Complex Number

PRELIMINARY

(i) The numbers of the form $x + iy$ are known as complex numbers, where $x, y \in R$ and $i = \sqrt{-1}$.

i is an imaginary unit and it is known as **iota**.

(ii) Complex numbers are denoted by z . Let $z = x + iy$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$

(iii) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,

$i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = i^2 = -1$, $i^{4n+3} = i^3 = -i$

(iv) The sum of four consecutive powers of i is always zero, i. e. $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

Note :

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) $\sqrt{-x} \times \sqrt{-y} \neq \sqrt{xy}$

So for two real numbers x and y, $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ possible if both x, y are non-negative.

(iii) ' i ' is neither positive, zero nor negative, Due to this reason order relations are not defined for imaginary numbers.

Solved Examples

Ex.1 If n is a positive integer, then which of the following relations is false

(A) $i^{4n} = 1$ (B) $i^{4n-1} = i$

(C) $i^{4n+1} = i$ (D) $i^{-4n} = 1$

Sol. We know that $i^2 = -1$

$\Rightarrow (i^2)^2 = (-1)^2 = 1$

$\Rightarrow i^{4n} = 1^n$ and therefore $i^{4n-1} = -i$

Ans.(B)

Ex.2 If $i^2 = -1$, then the value of $\sum_{n=1}^{200} i^n$ is

(A) 50 (B) –50

(C) 0 (D) 100

Sol. $\sum_{n=1}^{200} i^n = i + i^2 + i^3 + ... + i^{200} = \frac{i(1 - i^{200})}{1 - i}$

(since G.P.)

$= \frac{i(1-1)}{1-i} = 0$ **Ans.(C)**

Ex.3 The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$ is

(A) 2 (B) -2

(C) 1 (D) -1

Sol. Given expression

$$=\frac{i^{10}\left(i^{582}+i^{580}+i^{578}+i^{576}+i^{574}\right)}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$$

$$=i^{10}-1=\left(i^2\right)^5-1=(-1)^5-1 \quad =-1-1=-2$$

Ans.(C)

Ex.4 Find the value of $[i]^{198}$

Sol. $[i]^{198}=[i^2]^{99}=[-1]^{99}=-1$

Ex.5 Find the value of $i^n+i^{n+1}+i^{n+2}+i^{n+3}$

Sol. $i^n+i^{n+1}+i^{n+2}+i^{n+3}$

$= \quad i^n[1+i+i^2+i^3]$

$= \quad i^n[1+i-1-i] = i^n[0]=0$

Ex.6 The sum of series $i^2+i^4+i^6+$ ....$(2n+1)$ terms is -

(A) 0 (B) 1

(C) n (D) -1

Sol. Given series is a G.P. So, Sum of a G. P. is

$$=\frac{i^2[1-(i^2)^{2n+1}]}{1-i^2} = \frac{(-1)(1-(i)^{4n+2})}{1+1}$$

$$=\frac{(-1)(1+1)}{2} = -1$$

Ans.[D]

COMPLEX NUMBER

A number of the form $z=x+iy$ where $x, y \in R$ and $i=\sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

$Re(z)=x,\ Im(z)=y$

Here if $x=0$ the complex number is purely Imaginary and if $y=0$ the complex number is purely Real.

A complex number may also be defined as an ordered pair of real numbers any may be denoted by the symbol (a, b). If we write $z=(a, b)$ then a is called the real part and b the imaginary part of the complex number z.

Note :

(i) Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so $4+3i<1+2i$ or $i<0$ or $i>0$ is meaning less.

(ii) If two complex numbers are equal, then their real and imaginary parts are separately equal.

Thus if $a+ib=c+id \Rightarrow a=c$ and $b=d$

so if $z=0 \Rightarrow x+iy=0 \Rightarrow x=0$ and $y=0$

The student must note that

$x, y \in R$ and $x, y \neq 0$. Then if

$x+y=0 \Rightarrow x=y$ is correct

but $x+iy=0 \Rightarrow x=-iy$ is incorrect

Hence a real number cannot be equal to the imaginary number, unless both are zero.

