

HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. $\sin \frac{6\pi}{5} + i \left(1 + \cos \frac{6\pi}{5}\right)$

lies in 2nd quadrant and

$$\left| \frac{1 + \cos \frac{6\pi}{5}}{\sin \frac{6\pi}{5}} \right| = \left| \cot \left(\frac{3\pi}{5} \right) \right| = \left| \cot \left(\frac{\pi}{2} + \frac{\pi}{10} \right) \right| = \tan \frac{\pi}{10}$$

2nd quadrant $\Rightarrow \pi - \frac{\pi}{10}$

2. $z = 1 + \cos \frac{11\pi}{9} + i \sin \frac{11\pi}{9}$

$$= \left(-2 \cos \frac{11\pi}{18} \right) \left[\cos \frac{11\pi}{18} + i \sin \frac{11\pi}{18} \right] (-1)$$

$$|z| = -2 \cos \frac{11\pi}{18} = 2 \cos \frac{7\pi}{18}$$

$$\arg(z) = \frac{11\pi}{18} - \pi = -\frac{7\pi}{18}$$

4. $S = i + 2i^2 + 3i^3 + \dots + 100i^{100}$

$$iS = i^2 + 2i^3 + \dots + 100i^{101}$$

$$S(1-i) = i + i^2 + i^3 + \dots + i^{100} - 100i^{101}$$

$$S = \frac{-100i}{1-i} = \frac{-100i(1+i)}{2} = -50(i-1) = 50(1-i)$$

7. We have $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|} = k$ (let)

$$\Rightarrow \frac{9}{|z_2 - z_3|^2} = \frac{16}{|z_3 - z_1|^2} = \frac{25}{|z_1 - z_2|^2} = k^2$$

Now $\frac{9}{|z_2 - z_3|^2} = k^2$

$$\Rightarrow \frac{9}{z_2 - z_3} = k^2 (\bar{z}_2 - \bar{z}_3) \quad \dots(i)$$

$$[\text{As } |z|^2 = z \bar{z}]$$

$$\text{ly } \frac{16}{|z_3 - z_1|^2} = k^2 \Rightarrow \frac{16}{z_3 - z_1} = k^2 (\bar{z}_3 - \bar{z}_1) \quad \dots(ii)$$

$$\text{ly } \frac{25}{|z_1 - z_2|^2} = k^2 \Rightarrow \frac{25}{z_1 - z_2} = k^2 (\bar{z}_1 - \bar{z}_2) \quad \dots(iii)$$

$\therefore$ On adding (1), (2) and (3), we get

$$= \frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2} \\ = k^2 (\bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1 + \bar{z}_1 - \bar{z}_2) = 0$$

11. $G \rightarrow$ Centroid of $\Delta = \frac{z_1 + z_2 + z_3}{3}$

H $\rightarrow$ Orthocentre = z say, O $\rightarrow$ Circum centre = 0

$\rightarrow$ G divides HO in ratio 2 : 1 reckoning from

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2 \cdot 0 + 1 \cdot z}{2+1} \Rightarrow z = z_1 + z_2 + z_3$$

14. $z = \frac{\pi}{4}(1+i)^4 \left[\frac{1+\pi+\pi+1}{(\sqrt{\pi}+i)(1+\sqrt{\pi}i)} \right]$

$$= \frac{\pi}{4}(1+i)^4 \frac{2}{i}$$

$$= \frac{\pi}{2} \frac{(1+i)^4}{i} = \frac{\pi}{2} \frac{4e^{i\pi}}{e^{i\pi/2}} = 2\pi e^{i\pi/2}$$

$$|z| = 2\pi \quad \text{amp}(z) = \frac{\pi}{2}$$

$$\left(\frac{|z|}{\text{amp}(z)} \right) = \frac{2\pi}{\frac{\pi}{2}} = 4$$

15. Let $S = 1(\alpha_1 + \alpha_{2008}) + 2(\alpha_2 + \alpha_{2007}) + 3(\alpha_3 + \alpha_{2006}) + \dots + 2008(\alpha_{2008} + \alpha_1) \quad \dots(i)$

Also $S = 2008(\alpha_{2008} + \alpha_1) + 2007(\alpha_2 + \alpha_{2007}) + \dots + 2(\alpha_2 + \alpha_{2007}) + 1(\alpha_1 + \alpha_{2008}) \quad \dots(ii)$
(writing in reverse order)

$\therefore$ On adding (1) and (2), we get

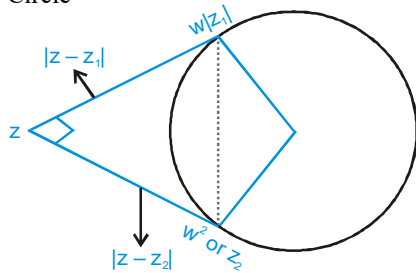
$$2S = 2009[2(\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{2008})]$$

$$2S = 2009[2(1 + \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{2008} - 1)]$$

Hence $S = -2009$



19. → Circle

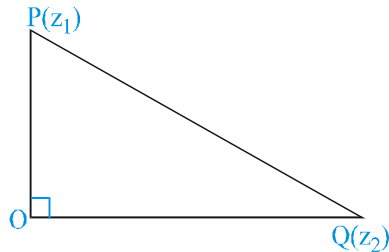


so by pythagorous theorem

$$\lambda = |w - w^2|^2 = |\sqrt{3}|^2 = 3$$

21. We have $\left| \frac{1}{z_2} + \frac{1}{z_1} \right| = \left| \frac{1}{z_2} - \frac{1}{z_1} \right|$

$$\Rightarrow |z_1 + z_2| = |z_1 - z_2| \Rightarrow z_1 \bar{z}_2 + z_2 \bar{z}_1 = 0$$



$$\Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary.}$$

Hence ΔPQR is right angled at O.

∴ Circumcentre of ΔPOQ is the mid point of PQ i.e.

$$\frac{1}{2}(z_1 + z_2)$$

24. $z^2 + z + 1$ is real so

$$z^2 + z + 1 = \bar{z}^2 + \bar{z} + 1$$

$$z^2 - \bar{z}^2 + z - \bar{z} = 0$$

$$(z - \bar{z})(z + \bar{z} + 1) = 0$$

$$\text{either } z = \bar{z} \text{ or } z + \bar{z} + 1 = 0$$

$$\Rightarrow \text{Im}(z) = 0 \quad \text{Let } z = \alpha + i\beta$$

$$\Rightarrow z \text{ is purely real}$$

$$\text{So } \alpha + i\beta + \alpha - i\beta + 1 = 0$$

$$\Rightarrow 2\alpha + 1 = 0$$

$$\Rightarrow \alpha = -\frac{1}{2}$$

$$\text{Also } (\alpha + i\beta)^2 + (\alpha + i\beta) + 1 > 0$$

$$\alpha^2 + \alpha + 1 - \beta^2 + i(2\alpha\beta + \beta) > 0$$

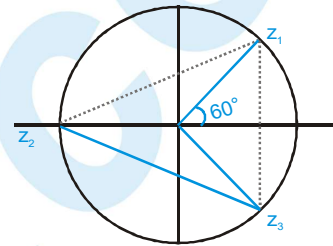
$$\text{if } \alpha = -1/2 \text{ then}$$

$$\frac{1}{4} - \frac{1}{2} + 1 - \beta^2 > 0$$

$$\Rightarrow \beta^2 - \frac{3}{4} < 0 \Rightarrow -\frac{\sqrt{3}}{2} < \beta < \frac{\sqrt{3}}{2}$$

25. $z_1 = 1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$

$$z_2 = z_1 e^{\frac{2\pi i}{3}} = \frac{-1-3}{2} = -2$$

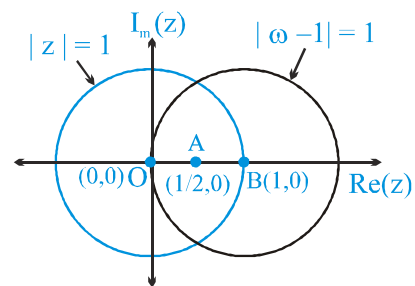


$$z_3 = z_1 e^{-\frac{2\pi i}{3}} = 2 e^{-\frac{\pi i}{3}} = 1 - i\sqrt{3}$$

27. Least distance and greatest distance of any z and ω from

the point $\left(\frac{1}{2}, 0\right)$ are $\frac{1}{2}$ and $\frac{3}{2}$ respectively.

$$\therefore \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \leq \left|z - \frac{1}{2}\right|^2 + \left|\omega - \frac{1}{2}\right|^2 \leq \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$



$$\text{Hence } 2 \leq |2z - 1|^2 + |2\omega - 1|^2 \leq 18$$

31. Let centre be origin & A_1 be Z_0 & $OA_1 = OA_2 = \dots = a$

$$\text{So } A_2 = z_0 e^{i\frac{2\pi}{n}}, \quad A_3 = z_0 e^{i\frac{4\pi}{n}} \dots$$

$$\text{Now } A_1 A_2 = \left| z_0 - z_0 e^{i\frac{2\pi}{n}} \right| = |z_0| \left| 1 - e^{i\frac{2\pi}{n}} \right| = a \left| e^{i\frac{\pi}{n}} \right|$$

$$\left| e^{-\frac{i\pi}{n}} - e^{\frac{i\pi}{n}} \right| = 2a \sin\left(\frac{\pi}{n}\right)$$

similarly $A_1 A_3 = 2a \sin \frac{2\pi}{n}$, $A_1 A_4 = 2a \sin \frac{3\pi}{n}$

Given $\frac{1}{2a \sin \frac{\pi}{n}} = \frac{1}{2a \sin \frac{2\pi}{n}} + \frac{1}{2a \sin \frac{3\pi}{n}}$

$$\Rightarrow \sin \frac{3\pi}{n} = \sin \frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n}$$

$$\Rightarrow n = 7 \text{ is only possible value.}$$

33. Let $z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$

So $\omega = \frac{1}{r} [\cos(\theta - \pi/2) + i \sin(\theta - \pi/2)] = \frac{1}{r} e^{i(\theta - \pi/2)}$

