CHAPTER 2

Complex Numbers

1. The set of all $\alpha \in R$, for which $\omega = \dfrac{1+(1-8\alpha)z}{1-z}$ is a purely imaginary number, for all $z \in C$ satisfying $|z| = 1$ and Re $z \neq 1$, is

(a) an empty set (b) equal to R

(c) $\{0\}$ (d) $\left\{0, \frac{1}{4}, -\frac{1}{4}\right\}$ *(Online 2018)*

2 If $|z - 3 + 2i| \leq 4$ then the difference between the greatest value and the least value of $|z|$ is

(a) $\sqrt{13}$ (b) $4+\sqrt{13}$ (c) 8 (d) $2\sqrt{13}$

(Online 2018)

3. The least positive integer n for which $\left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n = 1$, is :

(a) 3 (b) 5 (c) 2 (d) 6

(Online 2018)

4. Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

(a) z (b) -1 (c) 1 (d) $-z$ *(2017)*

5. Let $z \in C$, the set of complex numbers. Then the equation, $2|z + 3i| - |z - i| = 0$ represents

(a) a circle with radius $\frac{8}{3}$.

(b) a circle with diameter $\frac{10}{3}$.

(c) an ellipse with length of major axis $\frac{16}{3}$.

(d) an ellipse with length of minor axis $\frac{16}{9}$.

(Online 2017)

6. The equation $\text{Im}\left(\dfrac{iz-2}{z-i}\right) + 1 = 0, z \in C, z \neq i$ represents a part of a circle having radius equal to

(a) 1 (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) 2

(Online 2017)

7. A value of θ for which $\dfrac{7+8\ \text{-ız}\ \theta}{6-7\ \text{-ız}\ \theta}$ is purely imaginary is

(a) $\pi/3$ (b) $\pi/6$

(c) $\text{-ız}^{-6}\left(\dfrac{\sqrt{8}}{9}\right)$ (d) $\text{-ız}^{-6}\left(\dfrac{6}{\sqrt{8}}\right)$ *(2016)*

8. The point represented by $2 + i$ in the Argand plane moves 1 unit eastwards, then 2 units northwards and finally from there $2\sqrt{2}$ units in the south-westwards direction. Then its new position in the Argand plane is at the point represented by

(a) $1 + i$ (b) $2 + 2i$ (c) $-2 - 2i$ (d) $-1 - i$

(Online 2016)

9. Let $z = 1 + ai$ be a complex number, $a > 0$, such that z^3 is a real number. Then the sum $1 + z + z^2 + ... + z^{11}$ is equal to

(a) $1365\sqrt{3}i$ (b) $-1365\sqrt{3}i$

(c) $-1250\sqrt{3}i$ (d) $1250\sqrt{3}i$ *(Online 2016)*

10. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\dfrac{6 - 7\ \ 7}{7 - \overline{6\ \ 7}}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a

(a) circle of radius 2. (b) circle of radius $\sqrt{7}$.

(c) straight line parallel to x-axis.

(d) straight line parallel to y-axis. *(2015)*

11. The largest value of r for which the region represented by the set $\{\omega \in C : |\omega - 4 - i| \leq r\}$ is contained in the region represented by the set $\{z \in C : |z - 1| \leq |z + i|\}$, is equal to

(a) $\sqrt{6<}$ (b) $7\sqrt{7}$ (c) $\frac{8}{7}\sqrt{7}$ (d) $\frac{:}{7}\sqrt{7}$

(Online 2015)

12. If z is a non-real complex number, then the minimum value of $\dfrac{\text{Im } z^5}{(\text{Im } z)^5}$ is

(a) -1 (b) -2 (c) -4 (d) -5

(Online 2015)

13. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(a) lies in the interval (1, 2)

(b) is strictly greater than $\frac{5}{2}$

(c) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

(d) is equal to $\frac{5}{2}$ *(2014)*

14. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals

(a) $\frac{\pi}{2}-\theta$ (b) θ (c) $\pi-\theta$ (d) $-\theta$ *(2013)*

15. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(a) either on the real axis or on a circle not passing through the origin.

(b) on the imaginary axis.

(c) either on the real axis or on a circle passing through the origin.

