BINOMIAL THEOREM

BINOMIAL EXPRESSION

Any algebraic expression which contains two dissimilar terms is called Binomial expression.

For example : $x - y, xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

Terminology Used in Binomial Theorem

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

❖ $n! = n\,.\,(n-1)!$; $n \in \mathbb{N}$

BINOMIAL THEOREM

The formula by which any positive integral power of a Binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y $\in$ R and n $\in$ N, then : $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$

This theorem can be proved by induction.

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.
(b) The sum of the indices of x & y in each term is n.
(c) The Binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$.....) equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{n-r}$

(d) Symbol ${}^nC_r$ can also be denoted by $\binom{n}{r}$, C(n, r) or A_r^n .

❖ The coefficient of x^r in $(1+x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r\,.\,{}^nC_r$

Some Important Expansions :

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + ... + (-1)^n\,.\,{}^nC_nx^n$.

Ex. Expand the following Binomials :

(i) $(x - 3)^5$ **(ii)** $\left(1 - \frac{3x^2}{2}\right)^4$

Sol. **(i)** $(x - 3)^5 = {}^5C_0x^5 + {}^5C_1x^4(-3)^1 + {}^5C_2x^3(-3)^2 + {}^5C_3x^2(-3)^3 + {}^5C_4x(-3)^4 + {}^5C_5(-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1 - \frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^2}{2}\right) + {}^4C_2\left(-\frac{3x^2}{2}\right)^2 + {}^4C_3\left(-\frac{3x^2}{2}\right)^3 + {}^4C_4\left(-\frac{3x^2}{2}\right)^4$

$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$

Ex. Find the value of $\dfrac{\left(18^3+7^3+3.18.7.25\right)}{3^6+6.243.2+15.81.4+20.27.8+15.9.16+6.3.32+64}$

Sol. The numerator is of the form $a^3 + b^3 + 3ab(a + b) = (a + b)^3$

Where, a = 18 and b = 7 $\therefore$ $N^r = (18 + 7)^3 = (25)^3$

Denominator can be written as

$$3^6+{}^6C_1.3^5.2^1+{}^6C_2.3^4.2^2+{}^6C_3.3^3.2^3+{}^6C_4\,3^2.2^4+{}^6C_5\,3.2^5+{}^6C_6 2^6=(3+2)^6=5^6=(25)^3$$

$$\therefore \quad \frac{Nr}{Dr}=\frac{(25)^3}{(25)^3}=1$$

Pascal's Triangle

A triangular arrangement of numbers as shown. The numbers give the coefficients for the expansion of $(x + y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
etc.

IMPORTANT TERMS IN THE BINOMIAL EXPANSION

(a) General Term :

The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$$

Ex. Find **(i)** 28^{th} term of $(5x + 8y)^{30}$ **(ii)** 7^{th} term of $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$

Sol. **(i)** $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27} = \dfrac{30!}{3!\,27!}\,(5x)^3 \,.\, (8y)^{27}$

(ii) 7th term of $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$

$$T_{6+1}={}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6=\frac{9!}{3!\,6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6=\frac{10500}{x^3}$$

(b) Middle Term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\, x^{n/2}.\, y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Middle term has greatest Binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow$ ^nC_r will be maximum

- When $r = \frac{n}{2}$ if n is even
- When $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

$\Rightarrow$ The term containing greatest Binomial coefficient will be middle term in the expansion of $(1 + x)^n$

Ex. Find the middle term(s) in the expansion of

(i) $\left(1-\frac{x^2}{2}\right)^{14}$ **(ii)** $\left(3a-\frac{a^3}{6}\right)^9$

Sol. (i) $\left(1-\frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term

$$T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}.$$

(ii) $\left(3a-\frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2}+1\right)^{th}$.

It means T_5 & T_6 are middle terms

$$T_5 = {}^9C_4(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}a^{17}$$

$$T_6 = {}^9C_5(3a)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}a^{19}.$$

(c) Term Independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Ex. Find the term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$.

Sol. General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$ For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term.