So $\bar{z} \omega = r e^{-i\theta} \times \frac{1}{r} e^{i(\theta - \pi/2)} = e^{-i\pi/2} = -i$

35. $(z-1)(z-\alpha_1) \dots (z-\alpha_4) = z^5 - 1$

Put $z = \omega$, $z = \omega^2$ and divide

$$\frac{(\omega-1)(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-1)(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)}$$

$$= \frac{\omega^5 - 1}{\omega^{10} - 1} \frac{(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)}$$

$$= \frac{(\omega^2-1)^2}{(\omega-1)^2} = (\omega+1)^2 = \omega^4 = \omega$$

EXERCISE - 2

Part # I : Multiple Choice

2. $\alpha = -2 + 3z$

$$\alpha + 2 = 3z$$

$$|\alpha + 2| = 3|z|$$

$$(x+2)^2 + y^2 = 9$$

Similarly $\beta = -2 - 3z$

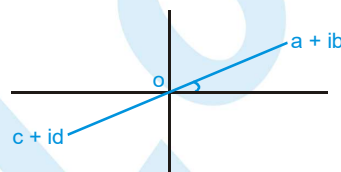
$$\Rightarrow \beta + 2 = -3z \Rightarrow |\beta + 2| = |-3z|$$

$$(x+2)^2 + y^2 = 9$$

Now $\alpha - \beta = 6z \Rightarrow |\alpha - \beta| = 6|z|$

so $(\alpha - \beta)$ moves on a circle with centre as origin and radius 6.

3.



$$|a + ib| = |c + id|$$

$$a + c = b + d$$

4. $z^3 + (1+i)z^2 + (1+i)z + i = 0$

$$\Rightarrow (z+i)(z^2 + z + 1) = 0$$

$$\Rightarrow (z+i)(z-\omega)(z-\omega^2) = 0 \Rightarrow z = -i, \omega, \omega^2$$

Now ω , and ω^2 satisfies the equation

$$z^{1993} + z^{1994} + 1 = 0$$

So ω and ω^2 are common roots

We have $|z|(z+1)^8 = z^8 |z+1| \dots (i)$

Taking modulus on both sides, we get

$$|z| |z+1|^8 = |z|^8 |z+1|$$

$$\therefore |z+1|^7 = |z|^7$$

$$\Rightarrow |z+1| = |z|$$

which represents the locus of z will be a straight line which is perpendicular bisector of the line segment joining

$(-1, 0)$ and $(0, 0)$ i.e. $\text{Re}(z) = \frac{-1}{2}$. Also there will be exactly 7 distinct 'z' satisfying given equation,

i.e. $z = \frac{-1}{2}, \frac{-1}{2} \pm ki$ where k has 3 distinct positive values k_1, k_2 and k_3 .

$$\therefore \sum_{r=1}^n \text{Re}(z_r) = \sum_{r=1}^7 z_r = \frac{-1}{2} \times 7 = -\frac{7}{2}$$

and $\sum_{r=1}^n \text{Im}(z_r) = 0 + (k_1 + k_2 + k_3) + (-k_1 - k_2 - k_3) = 0$

9. $\prod_{r=1}^n x_r = e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2^2}} \cdot e^{i\frac{\pi}{2^3}} \dots e^{i\frac{\pi}{2^n}}$

$$\lim_{n \rightarrow \infty} \prod_{r=1}^n x_r = e^{i\frac{\pi}{2}(1+\frac{1}{2}+\frac{1}{2^2}+\dots)}$$

$$\lim_{n \rightarrow \infty} \operatorname{Re} \left(\prod_{r=1}^n x_r \right) = \operatorname{Re} \left(e^{i\frac{\pi}{2} \cdot 2} \right) = -1$$

$$\lim_{n \rightarrow \infty} \operatorname{Im} \left(\prod_{r=1}^n x_r \right) = \operatorname{Im} \left(e^{i\frac{\pi}{2} \cdot 2} \right) = 0$$

10. All the three vertices lie on circle $|z| = 1$

take $z_1 = z_1, z_2 = z_1\omega, z_3 = z_1\omega^2$

So $z_1 + z_2 + z_3 = z_1(1 + \omega + \omega^2) = 0$

$z_1 z_2 z_3 = z_1^3$

$z_1 z_2 + z_2 z_3 + z_3 z_1 = z_1^2(\omega + \omega^3 + \omega^2) = 0$

11. $1 - \log_2 \frac{|x+1+2i|-2}{\sqrt{2}-1} \geq 0$

$$\Rightarrow \frac{|x+1+2i|-2}{\sqrt{2}-1} \leq 2 \Rightarrow |x+1+2i| \leq 2\sqrt{2}$$

$$\Rightarrow \sqrt{(x+1)^2 + 4} \leq 2\sqrt{2} \Rightarrow (x+1)^2 + 4 \leq 8$$

$$\Rightarrow (x+1)^2 \leq 4 \Rightarrow -3 \leq x \leq 1$$

But $x = -1$ not lie in the domain of function.

12. Since z_1 and z_2 lie on $|z| = 1$ and $|z| = 2$

then $|z_1| = 1$ and $|z_2| = 2$

$|2z_1 + z_2| \leq 2|z_1| + |z_2| \leq 4$

$\max |2z_1 + z_2| = 4$

$|z_1 - z_2| \geq |z_1| - |z_2| = |1-2| = 1$

$\min |z_1 - z_2| = 1$

$$\left| z_2 + \frac{1}{z_1} \right| \leq |z_2| + \frac{1}{|z_1|} = 2 + 1 = 3$$

$$\left| z_2 + \frac{1}{z_1} \right| \leq 3$$

13. $z = re^{i\theta}$

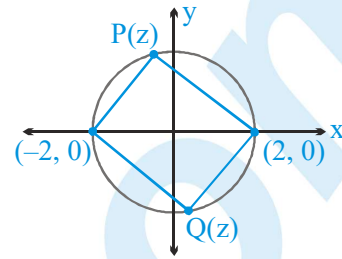
$r^2 e^{i\theta 2} + r^2 e^{i\theta} + r^2 = 0$

$r^2 [e^{i2\theta} + e^{i\theta} + 1] = 0$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$z = k\omega$ or $k\omega^2$ where $k > 0$

17. (A) For $P(z)$, $\arg \left(\frac{z-2}{z+2} \right) = \frac{\pi}{2}$



and for $Q(z)$, $\arg \left(\frac{z-2}{z+2} \right) = \frac{-\pi}{2}$

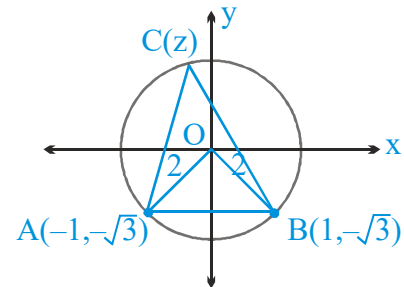
$\Rightarrow$ (A) is true.

(B) $\triangle AOB$ is equilateral.

$$\therefore \angle AOB = \frac{\pi}{3}$$

and $\angle ACB = \frac{\pi}{6}$

$$\therefore \arg \left(\frac{z-1+i\sqrt{3}}{z+1+i\sqrt{3}} \right) = \frac{\pi}{6} \quad (\text{By rotation})$$



$$\Rightarrow \arg \left(\frac{z+1+i\sqrt{3}}{z-1+i\sqrt{3}} \right) = -\frac{\pi}{6}$$

$\Rightarrow$ (B) is not true.

(C) and (D)

We have $|z^2 - 1| = (z^2 - 1)(\bar{z}^2 - 1) = z^2 \bar{z}^2 - z^2 - \bar{z}^2 + 1$

$|z|^4 - 2(x^2 - y^2 + 1) = 17 - 2(x^2 - y^2)$

$\Rightarrow x^2 + y^2 = 4 = 25 - 4x^2$

Now given $|z| = 2$

$\Rightarrow$ range of x is $[-2, 2]$

$\Rightarrow 9 \leq |z^2 - 1|^2 \leq 25$

Hence $3 \leq |z^2 - 1| \leq 5$

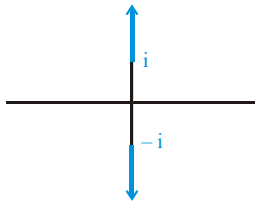
$\Rightarrow$ (C) and (D) are true.

19. $\|z+i\| - \|z-i\| = k$

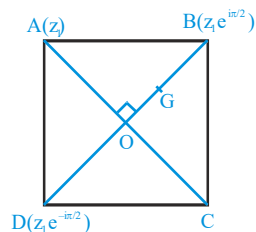
for $0 < k < 2$ its hyperbola having foci as i & $-i$.

for $k=0$ $\|z+i\| = \|z-i\|$ which is perpendicular bisector of line joining $i, -i$

for $k=2$ a pair of ray.