(d) on a circle with centre at the origin. *(2012)*

16. If $\omega(\neq 1)$ is a cube root of unity, and $(1+\omega)^7 = A + B\omega$. Then (A, B) equals

(a) $(1, 0)$ (b) $(-1, 1)$

(c) $(0, 1)$ (d) $(1, 1)$ *(2011)*

17. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals

(a) 0 (b) 1 (c) 2 (d) ∞ *(2010)*

18. If $\left|Z-\frac{4}{Z}\right|=2,$ then the maximum value of $|Z|$ is equal to

(a) $\sqrt{5}+1$ (b) 2 (c) $2+\sqrt{2}$ (d) $\sqrt{3}+1$ *(2009)*

19. The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is

(a) $\frac{1}{i-1}$ (b) $\frac{-1}{i-1}$ (c) $\frac{1}{i+1}$ (d) $\frac{-1}{i+1}$ *(2008)*

20. If $|z+4| \leq 3$, then the maximum value of $|z+1|$ is

(a) 6 (b) 0 (c) 4 (d) 10 *(2007)*

21. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$ is

(a) i (b) 1 (c) -1 (d) $-i$ *(2006)*

22. If $z^2+z+1=0$, where z is a complex number, then the value of $\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+....+\left(z^6+\frac{1}{z^6}\right)^2$ is

(a) 18 (b) 54 (c) 6 (d) 12 *(2006)*

23. If z_1 and z_2 are two non-zero complex numbers such that $|z_1+z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to

(a) $-\pi$ (b) $\pi/2$ (c) $-\pi/2$ (d) 0 *(2005)*

24. If $\omega = \frac{z}{z-(1/3)i}$ and $|\omega| = 1$, then z lies on

(a) a circle (b) an ellipse

(c) a parabola (d) a straight line *(2005)*

25. If the cube roots of unity are 1, ω, ω^2 then the roots of the equation $(x-1)^3+8=0$, are

(a) $-1, -1, -1$ (b) $-1, -1+2\omega, -1-2\omega^2$

(c) $-1, 1+2\omega, 1+2\omega^2$ (d) $-1, 1-2\omega, 1-2\omega^2$ *(2005)*

26. Let z, ω be complex numbers such that $\bar{z}+i\bar{\omega}=0$ and $z\omega=\pi$. Then $\arg z$ equals

(a) $3\pi/4$ (b) $\pi/2$ (c) $\pi/4$ (d) $5\pi/4$ *(2004)*

27. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p}+\frac{y}{q}\right)}{(p^2+q^2)}$ is equal to

(a) 2 (b) -1 (c) 1 (d) -2 *(2004)*

28. If $|z^2-1| = |z|^2+1$, then z lies on

(a) a circle (b) the imaginary axis

(c) the real axis (d) an ellipse *(2004)*

29. If $\left(\frac{1+i}{1-i}\right)^x=1$, then

(a) $x = 2n$, where n is any positive integer

(b) $x = 4n+1$, where n is any positive integer

(c) $x = 2n+1$, where n is any positive integer

(d) $x = 4n$, where n is any positive integer *(2003)*

30. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\text{Arg}(z) - \text{Arg}(\omega) = \pi/2$, then $\bar{z}\omega$ is equal to

(a) -1 (b) i (c) $-i$ (d) 1 *(2003)*

31. Let z_1 and z_2 be two roots of the equation $z^2+az+b=0$, z being complex further, assume that the origin, z_1 and z_2 form an equilateral triangle, then

(a) $a^2 = 2b$ (b) $a^2 = 3b$

(c) $a^2 = 4b$ (d) $a^2 = b$ *(2003)*

32. z and ω are two nonzero complex number such that $|z|=|\omega|$ and $\text{Arg } z + \text{Arg } \omega = \pi$ then z equals

(a) $\bar{\omega}$ (b) $-\bar{\omega}$ (c) ω (d) $-\omega$ *(2002)*

33. If $|z-4|<|z-2|$, its solution is given by

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$

(c) $\text{Re}(z) > 3$ (d) $\text{Re}(z) > 2$ *(2002)*

34. The locus of the centre of a circle which touches the circle $|z-z_1|=a$ and $|z-z_2|=b$ externally (z, z_1 & z_2 are complex numbers) will be

