REAL NUMBERS

CONTENTS

NUMBER SYSTEM

◈ **Natural Numbers :**

The simplest numbers are 1, 2, 3, 4....... the numbers being used in counting. These are called natural numbers.

◈ **Whole numbers :**

The natural numbers along with the zero form the set of whole numbers i.e. numbers 0, 1, 2, 3, 4 are whole numbers. W = {0, 1, 2, 3, 4....}

◈ **Integers :**

The natural numbers, their negatives and zero make up the integers.

$Z = \{....-4, -3, -2, -1, 0, 1, 2, 3, 4,....\}$

The set of integers contains positive numbers, negative numbers and zero.

◈ **Rational Number :**

(i) A rational number is a number which can be put in the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$.

(ii) A rational number is either a terminating or non-terminating and recurring (repeating) decimal.

(iii) A rational number may be positive, negative or zero.

◈ **Complex numbers :**

Complex numbers are imaginary numbers of the form a + ib, where a and b are real numbers and $i = \sqrt{-1}$, which is an imaginary number.

◈ **Factors :**

A number is a factor of another, if the former exactly divides the latter without leaving a remainder (remainder is zero) 3 and 5 are factors of 12 and 25 respectively.

◈ **Multiples :**

A multiple is a number which is exactly divisible by another, 36 is a multiple of 2, 3, 4, 9 and 12.

◈ **Even Numbers :**

Integers which are multiples of 2 are even number (i.e.) 2,4, 6, 8............... are even numbers.

◈ **Odd numbers :**

Integers which are not multiples of 2 are odd numbers.

◈ **Prime and composite Numbers :**

All natural number which cannot be divided by any number other than 1 and itself is called a prime number. By convention, 1 is not a prime number. 2, 3, 5, 7, 11, 13, 17 .... are prime numbers. Numbers which are not prime are called composite numbers.

◈ **The Absolute Value (or modulus) of a real Number :**

If a is a real number, modulus a is written as |a| ; |a| is always positive or zero.It means positive value of 'a' whether a is positive or negative

|3| = 3 and |0| = 0, Hence |a| = a ; if a = 0 or a > 0 (i.e.) $a \geq 0$

|–3| = 3 = – (–3) . Hence |a| = – a when a < 0

Hence, |a| = a, if a > 0 ; |a| = – a, if a < 0

◈ **Irrational number :**

(i) A number is irrational if and only if its decimal representation is non-terminating and non-repeating. e.g. $\sqrt{2}$, $\sqrt{3}$, π............... etc.

(ii) Rational number and irrational number taken together form the set of real numbers.

(iii) If a and b are two real numbers, then either (i) a > b or (ii) a = b or (iii) a < b

(iv) Negative of an irrational number is an irrational number.

(v) The sum of a rational number with an irrational number is always irrational.

(vi) The product of a non-zero rational number with an irrational number is always an irrational number.

(vii) The sum of two irrational numbers is not always an irrational number.

(viii) The product of two irrational numbers is not always an irrational number.

In division for all rationals of the form $\frac{p}{q}$ (q ≠ 0), p & q are integers, two things can happen either the remainder becomes zero or never becomes zero.

Type (1) : Eg : $\frac{7}{8} = 0.875$

```
8)70 (0.875
  64
  ---
   60
   56
  ---
    40
    40
  ---
    ×
```

This decimal expansion 0.875 is called **terminating**.

∴ If remainder is zero then decimal expansion ends (terminates) after finite number of steps. These decimal expansion of such numbers terminating.

Type (2) :

Eg : $\frac{1}{3} = 0.333$……….

$= 0.\overline{3}$

```
3)10 (0.33……
   9
  ---
   10
    9
  ---
    1……
```

or $\frac{1}{7} = 0.142857142857$…..

$= 0.\overline{142857}$

```
7)10 (0.14285….
   7
  ---
   30
   28
  ---
    20
    14
  ---
     60
     56
  ---
      40
      35
  ---
       50
       49
  ---
        1….
```

In both examples remainder is never becomes zero so the decimal expansion is never ends after some or infinite steps of division. These type of decimal expansions are called **non terminating.**

In above examples, after Ist step & 6 steps of division (respectively) we get remainder equal to dividend so decimal expansion is repeating (recurring).

So these are called **non terminating recurring decimal expansions**.

Both the above types (1 & 2) are rational numbers.