21.



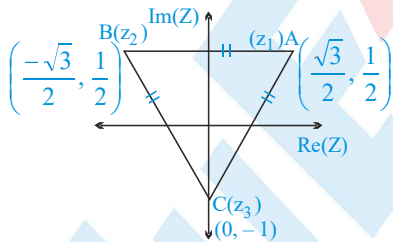
$\triangle ABC$ is isosceles triangle

So centroid divide median BO in ratio 2 : 1

$$\text{centroid } G = \frac{z_1 e^{i\pi/2}}{3} = \frac{z_1}{3} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$\text{Also centroid } G = \frac{z_1 e^{-i\pi/2}}{3} = \frac{z_1}{3} \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)$$

22. We have $z^4 = iz \Rightarrow z^3 = i \Rightarrow z = e^{i(4k+1)\frac{\pi}{6}}$
(Using D.M.T.)



Put $k=0, 1, 2$, we get

$$z_1 = e^{i\frac{\pi}{6}}, z_2 = e^{i\frac{5\pi}{6}} \text{ and } z_3 = e^{i\frac{3\pi}{2}}$$

Clearly triangle formed by z_1, z_2 and z_3 is equilateral.

$$\therefore \text{centroid of } \triangle ABC \text{ is } (0, 0) \text{ and Area } (\triangle ABC) = \frac{3\sqrt{3}}{4}$$

24. $\Rightarrow \frac{\pi}{2}$

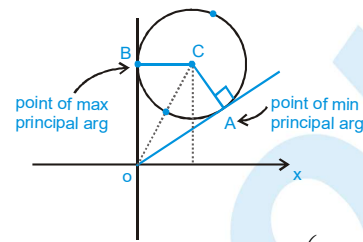
$$\max |z| = d + r$$

$$\min |z| = d - r$$

$$d = OC = \sqrt{5}$$

$$r = 1$$

$$\theta = \angle OCX = \tan^{-1} \frac{2}{1}$$



$$\alpha = \angle OCA = \tan^{-1} \frac{1}{2} \quad \left(Q \sin \alpha = \frac{1}{\sqrt{5}} \right)$$

$$\text{So principal Arg of } A = \theta - \alpha = \tan^{-1} 2 - \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} \frac{2 - \frac{1}{2}}{1 + 1} = \tan^{-1} \frac{3}{4}$$

Part # II : Assertion & Reason

1. $|z - 4 - 5i| = 4$ represents a circle with centre $(4, 5)$ and radius 4 and $\arg(z - 3 - 4i) = \frac{\pi}{4}$ represents a ray emanating from point $(3, 4)$. Ray will intersect the circle at only one point.
So statement (I) is false and statement (II) is true.

2. For statement-1

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Rightarrow z_1, z_2, z_3 \text{ are vertices of equilateral triangle}$$

For statement-2

$$|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$$

$$\therefore z_0 \text{ is circum centre}$$

4. Statement -1 $(1+z)^6 = -z^6$

take modulus

$$|1+z|^6 = |z|^6$$

$$\left| \frac{1+z}{z} \right| = 1 \quad \text{which is straight line}$$

$$\text{Statement -2 } z_2 = \frac{z_1 + z_3}{2}$$

$\therefore z_2$ is mid point of line joining z_1 & z_3 . Hence z_1, z_2, z_3 are collinear

EXERCISE - 3

Part # I : Matrix Match Type

1.(A) $\left| |z| - \frac{1}{|z|} \right| \leq \left| z + \frac{1}{z} \right|$

$$-2 \leq |z| - \frac{1}{|z|} \leq 2$$

$$|z|^2 + 2|z| - 1 \geq 0 \text{ and } |z|^2 - 2|z| - 1 \leq 0$$

$$|z| \geq \sqrt{2} - 1, |z| \leq \sqrt{2} + 1$$

$$|z|_{\min} = \sqrt{2} - 1$$

so minimum value of $\frac{|z|}{\tan \frac{\pi}{8}} = 1$

(B) $|z| = 1$

Let $z = \cos \theta + i \sin \theta$

$$\frac{z^n}{z^{2n} + 1} - \frac{\bar{z}^n}{\bar{z}^{2n} + 1}$$

$$= \frac{\cos n\theta + i \sin n\theta}{1 + \cos 2n\theta + i \sin 2n\theta} - \frac{\cos n\theta - i \sin n\theta}{1 + \cos 2n\theta - i \sin 2n\theta}$$

$$= \frac{\cos n\theta + i \sin n\theta}{2 \cos n\theta (\cos n\theta + i \sin n\theta)}$$

$$- \frac{\cos n\theta - i \sin n\theta}{2 \cos n\theta (\cos n\theta - i \sin n\theta)}$$

$$= \frac{1}{2 \cos n\theta} - \frac{1}{2 \cos n\theta} = 0$$

(C) $8iz^3 + 12z^2 - 18z + 27i = 0$

$$\Rightarrow (2iz + 3)(4z^2 + 9i) = 0$$

$$\Rightarrow z = \frac{3}{2}i, z^2 = -\frac{9}{4}i \Rightarrow 2|z| = 3$$

(D) $z^4 + z^3 + z^2 + z + 1$
 $= (z - z_1)(z - z_2)(z - z_3)(z - z_4)$

Put $z = -2$

$$\prod_{i=1}^4 (z_i + 2) = (-2)^4 + (-2)^3 + (-2)^2 + (-2) + 1 = 11$$

3. (A) Let $f(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$

$$f(-i) = (i + x_1)(i + x_2)(i + x_3)(i + x_4)$$

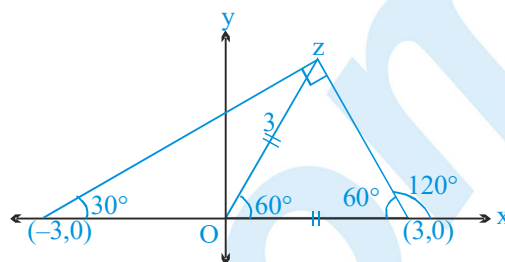
$$|f(-i)| = |(i + x_1)(i + x_2)(i + x_3)(i + x_4)|$$

$$= \sqrt{(x_1^2 + 1)} \sqrt{(x_2^2 + 1)} \sqrt{(x_3^2 + 1)} \sqrt{(x_4^2 + 1)} = 1$$

$$\Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

$$\Rightarrow a = b = c = d = 0$$

(B) From the figure $z = 3e^{i\frac{\pi}{3}}$

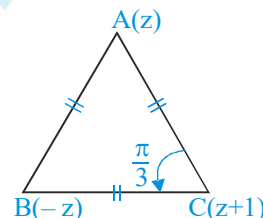


$$\therefore \arg z = \frac{\pi}{3}$$

Hence $\tan^2(\arg z) - 2 \cos(\arg z)$
 $= \tan^2 60^\circ - 2 \cos 60^\circ = 3 - 1 = 2$

(C) By rotation, we get

$$\frac{(z+1) - (-z)}{(z+1) - z} = e^{i\frac{\pi}{3}}$$



$$\Rightarrow 2z + 1 = \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \Rightarrow 2z = \left(\frac{-1}{2} + i \frac{\sqrt{3}}{2} \right)$$

$$\therefore z = \left(\frac{-1}{4} + i \frac{\sqrt{3}}{4} \right) \Rightarrow \operatorname{Re}(z) = \frac{-1}{4}$$

$$\Rightarrow 5 + 4 \left(\frac{-1}{4} \right) = 5 - 1 = 4$$

(D) We have $z_1^x + z_2^x = 2^x$

$$\Rightarrow (-2\omega^2)^x + (-2\omega)^x = 2^x$$

$$\Rightarrow (-1)^x [\omega^{2x} + \omega^x] = 1$$

$$\Rightarrow (-1)^x [\omega^{2x} + \omega^x + 1^x - 1] = 1$$

Clearly $x \neq 3n, n \in \mathbb{I}$

(Because if x is an integral multiple of 3 then LHS $\neq$ RHS)

$$\Rightarrow (-1)^x [\underbrace{\omega^{2x} + \omega^x + 1}_{\text{zero}} - 1] = 1$$

Now verify $\Rightarrow x = 1, 3$

Part # II : Comprehension

Comprehension # 2

Let $z = z_1 + z_2 + z_3 + z_4$

$$z_2 + z_3 + z_4 = z - z_1$$

$$\therefore (1-d)z_1 + z_2 + z_3 + z_4 = (1-d)z_1 + z - z_1 = z - dz_1$$

$$\text{Similarly } z_1 + (1-d)z_2 + z_3 + z_4 = z - dz_2$$

$$\text{and } z_1 + z_2 + (1-d)z_3 + z_4 = z - dz_3$$

$$\therefore |z - dz_1| = |z - dz_2| = |z - dz_3|$$

$$\therefore z \text{ is equidistant from } dz_1, dz_2, dz_3$$

but dz_1, dz_2, dz_3 lie on a circle with centre O and radius

$$|dz_1| = |dz_2| = |dz_3|$$

$$\rightarrow |dz_1| = |d| \cdot |z_1| \quad \therefore |dz_1| = |dz_2| = |dz_3|$$

$$|dz_2| = |d| \cdot |z_2| \quad \text{as } |z_1| = |z_2| = |z_3|$$

$$|dz_3| = |d| \cdot |z_3|$$

$$\therefore z = 0$$

$$\therefore \text{Arg } z \text{ not defined, Also } |z| = 0$$

(C) and (D)