(a) an ellipse (b) a hyperbola

(c) a circle (d) none of these *(2002)*

ANSWER KEY

1. (c)	**2.** (d)	**3.** (a)	**4.** (d)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (a)	**11.** (d)	**12.** (c)
13. (a)	**14.** (b)	**15.** (c)	**16.** (d)	**17.** (b)	**18.** (a)	**19.** (d)	**20.** (a)	**21.** (d)	**22.** (d)	**23.** (d)	**24.** (d)
25. (d)	**26.** (a)	**27.** (d)	**28.** (b)	**29.** (d)	**30.** (c)	**31.** (b)	**32.** (b)	**33.** (c)	**34.** (b)		

Explanations

1. **(c):** Given, $\omega = \dfrac{1+(1-8\alpha)z}{1-z}$

For ω to be purely imaginary, $\omega + \bar{\omega} = 0$

i.e., $\dfrac{1+(1-8\alpha)z}{1-z} + \dfrac{1+(1-8\alpha)\bar{z}}{1-\bar{z}} = 0$

$\Rightarrow \quad [1+(1-8\alpha)z][1-\bar{z}] + [1+(1-8\alpha)\bar{z}][1-z] = 0$

$\Rightarrow \quad [1-\bar{z}+(1-8\alpha)z-(1-8\alpha)z\bar{z}] + [1-z+(1-8\alpha)\bar{z} - (1-8\alpha)z\bar{z}] = 0$

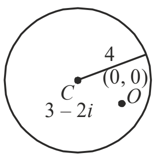
$\Rightarrow \quad 2-(z+\bar{z})+(1-8\alpha)(z+\bar{z})-2(1-8\alpha) = 0 \quad (\because z\bar{z} = |z| = 1)$

$\Rightarrow \quad 2-(z+\bar{z})+(z+\bar{z})-8\alpha(z+\bar{z})-2+16\alpha = 0$

$\Rightarrow \quad 16\alpha = 8\alpha(z+\bar{z})$

Either $z+\bar{z} = 2$ or $\alpha = \{0\}$. But $z+\bar{z} = 2$ is not possible $\therefore \alpha = \{0\}$

2. **(d):** Origin (O) lies inside the circle



Greatest value of $|z| = OC + r = \sqrt{13}+4$

Least value of $|z| = r - OC = 4-\sqrt{13}$

Required difference $= \sqrt{13}+4-4+\sqrt{13} = 2\sqrt{13}$

3. **(a):** $\left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n = 1 \Rightarrow \left(\dfrac{\frac{1+i\sqrt{3}}{2}}{\frac{1-i\sqrt{3}}{2}}\right)^n = 1$

$\Rightarrow \left(\dfrac{-\omega^2}{-\omega}\right)^n = 1 \quad \left[\begin{array}{l} \because \ \omega = -\dfrac{1-i\sqrt{3}}{2} \\ \text{and } \ \omega^2 = -\dfrac{(1+i\sqrt{3})}{2} \end{array}\right] \Rightarrow (\omega)^n = 1$

So, least positive integer value of n is 3.

4. **(d):** We have, $z = 1 + 2\omega$

i.e., $i\sqrt{3} = 1+2\omega \ \therefore \ \omega = \dfrac{-1+i\sqrt{3}}{2}$

Then ω is a cube root of unity. Also, $1+\omega+\omega^2 = 0$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k \Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3k$

$\Rightarrow \quad 3(\omega^2 - \omega^4) = 3k$

$\Rightarrow \quad k = \omega^2 - \omega = -1 - \omega - \omega = -1 - 2\omega = -z$

5. **(a):** We have, $2|z + 3i| - |z - i| = 0$

$\Rightarrow \quad 2|x + i(y+3)| = |x + i(y-1)| \quad (\because z = x+iy)$

$\Rightarrow \quad 2\sqrt{x^2+(y+3)^2} = \sqrt{x^2+(y-1)^2}$

$\Rightarrow \quad 4(x^2 + (y+3)^2) = x^2 + (y-1)^2$

$\Rightarrow \quad 3x^2 = y^2 - 2y + 1 - 4y^2 - 24y - 36$

$\Rightarrow \quad 3x^2 + 3y^2 + 26y + 35 = 0 \Rightarrow x^2+y^2+\dfrac{26}{3}y+\dfrac{35}{3} = 0$