(iii) The complex number 0 is purely real and purely imaginary both.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

(i) **Addition :** $(a+ib)+(c+id)=(a+c)+i(b+d)$

(ii) **Subtraction :** $(a+ib)-(c+id)=(a-c)+i(b-d)$

(iii) **Multiplication :**

$$(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$

(iv) **Division :** $\frac{a+ib}{c+id}=\frac{ac+bd}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}$ $(c+id\neq 0)$

Properties of Algebraic operations with Complex Number

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties-

Commutativity : $z_1+z_2=z_2+z_1$ & $z_1 z_2=z_2 z_1$

Associativity : $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ and $(z_1 z_2) z_3=z_1(z_2 z_3)$

Identity element : If $O=(0, 0)$ and $1=(1, 0)$ then $z+0=0+z=z$ and $z.1=1.z=z$. Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is $-z$ and multiplicative inverse of z is $\frac{1}{z}$.

Cancellation Law :

$$\left.\begin{array}{l} z_1+z_2=z_1+z_3 \\ z_2+z_1=z_3+z_1 \end{array}\right\} \Rightarrow z_2 = z_3$$

and $$z_1 \neq 0 \left.\begin{array}{l} z_1\ z_2=z_1\ z_3 \\ z_2\ z_1=z_3\ z_1 \end{array}\right\} \Rightarrow z_2 = z_3$$

Distributivity : $z_1(z_2+z_3)=z_1 z_2+z_1 z_3$

and $(z_2+z_3)z_1=z_2 z_1+z_3 z_1$

MULTIPLICATIVE INVERSE OF A NON-ZERO COMPLEX NUMBER

Multiplicative inverse of a non-zero complex number $z = x + iy$ is

$$z^{-1}=\frac{1}{z}=\frac{1}{x+iy}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}$$

$$=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$

i.e. $z^{-1}=\dfrac{\text{Re}(z)}{|z|^2}+i\dfrac{-\text{Im}(z)}{|z|^2}$

Solved Examples

Ex.7 If $z = -3 + 2i$, then $1/z$ is equal to

(A) $\frac{1}{13}(3+2i)$ (B) $-\frac{1}{13}(3+2i)$

(C) $\frac{1}{\sqrt{13}}(3+2i)$ (D) $-\frac{1}{\sqrt{13}}(3+2i)$

Sol. $z^{-1}=-\frac{3}{13}-i\frac{2}{13}=-\frac{1}{13}(3+2i)$ **Ans.(B)**

EQUALITY OF COMPLEX NUMBERS

Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal i.e.

If $a + ib = c + id$, then $a = c$ & $b = d$

Solved Examples

Ex.8 The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ are

(A) $x = -1, y = 3$ (B) $x = 3, y = -1$

(C) $x = 0, y = 1$ (D) $x = 1, y = 0$

Sol. $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

$\Rightarrow (4+2i)x+(9-7i)y-3i-3=10i$ **Ans.(B)**

Equating real and imaginary parts,

we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence $x = 3$ and $y = -1$.

Ex.9 If $a + ib = c + id$, then

(A) $a - c = i(b - d)$ (B) $a - ib = c - id$

(C) $a = d, b = c$ (D) none of these

Sol. If $a + ib = c + id$. Equating real and imaginary parts,

we get $a = c$ and $b = d$ $\Rightarrow$ $-b = -d$.

Therefore $a - ib = c - id$ **Ans.(B)**

Ex.10 $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

(A) $2n\pi\pm\frac{\pi}{3}$ (B) $n\pi+\frac{\pi}{3}$

(C) $n\pi\pm\frac{\pi}{3}$ (D) none of these

Sol. $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e., $\frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0$

$\Rightarrow 3-4\sin^2\theta=0$ (only if θ be real)

$\Rightarrow \sin\theta=\pm\frac{\sqrt{3}}{2}=\sin\left(\pm\frac{\pi}{3}\right)$

$\Rightarrow \theta=n\pi+(-1)^n\left(\pm\frac{\pi}{3}\right)=n\pi\pm\frac{\pi}{3}$.

