HINTS & SOLUTIONS

EXERCISE - 1
Single Choice

1. Given $\frac{x}{3} = \cos t + \sin t$ & $\frac{y}{4} = \cos t - \sin t$

Squaring these two,

$\Rightarrow \frac{x^2}{9} = 1 + 2\cos t \sin t$ .....**(i)**

$\frac{y^2}{16} = 1 - 2\sin t \cos t$ .....**(ii)**

Adding **(i)** & **(ii)**

$$\frac{x^2}{9} + \frac{y^2}{16} = 2 \quad \Rightarrow \quad \frac{x^2}{18} + \frac{y^2}{32} = 1$$

3. Consider P(θ), $Q\left(\theta + \frac{2\pi}{3}\right)$, and $R\left(\theta + \frac{4\pi}{3}\right)$. Then,

$P' \equiv (a\cos\theta, b\sin\theta)$

$$Q' \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

and $$R' \equiv \left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let the centroid of ΔP'Q'R' be (x', y').

$$x' = a\left[\frac{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3}\right]$$

$$= \frac{a}{3}\left[\cos\theta + 2\cos(\theta + \pi)\cos\frac{\pi}{3}\right] = 0$$

$$y' = \frac{a}{3}\left[\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right]$$

$$= \frac{a}{3}\left[\sin\theta + 2\sin(\theta + \pi)\sin\frac{\pi}{3}\right] = 0$$

$= 0$

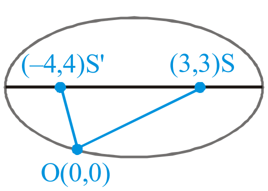
4. Here S is (3, 3) & S' is (–4, 4).

$\Rightarrow SS' = \sqrt{50} = 2ae$ .....**(i)**

Now $OS + OS' = 2a$

$3\sqrt{2} + 4\sqrt{2} = 2a$

$7\sqrt{2} = 2a$ .....**(ii)**



From **(i)** & **(ii)**

$$e = \frac{5}{7}$$

5. $$(5x - 10)^2 + (5y + 15)^2 = \frac{(3x + 4y + 7)^2}{4}$$

or $$(x - 2)^2 + (y + 3)^2 = \left(\frac{1}{2}\frac{3x - 4y - 7}{5}\right)^2$$

or $$\sqrt{(x - 2)^2 + (y + 3)^2} = \frac{1}{2}\frac{|3x - 4y - 7|}{5}$$

It is an ellipse, whose focus is (2, –3), directrix is $3x - 4y + 7 = 0$, and eccentricity is 1/2.

length of perpendicular from the focus to the directrix is

$$\frac{|3 \times 2 - 4(-3) + 7|}{5} = 5$$

or $\frac{a}{e} - ae = 5$

or $2a - \frac{a}{2} = 5$

or $a = \frac{10}{3}$

So, the length of the major axis is 20/3.

7. Since major axis is along y-axis.

∴ Equation of tangent is $x = my + \sqrt{b^2m^2 + a^2}$

slope of tangent $= \frac{1}{m} = \frac{-4}{3} \Rightarrow m = \frac{-3}{4}$

Hence equation of tangent is $4x + 3y = 24$

or $\frac{x}{6} + \frac{y}{8} = 1$

Its intercepts on the axes are 6 and 8.

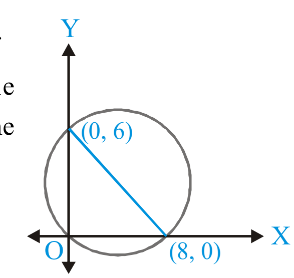
$$\text{Area }(\Delta AOB) = \frac{1}{2} \times 6 \times 8 = 24 \text{ sq.unit.}$$

8.

The center of the family of ellipse is (4, 3) and the distance of focus from the center is ae = 5/2.

Hence, the locus is

$$(x - 4)^2 + (y - 3)^2 = \frac{25}{4}$$



10. Let any tangent of ellipse is

$$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$$

Let it meets axes at $A\left(\frac{4}{\cos\theta},0\right)$ & $B\left(0,\frac{3}{\sin\theta}\right)$

Let mid point of AB is (h, k) then

$$2h=\frac{4}{\cos\theta},\ 2k=\frac{3}{\sin\theta}$$

Since $\cos^2\theta+\sin^2\theta=1$

$$\therefore\ \frac{16}{4h^2}+\frac{9}{4k^2}=1$$

$\Rightarrow\ 16k^2+9h^2=4h^2k^2$

Hence locus is $16y^2+9x^2=4x^2y^2$.

13. Let equations of tangent to the two ellipses are

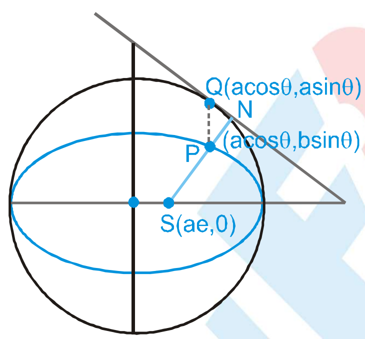
$$y=mx\pm\sqrt{(a^2+b^2)m^2+b^2} \qquad \text{.....(i)}$$

$$y=mx\pm\sqrt{a^2m^2+a^2+b^2} \qquad \text{.....(ii)}$$

On solving **(i)** and **(ii)** we get $m=\pm\frac{a}{b}$

Put solve of m in **(i)** to get the answer.

16.



Equation of tangent at P

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad \text{.....(i)}$$

Equation of tangent at Q

$x\cos\theta+y\sin\theta=a$

$$\Rightarrow\ \frac{x\cos\theta}{a}+\frac{y}{a}\sin\theta=1 \qquad \text{.....(ii)}$$

(i) – (ii)

$$y\sin\theta\left(\frac{1}{b}-\frac{1}{a}\right)=0$$

$\Rightarrow\ y=0 \qquad [\sin\theta\neq 0\ ;\ a\neq b]$

19. Positive end of latus rectum is $\left(ae,\frac{b^2}{a}\right)$

$\therefore$ Equation of normal is

$$\frac{a^2x}{ae}-\frac{b^2ay}{b^2}=a^2e^2$$

$\Rightarrow\ x-ey-e^3a=0$

22. Equation of normal at P $(3\cos\theta,\sin\theta)$ is

$3x\sec\theta-y\,\mathrm{cosec}\theta=8 \qquad$ **.....(i)**

Now equation of diameter through Q is

$3y\cos\theta+x\sin\theta=0 \qquad$ **.....(ii)**

Solving **(i)** & **(ii)** we get intersection point R,

$$\left(\frac{12}{5}\cos\theta,\frac{-4}{5}\sin\theta\right)$$

Let (h, k) be mid point of PR then

$$2h=\frac{27}{5}\cos\theta,\ 2k=\frac{1}{5}\sin\theta.$$

Now $\cos^2\theta+\sin^2\theta=1$

$$\therefore\ \frac{h^2}{(2.7)^2}+\frac{k^2}{(0.1)^2}=1$$

$\therefore$ Locus is ellipse.

25. $e=\sqrt{1-\frac{3}{5}}=\sqrt{\frac{2}{5}}$

$\therefore S_1=(\sqrt{2},0),\ S_2=(-\sqrt{2},0)$

Equation of tangent is $y=mx+\sqrt{5m^2+3}$

$$S_1F_1=\left|\frac{-\sqrt{2}m-\sqrt{5m^2+3}}{\sqrt{1+m^2}}\right|$$

$$S_2F_2=\left|\frac{\sqrt{2}m-\sqrt{5m^2+3}}{\sqrt{1+m^2}}\right|$$

Now $(S_1F_1)(S_2F_2)=\dfrac{5m^2+3-2m^2}{(1+m^2)}=3$.

26. The combined equation of the pair of lines through the origin joinin the points of intersection of the line $y=\sqrt{m}\,x+1$ with the given curve is $x^2+2xy+(2+\sin^2\alpha)$ $y^2-(y-\sqrt{m}\,x)^2=0$.

For the chord to subtend a right angle at the origin,

$$(1-m)+(2+\sin^2\alpha-1)=0$$

(As sum of the coefficient of $x^2+y^2=10$)

or $\sin^2\alpha = m-2$

or $0 \le m-2 \le 1$

or $2 \le m \le 3$

EXERCISE - 2

Part # I : Multiple Choice

1. $r^2 - r - 6 > 0$ and $r^2 - 6r + 5 > 0$

or $(r-3)(r+2) > 0$ and $(r-1)(r-5) > 0$

or $(r < -2 \text{ or } r > 3)$ and $(r < 1 \text{ or } r > 5)$

i.e., $r < -2$ or $r > 5$

Also, $r^2 - r - 6 \neq r^2 - 6r + 5$

or $r \neq \frac{11}{5}$

2. If both the foci are fixed, then the ellipse is fixed, that is, both the directrices can be decided (eccentricity is given). Similar is the case for option (c). Thus, (a) and (c) are the correct choices. In the remaining cases, the size of the ellipse is fixed, but its position is not fixed.