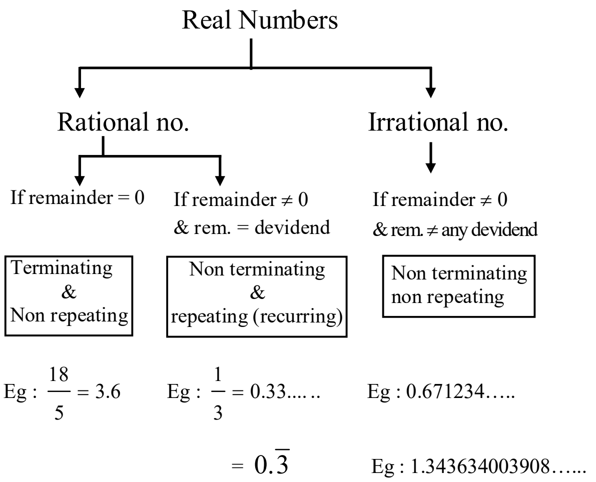
Types (3) :

Eg : The decimal expansion 0.327172398……is not ends any where, also there is no arrangement of digits (not repeating) so these are called **non terminating not recurring**.

These numbers are called **irrational numbers**.

Eg. :

0.1279312793	rational	terminating
0.1279312793….	rational	non terminating
or $0.\overline{12793}$		& recurring
0.32777	rational	terminating
$0.32\overline{7}$ or	rational	non terminating
0.32777…….		& recurring
0.5361279	rational	terminating
0.3712854043….	irrational	non terminating
		non recurring
0.10100100010000	rational	terminating
0.10100100010000….	irrational	non terminating
		non recurring.



❖ EXAMPLES ❖

Ex.1 Insert a rational and an irrational number between 2 and 3.

Sol. If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b. Also, if a,b are rational numbers, then $\frac{a+b}{2}$ is a rational number between them.

$\therefore$ A rational number between 2 and 3 is

$$\frac{2+3}{2} = 2.5$$

An irrational number between 2 and 3 is

$$\sqrt{2\times 3} = \sqrt{6}$$

Ex.2 Find two irrational numbers between 2 and 2.5.

Sol. If a and b are two distinct positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between 2 and 2.5 is

$$\sqrt{2\times 2.5} = \sqrt{5}$$

Similarly, irrational number between 2 and $\sqrt{5}$ is $\sqrt{2\times\sqrt{5}}$

So, required numbers are $\sqrt{5}$ and $\sqrt{2\times\sqrt{5}}$.

Ex.3 Find two irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$.

Sol. We know that, if a and b are two distinct positive irrational numbers, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between $\sqrt{2}$ and $\sqrt{3}$ is $\sqrt{\sqrt{2}\times\sqrt{3}} = \sqrt{\sqrt{6}} = 6^{1/4}$

Irrational number between $\sqrt{2}$ and $6^{1/4}$ is $\sqrt{\sqrt{2}\times 6^{1/4}} = 2^{1/4}\times 6^{1/8}$.

Hence required irrational number are $6^{1/4}$ and $2^{1/4}\times 6^{1/8}$.

Ex.4 Find two irrational numbers between 0.12 and 0.13.

Sol. Let a = 0.12 and b = 0.13. Clearly, a and b are rational numbers such that a < b.

We observe that the number a and b have a 1 in the first place of decimal. But in the second place of decimal a has a 2 and b has 3. So, we consider the numbers

c = 0.1201001000100001 ......

and, d = 0.12101001000100001.......

Clearly, c and d are irrational numbers such that a < c < d < b.

Theorem : Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.

Proof : Let the prime factorisation of a be as follows :

$a = p_1p_2.....p_n$, where $p_1,p_2,.....p_n$ are primes, not necessarily distinct.

Therefore,

$a^2 = (p_1p_2.....p_n)\ (p_1p_2\ .....\ p_n) = p_1^2 p_2^2 \ldots.. p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2,\ldots, p_n$. So p is one of $p_1, p_2,\ldots\ldots, p_n$.

Now, since $a = p_1\, p_2 \ldots\ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'.

Ex.5 Prove that

(i) $\sqrt{2}$ is irrational number

(ii) $\sqrt{3}$ is irrational number

Similarly $\sqrt{5}, \sqrt{7}, \sqrt{11}$ ...... are irrational numbers.

Sol. (i) Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s not having a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2. Now, by Theorem it following that 2 divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

(ii) Let us assume, to contrary, that $\sqrt{3}$ is rational. That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}$.