Ans.(C)

Ex.11 If $(x + iy)(2 - 3i) = 4 + i$, then-

(A) $x = -14/13, y = 5/13$

(B) $x = 5/13, y = 14/13$

(C) $x = 14/13, y = 5/13$

(D) $x = 5/13, y = -14/13$

Sol. $x+iy=\frac{4+i}{2-3i}=\frac{(4+i)(2+3i)}{13}=\frac{5+14i}{13}$

$\therefore$ $x = 5/13, y = 14/13$. **Ans.[B]**

Ex.12 If Complex Number $\frac{z-1}{z+1}$ is purely imaginary then locus of z is -

(A) a circle (B) a straight line
(C) a parabola (D) None of these

Sol. Let $z = x + iy$ then

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{x^2-1+iy(x-1)+iy(x+1)+y^2}{(x+1)^2+y^2}$$

$$= \frac{(x^2-1+y^2)+i[2xy]}{(x+1)^2+y^2}$$

If it is purely Imaginary

$$\frac{x^2-1+y^2}{(x+1)^2+y^2} = 0$$

$\Rightarrow$ $x^2 + y^2 - 1 = 0$

$\Rightarrow$ $x^2 + y^2 = 1$

which is the equation of a circle. **Ans.[A]**

CONJUGATE COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$ where $b \neq 0$ are said to be complex conjugate of each other (Here the complex conjugate is obtained by just changing the sign of i) e.g.conjugate of $z = -3 + 4i$ is $\bar{z} = -3 - 4i$.

Note : Image of any complex number in x-axis is called its conjugate.

Properties of Conjugate Complex Number

Let $z = a + ib$ and $\bar{z} = a - ib$ then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\,\text{Re}\,(z) =$ purely real

(iii) $z - \bar{z} = 2ib = 2i\,\text{Im}\,(z) =$ purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(ix) $\overline{z^n} = (\bar{z})^n$

(x) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(xi) $z + \bar{z} = 0$ or $z = -\bar{z}$

$\Rightarrow$ $z = 0$ or z is purely imaginary

(xii) $z = \bar{z}$ $\Rightarrow$ z is purely real

Solved Examples

Ex.13 The conjugate of $\frac{1}{3+4i}$ is -

(A) $(3 - 4i)$ (B) $\frac{1}{25}(3 + 4i)$
(C) $\frac{1}{25}(3 - 4i)$ (D) None of these

Sol. $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{1}{25}(3 - 4i)$

$\Rightarrow$ conjugate of $\left(\frac{1}{3+4i}\right) = \frac{1}{25}(3 + 4i)$.

Ans. [B]

Ex.14 If z is a complex number such that $z^2 = (\bar{z})^2$, then

(A) z is purely real
(B) z is purely imaginary
(C) Either z is purely real or purely imaginary
(D) None of these

Sol. Let $z = x + iy$, then its conjugate $\bar{z} = x - iy$

Given that $z^2 = (\bar{z})^2$

$\Rightarrow$ $x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy$

$\Rightarrow$ $4ixy = 0$.

If $x \neq 0$ then $y = 0$ and if $y \neq 0$ then $x = 0$.

Ans.(C)

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is denoted as mod (z) or $|z|$, is defined as

$|z| = \sqrt{x^2 + y^2}$, where $x = \text{Re}(z)$, $y = \text{Im}(z)$. Sometimes, $|z|$ is called absolute value of z. Note that $|z| \ge 0$.

For example, if $z = 3 + 2i$, then $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

Properties of modulus

(i) $|z| \ge 0$ and $|z| = 0$ if and only if $z = 0$, i.e., $x = 0$, $y = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$.

(iii) $z\bar{z} = |z|^2$

(iv) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (provide $z_2 \ne 0$)

(vii) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(viii) $|z_1 - z_2| \ge ||z_1| - |z_2||$

(ix) $|z^2| = |z|^2$ or $|z^n| = |z|^n$ also $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$

(xii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$

Solved Examples

Ex.15 The modulus of $z = \dfrac{(1+i\sqrt{3})(\cos\theta + i\sin\theta)}{2(1-i)(\cos\theta - i\sin\theta)}$ is-

(A) $\dfrac{1}{3\sqrt{2}}$ (B) $\dfrac{1}{\sqrt{3}}$

(C) $\dfrac{1}{\sqrt{2}}$ (D) 1

Sol. $|z| = \dfrac{|1+i\sqrt{3}|\,|\cos\theta + i\sin\theta|}{2|1-i|\,|\cos\theta - i\sin\theta|}$

