MATHEMATICAL INDUCTION

A. FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

The proof of proposition by mathematical induction consists of the following three steps :

Step I : (Verification step) : Actual verification of the proposition for the starting value "i".

Step II : (Induction step): Assuming the proposition to be true for "k", $k \geq i$ and proving that it is true for the value $(k + 1)$ which is next higher integer.

Step III : (Generalization step) : To combine the above two steps. Let p(n) be a statement involving the natural number n such that

(i) p (1) is true i.e. p(n) is true for n = 1.

(ii) p(m + 1) is true, whenever p(m) is true i.e. p(m) is true $\Rightarrow$ p(m + 1) is true.

Then p(n) is true for all natural numbers n.

B. SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

The proof of proposition by mathematical induction consists of following steps :

Step I : (Verification step): Actual verification of the proposition for the starting value i and (i + 1).

Step II : (Induction step) : Assuming the proposition to be true for k – 1 and k and then proving that it is true for the value k + 1; $k \geq i + 1$.

Step III : (generalization step) : Combining the above two steps. Let p(n) be a statement involving the natural number n such that

(i) p (1) is true i.e. p(n) is true for n = 1 and

(ii) p (m + 1) is true, whenever p(n) is true for all n, where $i \leq n \leq m$.

Then p(n) is true for all natural numbers. For $a \neq b$. The expression $a^n - b^n$ is divisible by

(a) a + b, if n is even.

(b) a – b, if n is odd or even.

C. DIVISIBILITY PROBLEMS

To show that an expression is divisible by an integer

(i) If a, p, n, r are positive integer, then first of all we write $a^{pn+r} = a^{pn} . a^r = (a^p)^n . a^r$.

(ii) If we have to show that the given expression is divisible by c.

Then express, $a^p = [1 + (a^p - 1)]$, if some power of $(a^p - 1)$ has c as a factor.

$a^p = [2 + (a^p - 2)$, if some power of $(a^p - 2)$ has c as a factor.

$a^p = [k + (a^p - k)]$, if some power of $(a^p - k)$ has c as a factor.

D. REVERSING TECHNIQUE

If $a_0, a_1, a_2.....a_n$ are in A.P. then sum of the series $a_0C_0 + a_1C_1 + ...+ a_nC_n$ can be obtained by the reversing technique explained below.

Let $S = a_0C_0 + a_1C_1 + a_2C_2 + ...+ a_{n-1} C_{n-1} + a_nC_n$ ....(i)

Using $C_r = C_{n-r}$ and reversing the order in which terms are written above, we obtain,

$S = a_nC_0 + a_{n-1}C_1 + a_{n-2}C_2 + ...+ a_1 C_{n-1} + a_nC_n$ ....(ii)

Adding (i) and (ii) we get,

$2S = (a_0 + a_n) C_0 + (a_1 + a_{n-1}) C_1 + (a_2 + a_{n-2}) C_2 + ...+ (a_{n-1} + a_1) C_{n-1} + (a_n + a_0) C_n$

As $a_0, a_1, a_2 .... a_n$ are in A.P., we have

$a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2} = ...$

So that, $2S = (a_0 + a_n) (C_0 + C_1 + C_2 + ...+ C_n) = (a_0 + a_n) 2^n \Rightarrow S = \frac{1}{2} (a_0 + a_n) 2^n = (a_0 + a_n) 2^{n-1}$

SOLVED PROBLEMS

Ex.1 By mathematical Induction prove that

$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ $\forall$ n $\in$ N.

Sol. We are to prove that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad \ldots(1)$$

For n = 1, L.H.S. = $\frac{1}{1.3}=\frac{1}{3}$

R.H.S = $\frac{1}{2+1}=\frac{1}{3}$

L.H.S. = R.H.S.

$\therefore$ Result that result (1) it true for n = 1

Assume that result (1) is true for n = m

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\frac{1}{(2m-1)(2m+1)}=\frac{m}{2m+1} \quad \ldots(2)$$

Adding next term i.e. $\frac{1}{(2m+1)(2m+3)}$ on both sides of (2), we get

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\frac{1}{(2m-1)(2m+1)}+\frac{1}{(2m+1)(2m+3)}$$

$$=\frac{m}{2m+1}+\frac{1}{(2m+1)(2m+3)}=\frac{1}{(2m+1)}\left[m+\frac{1}{2m+3}\right]$$

$$=\frac{1}{(2m+1)}\left[\frac{2m^2+3m+1}{2m+3}\right]=\frac{2m^2+3m+1}{(2m+1)(2m+3)}$$

$$=\frac{2m(m+1)+1(m+1)}{(2m+1)(2m+3)}=\frac{(m+1)(2m+1)}{(2m+1)(2m+3)}$$

$$=\frac{m+1}{2m+3}=\frac{m+1}{2(m+1)+1} \quad \ldots.(3)$$

Comparing (3) with (1) we see that the result is true for n = m +1

Hence by the principle of Mathematical Induction, result (1) is true $\forall$ n $\in$ N.