Suppose a and b not having a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

Ex.6 Prove that $7 - \sqrt{3}$ is irrational

Sol. **Method I :**

Let $7 - \sqrt{3}$ is rational number

$\therefore\ 7-\sqrt{3}=\frac{p}{q}$ (p, q are integers, $q \neq 0$)

$\therefore\ 7-\frac{p}{q}=\sqrt{3}$

$\Rightarrow\ \sqrt{3}=\frac{7q-p}{q}$

Here p, q are integers

$\therefore\ \frac{7q-p}{q}$ is also integer

$\therefore$ LHS = $\sqrt{3}$ is also integer but this is contradiction that $\sqrt{3}$ is irrational so our assumption is wrong that $(7-\sqrt{3})$ is rational

$\therefore\ 7-\sqrt{3}$ is irrational proved.

Method II :

Let $7-\sqrt{3}$ is rational

we know sum or difference of two rationals is also rational

$\therefore\ 7-(7-\sqrt{3})$

= $\sqrt{3}$ = rational

but this is contradiction that $\sqrt{3}$ is irrational

$\therefore\ (7-\sqrt{3})$ is irrational proved.

Ex.7 Prove that :

(i) $\frac{\sqrt{5}}{3}$ (ii) $2\sqrt{7}$ are irrationals

Sol. (i) Let $\frac{\sqrt{5}}{3}$ is rational

$\therefore\ 3\left(\frac{\sqrt{5}}{3}\right)$ = $\sqrt{5}$ is rational

($\because$ product of two rationals is also rational)

but this is contradiction that $\sqrt{5}$ is irrational

$\therefore\ \frac{\sqrt{5}}{3}$ is irrational proved.

(ii) Let $2\sqrt{7}$ is rational

$\therefore\ (2\sqrt{7})\times\frac{1}{2}=\sqrt{7}$

($\because$ division of two rational no. is also rational)

$\therefore\ \sqrt{7}$ is rational

but this is contradiction that $\sqrt{7}$ is irrational

$\therefore\ 2\sqrt{7}$ is irrational

proved

Theorem 1 :

Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

(A) Numbers are terminating (remainder = zero)

Eg : $\frac{32}{125}=\frac{2^5}{5^3}=\frac{2^8}{(2\times5)^3}=\frac{256}{10^3}=0.256$

Eg : $\frac{9}{25}=\frac{9\times2^2}{5^2\times2^2}=\frac{36}{(2\times5)^2}=\frac{36}{(10)^2}=0.36$

So we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m to an equivalent rational number of the form $\frac{a}{b}$ where b is a power of 10. These are terminates.

OR

Theorem 2 :

Let x = $\frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form $2^n\,5^m$, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

(B) Non terminating & recurring

Eg: $\frac{1}{7} = 0.\overline{142857} = 0.142857142857.....$

Since denominator 7 is not of the form $2^n\ 5^m$ so we zero (0) will not show up as a remainder.

Theorem 3 :

Let x = $\frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

From the discussion above, we can conclude that the decimal expansion of every rational number is either terminating or non-terminating repeating.

Eg : From given rational numbers check terminating or non terminating

(1) $\frac{13}{3125} = \frac{13}{(5)^5} = \frac{13 \times 2^5}{2^5 \times 5^5} = \frac{(13 \times 32)}{(10)^5}$

= terminating

(2) $\frac{17}{8} = \frac{17}{2^3} = \frac{17 \times 5^3}{(2 \times 5)^3} = \frac{17 \times 125}{(10)^3}$

= terminating

(3) $\frac{64}{455} = \frac{2^6}{5 \times 7 \times 13}$ ($\because$ we can not remove 7 & 13 from dinominator) non-terminating repeating ($\because$ no. is rational $\therefore$ it is always repeating or recurring)

(4) $\frac{15}{1600} = \frac{3 \times 5}{2^4 \times 10^2} = \frac{3 \times 5^5}{(2 \times 5)^4 \times 10^2} = \frac{3 \times 5^5}{10^6}$

= terminating

(5) $\frac{29}{343} = \frac{29}{(7)^3}$ = non terminating

(6) $\frac{23}{2^3 5^2} = \frac{23 \times 5}{(2 \times 5)^3} = \frac{23 \times 5}{(10)^3}$