This is the equation of circle with radius,

$r = \sqrt{0^2+\left(\dfrac{13}{3}\right)^2-\dfrac{35}{3}} = \sqrt{\dfrac{64}{9}} = \dfrac{8}{3}$

6. **(b):** Let $z = x + iy$

$\text{Im}\left[\left(\dfrac{ix-y-2}{x+i(y-1)}\right)\left(\dfrac{x-i(y-1)}{x-i(y-1)}\right)\right]+1 = 0 \Rightarrow \dfrac{(y-1)(y+2)+x^2}{x^2+(y-1)^2}+1 = 0$

$\Rightarrow \quad 2x^2 + 2y^2 - y - 1 = 0 \Rightarrow x^2 + y^2 - (1/2)y - (1/2) = 0$

$\therefore \quad$ Centre of circle is $\left(0, \dfrac{1}{4}\right)$

$\therefore \quad$ Radius $= \sqrt{0+\left(\dfrac{1}{4}\right)^2+\dfrac{1}{2}} = \sqrt{\dfrac{1}{16}+\dfrac{1}{2}} = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4}$

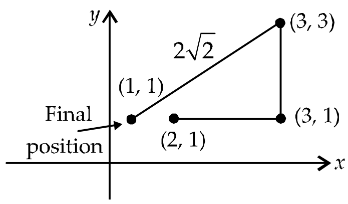
7. **(d) :** Let $\alpha = \dfrac{7+8 \text{ -ιz } \theta}{6-7 \text{ -ιz } \theta}$

$\Rightarrow \quad \alpha = \dfrac{\text{-}7+8 \text{ -ιz } \theta.\text{-}6+7 \text{ -ιz } \theta.}{\text{-}6-7 \text{ -ιz } \theta.\text{-}6+7 \text{ -ιz } \theta.} = \dfrac{\text{-}7-; \text{ -ιz}^7 \theta. + \text{ -<-ιz } \theta.}{6+9\text{-ιz}^7 \theta}$

As α is to be purely imaginary, we have

Re $(\alpha) = 0 \Rightarrow 2 = 6\sin^2\theta \quad 33\text{-ιz } \theta = \pm \dfrac{6}{\sqrt{8}}$

8. **(a) :**



Hence, the final position of the point is represented by $1 + i$.

9. **(b) :** $z = 1 + ai$, $z^2 = 1 - a^2 + 2ai$

$z^2 \cdot z = \{(1-a^2)+2ai\}\{1+ai\} = (1-a^2) + 2ai + (1-a^2)ai - 2a^2$

$\because \quad z^3$ is real $\Rightarrow 2a + (1-a^2)a = 0$

$\Rightarrow \quad a(3-a^2) = 0 \Rightarrow a = \sqrt{3} \quad (\because a > 0)$

Now, $= \dfrac{(1+\sqrt{3}i)^{12}-1}{1+\sqrt{3}i-1} = \dfrac{(1+\sqrt{3}i)^{12}-1}{\sqrt{3}i}$

$(1+\sqrt{3}i)^{12} = 2^{12}\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^{12} = 2^{12}\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)^{12}$

$= 2^{12}(\cos4\pi + i\sin4\pi) = 2^{12}$

$\therefore \quad 1 + z + z^2 + \ldots\ldots + z^{11} = \dfrac{2^{12}-1}{\sqrt{3}i} = \dfrac{4095}{\sqrt{3}i}$

$= -\dfrac{4095}{3}\sqrt{3}i = -1365\sqrt{3}i$

10. **(a) : 1st solution :** c qtmql $\left|\dfrac{6-7 \ 7}{7-\overline{6 \ 7}}\right| = 6$

$\Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow \quad (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow \quad |z_1|^2 - 4 - |z_1|^2|z_2|^2 + 4|z_2|^2 = 0$

$\Rightarrow \quad \{|z_1|^2 - 4\} - |z_2|^2\{|z_1|^2 - 4\} = 0$

$\Rightarrow \quad (1 - |z_2|^2)(|z_1|^2 - 4) = 0$

Thus $|z_1| = 2$ as $|z_2| \neq 1$ (given)

The point z lies on circle of radius 2.

