Sets & Relations

SETS

SET

A set is a collection of well defined objects which are distinct from each other. Set are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by small letters a, b, c ..... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$, e.g. the collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

METHODS TO WRITE A SET :

(i) Roster Method or Tabular Method :

In this method a set is described by listing elements, separated by commas and enclose then by curly brackets. Note that while writing the set in roster form, an element is not generally repeated e.g. the set of letters of word SCHOOL may be written as {S, C, H, O, L}.

(ii) Set builder form (Property Method) :

In this we write down a property or rule which gives us all the element of the set.

$A = \{x : P(x)\}$ or $\{x / P(x)\}$ where P(x) is the property by which $x \in A$ and colon (:) or Slash (/) stands for 'such that'

Solved Examples

Ex.1 Express set $A = \{x : x \in N \text{ and } x = 2n \text{ for } n \in N\}$ in roster form

Sol. A = {2, 4, 6,.........}

Ex.2 Express set $B = \{x^2 : x < 4, x \in W\}$ in roster form

Sol. B = {0, 1, 4, 9}

Ex.3 Express set A = {2, 5, 10, 17, 26} in set builder form

Sol. $A = \{x : x = n^2 + 1, n \in N, 1 \le n \le 5\}$

TYPES OF SETS

Null set or empty set : A set having no element in it is called an empty set or a null set or void set, it is denoted by ϕ or { }. A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton set : A set consisting of a single element is called a singleton set.

Finite set : A set which has only finite number of elements is called a finite set.

Order of a finite set : The number of elements in a finite set A is called the order of this set and denoted by O(A) or n(A). It is also called cardinal number of the set. e.g. A = {a, b, c, d}

$\Rightarrow$ n(A) = 4

Infinite set : A set which has an infinite number of elements is called an infinite set.

Equal sets : Two sets A and B are said to be equal if every element of A is member of B, and every element of B is a member of A. If sets A and B are equal, we write A = B and if A and B are not equal then $A \neq B$

Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same i.e. n(A) = n(B)

e.g. A = {1, 3, 5, 7}, B = {a, b, c, d}

$\Rightarrow$ n(A) = 4 and n(B) = 4

$\Rightarrow$ A and B are equivalent sets

Note - Equal sets are always equivalent but equivalent sets may not be equal

Solved Examples

Ex.4 Identify the type of set :

(i) $A = \{x \in N : 5 < x < 6\}$

(ii) A = {a, b, c}

(iii) A = {1, 2, 3, 4, ...}

(iv) A = {1, 2, 6, 7} and B = {6, 1, 2, 7, 7}

(v) A = {0}

Sol. (i) Null set (ii) finite set

(iii) infinite set (iv) equal sets

(v) singleton set

SUBSET AND SUPERSET :

Let A and B be two sets. If every element of A is an element B then A is called a subset of B and B is called superset of A. We write it as $A \subseteq B$. e.g. A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5, 6, 7} $\Rightarrow A \subseteq B$

If A is not a subset of B then we write $A \not\subseteq B$

PROPER SUBSET :

If A is a subset of B but $A \neq B$ then A is a proper subset of B and we write $A \subset B$. Set A is not proper subset of A so this is improper subset of A

Note :

(i) Every set is a subset of itself

(ii) Empty set ϕ is a subset of every set

(iii) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$

(iv) The total number of subsets of a finite set containing n elements is 2^n.

(v) Number of proper subsets of a set having n elements is $2^n - 1$.

(vi) Empty set ϕ is proper subset of every set except itself.

POWER SET :

Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

$X = \{x_1, x_2, x_3, \ldots\ldots\ldots x_n\}$ then $n(P(X)) = 2^n$; $n(P(P(x))) = 2^{2^n}$

Solved Examples

Ex.5 Examine whether the following statements are true or false :

(i) $\{a, b\} \not\subseteq \{b, c, a\}$

(ii) $\{a, e\} \not\subseteq \{x : x$ is a vowel in the English alphabet$\}$

(iii) $\{1, 2, 3\} \subseteq \{1, 3, 5\}$

(iv) $\{a\} \in \{a, b, c\}$

Sol. (i) False as {a, b} is subset of {b, c, a}

(ii) True as a, e are vowels

(iii) False as element 2 is not in the set {1, 3, 5}

(iv) False as $a \in \{a, b, c\}$ and $\{a\} \subseteq \{a, b, c\}$

Ex.6 If a set A = {a, b, c} then find the number of subsets of the set A and also mention the set of all the subsets of A.