(d) Numerically Greatest Term :

Binomial expansion of $(a + b)^n$ is as follows : –

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

If we put certain values of a and b in RHS, then each term of Binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively

$$T_r = {}^nC_{r-1}\; a^{n-(r-1)}\, b^{r-1}$$

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

Now, $$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\,\frac{a^{n-r}\, b^r}{a^{n-r+1} b^{r-1}}\right| = \frac{n-r+1}{r}\,.\left|\frac{b}{a}\right|$$

Consider $$\left|\frac{T_{r+1}}{T_r}\right| \ge 1$$

$$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \ge 1$$

$$\frac{n+1}{r} - 1 \ge \left|\frac{a}{b}\right|$$

$$r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$$

Case - I When $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 ...., m - 1)$

i.e. $T_2 > T_1, T_3 > T_2, ..., T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ...T_{n+1} < T_n$

Conclusion

When $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ $(r = 1, 2, 3,........, m-1, m)$

i.e. $T_2 > T_1, T_3 > T_2, ..., T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ $(r = m+1, m+2, \ldots\ldots\ldots..n)$

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

(i) In any Binomial expansion, the middle term(s) has greatest Binomial coefficient.

In the expansion of $(a + b)^n$

n	No. of Greatest Binomial Coefficient	Greatest Binomial Coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Ex. Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \dfrac{1}{5}$

Sol. Using $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \le r \le \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|} - 1 \le r \le \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

Solving we get $2 \le r \le 3$

$\therefore$ r = 2, 3

So, the greatest terms are T_{2+1} **and** T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2 . 3^9 (-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal.

Ex. If n is positive integer, then prove that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ ....**(i)**

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now, $0 < 7 - 4\sqrt{3} < 1$

$0 < (7 - 4\sqrt{3})^n < 1$

Let $(7 - 4\sqrt{3})^n = f'$ ....**(ii)**

$\Rightarrow \quad 0 < f' < 1$

Adding **(i)** and **(ii)**

$$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$= 2\,[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}\,(4\sqrt{3})^2 + \ldots\ldots\ldots]$$

$I + f + f' =$ even integer $\Rightarrow$ $(f + f'$ must be an integer$)$

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

$I + 1 =$ even integer

therefore I is an odd integer.

Ex. What is the remainder when 5^{99} is divided by 13.

Sol. $5^{99} = 5.5^{98} = 5.\,(25)^{49} = 5\,(26 - 1)^{49}$

$$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + \ldots\ldots\ldots + {}^{49}C_{48}\,(26)^1 - {}^{49}C_{49}\,(26)^0]$$

$$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + \ldots\ldots\ldots + {}^{49}C_{48}\,(26)^1 - 1]$$

$$= 5\,[{}^{49}C_0(26)^{49} - {}^{49}C_1(26)^{48} + \ldots\ldots\ldots + {}^{49}C_{48}\,(26)^1 - 13] + 60$$

$= 13\,(k) + 52 + 8$ (where k is a positive integer)

$= 13\,(k + 4) + 8$

Hence, remainder is 8.

Some Standard Expansions

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\; x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 + {}^nC_1\, x^{n-1}\, y^1 + \ldots\ldots\ldots + {}^nC_r\, x^{n-r}\, y^r + \ldots\ldots\ldots + {}^nC_n\, x^0\, y^n \qquad \ldots\text{(i)}$$

(ii) Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r\,(-1)^r\, x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 - {}^nC_1\, x^{n-1}\, y^1 + \ldots + {}^nC_r\,(-1)^r x^{n-r}\, y^r + \ldots + {}^nC_n\,(-1)^n\, x^0\, y^n \qquad \ldots\text{(ii)}$$

(iii) Adding **(i)** & **(ii)**, we get

$$(x + y)^n + (x - y)^n = 2[{}^nC_0 x^n\, y^0 + {}^nC_2\, x^{n-2}\; y^2 + \ldots\ldots\ldots]$$

(iv) Subtracting **(ii)** from **(i)**, we get

$$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1}\; y^1 + {}^nC_3\, x^{n-3}\; y^3 + \ldots\ldots\ldots]$$

PROPERTIES OF BINOMIAL COEFFICIENTS

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots\ldots\ldots + C_n x^n = \sum_{r=0}^{n} {}^nC_r r^r \;;\; n \in N \qquad \ldots\text{(i)}$$

where $C_0, C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are called combinatorial (Binomial) coefficients.

(a) The sum of all the Binomial coefficients is 2^n.

