HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. Since product of any r consecutive integers is divisible by r! and not divisible by r+1!.

So given product of 4 consecutive integers is divisible by 4! or 24.

2. Let three consecutive natural numbers are n, n+1, n+2,

$P(n) = (n)^3 + (n+1)^3 + (n+2)^3$

$P(1)=1^3 + 2^3 + 3^3=36$, which is divisible by 2 and 9

$P(2) = (2)^3 + (3)^3 + (4)^3 = 99$, which is divisible by 9 (not by 2).

Hence P(n) is divisible 9 $\forall$ n $\in$ N.

3. Let $P(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$

$P(1) = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = 1$ (integer)

$P(2) = 2^4\left(\frac{8}{7} + \frac{2}{5} + \frac{1}{3}\right) - \frac{2}{105} = 15$ (integer) etc.

Hence P(n) is an integer.

4. Let $P(n) = 10^n + 3.4^{n+2} + \lambda$ is divisible by 9 $\forall$ n $\in$ N

$P(1)=10+3.4^3+\lambda=202+\lambda=207+(\lambda-5)$

Which is divisible by 9 if $\lambda = 5$

9. Let $p(n) = n^2 + n = n(n + 1)$ is an odd integer since the product of two consecutive integers is always even. So hear principle of induction is not applicable.

10. Let $p(n) = 3^{4n+2} + 5^{2n+1}$

Here $P(1) = 3^6 + 5^3 = 9^3 + 5^3 = 14 \times 61$

Which is multiple of 14 but not of 16, 18 and 20.

14. $T_n = 1 + a + a^2 + \text{-------} + a^{n-1} = \frac{1-a^n}{1-a}$

$S_n = \Sigma T_n = \frac{1}{(1-a)}[\Sigma 1 - \Sigma a^n]$

$= \frac{1}{(1-a)}[n - (a + a^2 + a^3 + \text{----} a^n)]$

$= \frac{1}{(1-a)}\left[n - \frac{a(1-a^n)}{(1-a)}\right] = \frac{n}{1-a} - \frac{a(1-a^n)}{(1-a)^2}$

16. Since $x^n + y^n$ is divisible $(x + y)$ if n is odd.

Here $2n - 1$ is odd $\forall$ n $\in$ N.

18. n^{th} term of the given series

$$T_n = \frac{\frac{n}{2}\cdot\frac{n+1}{2}}{\Sigma n^3} = \frac{\frac{1}{4}n(n+1)}{\frac{1}{4}n^2(n+1)^2} = \frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\therefore S_n = \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \text{----} + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

25. Let $P(n) = 7^{2n} - 48n - 1$

$P(1) = 7^2 - 48.1 - 1 = 0$,

which is divisible by for all n $\in$ N

$P(2) = 7^4 - 48.2 - 1 = 2304$,

which is divisible by 2304 not by 25, 26 and 1234.

26. By Theorem-II

28. Let n is a positive integer.

$P(n) = n^3 - n$

P(1) = 0, which is divisible by for all n $\in$ N

P(2) = 6, which is divisible by 6 (not by 4 and 9)

30. Let $P(n): \quad n^p - n$

when $p = 2$

$P(n) = \quad n^2 - n$

$P(1) = \quad 0$ which is divisible all $n \in N$

$P(2) = \quad 2$ which is divisible by 2

$P(3) = \quad 6$ which is divisible by 2

Hence P(n) is divisible by 2 when n is greater than 1.