4. The ellipse is $16x^2 + 11y^2 = 256$

The equation of tangent is $\left(4\cos\theta, 16\sqrt{11}\cos\theta\right)$ is

$$16x(4\cos\theta) + 11y\left(\frac{16}{\sqrt{11}}\sin\theta\right) = 256$$

It touches $(x-1)^2 + y^2 = 4^2$ if

$$\left|\frac{4\cos\theta - 16}{\sqrt{16\cos^2\theta + 11\sin\theta}}\right| = 4$$

or $(\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

or $4\cos^2\theta + 8\cos\theta - 5 = 0$

or $\cos\theta = \frac{1}{2}$

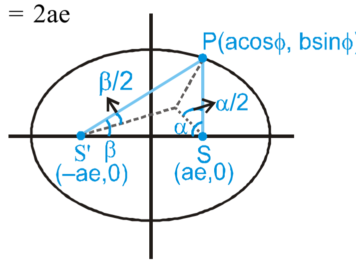
$\therefore$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

5. By definition of ellipse

$PS + PS' = 2a$ if $a > b$

$PS + PS' = 2b$ if $a < b$

and $SS' = 2ae$



Now by sine rule in ΔPSS'

$$\frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin[\pi-(\alpha+\beta)]}$$

or $$\frac{SP+S'P}{\sin\beta+\sin\alpha} = \frac{SS'}{\sin(\alpha+\beta)}$$

or $$\frac{2a}{\sin\beta+\sin\alpha} = \frac{2ae}{\sin(\alpha+\beta)}$$

or $$\frac{1}{e} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$\Rightarrow$ $$\frac{1-e}{1+e} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2} \quad \text{[By C \& D]}$$

7. The equation of the line joining θ and ϕ is

$$\frac{x}{5}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{3}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

If it passes through the point (4, 0). Then,

$$\frac{4}{5}\cos\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

or $$\frac{4}{5}=\frac{\cos\{(\theta-\phi)/2\}}{\cos\{(\theta+\phi)/2\}}$$

or $$\frac{4+5}{4-5}=\frac{\cos\{(\theta-\phi)2\}+\cos\{(\theta-\phi)2\}}{\cos\{(\theta-\phi)/2\}-\cos\{(\theta+\phi)/2\}}$$

$$=\frac{2\cos(\theta/2)\cos(\phi/2)}{2\sin(\phi/2)\sin(\theta/2)}$$

or $$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{1}{9}$$

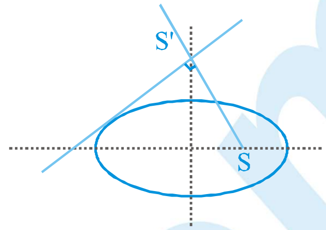
If it passes through the point (–5, 0), then

$$\tan\frac{\phi}{2}\tan\frac{\theta}{2}=9$$

13. Let S''(h, k) be the image.

SS' cuts a tangent at a point which lies on the auxiliary circle of the ellipse. Therefore,

$$\left(\frac{h\pm4}{2}\right)^2+\frac{k^2}{4}=25$$



14. Let P(h, k) be the point of intersection of E_1 and E_2.

Then,

$$\frac{h^2}{a^2}+k^2=1$$

or $h^2 = a^2(1-k^2)$ ....(i)

and $$\frac{h^2}{1}+\frac{k^2}{a^2}=1$$

or $k^2 = a^2(1-h^2)$ ....(ii)

Eliminating a from (i) and (ii), we get

$$\frac{h^2}{1-k^2}=\frac{k^2}{1-h^2}$$

or $h^2(1-h^2) = k^2(1-k^2)$

or $(h-k)(h+k)(h^2+k^2-1) = 0$

Hence, the locus is a set of curves consisting of the straigth lines y = x, y = –x, and the circle $x^2 + y^2 = 1$.

16. f(x) is a decreasing function and for the major axis to be the x-axis.

$f(k^2+2k+5) > f(k+11)$

or $k^2+2k+5 < k+11$

or $k \in (-3, 2)$

Then for the remaining values of k,

i.e., $k \in (-\infty, -3) \cup (2, \infty)$.

the major axis is the y-axis.

20. The equation of the tangent at $(t^2, 2t)$ to the parabola

$y^2 = 4x$ is

$2ty = 2(x+t^2)$

or $ty = x + t^2$

or $x - ty + t^2 = 0$ ....(i)

The equation of the normal at $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $5x^2 + 5y^2 = 20$ is

$(\sqrt{5}\sec\theta)x - (2\ \text{cosec}\theta)y = 5 - 4$

or $(\sqrt{5}\sec\theta)x - (2\ \text{cosec}\theta)y = 1$ ....**(ii)**

Given that **(i)** and **(ii)** represent the same line. Then

$$\frac{\sqrt{5}\sec\theta}{1} = \frac{-2\,\text{cosec}\,\theta}{-t} = \frac{-1}{t^2}$$

or $t = \frac{2}{\sqrt{5}}\cot\theta$ and $t = -\frac{1}{2}\sin\theta$

or $\frac{2}{\sqrt{5}}\cot\theta = -\frac{1}{2}\sin\theta$

or $4\cos\theta = -\sqrt{5}\sin^2\theta$

or $4\cos\theta = -\sqrt{5}(1 - \cos^2\theta)$

or $\sqrt{5}\cos^2\theta - 4\cos\theta - \sqrt{5} = 0$

or $(\cos\theta - \sqrt{5})(\sqrt{5}\cos\theta + 1) = 0$

or $\cos\theta = -\frac{1}{\sqrt{5}}$ $[\because \cos\theta \neq -\sqrt{5}]$

or $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

Putting $\cos\theta = -1/\sqrt{5}$ in $t = -(1/2)\sin\theta$, we get

$$t = -\frac{1}{2}\sqrt{1-\frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

Hence, $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ and $t = -\frac{1}{\sqrt{5}}$

Part # II : Assertion & Reason

2. The chord of contact of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

w.r.t. the point (8, 6) is

$$\frac{8x}{4} + \frac{6x}{2} = 1$$

or $2x + 3y = 1$

Hence, statement 1 is correct. Also, statement 2 is correct and explains statement 1.

3. $x^2 + y^2 + xy = 1$

Replacing x by –x & y by –y we get the same equation.

$\therefore$ Centre of conic is (0, 0) and every chord passing through the centre is bisected by the point. Hence st. I & st. II both are true & st I explains st. II.

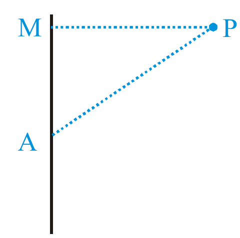
4. $|ay - bx| = c\sqrt{(x-a)^2 + (y-b)^2}$

or $\frac{|ay - bx|}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}\sqrt{(x-a)^2+(y-b)^2}$

or $PM = kPA$

where m is the length of perpendicular from P on the line $ay - bx = 0$, PA is the length of line segment joining P to the point A(a, b), and A lies on the line. So, the locus of P is a straight line through A inclined at an angle $\sin^{-1}(c/\sqrt{a^2+b^2})$ to the given line

(provided $c < \sqrt{a^2+b^2}$).



EXERCISE - 3

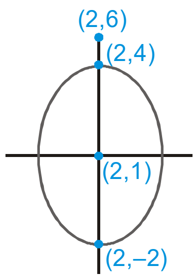
Part # I : Matrix Match Type

2. **(A)** $9(x^2-4x+4)+8(y^2-2y+1)=28+36+8$

$\Rightarrow\ 9(x-2)^2+8(y-1)^2=72$

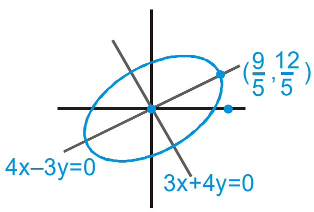
$$\Rightarrow\ \frac{(x-2)^2}{8}+\frac{(y-1)^2}{9}=1$$

Minimum distance of (2, 6) from the ellipse is 2 & maximum distance of (2, 6) from the ellipse is 8



(B) $$\frac{(3x+4y)^2}{225}+\frac{(4x-3y)^2}{100}=1$$

$$\frac{\left(\frac{3x+4y}{5}\right)^2}{9}+\frac{\left(\frac{4x-3y}{5}\right)^2}{4}=1$$



Point $\left(\frac{9}{5},\frac{12}{5}\right)$ lie on line $4x-3y=0$

$\therefore$ Minimum distance = 0

Maximum distance = 6

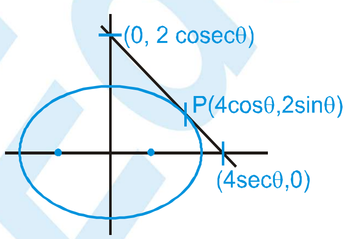
(C) $C_1 : x^2+y^2=3$

$C_2 : x^2+y^2=6$

$C_3 : x^2+y^2=12$

GM of 3, 6, 12 is $(3.6.12)^{1/3}=6$

(D) $$\frac{x^2}{16}+\frac{y^2}{4}=1$$



equation of tangent at P is

$$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{2}=1$$

$x\cos\theta+2y\sin\theta=4$

Area of triangle

$$\Delta=\frac{1}{2}.4\sec\theta.2\operatorname{cosec}\theta=\frac{8}{\sin 2\theta}$$

minimum area = 8 when $\sin 2\theta=1$

Part # II : Comprehension

Comprehension # 1

1. (c), 2. (b), 3. (d)

1. **(C)** $\lambda x-y+2(1+\lambda)=0$

or $\lambda(x+2)-(y-2)=0$

This line passing through (–2, 2).

$\mu x-y+2(1-\mu)=0$

or $\mu(x-2)-(y-2)=0$

This line passes through (2, 2).

Clearly, these represent the foci of the ellipse. So, $2ae=4$.

The circle $x^2+y^2-4y-5=0$, i.e., $x^2+(y-2)^2=9$ represents

an auxiliary circle. Thus, $a^2=9$ or $e=2/3$ and $b^2=5$.