$= \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ **Ans.[C]**

Ex.16 If for any complex number z, $|z - 4| < |z - 2|$, then

(A) $R(z) > 2$ (B) $R(z) < 0$

(C) $R(z) > 0$ (D) $R(z) > 3$

Sol. Let $z = x + iy$, then

$|z - 4| < |z - 2|$

$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow -4x < -12 \Rightarrow x > 3$

$\Rightarrow R(z) > 3$ **Ans.[D]**

Ex.17 If $\left|\dfrac{z-3i}{z+3i}\right| = 1$ then the locus of z is -

(A) x axis

(B) $x - y = 0$

(C) Circle passing through origin

(D) y axis

Sol. Let $z = x + iy$ then

$\left|\dfrac{z-3i}{z+3i}\right| = 1 \Rightarrow |z - 3i| = |z + 3i|$

$\Rightarrow |x + iy - 3i| = |x + iy + 3i|$

$\Rightarrow \sqrt{x^2 + (y-3)^2} = \sqrt{x^2 + (y+3)^2}$

$\Rightarrow 12y = 0$

$\Rightarrow y = 0$, which is equation of x - axis

Ans.[A]

Ex.18 For any two complex numbers z_1 and z_2 and any real numbers a and b;

$|(az_1 - bz_2)|^2 + |(bz_1 + az_2)|^2 =$

(A) $(a^2 + b^2)(|z_1| + |z_2|)$

(B) $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(C) $(a^2 + b^2)(|z_1|^2 - |z_2|^2)$

(D) none of these

Sol. $|(az_1 - bz_2)|^2 + |(bz_1 + az_2)|^2$

$= a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\text{Re}(z_1\bar{z}_2) + b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\text{Re}(z_1\bar{z}_2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ **Ans.(B)**

REPRESENTATION OF A COMPLEX NUMBER

(a) Cartesian Representation :

The complex number $z = x + iy = (x, y)$ is represented by a point P whose coordinates are refered to rectangular axis $x\,ox'$ and yoy', which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gussian plane.

Note :

(i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by $|z|$. Thus, $|z| = \sqrt{x^2 + y^2}$.

(ii) Angle of any complex number with positive direction of x-axis is called amplitude or argument of z. Thus, amp (z) = arg (z) = $\theta = \tan^{-1}\dfrac{y}{x}$.

(b) Polar Representation : If $z = x + iy$ is a complex number then $z = r(\cos\theta + i\sin\theta)$ is a polar form of complex number z where $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2} = |z|$.

(c) Exponential Form : If $z = x + iy$ is a complex number then its exponential form is $z = r\,e^{i\theta}$ where r is modulas and θ is amplitude of complex number.

(d) Vector Representation : If $z = x + iy$ is a complex number such that it represent point P(x, y) then its vector representation is $z = \vec{OP}$

Solved Examples

Ex.19 If $A \equiv 1 + 2i$, $B \equiv -3 + i$, $C \equiv -2 - 3i$ and $D \equiv 2 - 2i$ are vertices of a quadrilateral, then it is a

(A) rectangle (B) parallelogram
(C) square (D) rhombus

Sol. $\because A \equiv (1, 2); B \equiv (-3, 1); C \equiv (-2, -3); D \equiv (2, -2)$

$\therefore AB^2 = 16 + 1 = 17$, $BC^2 = 1 + 16 = 17$

$CD^2 = 16 + 1 = 17$, $AC^2 = 9 + 25 = 34$

$BD^2 = 25 + 9 = 34$.

Now since AB = BC = CD and AC = BD

$\therefore$ ABCD is square. **Ans.[C]**

Ex.20 The polar form of $-1 + i$ is-

(A) $\sqrt{2}\,(\cos \pi/4 + i \sin \pi/4)$

(B) $\sqrt{2}\,(\cos 5\pi/4 + i \sin 5\pi/4)$

(C) $\sqrt{2}\,(\cos 3\pi/4 + i \sin 3\pi/4)$

(D) $\sqrt{2}\,(\cos \pi/4 - i \sin \pi/4)$

Sol. $\because |-1 + i| = \sqrt{2}$, amp$(-1 + i) = \pi - \pi/4 = 3\pi/4$

$\therefore -1 + i = \sqrt{2}\,(\cos 3\pi/4 + i \sin 3\pi/4)$

Ans. [C]