We have

$$|z^2 - 1| = (z^2 - 1)(\bar{z}^2 - 1) = z^2 \bar{z}^2 - z^2 - \bar{z}^2 + 1$$

$$|z|^4 - 2(x^2 - y^2 + 1) = 17 - 2(x^2 - y^2)$$

$$\rightarrow x^2 + y^2 = 4 = 25 - 4x^2$$

$$\Rightarrow 9 \leq |z^2 - 1|^2 \leq 25$$

$$(z\bar{z})^2 - ((z + \bar{z})^2 - 2z\bar{z}) + 1 = |z|^4 + 2|z|^2 + 1 - (2\text{Re}(z))^2$$

$$= 2^4 + 2 \cdot 2^2 + 1 - 4|\text{Re}(z)|^2 = 25 - 4|\text{Re}(z)|^2$$

$$\text{Since } 0 \leq |\text{Re}(z)| \leq 2$$

$$\Rightarrow 25 - 4 \times 4 \leq |z^2 - 1|^2 \leq 25 - 4 \times 0$$

$$\Rightarrow 9 \leq |z^2 - 1|^2 \leq 25$$

$$\text{Hence } 3 \leq |z^2 - 1| \leq 5$$

$$\Rightarrow \text{(C) and (D) are true.}$$

Comprehension # 4

For A, $|z + 1| \leq 2 + \text{Re}(z)$

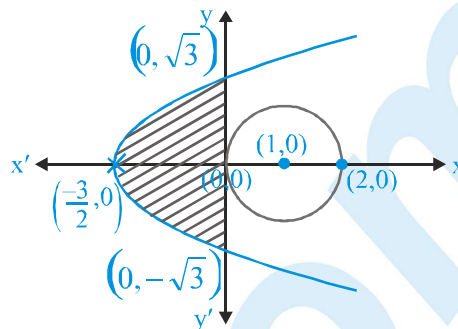
$$\Rightarrow (x+1)^2 + y^2 \leq 4 + 4x + x^2$$

$$\Rightarrow y^2 \leq 3 + 2x$$

$$\Rightarrow y^2 \leq 2\left(x + \frac{3}{2}\right) \quad \dots\dots\text{(i)}$$

For B, $|z - 1| \geq 1$

$$\Rightarrow (x-1)^2 + y^2 \geq 1 \quad \dots\dots\text{(ii)}$$



$$\text{For C, } |z - 1|^2 \geq |z + 1|^2$$

$$\Rightarrow (z - 1)(\bar{z} - 1) \geq (z + 1)(\bar{z} + 1)$$

$$\Rightarrow (z\bar{z} - \bar{z} - z + 1) \geq (z\bar{z} + \bar{z} + z + 1)$$

$$\Rightarrow z + \bar{z} \leq 0$$

$$\text{i.e. } x \leq 0 \quad \dots\dots\text{(3)}$$

$$\text{(i) } (-1, 0), (-1, 1), (-1, -1), (0, 0), (0, 1), (0, -1)$$

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 6.

$$\text{(ii) Required area} = 2 \int_{-\frac{3}{2}}^0 \sqrt{2\left(x + \frac{3}{2}\right)} dx = 2\sqrt{3} \text{ (square units)}$$

$$\text{(iii) Clearly } z = \frac{-3}{2} + i0 \text{ is the complex number in the region}$$

$A \cap B \cap C$ and having maximum amplitude.

$$\therefore \text{Re}(z) = \frac{-3}{2}$$

Comprehension # 5

$$1. \text{ AD} = x, \angle \text{ADC} = 180 - (C + \theta)$$

$$\begin{aligned} \text{Area of } \triangle \text{ABC} &= 2 \text{ area } \triangle \text{ADC} = \frac{1}{2} 2y \cdot x \sin(C + \theta) \\ &= xy \sin(C + \theta) \end{aligned}$$

$$2. \text{ Let affix of M is } z_m \text{ and } \angle \text{BOM} = \pi - 2B, \text{ then}$$

$$\frac{z_m - 0}{z_b - 0} = \frac{\text{OM}}{\text{OB}} e^{i(\pi - 2B)}$$

$$z_m = z_b e^{i(\pi - 2B)}$$

$$3. \text{ Let affix of L is } z_L \text{ and } \angle \text{BOL} = 2(A - \theta), \text{ then}$$

$$\frac{z_L - 0}{z_b - 0} = e^{i(2A - 2\theta)}$$

$$z_L = z_b e^{i(2A - 2\theta)}$$

EXERCISE - 4

Subjective Type

2. (A) The region between the concentric circles with centre at (0, 2) & radii 1 & 3 units
 (B) region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$
 (C) semi circle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$
 (D) a ray emanating from the point (3 + 4i) directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

3. (i) $z = 1 + e^{i\frac{18\pi}{25}} = e^{i\frac{9\pi}{25}} \left[e^{i\frac{9\pi}{25}} + e^{-i\frac{9\pi}{25}} \right]$

$$z = 2 \cos\left(\frac{9\pi}{25}\right) e^{i\frac{9\pi}{25}}$$

$$|z| = 2 \cos\left(\frac{9\pi}{25}\right) \quad \text{Arg } z = \frac{9\pi}{25}$$

(ii) $z = 2e^{i\pi} e^{i\pi/6} = 2e^{-i5\pi/6}$

$$|z| = 2 \quad \text{Arg } z = -\frac{5\pi}{6}$$

(iii) $|z| = \left(\sqrt{1 + \tan^2 1}\right)^2 = \sec^2 1$

$$\text{Arg } z = 2 \text{ Arg}(\tan 1 - i) = 2 \left(1 - \frac{\pi}{2}\right) = 2 - \pi$$

(iv) $z = \frac{(i-1)}{2 \sin\left(\frac{\pi}{5}\right) \left[\sin\left(\frac{\pi}{5}\right)i + \cos\left(\frac{\pi}{5}\right) \right]}$

$$|z| = \frac{\sqrt{2}}{2 \sin\left(\frac{\pi}{5}\right)} = \frac{1}{\sqrt{2}} \operatorname{cosec}\left(\frac{\pi}{5}\right)$$

$$\text{Arg}(z) = \pi - \frac{\pi}{4} - \frac{\pi}{5} = \frac{11\pi}{20}$$

4. If $x = \alpha + i\beta$ is a root then

$$\frac{A_1^2}{\alpha - a_1 + i\beta} + \frac{A_2^2}{\alpha - a_2 + i\beta} + \dots + \frac{A_n^2}{\alpha - a_n + i\beta} = K$$

& taking conjugate

$$\frac{A_1^2}{\alpha - a_1 - i\beta} + \frac{A_2^2}{\alpha - a_2 - i\beta} + \dots + \frac{A_n^2}{\alpha - a_n - i\beta} = K$$

Subtracting

$$\frac{2\beta A_1^2}{(\alpha - a_1)^2 + \beta^2} + \frac{2\beta A_2^2}{(\alpha - a_2)^2 + \beta^2} + \dots + \frac{2\beta A_n^2}{(\alpha - a_n)^2 + \beta^2} = 0$$

$$\Rightarrow \beta = 0 \quad \Rightarrow x = \alpha + i0$$

which is purely real. Hence true.

5. $|z|^2 \omega - |\omega|^2 z = z - \omega \dots (i)$

Put $z = \omega$ & $z = \frac{1}{\omega}$

we get L.H.S = R.H.S

Now, equation (i) be written as

$$\omega(1 + |z|^2) = z(1 + |\omega|^2)$$

$$\Rightarrow \frac{\omega}{z} = \frac{(1 + |\omega|^2)}{1 + |z|^2} = \lambda \quad \Rightarrow \quad \omega = \lambda z$$

But this is equation (i)

$$|z|^2 \lambda z - \lambda^2 |z|^2 z = z - \lambda z$$

$$\Rightarrow z \lambda |z|^2 (1 - \lambda) = z(1 - \lambda)$$

$$\Rightarrow (1 - \lambda)(\lambda |z|^2 - 1) = 0$$

$$\Rightarrow \lambda = 1 \quad ; \quad \lambda = \frac{1}{|z|^2}$$

from $\lambda = 1$ we get $z = \omega$

$$\lambda = \frac{1}{|z|^2} \text{ we get } \omega = \frac{1}{\bar{z}}$$

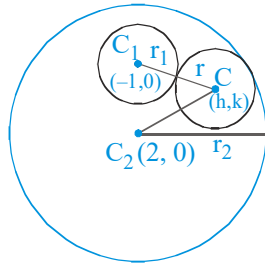
6. (i) $z_1 \bar{z}_1 = 1 \quad \Rightarrow \quad \bar{z}_1 = \frac{1}{z_1}$

(ii) $|z_1 + z_2 + \dots + z_n| = |\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$