2nd solution : Observe that if $\frac{\alpha-\beta}{6-\alpha\bar{\beta}}=61$ two complex numbers α and β of which $|\beta| \neq 1$, then $|\alpha| = 1$

Since $|\alpha - \beta| = |1 - \alpha\bar{\beta}| \Rightarrow |\alpha - \beta|^2 = |1 - \alpha\bar{\beta}|^2$

$\Rightarrow \quad |\alpha|^2 + |\beta|^2 - 2\text{Re}\,(\alpha\bar{\beta}) = 1 + |\alpha|^2\,|\beta|^2 - 2\,\text{Re}(\alpha\bar{\beta})$

$\Rightarrow \quad 1 - |\alpha|^2 - |\beta|^2 - |\alpha|^2\,|\beta|^2 = 0$

$\Rightarrow \quad (1 - |\alpha|^2)\,(1 - |\beta|^2) = 0$ As $|\beta| \neq 1 \quad \therefore |\alpha| = 1$

In our case take $\alpha = z_1/2$ and $\beta = z_2$

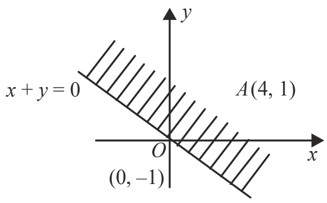
gives $|z_1/2| = 1 \quad \therefore \quad |z_1| = 2$

11. (d) : We have $|z - 1| \leq |z + i| \Rightarrow x + y \geq 0$

The region shaded is of the line $x + y = 0$

Co-ordinates of centre of circle $|\omega - 4 - i| = r$ is (4, 1) (say A)

The largest value of r would be the length of $\perp$ from $A(4, 1)$ on the line $x + y = 0$



$\Rightarrow \quad \left|\frac{9+6}{\sqrt{7}}\right| = \frac{:}{\sqrt{7}}$

12. (c): Let $z = re^{i\theta} \Rightarrow \frac{\text{Im}\,z^5}{(\text{Im}\,z)^5} = \frac{r^5(\sin 5\,\theta)}{r^5(\sin\theta)^5} = \frac{\sin 5\theta}{\sin^5\theta}.$

$\Rightarrow \frac{dz}{d\theta} = \frac{\sin^5\theta\cdot 5\cos 5\theta - 5\sin 5\theta\,\sin^4\theta\cos\theta}{(\sin^5\theta)^2}$ Put $\frac{dz}{d\theta} = 0$

$\Rightarrow \quad 5\sin^4\theta\,(\sin\theta\,\cos 5\theta - \cos\theta\,\sin 5\theta) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\sin(-4\theta) = 0$

$\Rightarrow \quad \theta = n\pi$ or $\theta = \frac{n\pi}{4}$, where $n \in Z$

As z is non-real complex number.

$\therefore$ only $\theta = \frac{n\pi}{4}$ is possible.

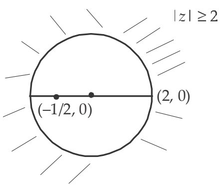
13. (a): 1st solution : $\left|z + \frac{1}{2}\right| \geq \left||z| - \frac{1}{2}\right|$

As $|z| \geq 2$ the minimum value of the expression occurs when $|z| = 2$

Thus $\left|z + \frac{1}{2}\right|_{\min} = \frac{3}{2}$

2nd solution :

Geometrically $|z| = 2$ is a circle and $|z| \geq 2$ is the boundary and exterior of the circle. The minimum distance between z and point (–1/2, 0) is realised at (2, 0) and is 3/2.



14. (b): Note that $\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = z$

Observe that $|z^2| = 1 = z\bar{z}$

Then the arg of the number $\frac{1+z}{1+\bar{z}}$ is just the argument of z and that's θ.

15. (c) : $z \neq 1$, $\frac{z^2}{z-1}$ is real.

If z is a real number, then $\frac{z^2}{z-1}$ is real.

Let $z = x + iy$

$\therefore \quad \frac{(x^2 - y^2 + 2xiy)\,((x-1) - iy)}{(x-1)^2 + y^2}$ is real

$\Rightarrow \quad -y(x^2 - y^2) + 2xy(x - 1) = 0$

$\Rightarrow \quad y(x^2 + y^2 - 2x) = 0 \Rightarrow y = 0$ or $x^2 + y^2 - 2x = 0$

$\therefore \quad z$ lies on real axis or on a circle passing through origin.