Sol. Since n(A) = 3

$\therefore$ number of subsets of A is $2^3 = 8$

and set of all those subsets is P(A) named as power set

P(A): {ϕ, {a}, {b}, {c}, {a,b}, {b,c}, {a,c}, {a,b,c}}

Ex.7 Find power set of set $A = \{1, 2\}$

Sol. $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Ex.8 If ϕ denotes null set then find $P(P(P(\phi)))$

Sol. Let $P(\phi) = \{\phi\}$

$P(P(\phi)) = \{\phi, \{\phi\}\}$

$P(P(P(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

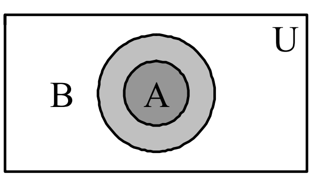
UNIVERSAL SET :

A set consisting of all possible elements which occur in the discussion is called a universal set i.e. it is a super set of each of the given set. Thus, a set that contains all sets in a given context is called the universal set. It is denoted by U. e.g. if $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 3, 5, 7\}$ then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

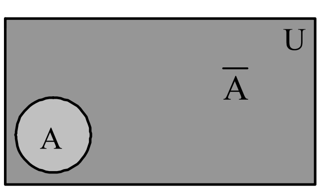
VENN (EULER) DIAGRAMS

The diagrams drawn to represent sets are called Venn diagram or Euler-Venn diagrams. Here we represents the universal U as set of all points within rectangle and the subset A of the set U is represented by the interior of a circle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B by two intersecting circles.

e.g. If A is subset of B then it is represented diagrammatically in fig.



e.g. If A is a set then the complement of A is represented in fig.

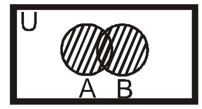


SOME OPERATION ON SETS :

(i) Union of two sets : If A and B are two sets then union ($\cup$) of A and B is the set of all those elements which belong either to A or to B or to both A and B.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$

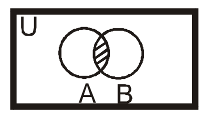


$A \cup B$

(ii) Intersection of two sets : If A and B are two sets then intersection ($\cap$) of A and B is the set of all those elements which belong to both A and B.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$

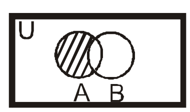


$A \cap B$

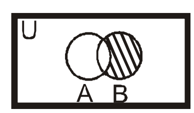
(iii) Difference of two sets : If A and B are two sets then the difference of A and B, is the set of all those elements of A which do not belong to B.

$A - B = \{x : x \in A \text{ and } x \notin B\}$. It is also written as $A \cap B'$.

Similarly $B - A = B \cap A'$ e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$; $A - B = \{1\}$

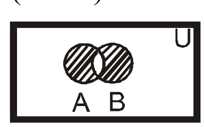


$A - B$



$B - A$

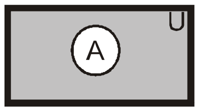
(iv) Symmetric difference of sets : Set of those elements which are obtained by taking the union of the difference of A & B is $(A - B)$ & the difference of B & A is $(B - A)$, is known as the symmetric difference of two sets A & B. It is denoted by $A \Delta B$ and $A \Delta B = (A - B) \cup (B - A)$



$(A \Delta B) = (A - B) \cup (B - A)$

(v) Complement of a set : Complement of a set A is a set containing all those elements of universal set which are not in A. It is denoted by $\overline{A}$, A^C or A'. So $A^C = \{x : x \in U \text{ but } x \notin A\}$. e.g. If set A = {1, 2, 3, 4, 5} and universal set

U = {1, 2, 3, 4,.......50} then $\overline{A}$ = {6, 7, ...50}



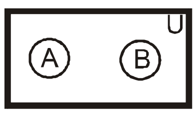
A'

NOTE :

All disjoint sets are not complementary sets but all complementary sets are disjoint.