7. (a) $\pi - 2$ (b) $1/2$

8. We have $C_1 : (x+1)^2 + y^2 = 9$

$$C_2 : (x-2)^2 + y^2 = 49$$



Now $CC_1 = r + r_1$

and $CC_2 = r_2 - r$

$$\Rightarrow CC_1 + CC_2 = r_1 + r_2$$

$\therefore$ Locus of C is an ellipse with focus at C_1 and C_2

Now $r_1 + r_2 = 2a = 10$ (1)

and $d_{C_1C_2}$ (focal length) $= 2ac = 3$ (2)

(1) and (2) $\Rightarrow$ eccentricity 'e' is $\frac{3}{10} \Rightarrow p + q = 13$

9. $\frac{z_1}{z_2} = e^{i\theta}$

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{e^{i\theta} + 1}{e^{i\theta} - 1} = \frac{\cos\theta + 1 + i\sin\theta}{\cos\theta - 1 + i\sin\theta}$$

$$= \frac{2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}}{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{\cot(\theta/2)}{i}$$

$$i \tan \frac{\theta}{2} = \left(\frac{z_1 - z_2}{z_1 + z_2} \right)$$

$$-\tan^2 \frac{\theta}{2} = \left(\frac{z_1 - z_2}{z_1 + z_2} \right)^2 \Rightarrow 1 - \sec^2 \frac{\theta}{2} = \frac{b^2 - 4c}{a^2}$$

$$\Rightarrow 1 - \sec^2 \frac{\theta}{2} = 1 - \frac{4ac}{b^2} \Rightarrow \cos^2 \frac{\theta}{2} = \frac{b^2}{4ac}$$

$$\Rightarrow \cos \frac{\theta}{2} = \sqrt{\frac{b^2}{4ac}} \Rightarrow \theta = 2\cos^{-1} \sqrt{\frac{b^2}{4ac}}$$

11. 51

$$12. D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$$

$$= e^{-iA} e^{-iB} e^{-iC} \begin{vmatrix} e^{-iA} & e^{i(A+C)} & e^{i(A+B)} \\ e^{i(B+C)} & e^{-iB} & e^{i(A+B)} \\ e^{i(B+C)} & e^{i(A+C)} & e^{-iC} \end{vmatrix}$$

As $A + B + C = \pi$

So $A + C = \pi - B, B + C = \pi - A, A + B = \pi - C$

$$D = e^{-i\pi} \begin{vmatrix} e^{-iA} & e^{i(\pi-B)} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{-iB} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{i(\pi-B)} & e^{-iC} \end{vmatrix}$$

$$D = - \begin{vmatrix} e^{-iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{-iB} & -e^{iC} \\ -e^{-iA} & -e^{-iB} & e^{-iC} \end{vmatrix}$$

$$= -e^{-iA} e^{-iB} e^{-iC} \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -e^{-i\pi} (-4) = -4$$

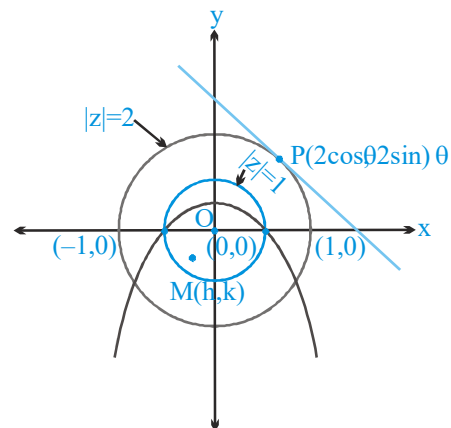
14. 18

15. Clearly the parabola should pass through (1, 0) and (-1, 0). Let directrix of this parabola be $x \cos\theta + y \sin\theta = 2$. If M(h, k) be the focus of this parabola, then distance of ($\pm 1, 0$) from 'M' and from the directrix should be same.

$$\Rightarrow (h-1)^2 + k^2 = (\cos\theta - 2)^2 \quad \dots(i)$$

$$\text{and } (h+1)^2 + k^2 = (\cos\theta + 2)^2 \quad \dots(ii)$$

Now (2) - (1) $\Rightarrow \cos\theta = \frac{h}{2} \quad \dots(iii)$



Also (2) + (1) $\Rightarrow (h^2 + k^2 + 1) = (\cos^2\theta + 4) \quad \dots(iv)$

∴ From (iii) and (iv), we get

$$h^2 + k^2 + 1 = 4 + \frac{h^2}{4} \Rightarrow \frac{3h^2}{4} + k^2 = 3$$

Hence locus of focus M(h, k) is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (Ellipse)

Also we know that area of the quadrilateral formed by

the tangents at the ends of the latus-rectum is $\frac{2a^2}{e}$

(where e is eccentricity of ellipse)

$$\therefore \text{Required area} = \frac{2(4)}{\frac{1}{2}} = 16 \text{ (square units)}$$

$$(\text{As } e^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2})$$

$$\begin{aligned} 16. |z_1 - 2z_2| &= |2 - z_1 \bar{z}_2| \\ |z_1 - 2z_2|^2 &= |2 - z_1 \bar{z}_2|^2 \\ (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) &= (2 - z_1 \bar{z}_2)(2 - \bar{z}_1 z_2) \\ z_1 \bar{z}_1 - 2\bar{z}_1 z_2 - 2z_1 \bar{z}_2 + 4z_2 \bar{z}_2 &= 4 - 2z_1 \bar{z}_2 - 2\bar{z}_1 z_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 \\ |z_1|^2 + 4|z_2|^2 - 4|z_1|^2 |z_2|^2 &= 0 \\ \Rightarrow (|z_1|^2 - 4)(1 - |z_2|^2) &= 0 \\ \Rightarrow |z_1| = 2 \quad (\text{as } |z_2| \neq 1) \end{aligned}$$

$$17. (i) |z| = 20 \quad (ii) OP = OQ = PR = QR = 20$$

$$18. \text{Let } z^{2m} + z^{2m-1} - 1 \dots + z + 1 = (z - z_1)(z - z_2) \dots (z - z_{2m})$$

Taking log on both the sides & differentiating w.r.t. z

$$\begin{aligned} \frac{2mz^{2m-1} + (2m-1)z^{2m-2} + \dots + 2z + 1}{z^{2m} + z^{2m-1} + \dots + z^2 + z + 1} \\ = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_{2m}} \\ \Rightarrow \frac{1 + 2 + 3 + \dots + 2m}{(2m+1)} \quad (\text{put } z = 1) \\ = \frac{1}{1 - z_1} + \frac{1}{1 - z_2} + \dots + \frac{1}{1 - z_{2m}} \\ \Rightarrow \sum_{r=1}^{2m} \frac{1}{z_r - 1} = - \left[\frac{2m(2m+1)}{2(2m+1)} \right] = -m \end{aligned}$$

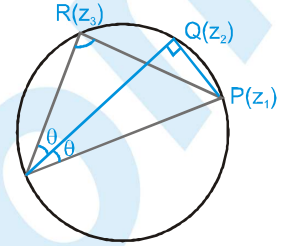
$$20. \bar{az} + az = 0$$

21. From the fig.

we have

$$z_2 = z_1 (\cos \theta e^{i\theta})$$

$$\text{and } z_3 = z_1 (\cos 2\theta e^{i2\theta})$$



$$\Rightarrow \frac{z_2^2}{z_3} = \frac{(z_1 \cos \theta)^2}{(z_1 \cos 2\theta)} \Rightarrow z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$$

$$22. \text{We have } f(w) = 2 + i\sqrt{3}, f(w^2) = 2 - i\sqrt{3}$$

$$\text{Let } f(z) = (z^2 + z + 1)g(z) + (az + b)$$

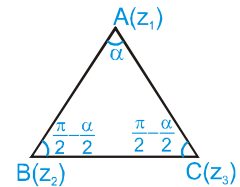
$$\therefore aw + b = 2 + i\sqrt{3} \quad \dots (i)$$

$$\text{and } aw^2 + b = 2 - i\sqrt{3} \quad \dots (ii)$$

∴ On solving (1) and (2), we get

$$a = 2, b = 3 \Rightarrow a + b = 5$$

$$23. \frac{z_3 - z_1}{z_2 - z_1} = e^{i\alpha}$$



$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \cos \alpha + i \sin \alpha$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = \cos \alpha - 1 + i \sin \alpha$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2 \sin^2 \frac{\alpha}{2} + 2i \sin \frac{\alpha}{2} \cdot \cos \frac{\alpha}{2}$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = 2i \sin \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right)$$

squaring both sides

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4 \sin^2 \frac{\alpha}{2} (\cos \alpha + i \sin \alpha)$$

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4 \sin^2 \frac{\alpha}{2} \left(\frac{z_3 - z_1}{z_2 - z_1} \right)$$

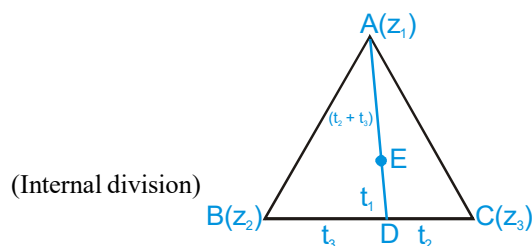
$$\Rightarrow (z_3 - z_2)^2 = 4 \sin^2 (\alpha/2) (z_3 - z_1) (z_1 - z_2)$$

24. Affixes of a point D which divides z_2, z_3 in the ratio $t_3 : t_2$

is $\frac{t_2 z_2 + t_3 z_3}{t_2 + t_3}$ (Internal division)