2. **(B)** Required area $=\frac{1}{2}(2ae)(b)=3\times\frac{2}{3}\times\sqrt{5}=2\sqrt{5}$

Comprehension # 2

$2ae=4$

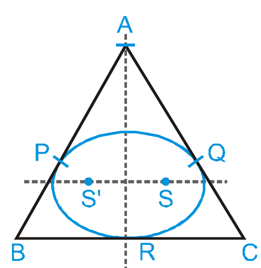
$2a=6$

$e=2/3$

$b^2=a^2(1-e^2)$

$=9(1-\frac{4}{9})$

$=5$



$\therefore$ Equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

1. If $\angle BAC=90°$ then locus of A is the director circle of the ellipse.

$x^2+y^2=14$

2. Let A be (h, k) then chord of contact PQ is

$$\frac{hx}{9}+\frac{ky}{5}=1$$

Homogenizing the equation of ellipse

$$\frac{x^2}{9}+\frac{y^2}{5}=\left(\frac{hx}{9}+\frac{ky}{5}\right)^2$$

$$x^2\left(\frac{h^2}{81}-\frac{1}{9}\right)+y^2\left(\frac{k^2}{25}-\frac{1}{5}\right)+\frac{2hk}{45}xy=0$$

coefficient of x^2 + coefficient of $y^2 = 0$

$$\frac{h^2}{81}-\frac{1}{9}+\frac{k^2}{25}-\frac{1}{5}=0 \Rightarrow 25x^2+81y^2=630$$

3. Chord of contact of A(h,k) is

$$\frac{hx}{9}+\frac{ky}{5}=1 \qquad \text{......(1)}$$

$$\frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{\sqrt{5}}.\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \text{.....(2)}$$

Comparing (1) & (2)

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\frac{h}{2\sqrt{3}}\ ;\ \sin\left(\frac{\alpha+\beta}{2}\right)=\frac{\sqrt{3}k}{2\sqrt{5}}$$

$$\Rightarrow \frac{x^2}{12}+\frac{3y^2}{20}=1$$

$$\Rightarrow 5x^2+9y^2=60$$

EXERCISE - 4
Subjective Type

2. $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

Since point $(7-\frac{5}{4}\alpha, \alpha)$ lies inside the ellipse,

$\therefore\ S_1<0$

$\Rightarrow\ 16\left(7-\frac{5}{4}\alpha\right)^2+25.\alpha^2<400$

$\Rightarrow\ (28-5\alpha)^2+25\alpha^2<400$

$\Rightarrow\ 50\alpha^2-280\alpha+384<0$

$\Rightarrow\ 25\alpha^2-140\alpha+192<0$

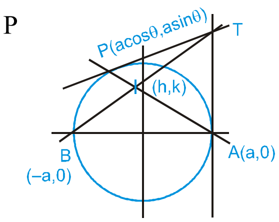
$\Rightarrow \alpha\in\left(\frac{12}{5},\frac{16}{5}\right)$

3. Equation of tangent at P

$x\cos\theta+y\sin\theta=a$

which meets $x = a$ at T

$\therefore$ T (a, a tan θ/2)



Equation of AP → $y=-\cot(\theta/2)(x-a)$ ....(1)

Equation of BT → $y=\dfrac{\tan(\theta/2)}{2}(x+a)$ ....(2)

From (1) & (2)

$y^2=-\frac{1}{2}(x^2-a^2)$

$x^2+2y^2=a^2$

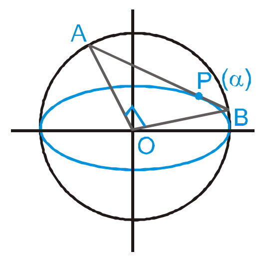
4. Equation of auxiliary circle is $x^2+y^2=a^2$ ....(1)

Equation of tangent at point P (acosα, b sinα)

is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$ ....(2)

Equation of pair of lines OA, OB is obtained by homogenous equation of (1) with the help of (2)

$$\therefore\ x^2+y^2=a^2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2$$



$\Rightarrow (1-\cos^2\alpha)x^2 - \frac{2xy\,a\sin\alpha\cos\alpha}{b} + y^2\left(1-\frac{a^2}{b^2}\sin\alpha\right)$

But $\angle AOB = 90°$

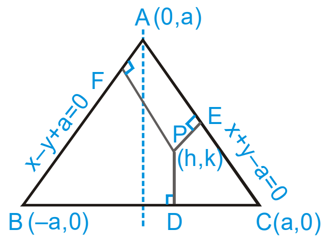
$\therefore$ Coeff. of x^2 + coeff. of $y^2 = 0$

$$1-\cos^2\alpha + 1 - \frac{a^2}{b^2}\sin^2\alpha = 0$$

$\Rightarrow\ 1 = \frac{a^2-b^2}{b^2}\sin^2\alpha \Rightarrow 1 = \frac{a^2e^2}{a^2(1-e^2)}\sin^2\alpha$

$\Rightarrow\ e = (1+\sin^2\alpha)^{-1/2}$

7.



$(PD)^2 = \frac{1}{2}PE.PF$

$$k^2 = \frac{1}{2}\left|\frac{h+k-a}{\sqrt{2}}\right|\left|\frac{h-k+a}{\sqrt{2}}\right|$$

$4k^2 = -(h+k-a)(h-k+a)$

$4k^2 = -\{h^2 - (k-a)^2\}$

$4k^2 = -\{h^2 - k^2 + 2ak - a^2\}$

$h^2 + 3k^2 + 2ak - a^2 = 0$

$\therefore$ Locus of (h, k) is

$x^2 + 3y^2 + 2ay - a^2 = 0$

$$x^2 + 3(y^2 + \frac{2}{3}ay + \frac{1}{9}a^2) = a^2 + \frac{a^2}{3}$$

$$x^2 + 3(y+\frac{1}{3}a)^2 = \frac{4a^2}{3}$$

$$\frac{x^2}{\frac{4a^2}{3}} + \frac{(y+\frac{1}{3}a)^2}{\frac{4a^2}{9}} = 1$$

$$\therefore\ e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}}$$

8. $A(ae + r_1\cos\theta, r_1\sin\theta)$

$B(ae - r_2\cos\theta, -r_2\sin\theta)$

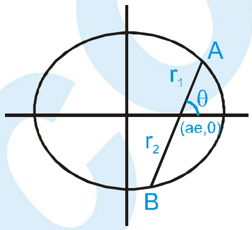
Both points lie on the ellipse

$$\therefore\ \frac{(ae+r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$b^2a^2e^2 + 2ab^2 er\cos\theta + b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$

$r^2(b^2\cos^2\theta + a^2\sin^2\theta) + 2ab^2e\cos\theta r + a^2b^2(e^2-1) = 0$

This is a quadratic equation in r with roots r_1 & $-r_2$.



$$|r_1 + r_2| = \sqrt{(r_1-r_2)^2 + 4r_1r_2}$$

$$= \sqrt{\left(\frac{2ab^2e\cos\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right)^2 - 4.\frac{a^2b^2(e^2-1)}{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{\sqrt{4a^2b^4e^2\cos^2\theta - 4(b^2\cos^2\theta + a^2\sin^2\theta)a^2b^2(e^2-1)}}{b^2\cos^2\theta + a^2\sin^2\theta}$$

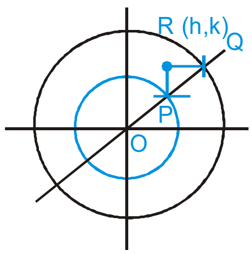
$$= \frac{\sqrt{4a^2b^2[b^2e^2\cos^2\theta - (e^2-1)(b^2\cos^2\theta + a^2\sin^2\theta)}}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{2ab^2}{b^2\cos^2\theta + a^2\sin^2\theta}\ \text{(by putting } e^2 = 1 - \frac{b^2}{a^2}\text{)}$$

9. $P(a\cos\theta, a\sin\theta)$ & $Q(b\cos\theta, b\sin\theta)$

$h = a\cos\theta,\ k = b\sin\theta$

$\therefore$ locus of R(h,k) is



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$b > a$

Foci (0, be) lies on inner circle, then $b^2e^2 = a^2$

$\Rightarrow\ e^2 = \frac{a^2}{b^2}$

$$e^2 = 1 - \frac{a^2}{b^2}$$

foci lie in the inner circle then

$$\frac{a^2}{b^2} = 1 - \frac{a^2}{b^2} \qquad [a = be]$$

$$\frac{a^2}{b^2} = \frac{1}{2} \qquad \Rightarrow \quad \frac{a}{b} = \frac{1}{\sqrt{2}} = e$$

10. The equation of any tangent PQ to the ellipse

$$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \text{....(i)}$$

This tangent cuts the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = a$ at the point P and Q.

Let the tangents at P and Q intersect the point R(h, k).

Then PQ becomes the chord of contact with respect to the point R for the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$, i.e., the equation of PQ is $\frac{hx}{c^2} + \frac{ky}{d^2} = 1$ ....(ii)

Equation (i) and (ii) represent same straight lines.

Therefore, $\frac{(\cos\theta)/a}{h/c^2} = \frac{(\sin\theta)/a}{k/d^2} = 1$

or $\quad \cos\theta = \frac{ah}{c^2}\sin\theta = \frac{bk}{d^2}$

Squaring and adding, we get

$$\frac{a^2h^2}{c^4} + \frac{b^2k^2}{d^4} = 1$$

$$\text{or} \quad \frac{a^2x^2}{c^4} + \frac{b^2y^2}{d^4} = 1 \qquad \text{....(iii)}$$

which is the locus of the point R(h, k)

If R(h, k) is the point of intersection of two perpendicular tangents, then the locus of R should be the director circle of the ellipse.

$$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$$

i.e., $x^2 + y^2 = c^2 + d^2$

$$\text{i.e.} \quad \frac{x^2}{c^2 + d^2} + \frac{y^2}{c^2 + d^2} = 1 \qquad \text{....(iv)}$$

Equations (iii) and (iv) represent the same locus. Therefore,

$$\frac{a^2}{c^4} = \frac{1}{c^2 + d^2}$$

$$\text{and} \quad \frac{b^2}{a^4} = \frac{1}{c^2 + d^2}$$

$$\text{or} \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$$

12. Tangent at point $(t^2, 2t)$ on parabola $y^2 = 4x$ is

$ty = x + t^2$ .....(i)

Normal at $(\sqrt{5}\cos\phi, 2\sin\phi)$ on ellipse $4x^2 + 5y^2 = 20$ is

$\sqrt{5}\, x\sec\phi - 2y\,\mathrm{cosec}\phi = 1$ .....(ii)

(i) & (ii) are same lines, hence by comparing

$$\frac{-\sqrt{5}}{\cos\phi} = \frac{-2}{t\sin\phi} = \frac{1}{t^2}$$

$\Rightarrow \quad \cos\phi = -\sqrt{5}\, t^2$ ..... (iii)

$\sin\phi = \frac{-2t^2}{t}$ ......(iv)