(vi) Disjoint sets : If $A \cap B = \phi$, then A, B are disjoint

e.g. If A = {1, 2, 3}, B = {7, 8, 9} then $A \cap B = \phi$



Disjoint

LAWS OF ALGEBRA OF SETS (PROPERTIES OF SETS):

(i) Idempotent Law :

For any set A, we have (i) $A \cup A = A$ and (ii) $A \cap A = A$

Proof :

(i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \text{ \& } x \in A\} = \{x : x \in A\} = A$

(ii) Identity Law :

For any set A, we have

(i) $A \cup \phi = A$ and

(ii) $A \cap U = A$ i.e. ϕ and U are identity elements for union and intersection respectively

Proof :

(i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\}$

$= \{x : x \in A\} = A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\}$

$= \{x : x \in A\} = A$

(iii) Commutative Law :

For any set A and B, we have

(i) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

(iv) Associative Law :

If A, B and C are any three sets then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

i.e. union and intersection are associative.

(v) Distributive Law :

If A, B and C are any three sets then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

(vi) De-Morgan's Principle :

If A and B are any two sets, then

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Proof : (i) Let x be an arbitrary element of $(A \cup B)'$. Then $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$ $\qquad \Rightarrow x \in A' \cap B'$

Again let y be an arbitrary element of $A' \cap B'$. Then $y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$ $\qquad \Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$ $\qquad \Rightarrow y \in (A \cup B)'$

$\therefore A' \cap B' \subseteq (A \cup B)'$.

Hence $(A \cup B)' = A' \cap B'$

Similarly (ii) can be proved.

NOTE :

(i) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A \cap \phi = \phi$, $A \cup U = U$

Solved Examples

Ex.9 Let A = {2, 4, 6, 8} and B = {6, 8, 10, 12} then find $A \cup B$

Sol. $A \cup B = \{2, 4, 6, 8, 10, 12\}$

Ex.10 Let A = {1, 2, 3, 4, 5, 6}, B = {2, 4, 6, 8}. Find A – B and B – A.

Sol. $A - B = \{x : x \in A \text{ and } x \notin B\} = \{1, 3, 5\}$

similarly $B - A = \{8\}$

Ex.11 State true or false :

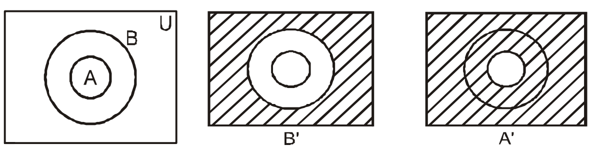
(i) $A \cup A' = \phi$ (ii) $\phi' \cap A = A$

Sol. (i) false because $A \cup A' = U$

(ii) true as $\phi' \cap A = U \cap A = A$

Ex.12 Use Venn diagram to prove that $A \subseteq B \Rightarrow B' \subseteq A'$.

Sol.



From venn diagram we can conclude that $B' \subseteq A'$.

Ex.13 Prove that if $A \cup B = C$ and $A \cap B = \phi$ then $A = C - B$.

Sol. Let $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in C$ $(\because A \cup B = C)$

Now $A \cap B = \phi \Rightarrow x \notin B$ $(\because x \in A)$

$\Rightarrow x \in C - B$ $(\because x \in C \text{ and } x \notin B) \Rightarrow A \subseteq C - B$

Let $x \in C - B \Rightarrow x \in C \text{ and } x \notin B$

$\Rightarrow x \in A \cup B \text{ and } x \notin B \Rightarrow x \in A$

$\Rightarrow C - B \subseteq A$ $\therefore A = C - B$

Ex.14 If $A = \{x : x = 2n + 1, n \in Z$ and $B = \{x : x = 2n, n \in Z\}$, then find $A \cup B$.

Sol. $A \cup B$ = {x : x is an odd integer} $\cup$ {x : x is an even integer} = {x : x is an integer} = Z

Ex.15 If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$ then find $A \cap B$.

Sol. We have,

$x \in A \cap B \Leftrightarrow x = 3n, n \in Z$ and $x = 4n, n \in Z$

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4

$\Leftrightarrow$ x is a multiple of 3 and 4 both

$\Leftrightarrow$ x is a multiple of 12 $\Leftrightarrow x = 12n, n \in Z$

Hence $A \cap B = \{x : x = 12n, n \in Z\}$

Ex.16 If A = {2, 3, 4, 5, 6, 7} and B = {3, 5, 7, 9, 11, 13} then find A – B and B – A..