32. Let $P(n) = \cos\theta \,.\, \cos2\theta \,.\, \cos4\theta$ ----- $\cos 2^{n-1}\theta$

$$P(1) = \cos\theta = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \frac{\sin 2\theta}{2\sin\theta}$$

$$P(2) = \cos\theta\cos2\theta = \frac{2(2\sin\theta\cos\theta)\cos2\theta}{4\sin\theta}$$

$$= \frac{2\sin2\theta\cos2\theta}{4\sin\theta}$$

$$= \frac{\sin4\theta}{4\sin\theta} = \frac{\sin 2^2\theta}{2^2\sin\theta}$$

Clearly, $P(n) = \dfrac{\sin 2^n\theta}{2^n\sin\theta}$

34. Here $T_n = n(n + 1)^2$

$\therefore \quad S_n = \Sigma T_n = \Sigma n^3 + 2\Sigma n^2 + \Sigma n$

$$= \frac{n^2(n+1)^2}{4} + 2\,.\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+5)$$

36. Let n^{th} term of the series is T_n and

$S_n = 4 + 14 + 30 + 52 + 80 + 114 + ---- + T_n$..**(i)**

$S_n = \quad 4 + 14 + 30 + 52 + 80 + ---- + T_n$ ..**(ii)**

Subtract **(ii)** from **(i)**

$0=(4 + 10 + 16 + 22 + 28 + 34 + ----$ n terms$) - T_n$

$T_n = 4 + 10 + 16 + 22 + ----$ n terms

$$= \frac{n}{2}[2 \times 4 + (n-1)6] = n(3n + 1) = 3n^2 + n$$

38. Let $p(n) = n^3 + (n+1)^3 + (n+2)^3$,

$p(A) = 36, p(B) = 99$ both are divisible by 99

Let it is true for $n = k$

$k^3 + (k+1)^3 + (k+2)^3 = 9q \;\; ; q \in I$

adding $9k^2 + 27k + 27$ both sides

$k^3 + (k+1)^3 + (k+2)^3 + 9k^2 + 27k + 27 = 9q + 9k^2 + 27k + 27$

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9r \,; r \in I$

39. Let $P(n) = 11^{n+2} + 12^{2n+1}$

$P(1) = 11^3 + 12^3 = 23 \times 133$, which is divisible by 133 but not by 113 and 123.

EXERCISE - 2

Subjective Type

2. (i) Given statement is true for n = 1

(ii) Let us assume that the statement is true for n = k

i.e. $1+2+3+.........+k<\frac{1}{8}(2k+1)^2$

$\Rightarrow 1+2+3+.........+k=\frac{1}{8}(2k+1)^2-\lambda$ where $\lambda \in R^+$

(iii) For $n=k+1$,

$1+2+.........+k+k+1=\frac{(2k+1)^2}{8}+(k+1)-\lambda$

$=\frac{(2k+3)^2}{8}-\lambda<\frac{(2k+3)^2}{8}$

So the result is true for n = k + 1

Hence by principle of mathematical induction the statement is true for all $n \in N$

3. $P(1): 1^3+1$ is divisible by 3

$P(4): 4^3+4$ is divisible by 3

5.

(i) Given statement is true for n = 1

(ii) Let us assume that the statement is true for n = k

i.e. $k(k+1)(k+2)=6\lambda$

(iii) For $n=k+1$,

$(k+1)(k+2)(k+3)=k(k+1)(k+2)+3(k+1)(k+2)$

$=6\lambda+3(k+1)(k+2)=$ multiple of 6 as $(k+1)(k+2)$ is even

So the result is true for n = k + 1

Hence by principle of mathematical induction the statement is true for all $n \in N$

6. Let $P(n)$; $\sin\theta+\sin 2\theta+.......+\sin n\theta$

$=\sin\left(\frac{n+1}{2}\right)\theta\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}$

P(A) is true

Let P(k) is also true

$\sin\theta+\sin 2\theta+..........+\sin k\theta$

$=\sin\left(\frac{k+1}{2}\right)\theta\sin\frac{k\theta}{2}\operatorname{cosec}\frac{\theta}{2}$

add $\sin(k+1)\theta$ both sides

$\sin\theta+\sin 2\theta+...........+\sin k\theta+\sin(k+1)\theta=\sin\left(\frac{k+1}{2}\right)\theta$

