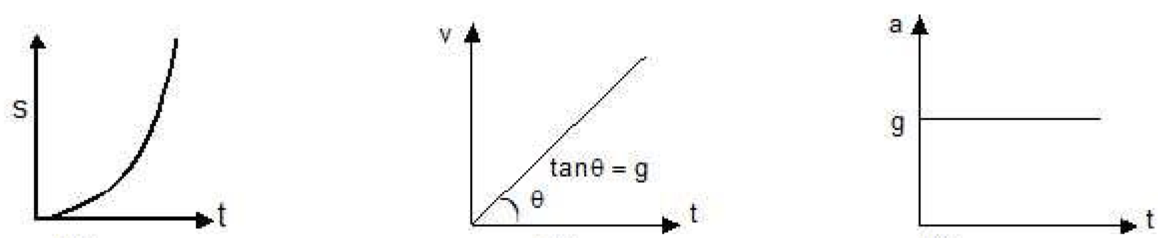
$v = u + at$; $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$

reduces to $v = gt$ ... (1) $h = \frac{1}{2}gt^2$ ..... (2)

and $v^2 = 2gh$ ..... (3)

These equations can be used to solve most of the problems of freely falling bodies as if.

t is given	h is given	v is given
From eqns. (1) and (2)	From eqns. (2) and (3)	From Eqns. (3) and (1)
$v = gt$	$t = \sqrt{\frac{2h}{g}}$	$t = \frac{v}{g}$
and $h = \frac{1}{2}gt^2$	$v = \sqrt{2gh}$	$h = \frac{v^2}{2g}$



(A) (B) (C)

(i) If the body is dropped from a height H, as in time t is has fallen a distance h from its initial position, the height of the body from the ground will be

$h' = H - h$ with $h = \left(\frac{1}{2}\right)gt^2$

(ii) As $h = \left(\frac{1}{2}\right)gt^2$, i.e., $h \propto t^2$, distance fallen in time, t,2t,3t etc., will be in the ratio of $1^2 : 2^2 : 3^2$, i.e., square of integers.

(iii) The distance fallen in the n^{th} sec,

$$h_{(n)} - h_{(n-1)} = \frac{1}{2}g(n)^2 - \frac{1}{2}g(n-1)^2 \quad = \frac{1}{2}g(2n-1)$$

So distances fallen in 1st, 2nd, 3rd sec etc. will be in the ratio of 1 : 3 : 5 i.e., odd integers only.

[2] Body projected vertically up :

Taking initial position as origin and direction of motion (i.e., vertically up) as positive.

here we have $v = 0$

[at highest point velocity = 0]

$a = -g$ [as acceleration is downwards while motion upwards]

If the body is projected with velocity u and reaches the highest point at a distance h above the ground in time t, the equations of motion viz.,

$v = u + at$; $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$

reduces to $0 = u - gt$ $h = ut - \frac{1}{2}gt^2$

and $0 = u^2 - 2gh$

Substituting the value of u from first equation in second and rearranging these,

$u = gt$ ... (1)

$h = \frac{1}{2}gt^2$ ... (2)

and $u^2 = 2gh$ ... (3)

These equations can be used to solve most of the problems of bodies projected vertically up as if.

t is given

From eqns. (1) and (2)

$u = gt$

and $h = \frac{1}{2}gt^2$

h is given

From eqns. (2) and (3)

$t = \sqrt{\frac{2h}{g}}$

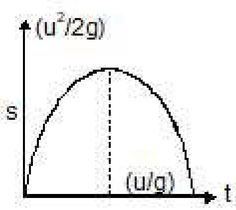
$v = \sqrt{2gh}$

u is given

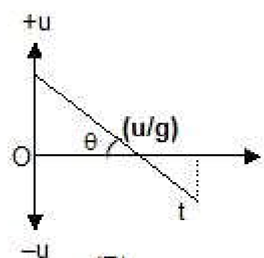
From Eqns. (3) and (1)

$t = \frac{u}{g}$

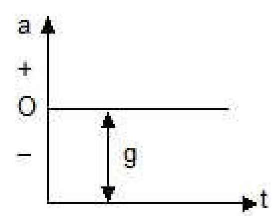
$h = \frac{u^2}{2g}$



(A)



(B)



(C)

IMPORTANT POINTS

(1) In case of motion under gravity for a given body, mass, acceleration, and mechanical energy remain constant while speed, velocity, momentum, kinetic energy and potential energy change.