= terminating

(7) $\frac{129}{2^5 \times 5^7 \times 7^5} = \frac{3 \times 43 \times 2^2}{(2 \times 5)^7 \times 7^5}$

= non terminating ($\because$ 7 cannot remove from denominator)

(8) $\frac{6}{15} = \frac{2 \times 3}{5 \times 3} = \frac{2}{5} = \frac{2 \times 2}{10}$

= terminating

(9) $\frac{35}{50} = \frac{35 \times 2}{100}$ = terminating

(10) $\frac{77}{210} = \frac{7 \times 11}{7 \times 30} = \frac{7 \times 11}{7 \times 2 \times 5 \times 3}$

= non terminating

➤ EUCLID'S DIVISION LEMMA OR EUCLID'S DIVISION ALGORITHM

For any two positive integers **a** and **b**, there exist unique integers q and r satisfying a = bq + r, where $0 \le r < b$.

For Example

(i) Consider number 23 and 5, then:

23 = 5 × 4 + 3

Comparing with a = bq + r; we get:

a = 23, b = 5, q = 4, r = 3

and $0 \le r < b$ (as $0 \le 3 < 5$).

(ii) Consider positive integers 18 and 4.

18 = 4 × 4 + 2

⇒ For 18 (= a) and 4(= b) we have q = 4, r = 2 and $0 \le r < b$.

In the relation a = bq + r, where $0 \le r < b$ is nothing but a statement of the long division of number a by number b in which q is the quotient obtained and r is the remainder.

Thus, dividend = divisor × quotient + remainder ⇒ a = bq + r

❖ H.C.F. (Highest Common Factor)

The H.C.F. of two or more positive integers is the largest positive integer that divides each given positive number completely.

i.e., if positive integer **d** divides two positive integers **a** and **b** then the H.C.F. of **a** and **b** is **d**.

For Example

(i) 14 is the largest positive integer that divides 28 and 70 completely; therefore H.C.F. of 28 and 70 is 14.

(ii) H.C.F. of 75, 125 and 200 is 25 as 25 divides each of 75, 125 and 200 completely and so on.

◈ Using Euclid's Division Lemma For Finding H.C.F.

Consider positive integers 418 and 33.

Step-1

Taking bigger number (418) as **a** and smaller number (33) as **b**

express the numbers as a = bq + r

$\Rightarrow$ 418 = 33 × 12 + 22

Step-2

Now taking the divisor 33 and remainder 22; apply the Euclid's division algorithm to get:

33 = 22 × 1 + 11 [Expressing as a = bq + r]

Step-3

Again with new divisor 22 and new remainder 11; apply the Euclid's division algorithm to get:

22 = 11 × 2 + 0

Step-4

Since, the remainder = 0 so we cannot proceed further.

Step-5

The last divisor is 11 and we say H.C.F. of 418 and 33 = 11

Verification :

(i) Using factor method:

$\therefore$ Factors of 418 = 1, 2, 11, 19, 22, 38, 209 and 418 and,

Factor of 33 = 1, 3, 11 and 33.

Common factors = 1 and 11

$\Rightarrow$ Highest common factor = 11 i.e., H.C.F. = 11

(ii) Using prime factor method:

Prime factors of 418 = 2, 11 and 19.

Prime factors of 33 = 3 and 11.

$\therefore$ **H.C.F.** = Product of all common prime factors = 11. For any two positive integers a and b which can be expressed as a = bq + r, where $0 \le r < b$, the, H.C.F. of (a, b) = H.C.F. of (q, r) and so on. For number 418 and 33

418 = 33 × 12 + 22

33 = 22 × 1 + 11

and 22 = 11 × 2 + 0

$\Rightarrow$ H.C.F. of (418, 33) = H.C.F. of (33, 22)

= H.C.F. of (22, 11) = 11.

EXAMPLES

Ex.8 Using Euclid's division algorithm, find the H.C.F. of **[NCERT]**

(i) 135 and 225 (ii) 196 and 38220

(iii) 867 and 255

Sol.(i) Starting with the larger number i.e., 225, we get:

225 = 135 × 1 + 90

Now taking divisor 135 and remainder 90, we get 135 = 90 × 1 + 45

Further taking divisor 90 and remainder 45, we get 90 = 45 × 2 + 0

$\therefore$ **Required H.C.F. = 45** **(Ans.)**

(ii) Starting with larger number 38220, we get:

38220 = 196 × 195 + 0

Since, the remainder is 0

$\Rightarrow$ **H.C.F. = 196** **(Ans.)**

(iii) Given number are 867 and 255

$\Rightarrow$ 867 = 255 × 3 + 102 **(Step-1)**

255 = 102 × 2 + 51 **(Step-2)**

102 = 51 × 2 + 0 **(Step-3)**

$\Rightarrow$ **H.C.F. = 51** **(Ans.)**

Ex.9 Show that every positive integer is of the form 2q and that every positive odd integer is of the from 2q + 1, where q is some integer.