Square & add (iii) & (iv) we get

$$t = \pm\frac{1}{\sqrt{5}},\ t = 0$$

when $t = \frac{-1}{\sqrt{5}}$, $\tan\phi = -2 \quad \Rightarrow \quad \phi = \pi - \tan^{-1}2$

$t = \frac{1}{\sqrt{5}}$, $\tan\phi = 2 \quad \Rightarrow \quad \phi = \pi + \tan^{-1}2$

$t = 0,\ \phi = \frac{\pi}{2}, \frac{3\pi}{2}$

13. Equation of tangent at point P $(a\cos\theta, b\sin\theta)$

on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

foci $F_1 \equiv (ae, 0)$, $F_2 = (-ae, 0)$

$$\text{and } d = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\text{Now } 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

$$= 4a^2\left(1 - \frac{b^2(a^2\sin^2\theta + b^2\cos^2\theta)}{a^2b^2}\right)$$

$= 4a^2\left(1 - \sin^2\theta - \frac{b^2}{a^2}\cos^2\theta\right)$

$= 4\cos^2\theta\,(a^2 - b^2)$

$= 4a^2e^2\cos^2\theta = (2ae\cos\theta)^2$

$= [(a - ae\cos\theta) - (a + ae\cos\theta)]^2$

$= (PF_1 - PF_2)^2$

14. Let point is P $(2\cos\theta, \sin\theta)$.

Equation of tangent is $\frac{x}{2}\cos\theta + \frac{y\sin\theta}{1} = 1$

Equation of normal is $2x\sec\theta - y\,\mathrm{cosec}\theta = 3$

Now tangent and normal meet major axis at

$Q\left(\frac{2}{\cos\theta}, 0\right)$ and $R\left(\frac{3}{2}\cos\theta, 0\right)$ respectively

Given QR = 2

$\Rightarrow \left|\frac{2}{\cos\theta} - \frac{3}{2}\cos\theta\right| = 2$

$\Rightarrow 3|\cos\theta|^2 + 4|\cos\theta| - 4 = 0$

$\Rightarrow |\cos\theta| = \frac{2}{3}, -2$ (reject)

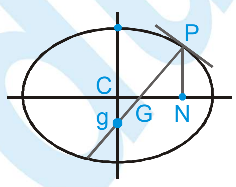
$\Rightarrow \cos\theta = \pm\left(\frac{2}{3}\right)$

17. Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let point is P $(a\cos\theta, b\sin\theta)$.

Equation of normal at P is

$ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$



$\Rightarrow \frac{x}{\frac{(a^2 - b^2)\cos\theta}{a}} + \frac{y}{\frac{-(a^2 - b^2)\sin\theta}{b}} = 1$

It meet major and minor axis at

$G\left(\frac{(a^2 - b^2)}{a}\cos\theta, 0\right)$ and $g\left(0, \frac{-(a^2 - b^2)}{b}\sin\theta\right)$

respectively.

$\therefore\ (CG)^2 = \left(\frac{a^2 - b^2}{a}\right)^2 \cos^2\theta$

and $(Cg)^2 = \left(\frac{a^2 - b^2}{b}\right)^2 \sin^2\theta$

$\therefore\ 7a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$

PN is ordinate,

$\therefore$ coordinate of N$(a\cos\theta, 0)$.

$e^2CN = \left(\frac{a^2 - b^2}{a^2}\right) a\cos\theta = \left(\frac{a^2 - b^2}{a}\right)\cos\theta = CG$

19. For point P, x-coordinate = 3

Given ellipse $9x^2 + 25y^2 = 225$

$9(3)^2 + 25y^2 = 225$

$y = \pm\frac{12}{5}$

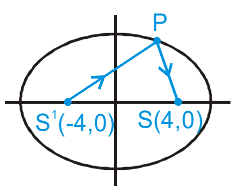
Coordinate of P is $\left(3, \pm\frac{12}{5}\right)$

Now $e = \frac{4}{5}$ & $ae = 4$

so foci is $(\pm 4, 0)$

Now equation of reflected ray (PS) is

$12x + 5y = 48$ or $12x - 5y = 48$



EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. Foci are $(\pm ae, 0)$. Therefore according to the condition,
$2ae = 2b$ or $ae = b$

Also, $b^2 = a^2(1 - e^2) \Rightarrow e^2 = (1 - e^2) \Rightarrow e = \dfrac{1}{\sqrt{2}}$

2. Since directrix is parallel to y-axis, hence axes of the ellipse are parallel to x-axis.
Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a > b)$$

$$e^2 = 1 - \frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2} = 1 - e^2 = 1 - \frac{1}{4} \Rightarrow \frac{b^2}{a^2} = \frac{3}{4}$$

Also, one of the directrices is $x = 4$

$$\Rightarrow \frac{a}{e} = 4 \quad \Rightarrow \quad a = 4e = 4.\frac{1}{2} = 2;$$

$$b^2 = \frac{3}{4}a^2 = \frac{3}{4}.4 = 3$$

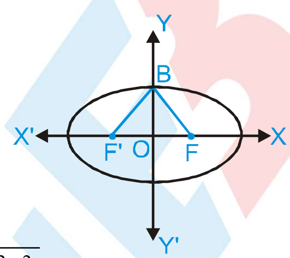
$\therefore$ Required ellipse is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

or $3x^2 + 4y^2 = 12$

4. $\angle F'BF = 90°$, $F'B \perp FB$
i.e., slope of (F'B) × Slope of (FB) = –1

$$\Rightarrow \frac{b}{ae} \times \frac{b}{-ae} = -1,$$

$b^2 = a^2e^2$ ..... **(i)**



We know that

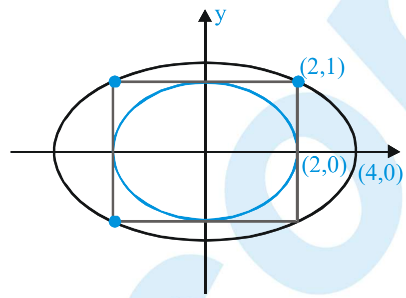
$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2e^2}{a^2}} = \sqrt{1 - e^2}$$

$$e^2 = 1 - e^2, \ 2e^2 = 1, \ e^2 = \frac{1}{2}, \ e = \frac{1}{\sqrt{2}}$$

5. Distance between foci = 6
$\Rightarrow$ ae = 3
Minor axis = 8
$\Rightarrow 2b = 8 \quad \Rightarrow b = 4 \quad \Rightarrow b^2 = 16$
$\Rightarrow a^2(1 - e^2) = 16 \quad \Rightarrow a^2 - a^2e^2 = 16$
$\Rightarrow a^2 - 9 = 16$
$\Rightarrow a = 5$
Hence ae = 3

$$\Rightarrow e = \frac{3}{5}$$

7.



Ellipse $x^2 + 4y^2 = 4$ (Given)
Eq^n. of the ellipse (required)

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

Ellipse passes through (2, 1)

therefore $\dfrac{4}{16} + \dfrac{1}{b^2} = 1 \quad \Rightarrow b^2 = \dfrac{4}{3}$

$$\frac{x^2}{16} + \frac{y^2}{4/3} = 1 \quad \Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1$$

$\Rightarrow x^2 + 12y^2 = 16$

9. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

from the given conditions
a = 4 and b = 2
$\therefore$ Eq of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

or $x^2 + 4y^2 = 16$

10. Let equation of any tangent to $y^2 = 16\sqrt{3}\,x$

be $y = mx + \dfrac{4\sqrt{3}}{m}$ ......**(i)**

and equation of any tangent to $2x^2 + y^2 = 4$

be $y = mx + \sqrt{2m^2 + 4}$ ......**(ii)**

but **(i)** and **(ii)** are same lines

$$\therefore \frac{4\sqrt{3}}{m} = \sqrt{2m^2 + 4}$$

$\Rightarrow m^4 + 2m^2 - 24 = 0 \quad \Rightarrow \quad m^2 = -6, 4$
$\therefore m = \pm 2$

11. $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

$e=\sqrt{1-\dfrac{b^2}{a^2}}=\dfrac{\sqrt{7}}{4}$

foci $(\pm ae, 0) \equiv (\pm\sqrt{7}, 0)$

centre of circle is (0, 3)

$x^2 + y^2 - 6y + c = 0$

passes through $(\sqrt{7}, 0)$

$7 + 0 - 0 + c = 0$

$c = -7$

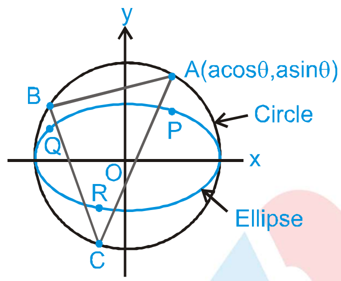
So $x^2 + y^2 - 6y - 7 = 0$

Part # II : IIT-JEE ADVANCED

1. Let $A \equiv (a\cos\theta, a\sin\theta)$ so the coordinates of

$B \equiv (a\cos(\theta+2\pi/3), a\sin(\theta+2\pi/3))$

$C \equiv (a\cos(\theta+4\pi/3), a\sin(\theta+4\pi/3))$



According to the given condition, coordinates of P are $(a\cos\theta, b\sin\theta)$ and that of Q are $(a\cos(\theta+2\pi/3), b\sin(\theta+2\pi/3))$ and that of R are $(a\cos(\theta+4\pi/3), b\sin(\theta+4\pi/3))$ (It is given that P, Q, R are on the same side of x-axis as A, B and C) Equation of the normal to the ellipse at P is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

or $ax\sin\theta - by\cos\theta = \dfrac{1}{2}(a^2-b^2)\sin 2\theta$ ....**(1)**

Equation of normal to the ellipse at Q is

$$ax\sin\left(\theta+\frac{2\pi}{3}\right)-by\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$=\frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{4\pi}{3}\right) \quad ....(2)$$

Equation of normal to the ellipse at R is

$ax\sin(\theta+4\pi/3) - by\cos(\theta+4\pi/3)$

$$=\frac{1}{2}(a^2-b^2)\sin(2\theta+8\pi/3) \quad ....(3)$$

But $\sin(\theta+4\pi/3)=\sin(2\pi+\theta-2\pi/3)=\sin(\theta-2\pi/3)$

and $\cos(\theta+4\pi/3)=\cos(2\pi+\theta-2\pi/3)=\cos(\theta-2\pi/3)$

and $\sin(2\theta+8\pi/3)=\sin(4\pi+2\theta-4\pi/3)=\sin(2\theta-4\pi/3)$

Now (3) can be written as

$ax\sin(\theta-2\pi/3) - by\cos(\theta-2\pi/3)$

$$=\frac{1}{2}(a^2-b^2)\sin(2\theta-4\pi/3) \quad ....(4)$$

For the lines (1), (2) and (4) to be concurrent, we must have determinant.