(2) The motion is independent of the mass of the body, as in any equation of motion, mass is not involved. This is why a heavy and lighter body when released from the same height, reach the ground simultaneously and with same velocity.

i.e $t = \sqrt{(2h/g)}$ and $v = \sqrt{2gh}$

However, momentum, kinetic energy or potential energy depend on the mass of the body (all $\propto$ mass)

(3) As from equation (2) time taken to reach a height h, $t_u = \sqrt{(2h/g)}$

Similarly, time taken to fall down through a distance h, $t_D = \sqrt{(2h/g)}$ so $t_u = t_D = \sqrt{(2h/g)}$

So in case of motion under gravity time taken to go up a height h is equal to the time taken to fall down through the same height h.

(4) If a body is projectd vertically up and it reaches a height h, then $u = \sqrt{(2gh)}$

and if a body falls freely through a height h, then

$v = \sqrt{(2gh)} = u$

So in case of motion under gravity, the speed with which a body is projected up is equal to the speed with which it comes back to the point of projection.

Solved Examples

Ex.30 A juggler throws balls into air. He throws one whenever the previous one is at its highest point. How high do the balls rise if he throws n balls each sec. Acceleration due to gravity is g.

Sol. Since the juggler is throwing n balls each second and he throws second ball when the first ball is at the highest point, so time taken by each ball to reach the highest point is t = 1/n

Taking vertical upward motion of ball up to the highest point, we have

$u = 0,\ a = -g,\ t = 1/n,\ u = ?$

As $v = u + at$ so $0 = 0\,u + (-g)\ 1/n$

or $u = g/n$ Also $v^2 = u^2 + 2as$,

so $0 = u^2 - 2gh$

i.e., $h = (u^2/2g) = g/(2n^2)$

Ex.31 A ball is projected vertically up with an initial speed of 20 m/s on a planet where acceleration due to gravity is 10 m/s^2.

(a) How long does it takes to reach the highest point?

(b) How high does it rise above the point of projection?

(c) How long will it take for the ball to reach a point 10 m above the point of projection?

Sol. As here motion is vertically upwards,

$a = -g$ and $v = 0$

(a) From 1st equation of motion, i.e., $v = u + at$,

$0 = 20 - 10t$

i.e. $t = 2$ sec. **Ans.**

(b) Using $v^2 = u^2 + 2as$

$0 = (20)^2 - 2 \times 10 \times h$

i.e $h = 20$ m. **Ans.**

(c) Using $s = ut + \frac{1}{2}at^2$, $10 = 20t\ (-\frac{1}{2}) \times 10 \times t^2$

i.e. $t^2 - 4t + 2 = 0$ or $t = 2 \pm \sqrt{2}$,

i.e. t = 0.59 sec. or 3.41 sec.

i.e., there are two times, at which the ball passes through h = 10 m, once while going up and then coming down.

Ex.32 A ball is thrown vertically upwards from a bridge with an initial velocity of 4.9 m/s. It strikes the water after 2s. If acceleration due to gravity is 9.8 m/s^2 (a) What is the height of the bridge? (b) With which velocity does the ball strike the water?

Sol. Taking the point of projection as origin and downward direction as positive,

(a) Using $s = ut + \left(\frac{1}{2}\right)at^2$ we have

$$h = -4.9 \times 2 + \left(\frac{1}{2}\right)9.8 \times 2^2 = 9.8\,m$$

(u is taken to be negative as it is upwards)

(b) Using $v = u + at$

$v = -4.9 + 9.8 \times 2 = 14.7$ m/s

Ex.33 A rocket is fired vertically up from the ground with a resultant vertical acceleration of 10 m/s^2. The fuel is finished in 1minute and it continues to move up.

(a) What is the maximum height reached?