Sol. A – B = {2, 4, 6} & B – A = {9, 11, 13)

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B & C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$ (if A & B are disjoint sets)

(iii) $n(A - B) = n(A) - n(A \cap B)$

(iv) $n(A \Delta B) = n[(A - B) \cup (B - A)]$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$

$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(vii) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Solved Examples

Ex.17 In a group of 40 students, 26 take tea, 18 take coffee and 8 take neither of the two. How many take both tea and coffee ?

Sol. $n(U) = 40, n(T) = 26,$ $n(C) = 18$

$n(T' \cap C') = 8 \Rightarrow n(T \cup C)' = 8$

$\Rightarrow n(U) - n(T \cup C) = 8$

$\Rightarrow n(T \cup C) = 32$

$\Rightarrow n(T) + n(C) - n(T \cap C) = 32$

$\Rightarrow n(T \cap C) = 12$

Ex.18 In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink tea and coffee both ?

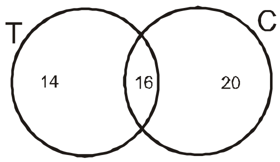
(ii) How many drink coffee but not tea ?

Sol. T : people drinking tea

C : people drinking coffee

(i) $n(T) = n(T - C) + n(T \cap C)$

$\Rightarrow 30 = 14 + n(T \cap C) \quad \Rightarrow n(T \cap C) = 16$



(ii) $n(C - T) = n(T \cup C) - n(T) = 50 - 30 = 20$

Ex.19 If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

Sol. We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case-I

When $n(A \cap B)$ is minimum, i.e., $n(A \cap B) = 0$

This is possible only when $A \cap B = \phi$. In this case,

$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$.

So, maximum number of elements in $A \cup B$ is 9.

Case-II

When $n(A \cap B)$ is maximum.

This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= (3 + 6 - 3) = 6$

So, minimum number of elements in $A \cup B$ is 6.

RELATIONS

ORDERED PAIR

A pair of objects listed in a specific order is called an ordered pair. It is written by listing the two objects in specific order separating them by a comma abd enclosing the pair in parantheses.

In the ordered pair (a, b), a is called the first element and b is called the second element.

Two ordered pairs are set to be equal if their corresponding elements are equal. i.e.

(a, b) = (c, d) if a = c and b = d.

CARTESIAN PRODUCT

The set of all possible ordered pairs (a, b), where a $\in$ A and b $\in$ B i.e. {(a, b) ; a $\in$ A and b $\in$ B} is called the cartesian product of A to B and is denoted by A × B.

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

$= \{(a_1, b_1),(a_1, b_2),(a_2, b_1),(a_2, b_2),(a_3, b_1),(a_3, b_2)\}$

If set $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ then

A × B and B × A can be written as :

A × B = {(a, b) : a $\in$ A and b $\in$ B and

B × A = {(b, a) ; b $\in$ B and a $\in$ A}

Clearly A × B $\neq$ B × A until A and B are equal

Note :

1. If number of elements in A : n(A) = m and n(B) = n then number of elements in (A × B) = m × n

2. Since A × B contains all such ordered pairs of the type (a, b) such that a $\in$ A & b $\in$ B, that means it includes all possibilities in which the elements of set A can be related with the elements of set B. Therefore, A × B is termed as largest possible relation defined from set A to set B, also known as universal relation from A to B.

Similarly A × B × C = {(a, b, c) : a $\in$ A, b $\in$ B, c $\in$ C} is called ordered triplet.

Solved Examples

Ex.1 If A = {1, 2} and B = {3, 4}, then find A × B.

Sol. A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}

Ex.2 Let A and B be two non-empty sets having elements in common, then prove that A × B and B × A have n^2 elements in common.

Sol. We have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

On replacing C by B and D by A, we get

$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

It is given that AB has n elements so $(A \cap B) \times (B \cap A)$ has n^2 elements

But $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$\therefore$ $(A \times B) \cap (B \times A)$ has n^2 elements

Hence A × B and B × A have n^2 elements in common.

RELATION

A relation R from set A to B (R : A $\to$ B) is a correspondence between set A to set B by which some or more elements of A are associated with some or more elements of B.

Therefore a relation R from A to B is a subset of A × B. Thus, R is a relation from A to B $\Rightarrow$ R $\subseteq$ A × B. The subsets is derived by describing a relationship between the first element and the second element of ordered pairs in A × B e.g. if A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 2, 3, 4, 5} and R = {(a, b) : a = b^2, a $\in$ A, b $\in$ B} then R = {(1, 1), (4, 2), (9, 3)}. Here a R b $\Rightarrow$ 1 R 1, 4 R 2, 9 R 3.