Affixes of a point E which divides AD in the ratio

$$(t_2 + t_3) : t_1 \text{ is } \left(\frac{t_1 z_1 + t_2 z_2 + t_3 z_3}{t_1 + t_2 + t_3} \right)$$



Hence E always lies in or on the ΔABC

26. (i) $a + b + c = \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$
 $= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$
 $= 0 + i \cdot 0 = 0$

(ii) $\rightarrow a + b + c = 0 \Rightarrow \bar{a} + \bar{b} + \bar{c} = 0$

$$\bar{a} = \frac{1}{a}$$

$$\bar{b} = \frac{1}{b}$$

$$\bar{c} = \frac{1}{c}$$

$$\text{L.H.S} = abc \left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b} \right) = abc(\bar{a} + \bar{b} + \bar{c}) = abc(0) = 0$$

(iii) Squaring and using

(ii) $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

(iv) by (iii) $\Rightarrow e^{2i\alpha} + e^{2i\beta} + e^{2i\gamma} = 0$

$$\cos 2\alpha + i\sin 2\alpha + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0$$

$$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

$$\text{and } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

(v) $2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

Similarly

$$1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$$

$$\frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$$

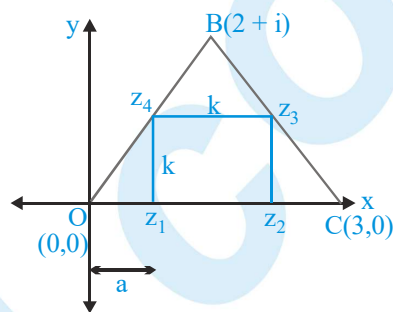
27. (B) one if n is even ; $-\omega^2$ if n is odd

28. Let $|z_1| = a$,
 and $|z_1 - z_2| = |z_1 - z_4| = k$

$$\Rightarrow \frac{z_4 - z_1}{z_2 - z_1} = e^{i\frac{\pi}{2}}$$

$$\Rightarrow z_4 = a + ik$$

$$\Rightarrow z_3 = a + ik + k = (a + k) + ik$$



Now, O, z_4, z_3 are collinear.

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ a & k & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0 \Rightarrow a - 2k = 0 \quad \dots(i)$$

Also z_B, z_3, z_C are collinear.

$$\Rightarrow \begin{vmatrix} 3 & 0 & 1 \\ a+k & k & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2k + a - 3 = 0$$

.....(ii)

$\therefore$ From (1) and (2), we get

$$a = \frac{3}{2}, k = \frac{3}{4}$$

Hence $z_3 = \frac{9}{4} + \frac{3}{4}i$ and $z_4 = \frac{3}{2} + \frac{3}{4}i$

$$\therefore \text{Required area of triangle} = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 9/4 & 3/4 & 1 \\ 3/2 & 3/4 & 1 \end{vmatrix} = \frac{9}{32}$$

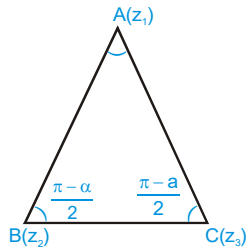
(square units)

29. $\frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC} e^{i\left(\frac{\pi - \alpha}{2}\right)}$

$$\frac{z_2 - z_3}{(z_1 - z_3)} = \frac{BC}{AC} e^{i\left(\frac{\pi - \alpha}{2}\right)}$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN



$$\Rightarrow \frac{(z_1 - z_2)(-z_1 + z_3)}{(z_3 - z_2)^2} = \left(\frac{AB}{BC}\right)^2 \cdot 1$$

$$\Rightarrow (z_2 - z_3)^2 = (z_3 - z_1)(z_1 - z_2) \cdot \left(\frac{BC}{AB}\right)^2$$

$$= (z_1 - z_2)(z_1 - z_2) \left[2 \cos\left(\frac{\pi - \alpha}{2}\right)\right]^2$$

$$= 4(z_3 - z_1)(z_1 - z_2) \sin^2 \alpha/2$$

$$30. \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} = -i \left[\cos \frac{2q\pi}{11} + i \sin \frac{2q\pi}{11} \right]$$

$$= -ie^{i \frac{2q\pi}{11}}$$

$$\therefore \sum_{q=1}^{10} \left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) = -i \sum_{k=1}^{10} e^{i \frac{2q\pi}{11}}$$

$$= -i \left[e^{\frac{2\pi i}{11}} + e^{\frac{4\pi i}{11}} + \dots + e^{\frac{20\pi i}{11}} \right] = -i(-1) = i$$

Given expression

$$\sum_{p=1}^{32} (3p+2) \left\{ \sum_{q=1}^{10} \left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right\}^p$$

$$= \sum_{p=1}^{32} (3p+2)i^p = 3 \sum_{p=1}^{32} pi^p + 2 \sum_{p=1}^{32} i^p = 3S_1 + 2S_2$$

where $S_1 = \sum_{p=1}^{32} pi^p$

$$S_1 = i + 2i^2 + 3i^3 + \dots + 32i^{32}$$

$$iS_1 = i^2 + 2i^3 + \dots + 32i^{33}$$

$$S_1(1-i) = i + i^2 + i^3 + \dots + i^{32} - 32i^{33}$$

$$S_1 = \frac{-32i}{(1-i)} = 16(1-i)$$

$$S_2 = 0$$

$$\therefore \sum_{p=1}^{32} (3p+2) \left\{ \sum_{q=1}^{10} \left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right\}^p$$

$$= 3S_1 + 2S_2 = 48(1-i) + 0 = 48(1-i)$$

6. Given that $\arg z\omega = \pi$... (i)

$$\bar{z} + i\bar{\omega} = 0 \Rightarrow \bar{z} = -i\bar{\omega}$$

$$\Rightarrow z = i\omega \Rightarrow \omega = -iz$$

$$\text{From (i) } \arg(-iz)^2 = \pi$$

$$\arg(-i) + 2 \arg(z) = \pi; \quad \frac{-\pi}{2} + 2 \arg(z) = \pi$$

$$2 \arg(z) = \frac{3\pi}{2}; \quad \arg(z) = \frac{3\pi}{4}$$

10. Given that $\omega = \frac{z}{z - \frac{i}{3}}$ and $|\omega| = 1$

$$\therefore |\omega| = \left| \frac{z}{z - \frac{i}{3}} \right| \Rightarrow \frac{|z|}{\left| z - \frac{i}{3} \right|} = 1$$

$$\Rightarrow |z| = \left| z - \frac{i}{3} \right| \Rightarrow -\frac{2}{3}y + \frac{1}{9} = 0$$

Which is a straight line.

13. $\left| z - \frac{4}{z} \right| \geq \left| z \right| - \left| \frac{4}{z} \right|$

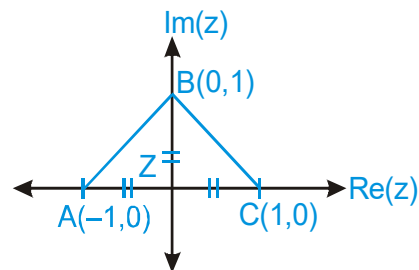
$$2 \geq \left| z \right| - \frac{4}{\left| z \right|}$$

$$2\left| z \right| \geq \left| z \right|^2 - 4$$

$$\left| z \right|^2 - 2\left| z \right| - 4 \leq 0$$

$$\left| z \right| \leq \sqrt{5} + 1$$

14. z is the circumcentre (0, 0) of triangle ABC so there exist only one complex number.



15. Let $z^2 + \alpha z + \beta = 0$ has $(1 + iy_1)$ and $(1 + iy_2)$

$$\text{so } z_1 z_2 = \beta$$

$$(1 + iy_1)(1 + iy_2) = \beta$$

$$\beta = 1 - y_1 y_2 + i(y_1 + y_2) \quad (\rightarrow \beta \text{ is purely real})$$

$$\text{here } y_1 + y_2 = 0$$

$$y_1 = -y_2$$

$$\beta = 1 - y_1 y_2$$

$$\beta = 1 + y_1^2$$

$$\beta > 1$$

$$\Rightarrow \beta \in (1, \infty)$$

16. $(1 + \omega)^7 = A + B\omega$

$$(-\omega^2)^7 = A + B\omega$$

$$-\omega^2 = A + B\omega$$

$$1 + \omega = A + B\omega$$

$$A = 1$$

$$B = 1 \quad (1, 1)$$

17. $\frac{z^2}{z-1}$ is purely real where $(Z \neq 1)$

$$\text{So } \frac{\bar{z}^2}{\bar{z}-1} = \frac{z^2}{z-1}$$

$$z\bar{z}^2 - \bar{z}^2 = \bar{z}z^2 - z^2$$

$$z\bar{z}(z - \bar{z}) = z^2 - \bar{z}^2$$

$$z\bar{z}(z - \bar{z}) = (z + \bar{z})(z - \bar{z})$$

$$\Rightarrow \bar{z} - z = 0 \text{ or } z + \bar{z} = z\bar{z}$$

$$\Rightarrow \bar{z} = z \text{ or } x^2 + y^2 - 2x = 0$$

$$(x-1)^2 + y^2 = 1$$

so either lie on z real axis or on a circle passing through the origin.