(b) After how much time from then will the maximum height be reached? (Take g = 10 m/s^2)

Sol. (a) The distance travelled by the rocket during burning interval (1 minute = 60 s) in which resultant acceleration is vertically upwards and 10 m/s^2 will be

$$h_1 = 0 \times 60 + (1/2) \times 10 \times 60^2 = 18000\,m \quad \text{... (1)}$$

Velocity acquired by it is

$v = 0 + 10 \times 60 = 600$ m/s ... (2)

After one minute the rocket moves vertically up with initial velocity of 600 m/s and continues till height h_2 till its velocity becomes zero.

$$0 = (600)^2 - 2gh_2$$

or $h_2 = 18000\,m$ [as g = 10 m/s^2] ... (3)

From equations (1) and (3) the maximum height reached by the rocket from the ground is

$$H = h_1 + h_2 = 18 + 18 = 36\,km$$

(b) The time to reach maximum height after burning of fuel is

$0 = 600 - gt$ $t = 60$ s

After finishing fuel the rocket goes up for 60 s.

Ex.34 A body is released from a height and falls freely towards the earth. Exactly 1 sec later another body is released. What is the distance between the two bodies after 2 sec the release of the second body, if g = 9.8 m/s^2.

Sol. The 2nd body falls for 2s, so

$$h_2 = \frac{1}{2}g(2)^2 \quad ... \quad (1)$$

While 1st has fallen for 2 + 1 = 3 sec so

$$h_1 = \frac{1}{2}g(3)^2 \quad ... \quad (2)$$

$\therefore$ Separation between two bodies after 2 sec the release of 2nd body, $d = h_1 - h_2$

$$= \frac{1}{2}g(3^2 - 2^2) \quad = 4.9 \times 5 = 24.5\,m$$

Ex.35 If a body travels half its total path in the last second of its fall from rest, find : (a) The time and (b) height of its fall. Explain the physically unacceptable solution of the quadratic time equation. (g = 9.8 m/s^2)

Sol. If the body falls a height h in time t, then

$$h = \frac{1}{2}gt^2$$

[u = 0 as the body starts from rest] ... (1)

Now, as the distance covered in (t – 1) second is

$$h' = \frac{1}{2}g(t-1)^2 \quad ... \quad (2)$$

So from Equations (1) and (2) distance travelled in the last second.

$$h - h' = \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2$$

i.e., $h - h' = \frac{1}{2}g(2t-1)$

But according to given problem as $(h - h') = \frac{h}{2}$

i.e., $\left(\frac{1}{2}\right)h = \left(\frac{1}{2}\right)g(2t-1)$ or $\left(\frac{1}{2}\right)gt^2 = g(2t-1)$

[as from equation (1) $h = \left(\frac{1}{2}\right)gt^2$]

or $t^2 - 4t + 2 = 0$

or $t = [4 \pm \sqrt{(4^2 - 4 \times 2)}]/2$

or $t = 2 \pm \sqrt{2}$ or t = 0.59 s or 3.41 s

0.59 s is physically unacceptable as it gives the total time t taken by the body to reach ground lesser than one sec while according to the given problem time of motion must be greater than 1s.

so t = 3.41 s

and h = ½ × (9.8) × $(3.41)^2$ = 57 m

EFFECT OF MEDIUM ON MOTION UNDER GRAVITY

Medium effects the motion of a body falling freely under gravity due to thrust and viscous drag.

Effect of Thrust :

If a body of volume V and density ρ is dropped in a medium of density σ, thrust will be $V\sigma g$ and will oppose the weight. So there are 3 possibilities:

(a) If $\sigma > \rho$, thrust will be greater than weight and so there will be a net force (Th–W) opposite to gravity and so the body will experience an acceleration.

$g' = \frac{Th - W}{m}$ opposite to gravity

$$= \frac{Th}{m} - g(as\; W = mg) = \frac{V\sigma g}{V\rho} - g$$

$$\Rightarrow g' = g\left(\frac{\sigma}{\rho} - 1\right)$$

$\therefore$ (as m = $V\rho$) ...... (1)

So, if a body is at rest it will be opposite to acceleration due to gravity (as in the case of a hydrogen filled balloon) and if the body falls through a liquid with an initial velocity the motion will be retarded [with retardation given by equation (1)], so body will stop after travelling distance and finally will move up [with acceleration given by equation (1)].