Ex.2 Use principle of Mathematical induction to prove that 1.3+2.4+3.5+ .... + n(n+2) $=\frac{n(n+1)(2n+7)}{6}$

Sol. We are to prove that

$$1.3+2.4+3.5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6} \quad \ldots(1)$$

For n = 1, L.H.S. = 1.3 = 3

R.H.S = $\frac{1(2)(9)}{6}=\frac{18}{6}=3$ L.H.S. = R.H.S

$\therefore$ Result (1) is true for n = 1

Assume that (1) is true for n = m

1.3 +2.4 + 3.5 + ... + m(m+2)

$$=\frac{m(m+1)(2m+7)}{6} \quad \ldots(2)$$

Adding next term i.e. (m +1)(m +3) on both sides of (2), we get

1.3+2.4+3.5+ .... +m(m+2)+(m+1)(m+3)

$$=\frac{m(m+1)(2m+7)}{6}+(m+1)(m+3)$$

$$=(m+1)\left[\frac{m(2m+7)}{6}+m+3\right]$$

$$=\frac{(m+1)(2m^2+7m+6m+18}{6}$$

$$=\frac{(m+1)(2m^2+7m+6m+18}{6}$$

$$=\frac{(m+1)(m+2)(2m+9)}{6}$$

$$=\frac{(m+1)(m+1+1)(2(m+1)+7)}{6} \quad \ldots.(3)$$

Comaring (3) with (1), we see that the result is true for = m + 1

Hence by the principle of Mathematical Induction result (1) is true $\forall$ n $\in$ N.

Ex.3 Use method of Induction to prove that $(1+x)^n \geq 1 + nx$ for x > –1 and for all natural number n.

Sol. We are to prove that

$(1+x)^n \geq 1+nx$ for n $\geq$1 and x>–1 ...(1)

For n=1, (1) becomes

1+x $\geq$ 1+x which is always true

$\therefore$ result (1) is true for n = m ....(2)

Now $x > -1 \Rightarrow 1 + x > 0$ .....(2)

$\therefore$ Multiplying both sides of (2) by $1 + x > 0$, we get

$(1+x)^{m+1} \geq (1+mx)(1+m)$

$\geq 1+x + mx + mx^2$

$\geq 1+ (m+1) m + mx^2$

$(1+x)^{m+1} \geq 1+ (m+1) x$ $[\because kx^2 \geq 0]$

Comparing this result with (1), we see that result (1) is true for $n = m + 1$

$\therefore$ By method of Induction, result (1) is true for all natural numbers n.

Ex.4 If x and y any two distinct integers then show that $x^n - y^n$ is integral multiple of $x-y$.

Sol. We want to show that $x^n - y^n$ is an integral multiple of $x-y$. In other words, we want to show that $x^n - y^n$ is divisible by $x-y \neq 0$

Let $P(n) = n^n - y^n$

$P(1) = x-y$ which is divisible by $x-y$

$\therefore$ result is true for n=1

Assume that result is true for n=m

$\therefore$ $P(m) = x^m - y^m$ is divisible by $x-y$

Let $x^m - y^m = \ell(x-y)$ ....(1)

where ℓ is an integer

Now $P(m+1) = x^{m+1} - y^{m+1} = x^{m+1} - yx^m + yx^m + yx^{m+1}$

$= x^m(x-y) + y(x^m - y^m)$

$= x^m(x-y) + y.\ell(x-y)$

$= (x-y)(x^m + \ell y)$

which is divisible by $x - y$

$\therefore$ result is true for $n = m + 1$

$\therefore$ It the result is true for $n = m$, then it is also true for $n = m + 1$

But the result is ture for $n = 1$

$\therefore$ by the method of Induction, the result is true for all $n \in N$.

For $n = 1$

L.H.S. = 1 R.H.S = $\frac{1(1+1)}{2} = \frac{1 \times 2}{2} = 1$

$\therefore$ L.H.S. = R.H.S.

Ex.5 Use the principle of Mathematical Induction to prove that

$10^n + 3.4^{n+2} + 5$ divisible by 9 $\forall$ $n \in N$.

Sol. Let $P(n) = 10^n + 3.4^{n+2} + 5$

$\therefore P(1) = 10^1 + 3 \times 4^3 + 5$

$= 10 + 192 + 5 = 207$

$= 9 \times 23$, which is divisible by 9.

$\therefore$ result it true for n=1

Assume that result is true for n=m.

$\therefore$ $P(m) = 10^m + 3.4^{m+2} + 5$ is divisible by 9.