18. $\bar{z} = \frac{1}{z}$

$$\arg\left(\frac{1+z}{1+\frac{1}{z}}\right) \Rightarrow \arg z \Rightarrow \theta$$

21. $\text{Re}((2 + 3i \sin \theta)(1 + 2i \sin \theta)) = 2 - 6 \sin^2 \theta = 0$

$$\Rightarrow \sin^2 \theta = \frac{1}{3}$$

2. (A) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2} = \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$

$$= \frac{1 - i^2 \cdot 3}{2(1 + i\sqrt{3})}$$

$$= \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$$

$$\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = \frac{1 + i\sqrt{3}}{2} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$\Rightarrow \left| \frac{z_2 - z_3}{z_1 - z_3} \right| = 1 \text{ and } \arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right) = \frac{\pi}{3}$$

Hence the Δ is equilateral,

(B) $\arg \frac{z_1}{z_2} = \frac{\pi}{2} \Rightarrow \frac{z_1}{z_2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$

$$(\rightarrow |z_2| = |z_1| = 1)$$

$$\therefore \frac{z_1^n}{z_2^n} = (i)^n$$

$$\text{Hence } i^n = 1 \Rightarrow n = 4k$$

3. (C) $z^{p+q} - z^p - z^q + 1 = 0$

$$\Rightarrow (z^p - 1)(z^q - 1) = 0$$

as α is root of (1), either $\alpha^p - 1 = 0$

$$\text{or } \alpha^q - 1 = 0$$

$$\Rightarrow \text{either } \frac{\alpha^p - 1}{\alpha - 1} = 0 \text{ or } \frac{\alpha^q - 1}{\alpha - 1} = 0 \text{ (as } \alpha \neq 1)$$

$$\Rightarrow \text{either } 1 + \alpha + \alpha^2 + \dots + \alpha^{p-1} = 0$$

$$\text{or } 1 + \alpha + \dots + \alpha^{q-1} = 0$$

$$\text{But } \alpha^p - 1 = 0 \text{ and } \alpha^q - 1 = 0$$

cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1 = \alpha^p = \alpha^q$.

5. Given $|z_1| < 1$ and $|z_2| > 1$

$$\text{Then to prove } \left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1 \left\{ \text{using } \left| \frac{z_1}{z_2} \right| = \left| \frac{z_1}{z_2} \right| \right\}$$

$$\Rightarrow |1 - z_1 \bar{z}_2| < |z_1 - z_2|$$

Squaring both sides, we get

$$(1 - z_1 \bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$\{\text{using } |z|^2 = z \bar{z}\}$$

$$\Rightarrow 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 < z_1 \bar{z}_1 - z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_2 \bar{z}_2$$

$$\Rightarrow 1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$$

$$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2 |z_2|^2 < 0$$

$$\Rightarrow (1 - |z_1|^2)(1 - |z_2|^2) < 0 \quad \dots (2)$$

which is true by (1) as $|z_1| < 1$ and $|z_2| > 1$

$$\therefore (1 - |z_1|^2) > 0 \text{ and } (1 - |z_2|^2) < 0$$

$\therefore$ (2) is true whenever (1) is true.

$$\Rightarrow \left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1$$

6. Given : $a_1 z + a_2 z^2 + \dots + a_n z^n = 1$

and $|z| < 1/3 \quad \dots (i)$

$$\{\text{using } |z_1 + z_2| \leq |z_1| + |z_2|\}$$

$$\Rightarrow |a_1 z| + |a_2 z^2| + |a_3 z^3| + \dots + |a_n z^n| \geq 1$$

$$\Rightarrow 2\{|z| + |z|^2 + |z|^3 + \dots + |z|^n\} > 1 \quad (\text{using } |a_r| < 2)$$

$$\Rightarrow \frac{2|z|(1 - |z|^n)}{1 - |z|} > 1$$

\{\text{using sum of } n \text{ terms of G.P.}\}

$$\Rightarrow 2|z| - 2|z|^{n+1} > 1 - |z|$$

$$\Rightarrow 3|z| > 1 + 2|z|^{n+1} \Rightarrow |z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1}$$

$$\Rightarrow |z| > \frac{1}{3}, \text{ which contradicts } \dots (i)$$

$\therefore$ There exists no complex number z such that

$$|z| < \frac{1}{3} \text{ and } \sum_{r=1}^n a_r z^r = 1$$

8. As we know; $|z|^2 = z \bar{z} \Rightarrow \frac{|z - \alpha|^2}{|z - \beta|^2} = k^2$

$$\Rightarrow (z - \alpha)(\bar{z} - \bar{\alpha}) = k^2(z - \beta)(\bar{z} - \bar{\beta})$$

$$|z|^2 - \alpha \bar{z} - \bar{\alpha} z + |\alpha|^2 = k^2(|z|^2 - \beta \bar{z} - \bar{\beta} z + |\beta|^2)$$

$$\text{or } |z|^2(1 - k^2) - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2)z + (|\alpha|^2 - k^2|\beta|^2) = 0$$

$$\Rightarrow z^2 - \frac{(\alpha - k^2\beta)}{(1 - k^2)}\bar{z} - \frac{(\bar{\alpha} - \bar{\beta}k^2)}{(1 - k^2)}z + \frac{|\alpha|^2 - k^2|\beta|^2}{(1 - k^2)} = 0$$

On comparing with equation of circle.

$$|z|^2 + a\bar{z} + \bar{a}z + b = 0$$

whose centre is $(-a)$ and radius = $\sqrt{|a|^2 - b}$

$\therefore$ centre for (i)

$$= \frac{\alpha - k^2\beta}{1 - k^2} \text{ and radius}$$

$$= \sqrt{\left(\frac{\alpha - k^2\beta}{1 - k^2}\right)\left(\frac{\bar{\alpha} - \bar{\beta}k^2}{1 - k^2}\right) - \frac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{1 - k^2}}$$

11. Here, centre of circle is $(1, 0)$ is also the mid-point of diagonals of square

$$\Rightarrow \frac{z_1 + z_2}{2} = z_0$$

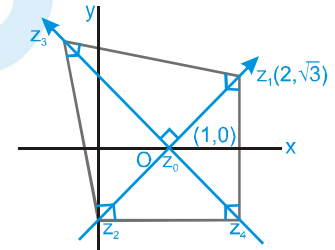
$$\Rightarrow z_2 = -\sqrt{3}i, \quad (\text{where } z_0 = 1 + 0i)$$

$$\text{and } \frac{z_3 - 1}{z_1 - 1} = e^{\pm i\pi/2}$$

$$\Rightarrow z_3 = 1 + (1 + \sqrt{3}i) \cdot \left(\cos \frac{\pi}{2} \pm i \sin \frac{\pi}{2}\right), \text{ as } z_1 = 2 + \sqrt{3}i$$

$$= 1 \pm i(1 + \sqrt{3}i) = (1 \mp \sqrt{3}) \pm i$$

$$z_3 = (1 - \sqrt{3}) + i \text{ and } z_4 = (1 + \sqrt{3}) - i$$



12. Let, $z_1 = \frac{w - \bar{w}z}{1 - z}$, be purely real

$$\Rightarrow z_1 = \bar{z}_1$$

$$\therefore \frac{w - \bar{w}z}{1 - z} = \frac{\bar{w} - w\bar{z}}{1 - \bar{z}}$$

$$\Rightarrow w - w\bar{z} - \bar{w}z + \bar{w}z\bar{z} = \bar{w} - w\bar{z} - w\bar{z} + w\bar{z}\bar{z}$$

$$\Rightarrow (w - \bar{w}) - (\bar{w} - w)|z|^2 = 0$$

$$\Rightarrow (w - \bar{w})(1 - |z|^2) = 0$$

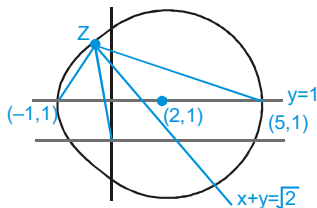
$$\Rightarrow |z|^2 = 1 \quad \{\text{as, } w - \bar{w} \neq 0, \text{ since } \beta \neq 0\}$$

$$\Rightarrow |z| = 1 \text{ and } z \neq 1.$$

15. $A = \{z : \text{Im } z \geq 1\} \quad y \geq 1$

$$B = \{z : |z - 2 - i| = 3\} \quad (x - 2)^2 + (y - 1)^2 = 9$$

$$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\} \quad x + y = \sqrt{2}$$



As we can see 3 curves intersects at only one point
So $A \cap B \cap C$ contains exactly one element

16. $|z+1-i|^2 + |z-5-i|^2 = (-1-5)^2 + (1-1)^2 = 36$
so exactly 36

17. As $3 - \sqrt{5} \leq |z| \leq 3 + \sqrt{5}$

As $-3 + \sqrt{5} \leq |\omega| \leq 3 + \sqrt{5}$

$-3 - \sqrt{5} \leq -|\omega| \leq 3 - \sqrt{5}$

$-\sqrt{5} \leq -|\omega| + 3 \leq 6 - \sqrt{5}$

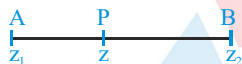
$-3 \leq |z| - |\omega| + 3 \leq 9$

22. $z = z_1 + t(z_2 - z_1)$

$\frac{z - z_1}{z_2 - z_1} = t, t \in (0,1) \Rightarrow z = \frac{z_1(1-t) + tz_2}{(1-t)+t}$

point P(z) divides point A(z_1) & B(z_2) internally in ratio $(1-t) : t$

Hence locus is a line segment such that P(z) lies between A(z_1) & B(z_2) as shown in figure.