(b) If $\sigma = \rho$ the net force

$$F = W - Th = V\rho g - V\sigma g = 0$$

or mg' = 0 , i.e. g ' = 0 ... (2)

So, the body will be in natural equilibrium i.e., it will remain at rest or in uniform motion as the case may be.

(c) If $\sigma < \rho$, then as thrust will be lesser than weight, net force will be downwards and will be

F = mg' = mg – Th or

$$g' = g - \frac{Th}{m} = g\left(1 - \frac{\sigma}{\rho}\right) \quad .... (3)$$

i.e., now body will be accelerated down with acceleration $g' = g\left[1 - \left(\frac{\sigma}{\rho}\right)\right]$, which is less than g i.e. the effect of thrust here is to decrease acceleration due to gravity.

Effect of Viscous Force :

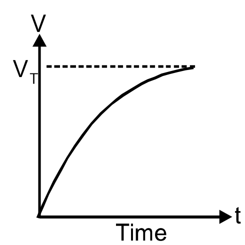
When a spherical body of radius r falls wtih a velocity in a medium of viscosity η, the viscous force opposing the motion, according to Stokes-law is given by $6\pi\eta rv$. This force is velocity dependent and non conservative. So when a spherical body falls freely in a medium $(\sigma < p)$ due to force mg the velocity will increase and as velocity increases the viscous force $(\propto v)$ will also increase. After some time a stage will be reached when viscous force exactly balances the net downward force mg. Then body will fall with a constant velocity called terminal velocity. Thus if v_T is the terminal velocity.

$6\pi\eta r\, v_T = mg$, i.e. $v_T = mg/6\pi\eta r$

But for a spherical body as

$$m = \left(\frac{4}{3}\right)\pi r^3 \rho \qquad v_T = \left(\frac{2}{9}\right) r^2 \frac{\rho g}{\eta} \quad .... (4)$$

However if, along with viscous force thrust is also taken into account which is usually the case, by placing g'



$\rightarrow g\left[1 - \frac{\sigma}{\rho}\right]$ in equation (4),

we get, $v_T = \left(\frac{2}{9}\right) r^2 \left[\frac{\rho - \sigma}{\eta}\right] g$ ... (5)

RELATIVE - VELOCITY

(i) There is nothing in absolute rest or absolute motion.

(ii) Motion is a combined property of the object under study and the observer.

Example

(i) A book placed on the table in a room is at rest, if it is viewed from the room but it is in motion, if it is viewed from the moon (another frame of reference). The moon is moving w.r.t. the book and the book w.r.t. the moon.

(ii) A robber enters a train moving at great speed with respect to the ground, brings out his pistol and says "Don't move, stand still".

The passengers stand still. The passengers are at rest with respect to the robber but are moving with respect to the rail track.

(iii) Relative motion means, the motion of a body with respect to another. Now if $\vec{V}_A$ and $\vec{V}_B$ are velocities of two bodies relative to earth, the velocity of B relative to A will be given by $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

Note :

(a) If two bodies are moving along the same line in same direction with velocities V_A and V_B relative to earth, the velocity of B relative to A will be given by $V_{BA} = V_B - V_A$.

If it is positive the direction of V_{BA} is that of B and if negative the direction of V_{BA} is opposite to that of B.

(b) However, if the bodies are moving towards or away from each other, as direction of V_A and V_B are opposite, velocity of B relative to A will have magnitude $V_{BA} = V_B - (-V_A) = V_B + V_A$ and directed towards A or away from A respectively.

(c) In dealing the motion of two bodies relative to each other $\vec{V}_{rel}$ is the difference of velocities of two bodies, if they are moving in same direction and is the sum of two velocities if they are moving in opposite direction.