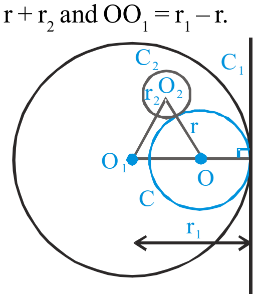
$$\Delta_1=\begin{vmatrix} a\sin\theta & -b\cos\theta & \frac{1}{2}(a^2-b^2)\sin 2\theta \\ a\sin\left(\theta+\frac{2\pi}{3}\right) & -b\cos\left(\theta+\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin(2\theta+4\pi/3) \\ a\sin\left(\theta-\frac{2\pi}{3}\right) & -b\cos\left(\theta-\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin(2\theta-4\pi/3) \end{vmatrix}=0$$

Thus lines (1), (2) and (4) are concurrent.

2. Let the given circles C_1 and C_2 have centres O_1 and O_2 and radii r_1 and r_2 respectively.

Let the variable circle C touching C_1 internally, C_2 externally have a radius r and centre at O.

Now, $OO_2 = r + r_2$ and $OO_1 = r_1 - r$.



$\Rightarrow$ $OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$

($\because$ C_2 lies inside C_1)

$\Rightarrow$ Locus of O is an ellipse with foci O_1 and O_2.

4. Given tangent is drawn at ($3\sqrt{3}\cos\theta, \sin\theta$) to

$$\frac{x^2}{27}+\frac{y^2}{1}=1$$

$\Rightarrow$ Equation of tangent is $\dfrac{x\cos\theta}{3\sqrt{3}}+\dfrac{y\sin\theta}{1}=1$

Thus sum of intercepts $=(3\sqrt{3}\sec\theta+\operatorname{cosec}\theta)=f(\theta)$ To minimise $f(\theta)$, $f'(\theta)=0$

$$\Rightarrow f'(\theta)=\frac{3\sqrt{3}\sin^3\theta-\cos^3\theta}{\sin^2\theta\cos^2\theta}=0$$

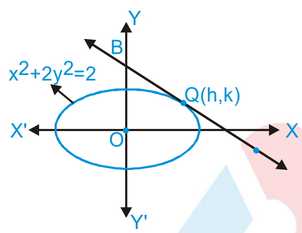
$\Rightarrow \sin^3\theta=\dfrac{1}{3^{3/2}}\cos^3\theta$ or $\tan\theta=\dfrac{1}{\sqrt{3}}$, i.e. $\theta=\dfrac{\pi}{6}$

7. Let the point of contact be

$R\equiv(\sqrt{2}\cos\theta, \sin\theta)$

Equation of tangent AB is

$$\frac{x}{\sqrt{2}}\cos\theta+y\sin\theta=1$$



$\Rightarrow A\equiv(\sqrt{2}\sec\theta, 0);\ B\equiv(0, \operatorname{cosec}\theta)$

Let the middle point Q of AB be (h, k)

$\Rightarrow h=\dfrac{\sec\theta}{\sqrt{2}},\ k=\dfrac{\operatorname{cosec}\theta}{2}$

$\Rightarrow \cos\theta=\dfrac{1}{h\sqrt{2}},\ \sin\theta=\dfrac{1}{2k} \Rightarrow \dfrac{1}{2h^2}+\dfrac{1}{4k^2}=1,$

$\therefore$ Required locus is $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$

Trick : The locus of mid-points of the portion of tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ intercepted between axes is $a^2y^2+b^2x^2=4x^2y^2$

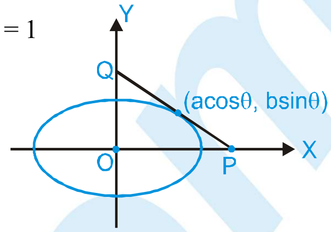
i.e., $\dfrac{a^2}{4x^2}+\dfrac{b^2}{4y^2}=1$ or $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$

8. Equation of tangent at ($a\cos\theta$, $b\sin\theta$) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$P=\left(\dfrac{a}{\cos\theta}, 0\right)$

$Q=\left(0, \dfrac{b}{\sin\theta}\right)$



Area of OPQ $=\dfrac{1}{2}\left|\left(\dfrac{a}{\cos\theta}\right)\left(\dfrac{b}{\sin\theta}\right)\right|=\dfrac{ab}{|\sin 2\theta|}$

$\therefore$ $(\text{Area})_{min}=ab$

10. Equation of ellipse is $\dfrac{x^2}{4}+y^2=1$

eccentricity $e=\dfrac{\sqrt{3}}{2}$

so focus are $(\sqrt{3},0)$ & $(-\sqrt{3},0)$

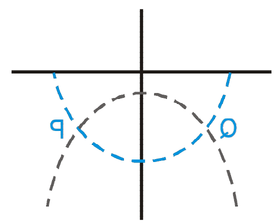
so end points of latus rectum will be

$\left(\sqrt{3},\frac{1}{2}\right)\left(\sqrt{3},-\frac{1}{2}\right),\left(-\sqrt{3},\frac{1}{2}\right)$ & $\left(-\sqrt{3},-\frac{1}{2}\right)$

$\Rightarrow$ $y_1<0$ & $y_2<0$

Hence coordinates of P & Q will be

$P\left(\sqrt{3},-\dfrac{1}{2}\right)$ & $Q\left(-\sqrt{3},-\dfrac{1}{2}\right)$.



So now equation of parabola taking these points as end points of latus rectum.

Focus will be (0, –1/2)

$4a=2\sqrt{3} \Rightarrow a=\dfrac{\sqrt{3}}{2}$

Hence vertex of the parabolas will be

$\left(0,-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\right),\ \left(0,-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\right)$

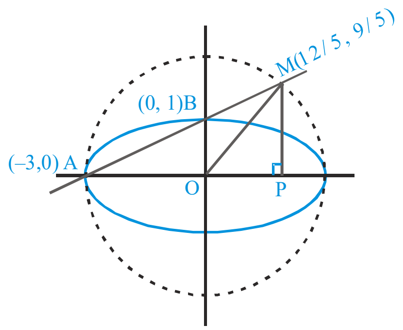
so eq. of parabolas will be

$x^2=-2\sqrt{3}\left(y+\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\right)$ &

$$x^2 = 2\sqrt{3}\left(y + \frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$$

11. Area of triangle AOM $= \frac{1}{2}$ AO. PM



$\Rightarrow$ Eqaution of AM is $y = \frac{1}{3}(x+3)$

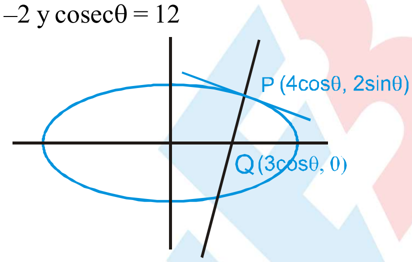
$x - 3y + 3 = 0$ which is chord of auxiliary circle $x^2 + y^2 = 9$, and PM is ordinate of point M

$\Rightarrow$ $(3y-3)^2 + y^2 = 9$ $\Rightarrow$ $y = \frac{9}{5} = PM$

$\Rightarrow$ Area of triangle $= \frac{1}{2}\,.\,3\,.\,\frac{9}{5} = \frac{27}{10}$.

12. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$4x\sec\theta - 2y\,\text{cosec}\theta = 12$



$x = 3\cos\theta$

$Q \equiv (3\cos\theta, 0)$

$2h = 7\cos\theta$

$2k = 2\sin\theta$

$\frac{4x^2}{49} + \frac{y^2}{1} = 1$ ...**(i)**

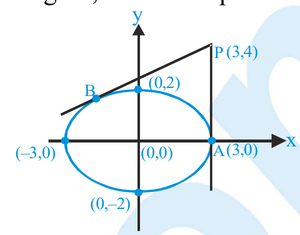
L.R $\Rightarrow$ $x = 2\sqrt{3}$

Putting in **(i)**

$y = \pm\frac{1}{7}$ $\therefore$ $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$

Paragraph for Question 13 to 15

13. As shown in figure, one of the point of contact is (3,0)



Let equation of other tangent,

$$y = mx + \sqrt{9m^2 + 4} \quad \text{as } c > 0$$

It passes through (3, 4), so

$$4 = 3m + \sqrt{9m^2 + 4}$$

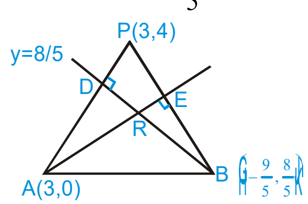
$$(4 - 3m)^2 = 9m^2 + 4$$

Solving, $m = \frac{1}{2}$

As we know that point of contact for the tangent given by $\left(-\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$

$\therefore$ Point of contact is $\left(-\frac{9}{5}, \frac{8}{5}\right)$

14. Equation of line BD : $y = \frac{8}{5}$



Equation of line AE : $2x + y = 6$

Now orthocentre R of ΔPAB will be intersection of line BD and line AE.