Let $10^m + 3.4^{m+2} + 5 = 9I$, where I is an integer

$\therefore$ $10^m = 9I - 3.4^{m+2} - 5$

$P(m+1) = 10^{m+1} + 3.4^{m+1+2} + 5$

$= 10^m.10 + 3.4^{m+2} . 4^1 + 5$

$= (9I - 3.4^{m+2} - 5).10 + 3.4^{m+2}12 + 5$

$= 90 I - 18 \times 4^{m+2} - 45$

$= 9(10I - 2.4^{m+2} - 5)$ which is divisible by9

$\therefore$ result is true for n=m+1

$\therefore$ by Mathematical Induction, result is true for all $n \in N$.

Ex.6 Use the method of Induction to prove that $n(n+1)(n+2)$ is a multiple of 6 $\forall$ $n \in N$.

Sol. Let $P(n) = n(n+1)(n+2)$

$\therefore$ $p(1) = 1(1+1)(1+2) = 6$ which is a multiple of 6

$\therefore$ result is true for n=1

Assume that result is true for n=m.

$\therefore$ $P(m) = m(m+1)(m+2)$ is a multiple of 6.

Let $m(m + 1)(m + 2) = 6I$ ....(1)

where I is an integer.

$p(m+1) = (m+1)(m+2)(m+3)$

$= (m+1)(m+2)(m) + (m+1)(m+2)(3)$

$= m(m+1)(m+2) + 3(m+1)(m+2)$

Now m+1 and m+2 are two consecutive integers and therefore, their product (m+1)(m + 2) is even.

Let $(k + 1)(m + 2) = 2m$ ....(2)

$\therefore$ $P(m+1) = 6I + 3(2k)$ [$\because$ of (1) and (2)]

$\therefore P(m+1) = 6(I+k)$, which is a multiple of 6.

$\therefore$ Result is true for $n = m + 1$

$\therefore$ By method of Induction the result is true for all $n \in N$.

EXERCISE

Q.1 $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Q.2 $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Q.3 $1 + 4 + 7 + ... + (3n-2) = \frac{1}{2}n(3n-1)$

Q.4 $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

Q.5 $1 + 3 + 5 + ... + (2n - 1) = n^2$

Q.6 $1.2.3 + 2.3.4 + 3.4.5 + ... + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$

Q.7 $1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

Q.8 $1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \frac{n}{3}(4n^2 + 6n - 1)$

Q.9 $2 + 3.2 + 4.2^2 + ... + (n+1)\,2^{n-1} = n.2^n$

Q.10 $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Q.11 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

Q.12 $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + .. + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Q.13 $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + .. + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

Q.14 $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$

Q.15 $a + (a + d) + (a + 2d) + ... + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

Q.16 $(1+x)^n > 1 + nx$, for $n > 1$, $x > -1$

Q.17 $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r-1}, r \neq 1$

Q.18 $n < 2^n$

Q.19 $3^n > 2^n$

Q.20 For each natural number, $n(n+1)$ is a multiple of 2.

Q.21 $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} ..... + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$.

Q.22 Sum, $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for any positive integer n.

Q.23 $n(n+1)(n+2)$ is a multiple of 6.

Q.24 $n(n+1)(2n+1)$ is divisible by 6.

Q.25 $2^{3n} - 1$ is divisible by 7.

Q.26 $12^n + 2.5^{n-1}$ is divisible by 7.

Q.27 $3^{2n} - 1$ is divisible by 8.

Q.28 $4^n + 15n - 1$ is divisible by 9.

Q.29 $10^n + 3.4^{n+2} + 5$ is divisible by 9.

Q.30 The sum of the cubes of three consecutive natural numbers is divisible by 9.

Q.31 $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Q.32 $10^{2n-1} + 1$ is divisible 11.

Q.33 $12^n + 25^{n-1}$ is divisible by 13.

Q.34 $15^{2n-1} + 1$ is divisible by 16.

Q.35 $5^{2n-1} + 1$ is divisible by 24.

Q.36 $2.7^n + 3.5^n - 5$ is divisible by 24.

Q.37 $n(n^2 - 1)$ is divisible by 24. When n is an odd number geater than 2.

Q.38 $7^{2n} + 2^{3n-3}, 3^{n-1}$ is divisible by 25.

Q.39 $6^{n-2} + 7^{2n+1}$ is divisible by 43.

Q.40 $7^{2n} - 1$ is divisible by 48.

Q.41 $3^{2n+2} - 8n - 9$ is divisible by 64.

Q.42 $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Q.43 $5^{2n+2} - 24n - 25$ is divisible by 578.

Q.44 $x^n - y^n$ is divisible by $x - y$.

Q.45 $x^n - y^n$ is divisible by $x+y$ when n is even.

Q.46 $p^{n+1} + (p+1)^{2n-1}$ is divisible by p^2+p+1 where p is a natural number.

Q.47 $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$.

Q.48 Show that if statement P (n) : $2 + 4 + 6 ......+ 2n = n(n + 1) + 2$ is true for n = k then is also true for n = k + 1. Can we apply principle of mathematical induction?

Q.49 Prove that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for $n \in N$.

Q.50 Prove that $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + ...+ \frac{1}{1+2+3.....+n} = \frac{2n}{n+1}$.