(d) If a man can swim reltive to water with velocity $\vec{V}$ and water is flowing relative to ground with velocity $\vec{V}_R$, velocity of man relative to ground $\vec{V}_m$ will be $\vec{V} = \vec{V}_m - \vec{V}_R$ i.e. $\vec{V}_m = \vec{V} + \vec{V}_R$

So, if the swimming is in the direction of flow of water $V_m = V + V_R$

And if the swimming is opposite to the flow of water $V_m = V - V_R$

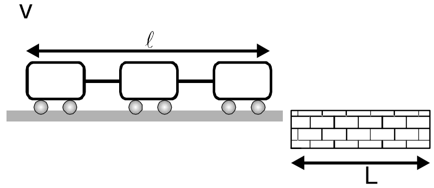
(e) If a boy is running with velocity $\vec{V}_{rel.}$ on a train moving with velocity $\vec{V}$ relative to the ground. The velocity of boy relative to ground, $\vec{V}$ will be given by $\vec{V}_{rel.} = \vec{v} - \vec{V} \Rightarrow \vec{v} = \vec{V}_{rel.} + \vec{V}$

So, if the boy is running on the train in the direction of motion of train $v = V_{rel.} + V$

And if the boy is running on the train in a direction opposite to the motion of train $v = V_{rel.} - V$

(f) Suppose a train having length ℓ is crossing a bridge of length L with constant speed v. Then the time taken by the train to cross the bridge will be :

$$t = \frac{\ell + L}{v}$$



Solved Examples

Ex.36 A steam boat goes across a lake and comes back : (a) on a quiet day when the water is still and (b) on a rough day when there is a uniform current so as to help the journey onward and to impede the journey backward. If the speed of launch on both days was same, in which case will it complete the journey in lesser time?

Sol. Let ℓ be the width of lake and v be the velocity of steam boat.

On a quiet day, time taken by steam boat in going and coming back is.

$$t_Q = \frac{\ell}{v} + \frac{\ell}{v} = \frac{2\ell}{v} \qquad ......(1)$$

on a rough day, let v' be the velocity of air current. As in going across the lake, the air current helps the motion, so time taken is $t_1 = \frac{\ell}{v + v'}$

In coming back, as the air current opposes the motion, so time taken is $t_2 = \frac{\ell}{v - v'}$

Total time in going and coming back on a rough day

$$t_R = t_1 + t_2 = \frac{2\ell v}{(v^2 - v'^2)} = \frac{2\ell}{v[1 - (v'/v)^2]} \qquad ....(2)$$

from (i) and (ii), we have

$$\frac{t_R}{t_Q} = \frac{1}{[1 - (v'/v')^2]} > 1 \text{ or } t_R > t_Q$$

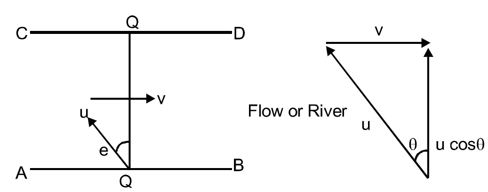
Therefore the time taken to complete the journey on quiet day is less than on a rough day.

Crossing of flowing river:

Let AB and CD be two banks of a river. The width of the river is PQ = h. It is flowing in the direction of arrow.

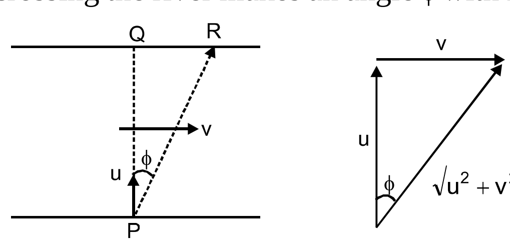
v is velocity of water flowing in river and u is velocity of swimmer or boat with respect to still water.

(a) If a boat rows with a speed u at an angle θ with the direction of width of the river, then the time taken by the man to cross the river along PQ is



$$t = \frac{h}{u\cos\theta} = \frac{h}{\sqrt{u^2 - v^2}} \text{ where } \sin\theta = \frac{v}{u}$$

(b) If a boat moves with a velocity u along the direcion of the width, then the resultant velocity of the boat crossing the river makes an angle ϕ with PQ.



$\therefore$ Resultant velocity $= \sqrt{u^2 + v^2}$

(c) Distance traversed by the man $= \dfrac{h}{\cos\phi} = t\sqrt{u^2+v^2}$

(d) Time taken in crossing the river - $t = \dfrac{h}{\cos\phi\sqrt{u^2+v^2}} = \dfrac{h}{u}$

This will be minimum time for crossing the river.