Sol. According to Euclid's division lemma, if a and b are two positive integers such that a is greater than b; then these two integers can be expressed as

a = bq + r; where $0 \le r < b$

Now consider

$b = 2$; then $a = bq + r$ will reduce to

$a = 2q + r$; where $0 \leq r < 2$,

i.e., $r = 0$ or $r = 1$

If $r = 0$, $a = 2q + r \Rightarrow a = 2q$

i.e., a is even

and, if $r = 1$, $a = 2q + r \Rightarrow a = 2q + 1$

i.e., a is add;

as if the integer is not even; it will be odd.

Since, a is taken to be any positive integer so it is applicable to the every positive integer that when it can be expressed as

$a = 2q$

$\therefore$ **a** is even and when it can expressed as

$a = 2q + 1$; a is odd.

Hence the required result.

Ex.10 Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.

Sol. Let **a** is **b** be two positive integers in which **a** is greater than **b**. According to Euclid's division algorithm; **a** and **b** can be expressed as

$a = bq + r$, where q is quotient and r is remainder and $0 \leq r < b$.

Taking $b = 4$, we get: $a = 4q + r$,

where $0 \leq r < 4$ i.e., $r = 0, 1, 2$ or 3

$\mathbf{r = 0} \Rightarrow a = 4q$, which is divisible by 2 and so is **even**.

$\mathbf{r = 1} \Rightarrow a = 4q + 1$, which is not divisible by 2 and so is **odd**.

$\mathbf{r = 2} \Rightarrow q = 4q + 2$, which is divisible by 2 and so is **even**.

and $\mathbf{r = 3} \Rightarrow q = 4q + 3$, which is not divisible by 2 and so is **odd**.

$\therefore$ **Any positive odd integer is of the form**

4q + 1 or 4q + 3; where q is an integer.

Hence the required result.

Ex.11 Show that one and only one out of n; $n + 2$ or $n + 4$ is divisible by 3, where **n** is any positive integer.

Sol. Consider any two positive integers **a** and **b** such that **a** is greater than **b**, then according to Euclid's division algorithm:

$a = bq + r$; where q and r are positive integers and $0 \leq r < b$

Let $a = n$ and $b = 3$, then

$a = bq + r \Rightarrow n = 3q + r$; where $0 \leq r < 3$.

$\mathbf{r = 0} \Rightarrow n = 3q + 0 = 3q$

$\mathbf{r = 1} \Rightarrow n = 3q + 1$ and $\mathbf{r = 2} \Rightarrow n = 3q + 2$

If $n = 3q$; **n is divisible by 3**

If $n = 3q + 1$; then $n + 2 = 3q + 1 + 2$

$= 3q + 3$; which is divisible by 3

$\Rightarrow$ **n + 2 is divisible by 3**

If $n = 3q + 2$; then $n + 4 = 3q + 2 + 4$

$= 3q + 6$; which is divisible by 3

$\Rightarrow$ **n + 4 is divisible by 3**

Hence, if **n** is any positive integer, then one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3.

Hence the required result.

Ex.12 Show that any positive integer which is of the form $6q + 1$ or $6q + 3$ or $6q + 5$ is odd, where q is some integer.

Sol. If **a** and **b** are two positive integers such that **a** is greater than **b**; then according to Euclid's division algorithm; we have

$a = bq + r$; where q and r are positive integers and $0 \leq r < b$.

Let $b = 6$, then

$a = bq + r \Rightarrow \mathbf{a = 6q + r}$; where $0 \leq r < 6$.

When $r = 0 \Rightarrow a = 6q + 0 = 6q$;

which is even integer

When $r = 1 \Rightarrow a = 6q + 1$

which is odd integer

When $r = 2 \Rightarrow a = 6q + 2$ **which is even.**

When $r = 3 \Rightarrow a = 6q + 3$ **which is odd.**

When $r = 4 \Rightarrow a = 6q + 4$ **which is even.**

When $r = 5 \Rightarrow a = 6q + 5$ **which is odd.**

This verifies that when r = 1 or 3 or 5; the integer obtained is 6q + 1 or 6q + 3 or 6q + 5 and each of these integers is a positive odd number.