$\sin\frac{k\theta}{2}\operatorname{cosec}\frac{\theta}{2}+\sin(k+1)\theta$

$=\sin\left(\frac{k+1}{2}\right)\theta\left[\frac{\sin\frac{k\theta}{2}+2\cos\left(\frac{k+1}{2}\right)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right]$

$=\sin\left(\frac{k+1}{2}\right)\theta\left[\frac{\frac{\sin k\theta}{2}+\frac{\sin(k+2)\theta}{2}-\frac{\sin k\theta}{2}}{\sin\frac{\theta}{2}}\right]$

$=\sin\left(\frac{k+1+1}{2}\right)\theta\sin\left(\frac{k+1}{2}\right)\theta \cdot \operatorname{cosec}\frac{\theta}{2}$

$\Rightarrow$ P(k + 1) is true

8.

(i) Given statement is true for n = 1

(ii) Let us assume that the statement is true for n = k

i.e. $1.3+2.3^2+3.3^3+.......+k.3^k=\frac{(2k-1)3^{k+1}+3}{4}$

(iii) For $n=k+1$,

L.H.S. $=1.3+2.3^2+3.3^3+.......+k.3^k+(k+1)3^{k+1}$

$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k+1}=\frac{(2k+1)3^{k+2}+3}{4}$

= R.H.S.

so by principle of mathematical induction the statement is true for all $n \in N$

11. (i) Given statement is true for n = 1

(ii) Let us assume that the statement is true for n = k

i.e. $2k+7=(k+3)^2-\lambda$ where $\lambda \in R^+$

(iii) For $n=k+1$,

$2(k+1)+7=2k+7+2=(k+3)^2-\lambda+2$

$=(k+4)^2-2k-\lambda-5<(k+4)^2$

So the result is true for n = k + 1

Hence by principle of mathematical induction the statement is true for all $n \in N$

EXERCISE - 3

Part # I : AIEEE/JEE-MAIN

1. $S(K) = 1 + 3 + 5 + ... + (2K - 1) = 3 + K^2$

S(1) is not true

let S(K) is true then

$S(K + 1) = 1 + 3 + 5 + ... + (2K - 1) + (2K + 1)$

$= S(K) + (2K + 1)$

$= 3 + K^2 + 2K + 1 = 3 + (K + 1)^2$

Hence $S(K) \Rightarrow S(K + 1)$

2. Since $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ----$

n terms $= \dfrac{n(n+1)^2}{2}$, when n is even

When n is odd the n^{th} term of series will be n^2 in this case, $(n - 1)$ is even

so for finding sum of first $(n - 1)$ terms of the series, we replacing n by $(n - 1)$ in the given formula.

So sum of first $(n - 1)$ terms $= \dfrac{(n-1)n^2}{2}$

Hence sum of n terms of the series

= (the sum of $(n - 1)$ terms + the n^{th} term)

$$= \frac{(n-1)n^2}{2} + n^2 = \frac{(n+1)n^2}{2}$$

3. $$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\therefore \quad A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

Now $nA-(n-1)I = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$

4. ∵ $\sqrt{n(n+1)} < \sqrt{(n+1)(n+1)}$

i.e. $\sqrt{n(n+1)} < n + 1 \quad \forall\, n \in N$

Hence statement-2 is true.

For n = 2 given result is true.

let it is true for $n = K \in N, K \geq 2$ then

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{K}} > \sqrt{K}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}}$$

$$= \frac{\sqrt{K(K+1)}+1}{\sqrt{K+1}} > \frac{\sqrt{KK}+1}{\sqrt{K+1}} = \sqrt{K+1}$$

(∵ by statement-2 $\sqrt{n(n+1)} < n+1 \Rightarrow \sqrt{n} < \sqrt{n+1}$)

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{K+1}} > \sqrt{K+1}$$

Hence statement-1 is true for every natural number $n \geq 2$.