Put x = 1, in **(i)** we get

$$C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n = 2^n \quad \Rightarrow \quad \sum_{r=0}^{n} {}^nC_r = 0 \qquad \ldots\text{(ii)}$$

(b) Put x= –1 in **(i)** we get

$$C_0 - C_1 + C_2 - C_3 \ldots\ldots\ldots\ldots + C_n = 0 \quad \Rightarrow \quad \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0 \qquad \ldots\text{(iii)}$$

(c) The sum of the Binomial coefficients at odd position is equal to the sum of the Binomial coefficients at even position and each is equal to 2^{n-1}.

From **(ii)** & **(iii)**, $C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots 1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

Ex. Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Sol. LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{12}C_{11} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{13}C_{11} + {}^{13}C_{10} + \ldots\ldots\ldots {}^{25}C_{10}$

and so on.

$\therefore$ LHS $= {}^{26}C_{11}$

Aliter

LHS = coefficient of x^{10} in $\{(1+x)^{10} + (1+x)^{11} + \ldots\ldots\ldots\ldots\ldots (1+x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $\left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Ex. Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n\,.\,2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Sol. **(i)** L.H.S. $= \sum_{r=1}^{n} r.\,{}^nC_r = \sum_{r=1}^{n} r.\dfrac{n}{r}.\,{}^{n-1}C_{r-1}$

$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots\ldots + {}^{n-1}C_{n-1}\right]$

$= n\,.\,2^{n-1}$

Aliter : (Using method of differentiation)

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n \quad ....\text{(A)}$$

Differentiating (A), we get

$$n(1+x)^{n-1} = C_1 + 2C_2 x + 3C_3 x^2 + ... + n.C_n x^{n-1}.$$

Put $x = 1$,

$$C_1 + 2C_2 + 3C_3 + ... + n.C_n = n.2^{n-1}$$

(ii) $\text{L.H.S.} = \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^{n} \frac{n+1}{r+1} {}^nC_r$

$$= \frac{1}{n+1} \sum_{r=0}^{n} {}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1} - 1\right]$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1} \quad \text{(where C is a constant)}$$

Put $x = 0$, we get, $C = -\frac{1}{n+1}$

$$\therefore \quad \frac{(1+x)^{n+1} - 1}{n+1} = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}$$

Put $x = 1$, we get

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

Put $x = -1$, we get

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... = \frac{1}{n+1}$$

Ex. Prove that $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin \frac{n\pi}{4}$.

Sol. Consider the expansion $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ ....**(i)**

putting $x = -i$ in **(i)** we get

$$(1-i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + ... (-1)^n C_n i^n$$

or $2^{n/2}\left[\cos\left(-\frac{n\pi}{4}\right) + i \sin\left(-\frac{n\pi}{4}\right)\right] = (C_0 - C_2 + C_4 - ....) - i\,(C_1 - C_3 + C_5 - ......)$ ....**(ii)**

Equating the imaginary part in **(ii)** we get $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin \frac{n\pi}{4}$.

Ex. If $(1 + x)^n = C_0 + C_1x + C_2 x^2 + ... + C_nx^n$ then prove that $\sum\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2 = (n-1)^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + ... + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + ...$

$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + ... + (C_2 + C_n)^2 + ... + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2 \sum\sum_{0 \le i < j \le n} C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \frac{2n!}{2.n!n!}\right\}$ $\Rightarrow$ $n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n}$ = R.H.S.

Multinomial Theorem

Using Binomial theorem, we have $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r,\ n \in N$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r \text{, where } s + r = n$$

This result can be generalized in the following form.

$$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1 + r_2 + ... + r_k = n} \frac{n!}{r_1!r_2!.....r_k!} x_1^{r_1} x_2^{r_2} ......x_k^{r_k}$$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!.....r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} ......x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular Cases

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Ex. Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Sol. $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!} (a)^{r_1} (-b)^{r_2} (-c)^{r_3} (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\frac{(10)!}{2!3!4!1!} (-1)^3 (-1)^4 = -12600$

Ex. Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Sol. Total number of terms = ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \dfrac{10!}{2! \times 3! \times 5!} = 2520$

BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots \infty$ provided $|x| < 1$.