Ex.21 The polar form of $\dfrac{1+7i}{(2-i)^2}$ is

(A) $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

(B) $\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

(C) $\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

(D) none of these

Sol. $\dfrac{1+7i}{(2-i)^2} = \dfrac{(1+7i)}{(3-4i)}\dfrac{(3+4i)}{(3+4i)} = \dfrac{-25+25i}{25} = -1 + i$

Let $z = x + iy = -1 + i$

$\therefore r\cos\theta = -1$ and $r\sin\theta = 1$

$\therefore \theta = \dfrac{3\pi}{4}$ and $r = \sqrt{2}$

Thus $\dfrac{1+7i}{(2-i)^2} = \sqrt{2}\left[\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right]$

Ans.(A)

ARGUMENT OF A COMPLEX NUMBER

The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis. If $z = x + iy$ then

$$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

The argument of any complex number is not unique. $2n\pi + \theta$ (n integer) is also argument of z for various values of n. The value of θ satisfying the inequality $-\pi < \theta \le \pi$ is called the principle value of the argument.

For finding the principle argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.

Note :

(i) If a complex number is multiplied by iota(i) its amplitude will be be increased by $\pi/2$ and will be decreased by $\pi/2$, if is multiplied by –i.

(ii) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

(iii) Argument of zero is not defined.

(iv) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is

$$\theta = \theta_2 - \theta_1 = \arg \overline{PR} - \arg \overline{PQ} = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

Properties of Arguments

(i) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

(iii) $\arg(z^n) = n \arg(z)$

(iv) If $\arg(z) = 0 \Rightarrow z$ is a positive real number

(v) $\arg(z) + \arg(\bar{z}) = 0$

(vi) $\arg(z - \bar{z}) = \pm \pi/2$

(vii) $\arg(z) = \pi \Rightarrow z$ is a negative real number

(viii) $\arg(\bar{z}) = -\arg(z) = \arg(1/z)$

(ix) $\arg(-z) = \arg(z) \pm \pi$

(x) $\arg(i\,y) = \pi/2$ if $y > 0$

$= -\pi/2$ if $y < 0$ (where $y \in R$)

Solved Examples

Ex.22 Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then principal $\arg(z_1 z_2)$ is given by

(A) $\alpha + \beta + \pi$ (B) $\alpha + \beta - \pi$

(C) $\alpha + \beta - 2\pi$ (D) $\alpha + \beta$

Sol. We know that principal argument of a complex number lie between $-\pi$ and π, but $\alpha + \beta > \pi$, therefore principal $\arg(z_1z_2) = \arg z_1 + \arg z_2 = \alpha + \beta$, is given by $\alpha + \beta - 2\pi$. **Ans.(C)**

Ex.23 The amplitude of the complex number $z = \sin\alpha + i(1 - \cos\alpha)$ is

(A) $2\sin\frac{\alpha}{2}$ (B) $\frac{\alpha}{2}$

(C) α (D) None of these

Sol. $z = \sin\alpha + i(1 - \cos\alpha)$

$$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$ **Ans.(B)**

Ex.24 The amplitude of $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$

(C) $\frac{\pi}{10}$ (D) $\frac{\pi}{15}$

Sol.

$$\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$$

$$= 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i2\sin^2\frac{\pi}{10}$$

$$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

For amplitude, $\tan\theta = \dfrac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10}$

$\Rightarrow \quad \theta = \frac{\pi}{10}$. **Ans.(C)**

Ex.25 The amplitude of $\frac{a+ib}{a-ib}$ is equal to-

(A) $\tan^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (B) $\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

(C) $\tan^{-1}\left(\frac{2ab}{a^2+b^2}\right)$ (D) $\tan^{-1}\left(\frac{a^2-b^2}{2ab}\right)$

Sol. amp $\left(\frac{a+ib}{a-ib}\right)$ = amp (a + ib) – amp (a – ib)

$= \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(-\frac{b}{a}\right)$

$= \tan^{-1}\left[\frac{2(b/a)}{1-(b^2/a^2)}\right] = \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