Solving for R, we get $R \equiv \left(\frac{11}{5}, \frac{8}{5}\right)$

15. Equation of line AB is $x + 3y = 3$

Now let the point be (h,k)

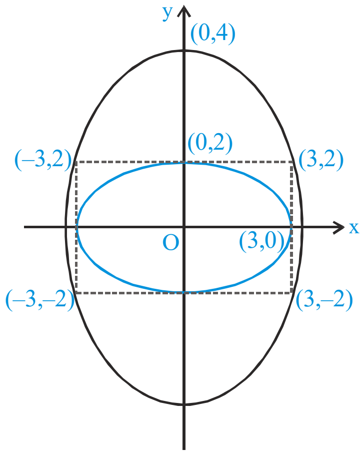
According to question,

$$\left|\frac{h + 3k - 3}{\sqrt{1^2 + 3^2}}\right| = \sqrt{(h-3)^2 + (4-k)^2}$$

After solving, we get

$$9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$$

16. Let equation of E_2 be



$\frac{x^2}{a^2}+\frac{y^2}{16}=1$ ($\because$ E_2 passes through (0, 4))

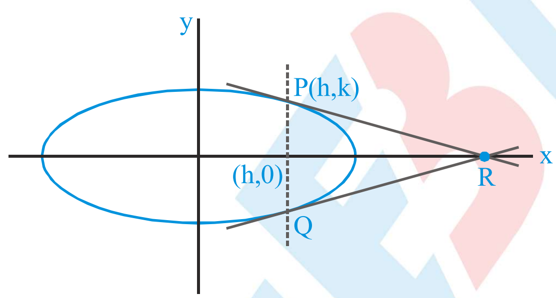
$\because$ E_2 passes through (3,2)

$\therefore$ $\frac{9}{a^2}+\frac{4}{16}=1$

$\Rightarrow$ $a^2 = 12$

$\therefore$ $e^2=1-\frac{a^2}{16}=1-\frac{3}{4} \Rightarrow e=\frac{1}{2}$

17.



Tangent at P(h, k) is $\frac{xh}{4}+\frac{ky}{3}=1$

$\Rightarrow$ $R\left(\frac{4}{h},0\right)$

$\Delta PQR = k\left(\frac{4}{h}-h\right)$

$=\sqrt{3\left(1-\frac{h^2}{4}\right)}\left(\frac{4}{h}-h\right)$

which is a decreasing function in $\left[\frac{1}{2},1\right]$

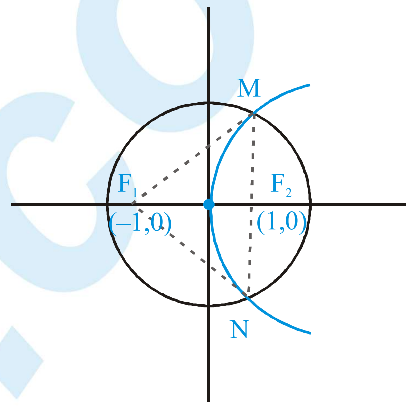
$\Rightarrow$ $\Delta_1=\sqrt{3\left(1-\frac{1}{16}\right)}\left(8-\frac{1}{2}\right)=\frac{45\sqrt{5}}{8}$

& $\Delta_2=\sqrt{3\left(1-\frac{1}{4}\right)}(4-1)=\frac{9}{2}$

$\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=45-36=9$

20. $a=3$

$e=\frac{1}{3}$



$\therefore$ $F_1\equiv(-1,0)$

$F_2\equiv(1,0)$

So, equation of parabola is $y^2=4x$

Solving simultaneously, we get $\left(\frac{3}{2},\pm\sqrt{6}\right)$

$\therefore$ Orthocentre is $\left(\frac{-9}{10},0\right)$

21. Equation of tangent at M is $\frac{x\times3}{2\times9}+\frac{y\sqrt{6}}{8}=1$

Put $y=0$ as intersection will be on x-axis.

$\therefore$ $R\equiv(6,0)$

Equation of normal at M is $\sqrt{\frac{3}{2}}x+y=2\sqrt{\frac{3}{2}}+\left(\sqrt{\frac{3}{2}}\right)^3$

Put $y=0$, $x=2+\frac{3}{2}=\frac{7}{2}$

$\therefore Q\equiv\left(\frac{7}{2},0\right)$

$\therefore$ Area $(\Delta MQR)=\frac{1}{2}\times\left(6-\frac{7}{2}\right)\times\sqrt{6}=\frac{5}{4}\sqrt{6}$

Area of quadrilateral $(MF_1NF_2)=2\times$ Area (ΔF_1F_2M)

$=2\times\frac{1}{2}\times2\times\sqrt{6}=2\sqrt{6}$ sq. units

$\therefore$ Required Ratio $=\frac{5/4}{2}=\frac{5}{8}$

MOCK TEST

1. (B)

Given $\quad b\sin\alpha + b\sin\beta = b$

$\sin\alpha + \sin\beta = 1$

$$h = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \qquad \text{.......(i)}$$

$$k = \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \qquad \text{.......(ii)}$$

$(1)^2 + (2)^2$

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = \frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\left(\cos^2\frac{\alpha-\beta}{2}\right) = 1 \qquad \text{.......(iii)}$$

given $\sin\alpha + \sin\beta = 1$

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 1 \qquad \text{.......(iv)}$$

from **(ii)** & **(iv)**

$$\frac{k\cos\left(\frac{\alpha-\beta}{2}\right)}{b} = \frac{1}{2\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \frac{2k}{b}\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1 \qquad \text{.......(v)}$$

from **(iii)** & **(v)**

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\frac{b}{2y} = 1$$

3. (B)

f(x) is a decreasing function & for major axis to be x-axis

$$f(k^2 + 2k + 5) > f(k + 11)$$

$\Rightarrow \; k^2 + 2k + 5 < k + 11$

$\Rightarrow \; k \in (-3, 2)$

5. (A)

Tangent at Q is, $x\cos\theta + y\sin\theta = a$

$ST = |ae\cos\theta - a| = a(1 - e\cos\theta)$

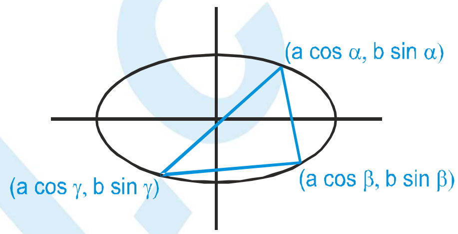
Also $SP = e\,PM = e\left(\frac{a}{e} - a\cos\theta\right) = a(1 - e\cos\theta)$

$ST = SP$

$\Rightarrow$ isosceles

7. (D)

$$x_1 = \frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma)$$



$$y_1 = \frac{b}{3}(\sin\alpha + \sin\beta + \sin\gamma)$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$= \left(\frac{\cos\alpha + \cos\beta + \cos\gamma}{3}\right)^2 + \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{3}\right)^2$$

$$\Rightarrow \frac{9x_1^2}{a^2} + \frac{9y_1^2}{b^2}$$

$= 1 + 1 + 1 + 2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha + \sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha)$

$$\Rightarrow \sum\cos\alpha + \cos\beta + \sum\sin\alpha\sin\beta = \frac{9x_1^2}{2a^2} + \frac{9y_1^2}{2b^2} - \frac{3}{2}$$

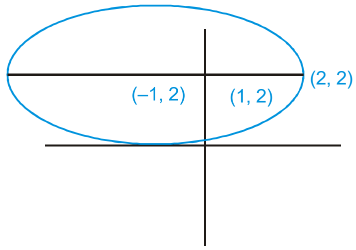
9. (A)

S_1 : Equation normal at P(θ)is $5\sec\theta\, x - 4\,\text{cosec}\,\theta\, y = 25 - 16$ and it passes through $P(0, \alpha)$

$$\alpha = \frac{-9}{4\,\text{cosec}\,\theta} \quad \text{i.e. } \alpha = \frac{-9}{4}\sin\theta \qquad |\alpha| \le \frac{9}{4}$$

S_2 : $4x^2 + 8x + 9y^2 - 36y = -4$

$4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = 36$



$$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$$

$\therefore$ G = 5, L = 1

S_3 : $\dfrac{2b^2}{a}=\dfrac{2a}{3}$ $\Rightarrow$ $3b^2=a^2$

$\therefore$ $b^2=a^2(1-e^2)$ $\Rightarrow$ $1=3(1-e^2)$

$\Rightarrow$ $e=\sqrt{2/3}$

S_4 : The equation is $\left(x-\dfrac{1}{13}\right)^2+\left(y-\dfrac{2}{13}\right)^2$

$$=\frac{1}{a^2}\left(\frac{5x+12y-1}{13}\right)^2$$

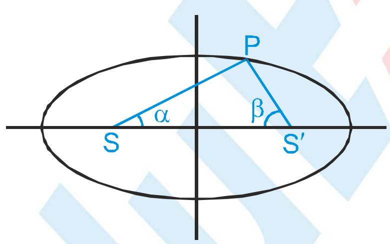
$\therefore$ $\dfrac{1}{a^2}<1$ i.e. $a^2>1$

11. (A,B,C)

Focal property of ellipse

$PS+PS'=2a$; if $a>b$

$PS+PS'=2b$; if $a<b$



$PS\cos\alpha+PS'\cos\beta=2ae$ ......(i)

$PS\sin\alpha-PS'\sin\beta=0$ ......(ii)

$PS+PS'=2a$ ......(iii)

from (i) and (ii), we get $PS=\dfrac{2ae\sin\beta}{\sin(\alpha+\beta)}$,

$PS'=\dfrac{2ae\sin\alpha}{\sin(\alpha+\beta)}$ ......(iv)

from (iii) and (iv)

$e(\sin\alpha+\sin\beta)=\sin(\alpha+\beta)$

$\therefore$ $e\cdot 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}=2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha+\beta}{2}$

$\therefore$ $e\left(\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}+\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\right)$

$=\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}-\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}$

$\therefore$ $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{1-e}{1+e}=\dfrac{2a\left(a-\sqrt{a^2-b^2}\right)-b^2}{b^2}$

$\therefore$ C is correct option & D is incorrect

12. (A, B, C)

Equation of tangent at θ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

$\Rightarrow$ $\dfrac{x}{a\sec\theta}+\dfrac{y}{b\,\text{cosec}\,\theta}=1$

$\therefore$ Length of intercept,

$d=\sqrt{a^2\sec^2\theta+b^2\,\text{cosec}^2\theta}$

$\therefore$ $d^2=a^2\sec^2\theta+b^2\,\text{cosec}^2\theta$

$\Rightarrow$ $\dfrac{d(d^2)}{d\theta}=2a^2\dfrac{\sin\theta}{\cos^3\theta}-2b^2\dfrac{\cos\theta}{\sin^3\theta}=0$

i.e., $\dfrac{a^2\sin\theta}{\cos^3\theta}=\dfrac{b^2\cos\theta}{\sin^3\theta}$

$\therefore$ $\tan^4\theta=\dfrac{b^2}{a^2}$

$\therefore$ $\tan\theta=\pm\sqrt{\dfrac{b}{a}}$

$\Rightarrow$ $\theta=\tan^{-1}\sqrt{\dfrac{b}{a}}, -\tan^{-1}\sqrt{\dfrac{b}{a}}$

$\Rightarrow$ $\theta=\pi-\tan^{-1}\sqrt{\dfrac{b}{a}}, -\pi+\tan^{-1}\sqrt{\dfrac{b}{a}}$

14. (A, C)

The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P.