Hence the required result.

Ex.13 Use Euclid's Division Algorithm to show that the square of any positive integer is either of the form 3m or 3m + 1 for some integer m.

Sol. Let **a** and **b** are two positive integers such that **a** is greater than **b**; then:

a = bq + r; where q and r are also positive integers and $0 \le r < b$

Taking b = 3, we get:

$a = 3q + r$; where $0 \le r < 3$

$\Rightarrow$ The value of positive integer **a** will be 3q + 0, 3q + 1 or 3q + 2

i.e., 3q, 3q + 1 or 3q + 2.

Now we have to show that the squares of positive integers 3q, 3q + 1 and 3q + 2 can be expressed as 3m, or 3m + 1 for some integer m.

$\therefore$ **Square of 3q** $= (3q)^2$

$= 9q^2 = 3(3q^2) = 3m$; 3 where m is some integer.

Square of 3q + 1 $= (3q + 1)^2$

$= 9q^2 + 6q + 1$

$= 3(3q^2 + 2q) + 1 = 3m + 1$ for some integer m.

Square of 3q + 2 $= (3q + 2)^2$

$= 9q^2 + 12q + 4$

$= 9q^2 + 12q + 3 + 1$

$= 3(3q^2 + 4q + 1) + 1 = \mathbf{3m + 1}$ for some integer m.

$\therefore$ The square of any positive integer is either of the form 3m or 3m + 1 for some integer m.

Hence the required result.

Ex.14 Use Euclid's Division Algorithm to show that the cube of any positive integer is either of the 9m, 9m + 1 or 9m + 8 for some integer m.

Sol. Let **a** and **b** be two positive integers such that **a** is greater than **b**; then:

a = bq + r; where q and r are positive integers and $0 \le r < b$.

Taking b = 3, we get:

$a = 3q + r$; where $0 \le r < 3$

$\Rightarrow$ Different values of integer **a** are

3q, 3q + 1 or 3q + 2.

Cube of 3q $= (3q)^3 = 27q^3 = 9(3q^3) = 9m$; where **m** is some integer.

Cube of 3q + 1 $= (3q + 1)^3$

$= (3q)^3 + 3(3q)^2 \times 1 + 3(3q) \times 1^2 + 1^3$

$[\because (q + b)^3 = a^3 + 3a^2b + 3ab^2 + 1]$

$= 27q^3 + 27q^2 + 9q + 1$

$= 9(3q^3 + 3q^2 + q) + 1$

$= \mathbf{9m + 1}$; where m is some integer.

Cube of 3q + 2 $= (3q + 2)^3$

$= (3q)^3 + 3(3q)^2 \times 2 + 3 \times 3q \times 2^2 + 2^3$

$= 27q^3 + 54q^2 + 36q + 8$

$= 9(3q^3 + 6q^2 + 4q) + 8$

$= \mathbf{9m + 8}$; where m is some integer.

$\therefore$ Cube of any positive integer is of the form 9m or 9m + 1 or 9m + 8.

Hence the required result.

➤ THE FUNDAMENTAL THEOREM OF ARITHMETIC

Statement : Every composite number can be decomposed as a product prime numbers in a unique way, except for the order in which the prime numbers occur.

For example :

(i) $30 = 2 \times 3 \times 5, 30 = 3 \times 2 \times 5, 30 = 2 \times 5 \times 3$ and so on.

(ii) $432 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^4 \times 3^3$

or $432 = 3^3 \times 2^4$.

(iii) $12600 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7$

$= 2^3 \times 3^2 \times 5^2 \times 7$

In general, a composite number is expressed as the product of its prime factors written in ascending order of their values.

e.g., (i) $6615 = 3 \times 3 \times 3 \times 5 \times 7 \times 7$

$= 3^3 \times 5 \times 7^2$

(ii) $532400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 11 \times 11 \times 11$

□ EXAMPLES □

Ex.15 Consider the number 6^n, where n is a natural number. Check whether there is any value of $n \in N$ for which 6^n is divisible by 7.