16. (d) : $(1 + \omega)^7 = (-\omega^2)^7 = -\omega^{14} = -\omega^{12}\,\omega^2$

$= -\omega^2 = 1 + \omega = A + B\omega$ given

Hence, on comparison, we have $(A, B) = (1, 1)$.

17. (b) : 1st solution :

$|z - 1| = |z + 1| = |z - i|$ reads that the distance of desired complex number z is same from three points in the complex plane –1, 1 and i. These points are non-collinear, hence the desired number is the centre of the (unique) circle passing through these three non-collinear points.

2nd solution :

We resort to definition of modulus.

$|z - 1| = |z + 1| \quad \Rightarrow \quad |z - 1|^2 = |z + 1|^2$

$\Rightarrow \quad (z-1)(\bar{z}-1) = (z+1)(\bar{z}+1)$

$\Rightarrow \quad z\bar{z} - z - \bar{z} + 1 = z\bar{z} + z + \bar{z} + 1$

$\Rightarrow \quad z + \bar{z} = 0$ (z being purely imaginary)

Thus $x = 0$

Again, $|z - 1|^2 = |z - i|^2 \Rightarrow (x - 1)^2 + y^2 = x^2 + (y - 1)^2$

$\Rightarrow \quad 1 + y^2 = (y - 1)^2$ (because $x = 0$)

$\Rightarrow \quad 1 + y^2 = y^2 - 2y + 1 \therefore y = 0$

Thus, (0, 0) is the desired point.

18. (a): We have for any two complex numbers α and β

$||\alpha| - |\beta|| \leq |\alpha - \beta|$

Now $\left||Z| - \frac{4}{|Z|}\right| \leq \left|Z - \frac{4}{Z}\right| \Rightarrow \left||Z| - \frac{4}{|Z|}\right| \leq 2$

Set $|Z| = r > 0$, then $\left|r - \frac{4}{r}\right| \leq 2 \Rightarrow \quad -2 \leq r - \frac{4}{r} \leq 2$

The left inequality gives $r^2 + 2r - 4 \geq 0$

The corresponding roots are $r = \frac{-2 \pm \sqrt{20}}{2} = -1 \pm \sqrt{5}$

Thus $r \geq \sqrt{5} - 1$ or $r \leq -1 - \sqrt{5}$

implies that $r \geq \sqrt{5} - 1$ (As $r > 0$) ...(i)

Again consider the right inequality

$r - \frac{4}{r} \leq 2 \quad \Rightarrow \quad r^2 - 2r - 4 \leq 0$

The corresponding roots are $r = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$

Thus $1 - \sqrt{5} \leq r \leq 1 + \sqrt{5}$

But $r > 0$, hence $r \leq 1 + \sqrt{5}$ ...(ii)

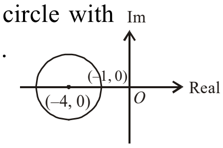
(i) and (ii) gives $\sqrt{5} - 1 \leq r \leq \sqrt{5} + 1$

So, the greatest value is $\sqrt{5} + 1$.

19. (d) : $\bar{z} = \frac{1}{i-1}$

We have $z = (\bar{\bar{z}})$ giving $z = \frac{1}{\bar{i} - 1} = \frac{1}{-i-1} = \frac{-1}{i+1}$

20. (a) : z lies on or inside the circle with centre (–4, 0) and radius 3 units. Hence maximum distance of z from (–1, 0) is 6 units.



21. (d) : $\sum_{k=1}^{n}\left(\sin\frac{2k\pi}{n+1} + i\cos\frac{2k\pi}{n+1}\right)$

$\therefore \quad = \sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right) = -i$

22. (d) : $z^2 + z + 1 = 0 \quad \Rightarrow \quad z = \omega, \omega^2$

$\therefore \quad \left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \ldots..\left(z^6+\frac{1}{z^6}\right)^2$

$= 4\,(\omega + \omega^2)^2 + 2(\omega^3 + \omega^3)^2 = 4\,(-1)^2 + 2(2^2) = 4 + 8 = 12$

23. (d) : Let $z_1 = \cos\theta_1 + i\sin\theta_1$,

$z_2 = \cos\theta_2 + i\sin\theta_2$

$\therefore \quad z_1 + z_2 = (\cos\theta_1 + \cos\theta_2) + i(\sin\theta_1 + \sin\theta_2)$