(e) Distance traversed in the direction of flow of river $\Rightarrow \dfrac{h}{u}v = h\left(\dfrac{v}{u}\right)$

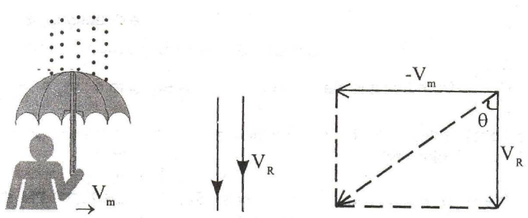
(f) Thus to cross the river in minimum time the boat is rowed in a direction perpendicular to the velocity of water. To cross the river from one point P to an opposite point O on the other bank the direction of rowing makes an angle θ with PQ such that $\theta = \sin^{-1}\left(\dfrac{v}{u}\right)$

(g) To across the river from one point on one bank to an opposite point on the other bank the velocity of boat $\vec{u}$ makes an angle $\left(\dfrac{\pi}{2}+\theta\right)$

To cross the river in minimum time $\theta = 0$ and $t = \dfrac{h}{u}$

Rain problems

If rain is falling vertically with a velocity $\vec{V}_R$ and an observer is moving horizontally with speed $\vec{V}_m$, the velocity of rain relative to observer :



$\vec{v}_{RM} = \vec{V}_R - \vec{V}_m$

which by law of vector addition has magnitude

$\vec{V}_{RM} = \sqrt{V_R^2 + V_M^2}$

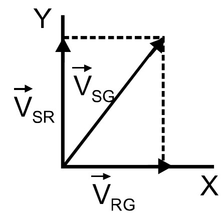
and direction $\theta = \tan^{-1}\left(\dfrac{V_M}{V_R}\right)$ with vertical.

Solved Examples

Ex.37 A swimmer can swim in still water at a rate 4 km/hour. If he swims in a river flowing at 3 km/h and keeps his direction (with respect to water.) perpendicular to the current. Find his velocity with respect to the ground.

Sol. The velocity of the swimmer with respect to water is $\vec{v}_{SR} = 4.0\,\text{km/hr}$ in the direction perpendicular to the river. The velocity of river with respect to the ground is $\vec{v}_{RG} = 3.0\,\text{km/hr}$ along the length of river.

The velocity of the swimmer with respect to the ground is $\vec{v}_{SG}$ where



$\vec{V}_{SG} = \vec{V}_{SR} + \vec{V}_{RS}$ $\quad V_{SG} = \sqrt{V_{SR}^2 + V_{RG}^2} = \sqrt{4^2+3^2}$

$= \sqrt{16+9} = \sqrt{25} = 5\,\text{km/hr}$

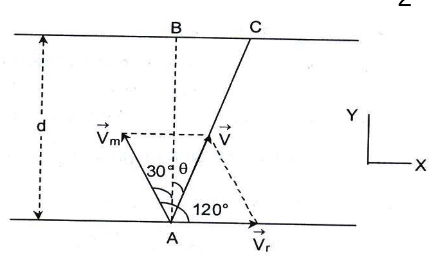
The angle θ made with the direction of flow is

$\theta = \tan^{-1}\left[\dfrac{V_{SR}}{V_{RG}}\right] = \tan^{-1}\left(\dfrac{4}{3}\right)$

Ex.38 A man can swim in still water at a speed of 3 km/hr. He wants to, cross a 500 m wide river flowing at 2 km/hour. He keeps himself always at an angle of 120^0 with the river flow while swimming.

(a) Find the time he takes to cross the river.
(b) At what point on the opposite bank will he arrive.

Sol. Width of river AB = d = 500 m $= \dfrac{1}{2}$ km.

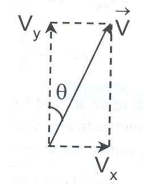


V_m = 3 km/hr. Velocity of man in still water
V_r = 2 km/hr. Velocity of river
V = resultant velocity of man in flowing river.