Sol. Since, $6 = 2 \times 3$; $6^n = 2^n \times 3^n$

$\Rightarrow$ The prime factorisation of given number 6^n

$\Rightarrow$ **6^n is not divisible by 7.** **(Ans)**

Ex.16 Consider the number 12^n, where n is a natural number. Check whether there is any value of $n \in N$ for which 12^n ends with the digit zero.

Sol. We know, if any number ends with the digit zero it is always divisible by 5.

$\Rightarrow$ If 12^n ends with the digit zero, it must be divisible by 5.

This is possible only if prime factorisation of 12^n contains the prime number 5.

Now, $12 = 2 \times 2 \times 3 = 2^2 \times 3$

$\Rightarrow$ $12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n$

i.e., prime factorisation of 12^n does not contain the prime number 5.

$\Rightarrow$ **There is no value of $n \in N$ for which 12^n ends with the digit zero.** **(Ans)**

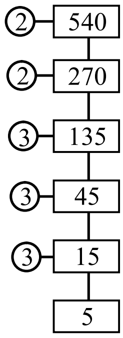
➤ USING THE FACTOR TREE

□ EXAMPLES □

Ex.17 Find the prime factors of :

(i) 540 (ii) 21252 (iii) 8232

(i)



540 divided by 2 gives 270

270 divided by 2 gives 135

135 divided by 3 gives 45

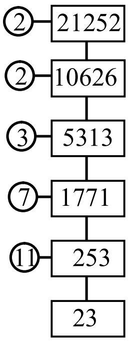
45 divided by 3 gives 15

15 divided by 3 gives 5

5 is a prime number and so cannot be further divided by any prime number

$\therefore$ $540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$

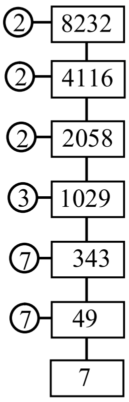
(ii)



$\therefore$ $21252 = 2 \times 2 \times 3 \times 7 \times 11 \times 23$

$= \mathbf{2^2 \times 3 \times 11 \times 7 \times 23.}$

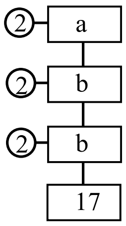
(iii)



$\therefore$ $8232 = 2 \times 2 \times 2 \times 3 \times 7 \times 7 \times 7$

$= \mathbf{2^3 \times 3 \times 7^3.}$

Ex.18 Find the missing numbers a, b and c in the following factorisation:



Can you find the number on top without finding the other ?

Sol. $c = 17 \times 2 = 34$

$b = c \times 2 = 34 \times 2 = 68$ and

$a = b \times 2 = 68 \times 2 = 136$

i.e., $a = 136$, $b = 68$ and $c = 34$. **(Ans)**

Yes, we can find the number on top without finding the others.

Reason: The given numbers 2, 2, 2 and 17 are the only prime factors of the number on top and so the number on top $= 2 \times 2 \times 2 \times 17 = 136$

➢ USING THE FUNDAMENTAL THEOREM OF ARITHMETIC TO FIND H.C.F. AND L.C.M.

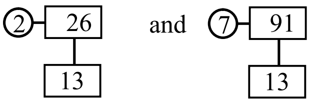
EXAMPLES

Ex.19 Find the L.C.M. and H.C.F. of the following pairs of integers by applying the Fundamental theorem of Arithmetic method i.e., using the prime factorisation method.

(i) 26 and 91 (ii) 1296 and 2520

(iii) 17 and 25

Sol. (i) Since, $26 = 2 \times 13$ and, $91 = 7 \times 13$



∴ **L.C.M.** = Product of each prime factor with highest powers. $= 2 \times 13 \times 7 = 182$. **(Ans)**

i.e., **L.C.M.** (26, 91) = 182. **(Ans)**

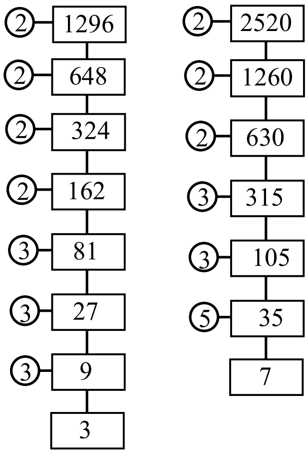
H.C.F. = Product of common prime factors with lowest powers. = 13.

i.e., **H.C.F** (26, 91) = 13.