Now $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \sqrt{(\cos\theta_1+\cos\theta_2)^2+(\sin\theta_1+\sin\theta_2)^2} = 1+1$

$\Rightarrow 2(1 + \cos(\theta_1 - \theta_2)) = 4$ (by squaring)

$\Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0 \ (\because \cos 0° = 1)$

$\Rightarrow \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = 0.$

24. (d) : Given $\omega = \dfrac{3z}{3z-i} \quad \therefore \ |\omega| = \dfrac{3|z|}{|3z-i|}$

$\Rightarrow |3z-i| = 3|z|$

$\Rightarrow |3(x)+i(3y-1)| = |3(x+iy)| \quad (\because z = x+iy)$

$\Rightarrow (3x)^2 + (3y-1)^2 = 9(x^2+y^2) \Rightarrow 6y - 1 = 0$ which is straight line.

25. (d) :1st solution : (By making the equation from the given roots)

Let us consider $x = -1, -1, -1$

$\therefore$ Required equation from given roots is

$(x + 1)(x + 1)(x + 1) = 0$

$(x + 1)^3 = 0$ which does not match with the given equation $(x - 1)^3 + 8 = 0$ so $x = -1, -1, -1$ cannot be the proper choice.

Again consider $x = -1, -1 + 2\omega, -1 - 2\omega^2$

$\therefore$ Required equation from given roots is

$\Rightarrow \quad (x + 1)(x + 1 - 2\omega)(x + 1 + 2\omega^2) = 0$

$\Rightarrow \quad (x + 1)[(x + 1)^2 + (x + 1)(2\omega^2 - 2\omega) - 4\omega^3] = 0$

$\Rightarrow \quad (x + 1)[(x + 1)^2 + 2(x + 1)(\omega^2 - \omega) - 4] = 0$

$\Rightarrow \quad (x + 1)^3 + 2(x + 1)^2(\omega^2 - \omega) - 4(x + 1) = 0$

which cannot be expressed in the form of given equation $(x - 1)^3 + 8 = 0$. Now consider the roots $x_i = -1, 1 - 2\omega, 1 - 2\omega^2$ $(i = 1, 2, 3)$ and the equation with these roots is given by x^3 – (sum of the roots)x^2 + x(Product of roots taken two at a time) – Product of roots taken all at a time = 0

Now sum of roots $x_1 + x_2 + x_3 = -1 + 1 - 2\omega + 1 - 2\omega^2 = 3$

Product of roots taken two at a time

$= -1 + 2\omega - 1 + 2\omega^2 + 1 + 2(\omega^2 + \omega) + 4\omega^3 = 3$

Product of roots taken all at a time = $(-1)[(1 - 2\omega)(1 - 2\omega^2)] = -7$

$\therefore$ Required equation is $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow x^3 - 3x^2 + 3x - 1 + 8 = 0 \Rightarrow (x - 1)^3 + 8 = 0$ which matched with given equation.

2nd solution : (by taking cross checking)

As $(x - 1)^3 + 8 = 0$ ...(*)

and $x = -1$ satisfies $(x - 1)^3 + 8 = 0$

i.e. $(-2)^3 + 8 = 0 \Rightarrow 0 = 0$

Similarly for $1 - 2\omega$ we have $(x - 1)^3 + 8 = 0$

$\Rightarrow (1 - 2\omega - 1)^3 + 8 = 0$

$\Rightarrow (-2\omega)^3 + 8 = 0 \Rightarrow -8 + 8 = 0$ and for $1 - 2\omega^2$

we have $(1 - 2\omega^2 - 1)^3 + 8 = 0 \Rightarrow \omega^6(-8) + (8) = 0 \Rightarrow 0 = 0$

$\therefore \quad -1, 1 - 2\omega, 1 - 2\omega^2$ are roots of $(x - 1)^3 + 8 = 0$ and on the other hand the other roots does not satisfy the equation $(x - 1)^3 + 8 = 0$.