NOTE :

(i) If a is related to b then symbolically it is written as a R b where a is pre-image and b is image

(ii) If a is not related to b then symbolically it is written as a $\not R$ b.

(iii) Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A × B consists of mn ordered pairs. So total number of subsets of A × B i.e. number of relations from A to B is 2^{mn}.

(iv) A relation R from A to A is called a relation on A.

Solved Examples

Ex.3 If the number of elements in A is m and number of element in B is n then find

(i) The number of elements in the power set of A × B.

(ii) number of relation defined from A to B

Sol. (i) Since n(A) = m; n(B) = n then n(A × B) = mn

So number of subsets of A × B = 2^{mn}

$\Rightarrow$ n (P(A × B)) = 2^{mn}

(ii) number of relation defined from A to B = 2^{mn}

Any relation which can be defined from set A to set B will be subset of A × B

$\because$ A × B is largest possible relation A → B

$\therefore$ no. of relation from A → B = no. of subsets of set (A × B)

DOMAIN, CO-DOMAIN AND RANGE OF A RELATION :

Let R be a relation from a set A to a set B. Then the set of all first components of coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components of corrdinates of the ordered pairs in R is called the range of R.

Thus, Domain (R) = {a : (a, b) ∈ R},

Co-domain (R) = B and

Range (R) = {b : (a, b) ∈ R}

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of Co-domain (B).

Solved Examples

Ex.4 Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "(x, y) ∈ R ⇒ x > y". Find relation R and its domain and range.

Sol. Under relation R, we have 3R2, 5R2, 5R4, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and range (R) = {2, 4, 6}

Ex.5 Let A = {2, 3, 4, 5, 6, 7, 8, 9}. Let R be the relation on A defined by

{(x, y) : x ∈ A, y ∈ A and x divides y}.

Find domain and range of R.

Sol. The relation R is

R = {(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}

Domain of R = {2, 3, 4, 5, 6, 7, 8, 9} = A

Range of R = {2, 3, 4, 5, 6, 7, 8, 9} = A

Ex.6 Let R be the relation on the set N of natural numbers defined by

R : {(x, y)} : x + 3y = 12 x ∈ N, y ∈ N} Find

(i) R **(ii)** Domain of R **(iii)** Range of R

Sol. (i) We have, x + 3y =12 ⇒ x = 12 – 3y

Putting y = 1, 2, 3, we get x = 9, 6, 3 respectively

For y = 4, we get x = 0 ∉ N. Also for y > 4, x ∉ N

$\therefore$ R = {(9, 1), (6, 2), (3, 3)}

(ii) Domain of R = {9, 6, 3}

(iii) Range of R = {1, 2, 3}

INVERSE OF A RELATION

Let A, B be two sets and let R be a relation from a set A to B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by R^{-1} = {(b, a) : (a, b) ∈ R}, Clearly,

(a, b) ∈ R ⇔ (b, a) ∈ R^{-1} Also,

Dom of R = Range of R^{-1} and

Range of R = Dom of R^{-1}

Solved Examples

Ex.7 Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$ i.e., $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Determine also :

(i) Domains of R and R^{-1}

(ii) Range of R and R^{-1}

Sol. We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10-x}{2}$, $x, y \in A$

where $A = \{1,2,3,4,5,6,7,8,9,10\}$

Now, $x = 1 \Rightarrow y = \frac{10-1}{2} = \frac{9}{2} \notin A$

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of a under the defined relation. Further we find that

for $x = 2$, $y = \frac{10-2}{2} = 4 \in A \quad \therefore (2, 4) \in R$

for $x = 4$, $y = \frac{10-4}{2} = 3 \in A \quad \therefore (4, 3) \in R$

for $x = 6$, $y = \frac{10-6}{2} = 2 \in A \quad \therefore (6, 2) \in R$

for $x = 8$, $y = \frac{10-8}{2} = 1 \in A \quad \therefore (8, 1) \in R$

Thus $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom (R) = {2, 4, 6, 8} = Range (R^{-1})

and Range (R) = {4, 3, 2, 1} = Dom (R^{-1})

TYPES OF RELATIONS

In this section we intend to define various types of relations on a given set A.