So answer are (A) and (C)

15. (A, C)

We have

Slope of AB = Slope of tangent at C

$$\Rightarrow \frac{b(\sin\beta-\sin\alpha)}{a(\cos\beta-\cos\alpha)}=\frac{-b\cos\theta}{a\sin\theta}$$

$$\left[\text{for } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{dy}{dx}=\frac{-b^2x}{a^2y}\right]$$

$$\Rightarrow \frac{-\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{-\cos\theta}{\sin\theta}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right)=\tan\theta$$

$$\Rightarrow \theta=\frac{\alpha+\beta}{2}+n\pi\ (n\in I).$$

16. (A)

Statement - 2 is true (standard result)

The ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\therefore$ auxiliary circle is $x^2+y^2=9$ and $\left(-\sqrt{5},0\right)$ and $\left(\sqrt{5},0\right)$ are focii

$\therefore$ statement - 1 is true

statement -2 is true

17. $x^2+y^2=5$ is the director circle of ellipse $4x^2+y^2=4$

18. (D)

Statement -I : (False) Since $\angle PQT=\frac{\pi}{2}$

$\therefore$ PQ $\perp$ to the directrix.

$\therefore$ By definition PS = ePQ

$\therefore$ S-1 is false.

Statement - II : True (Standard results)

19. Congruent $\Rightarrow$ same eccentricity

equal eccentricity $\Rightarrow$ similar but not congruent.

20. (D)

Statement-1 Locus of point of intersection of only perpendicular lines is a circle and other vertices B and C does not form a circle

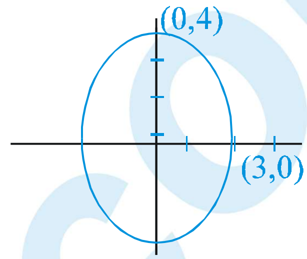
Statement-2 Obvious (Standrad definition)

21. (A) → S; (B) → R ; (C) → P ; (D) → Q

(A) $e^2=1-\frac{9}{16}=\frac{7}{16}$

ellipse touch the x-axis at (3, 0) and y-axis at (0, 4)

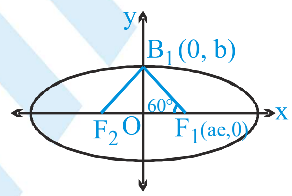
$e=\frac{\sqrt{7}}{4}$



(B) Locus is as ellipse

(C) We have $\frac{b}{ae}=\sqrt{3}$

$\Rightarrow a^2(1-e^2)=3a^2e^2$ (As $b^2=a^2(1-e^2)$)



$\Rightarrow e^2=\frac{1}{4}$

Hence $e=\frac{1}{2}$

(D) The given distance is clearly the length of semi major axis Thus, $\sqrt{\frac{a^2+2b^2}{2}}=a$

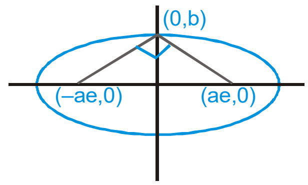
$\Rightarrow 2b^2=a^2 \qquad \Rightarrow 2a^2(1-e^2)=a^2$

$\Rightarrow e^2=\frac{1}{2} \qquad \Rightarrow e=\frac{1}{\sqrt{2}}$

22. (A) → R ; (B) → P ; (C) → S, T ; (D) → Q

(A) $\Rightarrow \frac{b-0}{0-ae}\times\frac{b-0}{0+ae}=-1$

$\Rightarrow -\frac{b^2}{a^2e^2}=-1$



$$\Rightarrow \quad \frac{a^2(1-e^2)}{a^2e^2} = 1 \qquad \Rightarrow \quad 1 - e^2 = e^2$$

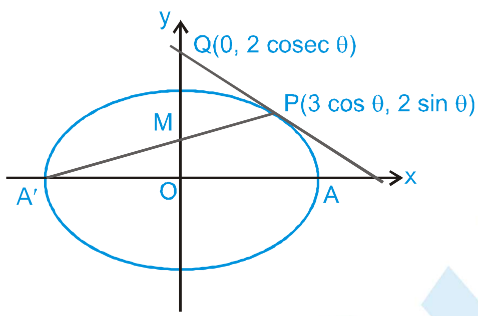
$$\Rightarrow \quad 2e^2 = 1 \qquad \Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

(B) Here $a = 3; b = 2$

and $T : \dfrac{x\cos\theta}{3} + \dfrac{y\sin\theta}{2} = 1$

$x = 0; y = 2\operatorname{cosec}\theta$

Equation of chord A′P, $y = \dfrac{2\sin\theta}{3(\cos\theta+1)}(x+3)$



Put $x = 0,\ y = \dfrac{2\sin\theta}{1+\cos\theta} = OM$

Now,

$OQ^2 - MQ^2 = OQ^2 - [OQ - OM]^2$

$= 2(OQ)(OM) - OM^2 = OM\,\{2(OQ) - (OM)\}$

$$= \frac{2\sin\theta}{1+\cos\theta}\left[\frac{4}{\sin\theta} - \frac{2\sin\theta}{1+\cos\theta}\right] = 4$$

(C) $\left(y + \dfrac{1}{2}\right)^2 = 8(x-2)$

$\Rightarrow \quad Y^2 = 8X$

for three normal $X > 2a$

$\Rightarrow \quad x - 2 > 4 \ \therefore\ x > 6$

(D) Area of parallelogram $= 2ab$

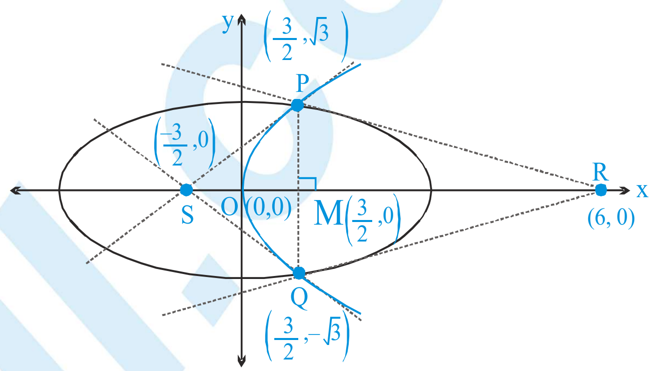
$$= 2 \times (2) \times \left(\frac{1}{2}\right) = 2$$

23. Solving the curves $y^2 = 2x$ and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ for the points of intersection, we have

$$4x^2 + 18x - 36 = 0 \qquad \Rightarrow \quad x = \frac{3}{2}, -6$$

But from $y^2 = 2x$ we have $x > 0$

$\therefore \quad x = \dfrac{3}{2}$ at which $y^2 = 2 \cdot \dfrac{3}{2}$



$\Rightarrow \quad y = \pm\sqrt{3}$

$\therefore \quad P\left(\dfrac{3}{2}, \sqrt{3}\right)$ and $Q\left(\dfrac{3}{2}, -\sqrt{3}\right)$

Now equation of tangents at P and Q to ellipse

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is $\dfrac{x}{9}\left(\dfrac{3}{2}\right) + \dfrac{y}{4}\left(\pm\sqrt{3}\right) = 1$

which intersect at R(6, 0)

Equation of tangents at P and Q to parabola $y^2 = 2x$ will be $y\left(\pm\sqrt{3}\right) = x + \dfrac{3}{2}$ which cut x-axis $S\left(\dfrac{-3}{2}, 0\right)$

$$\therefore \quad \frac{\text{Area }\Delta PQS}{\text{Area }\Delta PQR} = \frac{\frac{1}{2}PQ\cdot MS}{\frac{1}{2}PQ\cdot MR} = \frac{MS}{MR}$$

$$= \frac{\frac{3}{2} - \left(\frac{-3}{2}\right)}{6 - \frac{3}{2}} = \frac{3}{\frac{9}{2}} = \frac{2}{3}$$

Area of quadrilateral PRQS $= \dfrac{1}{2}PQ(MS + MR) = \dfrac{1}{2} \cdot$

$2\sqrt{3}\ (6-(-3/2)) = \dfrac{15\sqrt{3}}{2}$

Clearly upper end of latus rectum of parabola is $\left(\dfrac{1}{2},1\right)$.