Ans.[B]

Ex.26 If $z = \frac{1}{i}$ then arg ($\bar{z}$) is -

(A) π (B) $-\frac{\pi}{2}$

(C) 0 (D) $\frac{\pi}{2}$

Sol. $z = \frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{+1} = -i$

$\therefore$ $\bar{z} = i$, which is the positive Imaginary quantity

$\therefore$ arg ($\bar{z}$) $= \frac{\pi}{2}$ **Ans.[D]**

Ex.27 If $z = \frac{3-i}{2+i} + \frac{3+i}{2-i}$ then arg (zi) is-

(A) $-\pi$ (B) π

(C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{2}$

Sol. $z = \frac{3-i}{2+i} + \frac{3+i}{2-i}$

$= \frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$

$\Rightarrow$ z = 2

$\Rightarrow$ (iz) = 2i,

which is the positive Imaginary quantity

$\therefore$ arg (iz) $= \frac{\pi}{2}$ **Ans.[D]**

DIFFERENT WAYS OF WRITING A COMPLEX NUMBER

z = a + ib (Algebraic form)

z = r (cos θ + i sin θ) (Polar form)

where r is modulus of complex number and θ is it's argument.

$z = r\,e^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{i\theta}}{2i}$

Solved Examples

Ex.28 If $z = re^{i\theta}$, then $|e^{iz}|$ =

(A) $e^{r\sin\theta}$ (B) $e^{-r\sin\theta}$

(C) $e^{-r\cos\theta}$ (D) $e^{r\cos\theta}$

Sol. If $z = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = ir(\cos\theta + i\sin\theta) = -r\sin\theta + ir\cos\theta$

or $e^{iz} = e^{(-r\sin\theta + ir\cos\theta)} = e^{-r\sin\theta}\,e^{ri\cos\theta}$

or $|e^{iz}| = |e^{-r\sin\theta}|\,|e^{ri\cos\theta}|$

$= e^{-r\sin\theta}\,[\cos^2(r\cos\theta) + \sin^2(r\cos\theta)]^{1/2}$

$= e^{-r\sin\theta}$ **Ans.(B)**

ROTATION THEOREM

$$\frac{z_2-z_0}{|z_2-z_0|} = \frac{z_1-z_0}{|z_1-z_0|}e^{i\theta}$$

or $$\frac{z_1-z_0}{|z_1-z_0|} = \frac{z_2-z_0}{|z_2-z_0|}\,e^{i(2\pi-\theta)} = \frac{z_2-z_0}{|z_2-z_0|}\,e^{-i\theta}$$

Solved Examples

Ex29 If the points z_1, z_2, z_3 are the vertices of an equilateral triangle in the complex plane, then the value of $z_1^2 + z_2^2 + z_3^2$ is equal to

(A) $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1}$ (B) $z_1z_2 + z_2z_3 + z_3z_1$

(C) $z_1z_2 - z_2z_3 - z_3z_1$ (D) $-\frac{z_1}{z_2} - \frac{z_2}{z_3} - \frac{z_3}{z_1}$

Sol. $\overline{AC} = \overline{AB}\ e^{i\pi/3}$

By rotating $\pi/3$ in clockwise sense

$\Rightarrow (z_3 - z_1) = (z_2 - z_1)e^{i\pi/3}$ ....(i)

Also $(z_1 - z_2) = (z_3 - z_2)\ e^{i\pi/3}$ ....(ii)

DIviding (i) by (ii) we get

$$\Rightarrow \frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Ans.(B)

SQUARE ROOT OF A COMPLEX NUMBER

If $z = x + iy$

suppose $\sqrt{z} = \sqrt{x + iy} = a + ib$

$\Rightarrow x + iy = a^2 - b^2 + 2iab$

On comparing the real and imaginary parts

$x = a^2 - b^2$, $y = 2ab$

Now, $a^2 + b^2 = \sqrt{x^2 + y^2} = |z|$ .....(i)

$a^2 - b^2 = x$ .....(ii)

From Equation (i) and (ii)

$$a = \pm\sqrt{\frac{|z|+x}{2}},\ b = \pm\sqrt{\frac{|z|-x}{2}}$$

Solving these two equations we shall get the required square roots as follows :

$$\pm\left[\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right] \text{ if } y > 0 \quad \text{and}$$

$$\pm\left[\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right] \text{ if } y < 0$$

Note :

(i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ (Here b = 1)

(ii) The square root of $-$ i is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$ (Here b = -1)

(iii) The square root of ω is $\pm\omega^2$

(iv) The square root of ω^2 is $\pm\omega$

Solved Examples

Ex.30 The square roots of $7 + 24i$ is.