$\vec{V} = V_x\hat{i} + V_y\hat{j}$

Now, $V_x = V_r - V_m \sin 30^0 \;=\; 2 - 3 \times \frac{1}{2} = \frac{1}{2}\text{km/hr}$

$V_y = V_m \cos 30^0$



$= \frac{3\sqrt{3}}{2}\text{km/hr}$

Displacement along Y-axis, d = V_y x t

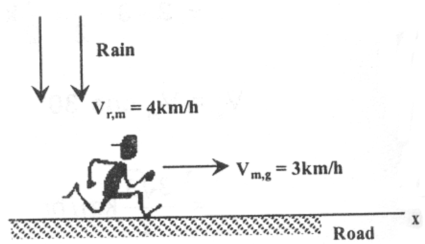
or $t = \frac{d}{V_y} = \frac{\frac{1}{2}}{\frac{3\sqrt{3}}{2}}$ $\quad \therefore \quad t = \frac{1}{3\sqrt{3}}\text{hr}$

Displacement along X-axis, $BC = V_x \times t$

$= \frac{1}{2} \times \frac{1}{3\sqrt{3}} \;=\; \frac{1}{6\sqrt{3}}\text{km}$

Ex.39 A man is walking on a level road at a speed of 3 km/hr. Raindrops fall vertically with a speed of 4 km/hr. Find the velocity of raindrops with respect to the men.

Sol. If we consider velocity of rain with respect to the man is v km/h.



Relative velocity of man w.r.t. ground

$\vec{v}_{mg} = \vec{v}_m - \vec{v}_g$ .... (1)

Velocity of rain w.r.t. ground $\vec{v}_{rg} = \vec{v}_r - \vec{v}_g$ .... (2)

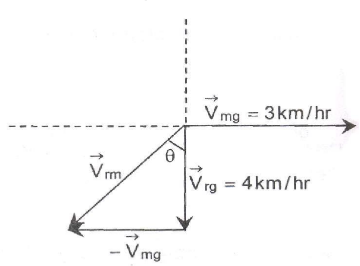
Velocity of rain w.r.t. man $\vec{v}_{rm} = \vec{v}_r - \vec{v}_m$

On subtracting equation (1) from equation (2)

$\vec{v}_{rm} = \vec{v}_{rg} - \vec{v}_{mg}$

$|v_{rm}| = \sqrt{v_{rg}^2 + v_{mg}^2} = \sqrt{4^2 + 3^2} = 5\,\text{km/hr.}$

Direction : $\tan\theta = \frac{3}{4}$ or $\theta = \tan^{-1}\left(\frac{3}{4}\right)$



Ex.40 A girl standing on a road has to hold her umbrella at 30^0 with the vertical to keep the rain away. She throws the umbrella and starts running at 10 km/hr. She finds that raindrops are hitting her head vertically. Find the speed of raindrops with respect to (a) the road (b) the moving girl.

Sol. Suppose, the velocity of rain with respect to girl

$= V_{RG}$

The velocity of rain with respect to the ground

$= V_{RG}$

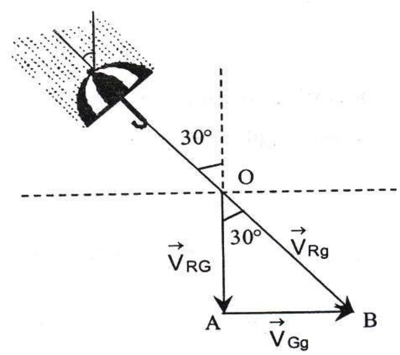
The velocity of girl with respect to ground

$= V_{Gg} = 10\,\text{km/hr.}$

$\vec{v}_{RG} = \vec{v}_R - \vec{v}_G$ ... (1)

$\vec{v}_{RG} = \vec{v}_R - \vec{v}_g$ ... (2)

$\vec{v}_{Gg} = \vec{v}_G - \vec{v}_g$ ... (3)



On adding equation (1) and equation (3)

$\vec{v}_{RG} + \vec{v}_{Gg} = \vec{v}_R - \vec{v}_g = \vec{v}_{Rg}$

(a) By triangle AOB

$\sin 30^0 = \frac{AB}{OB} = \frac{10}{V_{Rg}}$

$V_{Rg} = \frac{10}{\sin 30^0} = \frac{10}{1/2} = 20\,\text{km/hr}$

(b) Now, taking

$\frac{V_{RG}}{V_{Gg}} = \cot 30^0$

$\frac{V_{RG}}{10} = \sqrt{3}$ or $V_{RG} = 10\sqrt{3}\,\text{Km/hr}$