(ii) Since, $1296 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 = 2^4 \times 3^4$

and, $2520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7$

$= 2^3 \times 3^2 \times 5 \times 7$



∴ **L.C.M.** = Product of each prime factor with highest powers

$= 2^4 \times 3^4 \times 5 \times 7 =$ **45,360**

i.e., **L.C.M.** (1296, 2520) = 45,360 **(Ans)**

H.C.F. = Product of common prime factors with lowest powers.

$= 2^3 \times 3^2 = 8 \times 9 = 72$

i.e., H.C.F. (1296, 2520) = **72.** **(Ans)**

(iii) Since, $17 = 17$

and, $25 = 5 \times 5 = 5^2$

∴ **L.C.M.** $= 17 \times 5^2 = 17 \times 25 = 425$

and, **H.C.F.** = Product of common prime factors with lowest powers

= 1, as given numbers do not have any common prime factor.

In example 19 (i) :

Product of given two numbers $= 26 \times 91 = 2366$

and, product of their

L.C.M. and **H.C.F.** $= 182 \times 13 = 2366$

∴ Product of L.C.M and H.C.F of two given numbers = Product of the given numbers

In example 19 (ii) :

Product of given two numbers

$= 1296 \times 2520 = 3265920$

and, product of their

L.C.M. and **H.C.F.** $= 45360 \times 72 = 3265920$

$\therefore$ L.C.M. (1296, 2520) × H.C.F. (1296, 2520)

= 1296 × 2520

In example 19 (iii) :

The given numbers 17 and 25 do not have any common prime factor. Such numbers are called co-prime numbers and their H.C.F. is always equal to 1 (one), whereas their L.C.M. is equal to the product of the numbers.

But in case of two co-prime numbers also, the product of the numbers is always equal to the product of their L.C.M. and their H.C.F.

As, in case of co-prime numbers 17 and 25;

H.C.F. = 1; L.C.M. = 17 × 25 = 425;

product of numbers = 17 × 25 = 425

and product of their H.C.F. and L.C.M.

= 1 × 425 = 425.

> ➢ For any two positive integers :
> Their L.C.M. × their H.C.F.
> = Product of the number
>
> $\Rightarrow$ (i) L.C.M. = $\frac{\text{Product of the numbers}}{\text{H.C.F.}}$
>
> (ii) H.C.F. = $\frac{\text{Product of the numbers}}{\text{L.C.M.}}$
>
> (iii) One number = $\frac{\text{L.C.M.} \times \text{H.C.F.}}{\text{Other number}}$

Ex.20 Given that H.C.F. (306, 657) = 9, find L.C.M. (306, 657)

Sol. H.C.F. (306, 657) = 9 means H.C.F. of 306 and 657 = 9

Required L.C.M. (306, 657) means required L.C.M. of 306 and 657.

For any two positive integers;

their L.C.M. = $\frac{\text{Product of the numbers}}{\text{Their H.C.F.}}$

i.e., L.C.M. (306, 657) = $\frac{306 \times 657}{9}$ = 22,338.

Ex.21 Given that L.C.M. (150, 100) = 300, find H.C.F. (150, 100)

Sol. L.C.M. (150, 100) = 300

$\Rightarrow$ L.C.M. of 150 and 100 = 300

Since, the product of number 150 and 100

= 150 × 100

And, we know :

H.C.F. (150, 100) = $\frac{\text{Product of 150 and 100}}{\text{L.C.M.(150,100)}}$

= $\frac{150 \times 100}{300}$ = 50.

Ex.22 The H.C.F. and L.C.M. of two numbers are 12 and 240 respectively. If one of these numbers is 48; find the other numbers.

Sol. Since, the product of two numbers

= Their H.C.F. × Their L.C.M.

$\Rightarrow$ One no. × other no. = H.C.F. × L.C.M.

$\Rightarrow$ Other no. = $\frac{12 \times 240}{48}$ = 60.

Ex.23 Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 + 5$ are composite numbers.

Sol. Since,

$7 \times 11 \times 13 + 13 = 13 \times (7 \times 11 + 1)$

$= 13 \times 78 = 13 \times 13 \times 3 \times 2;$

that is, the given number has more than two factors and it is a composite number.

Similarly, $7 \times 6 \times 5 \times 4 \times 3 + 5$

$= 5 \times (7 \times 6 \times 4 \times 3 + 1)$

$= 5 \times 505 = 5 \times 5 \times 101$

$\Rightarrow$ The given no. is a composite number.