And equation of tangent at $\left(\dfrac{1}{2},1\right)$ to

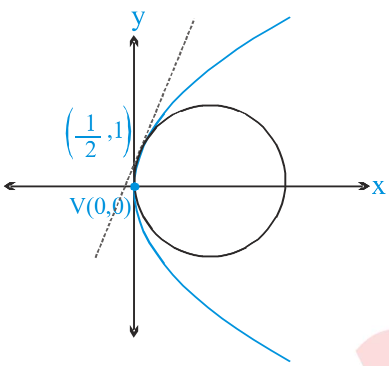
$y^2 = 2x$ is $y = x + \dfrac{1}{2}$

$\therefore$ The equation of circle is

$$\left(x-\frac{1}{2}\right)^2 + (y-1)^2 + \lambda\left(y - x - \frac{1}{2}\right) = 0$$

As above circle passes through V (0, 0),

So $\dfrac{1}{4} + 1 - \dfrac{\lambda}{2} = 0 \quad \Rightarrow \quad \lambda = \dfrac{5}{2}$



$\Rightarrow$ The equation of required circle is

$$\left(x-\frac{1}{2}\right)^2 + (y-1)^2 + \frac{5}{2}\left(y - x - \frac{1}{2}\right) = 0$$

$\Rightarrow \quad 2x^2 + 2y^2 - 7x + y = 0$

24.

1. (D) Using conditions for ellipse

$-2 < \lambda < 2 \quad \Rightarrow \quad \lambda = -1, 0, 1$

2. (C) Consider $f = x^2 + xy + y^2 - 1 = 0$

3. (C) Solving $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$ centre of ellipse comes out to be (0, 0) Longest chord of ellipse is – major Axis.

While chord perpendicular to it and through centre of ellipse is minor axis, so is the smallest chord. Let P be a point on ellipse such that OP = r and $\angle POX = \theta$ = then P $(r\cos\theta, r\sin\theta)$ as P lies on ellipse $r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta - 1 = 0$

$\Rightarrow \quad r^2 = \dfrac{1}{1+\sin\theta\cos\theta} = \dfrac{1}{1+\dfrac{1}{2}\sin 2\theta}$

$\therefore$ r is smallest if $\sin 2\theta$ is greatest

i.e. $\sin 2\theta = 1$

$r^2_{max} = OA^2 = 2 \quad \Rightarrow \quad OA = \sqrt{2}$

$r^2_{min} = OB^2 = \dfrac{2}{3} \quad \Rightarrow \quad OB = \sqrt{\dfrac{2}{3}}$

25.

1. (B)

Distance between Q′ and R′ is maximum so QR is major axis and height of the bird's path above the ground level is $5\sqrt{3}$. $PR' = 15$, $PQ' = 5$, $RQ = R'Q' = 10$

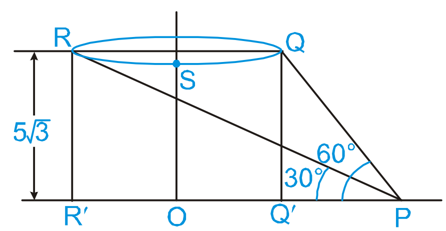
$\therefore$ coordinates of $Q \equiv (5, 0, 5\sqrt{3})$

$OP = OR' + R'P = 5 + 5 = 10$

$\therefore$ coordinates of P are (10, 0, 0)

Equation of the line joining P (10, 0, 0) and R (– 5, 0 , $5\sqrt{3}$) is

$$\frac{x-10}{15} = \frac{y-0}{0} = \frac{z-0}{-5\sqrt{3}}$$



i.e. $\dfrac{x-10}{\sqrt{3}} = \dfrac{z}{-1}, y = 0$

Since the line passes through the point $(13, 0, -\sqrt{3})$

$\therefore$ the equation of the line can be written as

or $\dfrac{x-13}{\sqrt{3}} = \dfrac{z+\sqrt{3}}{-1}, \ y = 0$

2. (C)

Equation of plane in which ellipse lies is $z = 5\sqrt{3}$ ....**(i)**

equation of vertical plane tangent at Q is $x = 5$ ....**(ii)**

$\therefore$ the required plane is $((z - 5\sqrt{3}) + \lambda(x-5) = 0$

as this plane passes through P (10, 0, 0)

$\therefore$ equation of this plane is $\sqrt{3}\,x + z - 10\sqrt{3} = 0$.

3. (C)

Let the bird is at S and projection of S on the ground is S′

$\therefore$ $OS' = \text{semi minor axis} = \dfrac{1}{\sqrt{b}}$

$\therefore$ $\left(\dfrac{1}{\sqrt{2}}\right)^2 = 1 - \dfrac{\frac{1}{b}}{25} \quad \Rightarrow \quad \dfrac{1}{\sqrt{b}} = \dfrac{5}{\sqrt{2}}$

$\therefore$ $PS' = \sqrt{\left(\dfrac{5}{\sqrt{2}}\right)^2 + (10)^2} = \sqrt{\dfrac{25+200}{2}}$

$= \sqrt{\dfrac{225}{2}} = \dfrac{15}{\sqrt{2}}$ and $SS' = 5\sqrt{3}$

$\therefore$ $\tan\theta = \dfrac{SS'}{PS'} = \sqrt{\dfrac{2}{3}}$

26. (3)

$\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$

equation of normal at the point $(13\cos\theta, 5\sin\theta)$

$(y - 5\sin\theta) = \dfrac{13\sin\theta}{5\cos\theta}(x - 13\cos\theta)$ it passes through (0, 6)

then $\cos\theta(15 + 72\sin\theta) = 0$

or $\cos\theta = 0, \quad \sin\theta = -\dfrac{5}{24}$

$\theta = \dfrac{\pi}{2}, 2\pi - \sin^{-1}\dfrac{5}{24}, \pi + \sin^{-1}\dfrac{5}{24}$

equation has three roots hence three normal can be drawn.

27. Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. Equation of chord of the ellipse $\dfrac{x^2}{2a^2} + \dfrac{y^2}{a^2} = 1$, whose mid-point is $(at^2, 2at)$ is $\dfrac{x\cdot at^2}{2a^2} + \dfrac{y\cdot 2at}{a^2} = \dfrac{a^2t^4}{2a^2} + \dfrac{4a^2t^2}{a^2}$

$\Rightarrow$ $tx + 4y = at^3 + 8at \qquad (\because t \neq 0)$

As it passes through $\left(11a, -\dfrac{a^2}{4}\right)$,

$\Rightarrow$ $11at - 4\left(\dfrac{a^2}{4}\right) = at^3 + 8at$

$\Rightarrow$ $at^3 - 3at + a^2 = 0$

$\Rightarrow$ $t^3 - 3t + a = 0 \quad (a \neq 0)$

Now, three chords of the ellipse will be bisected by the parabola if the equation (1) has three real and distinct roots.

Let $f(t) = t^3 - 3t + a$

$f'(t) = 3t^2 - 3 = 0 \quad \Rightarrow \quad t = \pm 1$

So, $f(1)\,f(-1) < 0$

$\Rightarrow$ $a \in (-2, 2)$

But $a \neq 0$, so $a \in (-2, 0) \cup (0, 2)$

$\therefore$ Number of integral values of 'a' = 2.

28. (1/2)

Let line OPQ makes angle θ with x-axis so

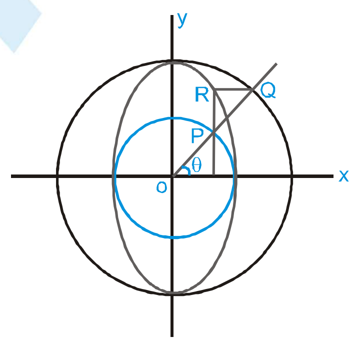
$P \equiv (a\cos\theta, a\sin\theta)$, $Q\,(b\cos\theta, b\sin\theta)$

and Let R (x, y)

So $X = a\cos\theta \quad Y = b\sin\theta$

eliminating θ, we get

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, locus of R is an ellipse.



Also a < b so vertices are (0,b) and (0, – b)

and extremities of minor axis are (±a, 0).

So ellipse touches both inner circle and outer circle if focii are (0, ± a)

$\Rightarrow$ $a = be \quad$ i.e $\quad e = a/b$

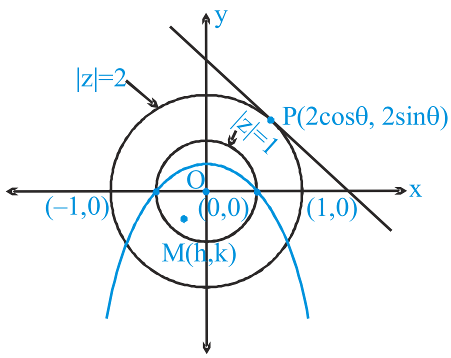
Also $\quad e = \sqrt{1-e^2} \quad \Rightarrow \quad e^2 = 1 - e^2 \quad \Rightarrow \quad e = 1/\sqrt{2}$

and ratio of radii is $\dfrac{a}{b} = e = \dfrac{1}{\sqrt{2}}$.

29. Clearly the parabola should pass through (1, 0) and (–1, 0). Let directrix of this parabola be $x\cos\theta + y\sin\theta = 2$. If M (h,k) be the focus of this parabola, then distance of (±1, 0) from 'M' and from the directrix should be same.

$\Rightarrow$ $(h-1)^2 + k^2 = (\cos\theta - 2)^2 \qquad$ ....(i) and $(h+1)^2 + k^2 = (\cos\theta + 2)^2$ ....(ii)

Now (ii) – (i) $\Rightarrow$ $\cos\theta = \frac{h}{2}$ ....(iii)



Also (ii) + (i)

$\Rightarrow$ $(h^2 + k^2 + 1) = (\cos^2\theta + 4)$ ....(iv)

$\therefore$ From (iii) and (iv), we get

$h^2 + k^2 + 1 = 4 + \frac{h^2}{4} \Rightarrow \frac{3h^2}{4} + k^2 = 3$

Hence locus of focus M(h, k) is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (Ellipse)

Also we know that area of the quadrilateral formed by the tangents at the ends of the latus-rectum is $\frac{2a^2}{e}$

(where e is eccentricity of ellipse)

$\therefore$ Requred area = $\frac{2(4)}{1/2} = 16$ (square units)

(As $e^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$)

30. (4)

For ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ equation of director circle is $x^2 + y^2 = 25$. This director circle will cut the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ at 4 points

hence number of point = 4.