(A) $\pm(4 + 3i)$ (B) $\pm(3 + 4i)$

(C) $\pm(2 + 3i)$ (D) $\pm(4 - 3i)$

Sol. Here $|z| = 25$, $x = 7$,

Hence square root =

$$\pm\left[\left(\frac{25+7}{2}\right)^{1/2} + i\left(\frac{25-7}{2}\right)^{1/2}\right] = \pm(4 + 3i)$$

Ans.(A)

MISCELLANEOUS RESULTS

(i) If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

(ii) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$.

(iii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by points P and Q respectively in Argand Plane then -

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

(iv) If a point P divides AB in the ratio of m : n, then $z = \frac{mz_2 + nz_1}{m+n}$ where z_1, z_2 and z represents the point A, B and P respectively.

(v) $|z - z_1| = |z - z_2|$ represents a perpendicular bisector of the line segmentjoining the points z_1 and z_2.

(vi) Let P be any point on a circle whose centre C and radius r, let the affixes of P and C be z and z_0 then $|z - z_0| = r$.

(a) Again if $|z - z_0| < r$ represent interior of the circle of radius r.

(b) $|z - z_0| > r$ represent exterior of the circle of radius r.

(vii) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is.

$$\theta = \theta_2 - \theta_1$$
$$= \arg. \vec{PR} - \arg \vec{PQ}$$
$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

(a) If z_1, z_2, z_3 are collinear, thus $\theta = 0$

therefore $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(b) If z_1, z_2, z_3 are such that PR $\perp$ PQ,

$\theta = \pi/2$ So $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.

Solved Examples

Ex.31 The points represented by the complex numbers $1 + i, -2 + 3i, \frac{5}{3}i$ on the Argand diagram are

(A) Vertices of an equilateral triangle

(B) Vertices of an isosceles triangle

(C) Collinear

(D) None of these

Sol. Let $z_1 = 1 + i$, $z_2 = -2 + 3i$ and $z_3 = 0 + \frac{5}{3}i$

Then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & 5/3 & 1 \end{vmatrix}$

$$= 1\left(3 - \frac{5}{3}\right) + 1(2) + 1\left(\frac{-10}{3}\right)$$

$= \frac{4}{3} + 2 - \frac{10}{3} = \frac{4 + 6 - 10}{3} = 0$ **Ans.(C)**

Ex.32 If the complex numbers, z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 =$

(A) 0 (B) 1

(C) –1 (D) None of these

Sol. Let the complex number z_1, z_2, z_3 denote the vertices A, B, C of an equilateral triangle ABC. Then, if O be the origin we have **OA** = z_1, **OB** = z_2, **OC** = z_3

Therefore $|z_1| = |z_2| = |z_3| \Rightarrow$ OA = OB = OC

i.e., O is then circumcentre of ΔABC

Hence $z_1 + z_2 + z_3 = 0$. **Ans.(A)**

Ex.33 If $|z| = 2$, then the points representing the complex numbers $-1 + 5z$ will lie on a

(A) Circle (B) Straight line

(C) Parabola (D) None of these

Sol. Let $\omega = -1 + 5z$, then $\omega + 1 = 5z$

$\Rightarrow |\omega + 1| = 5|z| = 5 \times 2 = 10$

($\because |z| = 2$, given value)

Thus ω lies on a circle. **Ans.(A)**

Ex.34 The equation $z\bar{z} + (2-3i)z + (2+3i)\bar{z} + 4 = 0$ represents a circle of radius

(A) 2 (B) 3

(C) 4 (D) 6

Sol. Here $a = 2 - 3i$, $\bar{a} = 2 + 3i$ and $b = 4$.

Hence radius $= \sqrt{a\bar{a} - b} = \sqrt{(2-3i)(2+3i) - 4} = 3$