26. (a) : $\bar{z} + i\bar{\omega} = 0$

$\Rightarrow \quad \bar{z} = -i\bar{\omega} \Rightarrow z = i\omega \Rightarrow \quad \omega = -iz \ \therefore \ \arg(-iz^2) = \pi$

$\Rightarrow \quad \arg(-i) + 2\arg(z) = \pi$

$\Rightarrow \quad 2\arg(z) = \pi + \pi/2 = 3\pi/2 \Rightarrow \arg(z) = 3\pi/4$

27. (d) : $z^{1/3} = p + iq$

$\Rightarrow \quad x - iy = (p + iq)^3 \ \Rightarrow x - iy = p^3 - 3pq^2 + i(3p^2q - q^3)$

$\Rightarrow \quad x = p^3 - 3pq^2$ and $y = -(3p^2q - q^3)$

$\dfrac{x}{p} = p^2 - 3q^2$ and $\dfrac{y}{q} = -(3p^2 - q^2)$ ...(*)

Adding the equations of (*) we get $\dfrac{x}{p} + \dfrac{y}{q} = -2(p^2 + q^2)$

28. (b) : $|z^2 - 1| = |z|^2 + 1$

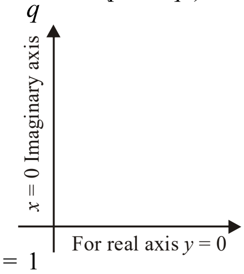
$\Rightarrow \quad$ Let $z = x + iy$

$\Rightarrow \quad (x - 1)^2 + y^2 = (x^2 + y^2) + 1$

$\Rightarrow \quad 2x = 0$

$\Rightarrow \quad x = 0$

$\Rightarrow \quad z$ lies on imaginary axis.



29. (d) : Given $\left(\dfrac{1+i}{1-i}\right)^x = 1 \Rightarrow \left(\dfrac{2i}{2}\right)^x = 1$

$\Rightarrow i^x = 1 \quad \Rightarrow \quad i^x = (i)^{4n}$

$\Rightarrow x = 4n, \ n \in I^+$

30. (c) : $|z\omega| = 1 \Rightarrow |z||\omega| = 1$ So $|z| = \dfrac{1}{|\omega|}$ ... (1)

Again $Arg(z) - Arg(\omega) = \dfrac{\pi}{2}$

$\therefore \quad \dfrac{z}{\omega} = \left|\dfrac{z}{\omega}\right| i = |z|^2 i$ from (1)

$\therefore \quad \dfrac{z}{\omega} = z\bar{z}\,i \ \Rightarrow \bar{z}\,\omega = \dfrac{1}{i} = -i.$

31. (b) : As z_1, z_2 are roots of $z^2 + az + b = 0$

$\therefore \quad z_1 + z_2 = -a, \ z_1z_2 = b$

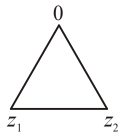
Again $0, z_1, z_2$ are vertices of an equilateral triangle

$\therefore \quad 0^2 + z_1^2 + z_2^2 = 0z_1 + z_1z_2 + z_20 = 0$

$z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow \quad (z_1 + z_2)^2 = 3z_1z_2$

$a^2 = 3b$



32. (b) : Let $|z| = |\omega| = r$

$\therefore \ z = re^{i\alpha}$ and $\omega = re^{i\beta}$ where $\alpha + \beta = \pi$ (given)

Now $z = re^{i\alpha} = re^{i(\pi - \beta)} = re^{i\pi} \cdot e^{-i\beta} = -re^{-i\beta} = -\bar{\omega}$

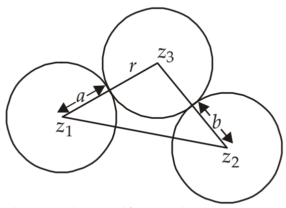
33. (c) : $|z - 4| < |z - 2|$

or $|a - 4 + ib| < |(a - 2) + ib|$ by taking $z = a+ib$

$\Rightarrow \quad (a - 4)^2 + b^2 < (a - 2)^2 + b^2$

$\Rightarrow \quad -8a + 4a < -16 + 4 \Rightarrow 4a > 12 \Rightarrow a > 3 \Rightarrow \operatorname{Re}(z) > 3$

34. (b) :



$z_1z_3 - z_3\,z_2 = (a + r) - (b + r)$

$= a - b$ = a constant, which represent a hyperbola

Since, A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points (foci) is always constant.