(i) Void relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(ii) Universal relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(iii) Identity relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

(iv) Reflexive relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Note :

Every identity relation is reflexive but every reflexive relation in not identity.

(v) Symmetric relation : A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$. i.e. $a R b \Rightarrow b R a$ for all $a, b \in A$.

(vi) Transitive relation : Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

Note :

(i) Every null relation is a symmetric and transitive relation.

(ii) Every singleton relation is a transitive relation.

(iii) Universal and identity relations are reflexive, symmetric as well as transitive.

(iv) Anti-symmetric Relation

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$ e.g. Relations "being subset of"; "is greater than or equal to" and "identity relation on any set A" are antisymmetric relations.

(v) Equivalence relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b \in A$

Solved Examples

Ex.8 Which of the following are identity relations on set $A = \{1, 2, 3\}$.

$R_1 = \{(1, 1), (2, 2)\}$, $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$, $R_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Sol. The relation R_3 is idenity relation on set A.

R_1 is not identity relation on set A as $(3, 3) \notin R_1$.

R_2 is not identity relation on set A as $(1, 3) \in R_2$

Ex.9 Which of the following are reflexive relations on set $A = \{1, 2, 3\}$.

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$, $R_2 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$.

Sol. R_1 is a reflexive relation on set A.

R_2 is not a reflexive relation on A because $2 \in A$ but $(2, 2) \notin R_2$.

Ex.10 Prove that on the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive.

Sol. Because for any $x, y, z \in N$ $x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$. so R is transitive.

Ex.11 Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Sol. Since a relation R in T is said to be an equivalenece relation if R is reflexive, symmetric and transitive.

(i) Since every triangle is congruent to itself

$\therefore$ R is reflexive

(ii) $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2

$\Rightarrow T_2$ is congruent to $T_1 \quad \Rightarrow (T_2, T_1) \in R$

Hence R is symmetric

(iii) Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to T_3

$\Rightarrow T_1$ is congruent to $T_3 \quad \Rightarrow (T_1, T_3) \in R$

$\therefore$ R is transitive

Hence R is an equivalence relation.

Ex.12 Show that the relation R in R defined as $R = \{(a, b) : a \le b\}$ is transitive.

Sol. Let $(a, b) \in R$ and $(b, c) \in R$

$\therefore$ $(a \le b)$ and $b \le c \Rightarrow a \le c$

$\therefore$ $(a, c) \in R$

Hence R is transitive.

Ex.13 Show that the relation R in the set {1, 2, 3} given by $R = \{(1, 2), (2, 1)\}$ is symmetric.

Sol. Let $(a, b) \in R$ $\qquad [\because (1, 2) \in R]$

$\therefore$ $(b, a) \in R$ $\qquad [\because (2, 1) \in R]$

Hence R is symmetric.

Ex.14 If $X = \{x_1, x_2, x_3\}$ and $y = (x_1, x_2, x_3, x_4, x_5\}$ then find which is a reflexive relation of the following :

(a) $R_1 : \{(x_1, x_1), (x_2, x_2)$

(b) $R_1 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3)$

(c) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_3), (x_2, x_4)$

(d) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)$

Sol. (a) non-reflexive because $(x_3, x_3) \notin R_1$

(b) Reflexive

(c) Reflexive

(d) non-reflexive because $x_4 \notin X$

Ex.15 If $x = \{a, b, c\}$ and $y = \{a, b, c, d, e, f\}$ then find which of the following relation is symmetric relation :

$R_1 : \{ \}$ i.e. void relation

$R_2 : \{(a, b)\}$

$R_3 : \{(a, b), (b, a)(a, c)(c, a)(a, a)\}$

Sol. R_1 is symmetric relation because it has no element in it.

R_2 is not symmetric because $(b, a) \in R_2$ & R_3 is symmetric.

Ex.16 If x = {a, b, c} and y = (a, b, c, d, e} then which of the following are transitive relation.

(a) R_1 = { }

(b) R_2 = {(a, a)}

(c) R_3 = {(a, a}.(c, d)}

(d) R_4 = {(a, b), (b, c)(a, c),(a, a), (c, a)}

Sol. (a) R_1 is transitive relation because it is null relation.

(b) R_2 is transitive relation because all singleton relations are transitive.

(c) R_3 is transitive relation

(d) R_4 is also transitive relation