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots.\ \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots.\ \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots.\ \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.\ \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots.. + \dfrac{(-1)^r(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots.. + \dfrac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

Approximations

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 \ldots\ldots$$

If **$x < 1$**, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately.
This is an approximate value of $(1 + x)^n$

Ex. Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Sol. $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{r!}(-x)^r$$

$$= (-1)^r\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}(-x)^r = \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}x^r$$

$$= \frac{(n-1)!\,n(n+1)\ldots\ldots(n+r-1)}{(n-1)!\,r!}x^r$$

Hence, coefficient of x^r is $\dfrac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$ Proved

Ex. If x is so small such that its square and higher powers may be neglected then find the approximate value of $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Sol. $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \dfrac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$

EXPONENTIAL SERIES

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1+\frac{x}{1!}\ln a+\frac{x^2}{2!}\ln^2 a+\frac{x^3}{3!}\ln^3 a+.......\infty$, where a > 0

(e) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.......\infty$

Ex. Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Sol. In the exponential series $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+....$

Hence $e^{2x+3} = e^3 \cdot e^{2x} = e^3\left[1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right]$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3 \cdot \frac{2^2}{2!} = 2e^3$.

LOGARITHMIC SERIES

(a) $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.......\infty$, where $-1 < x \le 1$

(b) $\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+.......\infty$, where $-1 \le x < 1$

❖ $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.......\infty = \ln 2$ ❖ $e^{\ln x} = x$; for all x > 0 ❖ $\ln 2 = 0.693$ ❖ $\ln 10 = 2.303$

Ex. If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e(1+px+qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - .....$$

Sol. Right hand side $= \left[\alpha x-\frac{\alpha^2x^2}{2}+\frac{\alpha^3x^3}{3}-.....\right]+\left[\beta x-\frac{\beta^2x^2}{2}+\frac{\beta^3x^3}{3}-....\right]$

$= \log_e(1+\alpha x)+\log_e(1+\beta x)$ $= \log_e(1+(\alpha+\beta)x+\alpha\beta x^2)$ $= \log_e(1+px+qx^2) =$ Left hand side.

Here, we have used the facts α + β = p and αβ = q. We know this from the given roots of the quadratic equation. We have also assumed that that both $|\alpha.x| < 1$ and $|\beta x| < 1$.

If x, y $\in$ R and and n $\in$ N, then : $(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ...+ {}^nC_rx^{n-r}y^r + ...+ {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$

1. Important Terms in the Binomial Expansion

(a) General term : The general terms or the (r + 1)th term in the expansion of (x + y)n is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}.y^r$$

(b) Middle Term

The middle term (s) is the expansion of $(x+y)^n$ is (are) :

(i) If n is even , there is only one middle terms which is given by $T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$.

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term Independent of x

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

2. If $\left(\sqrt{A}+B\right)^n = I + f$, where I & n are positive integers & $0 \le f < 1$, then

(a) $(I+f).\,f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I+f)(1-f) = k^n$ if n is even & $\sqrt{A} - B < 0$

3. Some Results on Binomial Coefficients

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ...\frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4}......+ \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + ...= C_n = 2^n$

(f) $C_0 + C_2 + C_4 + ...= C_1 + C_3 + C_5 + ... = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ...+ C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

Remember : $(2n)! = 2^n.\, n!\, [1.3.5.........(2n-1)]$

4. Greatest Coefficient & Greatest Term in Expansion of $(x + a)^n$

(a) If n is even greatest coefficient is ${}^nC_{n/2}$

If n is odd greatest coefficient is $\dfrac{{}^nC_{n-1}}{2}$ or $\dfrac{{}^nC_{n+1}}{2}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \& \ T_{P+1} \ \text{if} \ \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{is an integer} \\ T_{q+1} \ \text{if} \ \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{is non integer and} \in (q, q+1), q \in I \end{cases}$$

5. Binomial Theorem for Negative or Fractional Indices

If n $\in$ Q, then $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots \infty$ provided $|x| < 1$

Note

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots \infty$

(ii) $(1+x)^{-1} = 1 - x + x^2 + x^3 + \ldots\ldots \infty$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \infty$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots \infty$

6. Exponential Series

(a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots\ldots\infty$; where x may be any real or complex number & $e = \lim\limits_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$

(b) $a^x = 1 + \dfrac{x}{1!}\ln a + \dfrac{x^2}{2!}\ln^2 a + \dfrac{x^3}{3!}\ln^3 a + \ldots\ldots \infty$, where $a > 0$

7. Logarithmic Series

(a) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots.. \infty$, where $-1 < x \le 1$

(b) $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots.. \infty$, where $-1 \le x < 1$

(c) $\ln\dfrac{(1+x)}{(1-x)} = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots\ldots\infty\right) |x| < 1$