Hence options A,C & D are correct.

24. (A) $|z - i| = |z| \Rightarrow |z - i|^2 = |z|^2$

$\Rightarrow (z - i|z|)(\bar{z} + i|z|) = (z + i|z|)(\bar{z} - i|z|)$

$\Rightarrow 2i|z|z = 2i|z|\bar{z}$

$\Rightarrow z = \bar{z} \therefore z$ is purely real.

$\therefore z$ lies on real axis.

(B) Locus is ellipse having focii $(-4, 0)$ & $(4, 0)$

$2ae = 8$ & $2a = 10$

$\Rightarrow a = 5$ & $e = 4/5$

It is ellipse having eccentricity $4/5$.

(C) $w = 2(\cos\theta + i\sin\theta)$

$z = 2(\cos\theta + i\sin\theta) - \frac{1}{2(\cos\theta + i\sin\theta)}$

$x + iy = \frac{3}{2}\cos\theta + \frac{i5}{2}\sin\theta$

$\Rightarrow x = \frac{3}{2}\cos\theta$ & $y = \frac{5}{2}\sin\theta$

It is a locus $\frac{x^2}{9/4} + \frac{y^2}{25/4} = 1$

$\frac{9}{4} = \frac{25}{4}(1 - e^2) \Rightarrow e = \frac{4}{5}$

since $x = \frac{3}{2}\cos\theta \Rightarrow |\operatorname{Re}(z)| \leq \frac{3}{2}$

$|\operatorname{Re}(z)| \leq \frac{3}{2} \Rightarrow |\operatorname{Re}(z)| \leq 2$

Consider the circle $x^2 + y^2 - 9 = 0$

By putting $x = \frac{3}{2}\cos\theta$

& $y = \frac{5}{2}\sin\theta$ into $x^2 + y^2 - 9 < 0$

$\frac{9\cos^2\theta}{4} + \frac{25\sin^2\theta}{4} - 9 < 0$

(D) $z = (\cos\theta + i\sin\theta) + \frac{1}{(\cos\theta + i\sin\theta)}$

$z = 2\cos\theta$

where z is real value & $z \in [-2, 2]$

25. Comprehension (3 questions together)

$a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$7a + 7b + 7c = 0$

$\Rightarrow a = K, b = 6K, c = -7K$

(i) $(K, 6K, -7K)$

$2x + y + z = 1$

$2K + 6K - 7K = 1$

($\rightarrow$ point lies on the plane)

$\Rightarrow K = 1$

$\Rightarrow 7a + b + c = 7K + 6K - 7K = 6$

(ii) $x^3 - 1 = 0$

$\Rightarrow x = 1, \omega, \omega^2$

$\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ since $\operatorname{Im}(\omega) > 0$

If $a = 2 = K \Rightarrow b = 12$ & $c = -14$

Hence $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}}$

$= 3\omega + 1 + 3\omega^2 = -3 + 1 = -2$

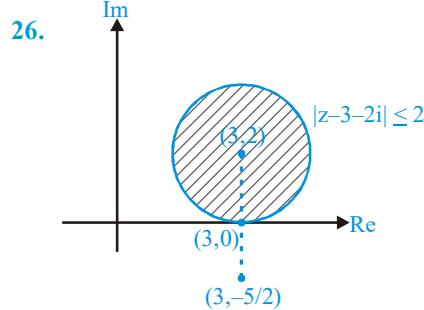
(iii) $\rightarrow b = 6 \Rightarrow 6K = 6 \Rightarrow K = 1$

$\Rightarrow a = 1, b = 6$ & $c = -7$

$x^2 + 6x - 7 = 0$

$\Rightarrow \alpha + \beta = -6, \alpha\beta = -7$

$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\alpha + \beta}{\alpha\beta} \right)^n = \sum_{n=0}^{\infty} \left(\frac{6}{-7} \right)^n = \frac{1}{1 - \frac{6}{-7}} = 7$



We have to find minimum value of

$$2 \left| z - \left(3 - \frac{5}{2}i \right) \right|$$

$= 2 \times (\text{minimum distance between } z \text{ and point } (3, -\frac{5}{2}))$

$= 2 \times (\text{distance between } (3, 0) \text{ and } (3, -\frac{5}{2}))$

$= 2 \times \frac{5}{2} = 5 \text{ units.}$

27. Ans. 3 (Bonus)

(Comment : If $\omega = e^{i\pi/3}$

then $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is not always an integer.

For example if $a = b = c = 1$ then the value of

$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is $\frac{17}{3}$. Now if we consider $\omega = e^{i\frac{2\pi}{3}}$ then the solution is)

$$|x|^2 = (a + b + c)(\bar{a} + \bar{b} + \bar{c})$$

$$= |a|^2 + |b|^2 + |c|^2 + a\bar{b} + a\bar{c} + b\bar{a} + b\bar{c} + c\bar{a} + c\bar{b}$$

$$|y|^2 = (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega)$$

$$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega^2 + a\bar{c}\omega + b\bar{a}\omega + b\bar{c}\omega^2 + c\bar{a}\omega^2 + c\bar{b}\omega$$

$$|z|^2 = (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$$

$$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega + a\bar{c}\omega^2 + b\bar{a}\omega^2 + b\bar{c}\omega + c\bar{a}\omega + c\bar{b}\omega^2$$

$$\therefore |x|^2 + |y|^2 + |z|^2 = 3(|a|^2 + |b|^2 + |c|^2)$$

$$\Rightarrow \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$$

28. (A)



$$|\vec{a}| = 2$$

$$|\vec{b}| = 2$$

$$|\vec{c}| = 2\sqrt{3}$$

$$\cos \theta = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{a}||\vec{b}|}$$

$$= \frac{4 + 4 - 12}{2 \cdot 2 \cdot 2} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

(B) $\int_a^b (f(x) - 3x) dx = a^2 - b^2 = \int_a^b (-2x) dx$

$$\Rightarrow \int_a^b (f(x) - x) dx = 0$$

$\Rightarrow$ one of the possible solution of this equation is

$$f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

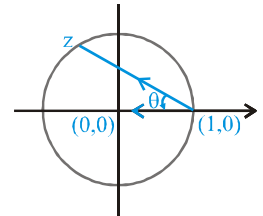
(C) $\frac{\pi^2}{\ln 3} \int_{7/6}^{5/6} (\sec \pi x) dx$

$$= \frac{\pi^2}{\ln 3} \frac{1}{\pi} \left[\ln |\sec \pi x + \tan \pi x| \right]_{7/6}^{5/6}$$

$$= \frac{\pi}{\ln 3} \ln \left| \frac{\sec \frac{5\pi}{6} + \tan \frac{5\pi}{6}}{\sec \frac{7\pi}{6} + \tan \frac{7\pi}{6}} \right| = \frac{\pi}{\ln 3} \bullet \ln 3 = \pi$$

(D) Let $\theta = \text{Arg}\left(\frac{1}{1-z}\right)$

$$\Rightarrow \theta = \text{Arg}\left(\frac{0-1}{z-1}\right)$$



which is shown in adjacent diagram.

$\Rightarrow$ Maximum value of θ is

approaching to $\frac{\pi}{2}$ but θ will never

obtained the value equal to $\frac{\pi}{2}$.

Hence there is an error in asking the problem.

29. (A) Let $z = \cos \theta + i \sin \theta$

$$\text{Re}\left(\frac{2i(\cos \theta + i \sin \theta)}{1 - (\cos \theta + i \sin \theta)^2}\right) = \text{Re}\left(\frac{\cos \theta i - \sin \theta}{\sin^2 \theta - i \cos \theta \sin \theta}\right)$$

$$= \operatorname{Re}\left(-\frac{1}{\sin \theta}\right) = \frac{-1}{\sin \theta}$$

$\therefore$ Set will be $(-\infty, -1] \cup [1, \infty)$

$$(B) -1 \leq \frac{8 \cdot 3^{(x-2)}}{1-3^{2(x-1)}} \leq 1 \quad x \neq 1$$

$$\Rightarrow -1 \leq \frac{8 \cdot 3^x}{(3-3^x)(3+3^x)} \leq 1$$

$$3^x = t \quad \therefore t > 0$$

$$\frac{8t}{(3-t)(t+3)} \geq -1$$

$$\Rightarrow t \in (0, 3) \cup [9, \infty)$$

$$\Rightarrow x \in (-\infty, 1) \cup [2, \infty)$$

$$\frac{8t}{(3-t)(t+3)} \leq 1$$

$$\Rightarrow t \in (0, 1] \cup (3, \infty)$$

$$\Rightarrow x \in (-\infty, 0] \cup (1, \infty)$$

Taking intersection,

$$x \in (-\infty, 0] \cup [2, \infty)$$

$$(C) f(\theta) = \begin{vmatrix} 1 & \tan \theta & 1 \\ -\tan \theta & 1 & \tan \theta \\ -1 & -\tan \theta & 1 \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_3$$

$$\Rightarrow f(\theta) = \begin{vmatrix} 2 & \tan \theta & 1 \\ 0 & 1 & \tan \theta \\ 0 & -\tan \theta & 1 \end{vmatrix}$$

$$\Rightarrow f(\theta) = 2 \sec^2 \theta$$

$$\Rightarrow f(\theta) \in [2, \infty)$$

$$(D) f(x) = 3x^{5/2} - 10x^{3/2}$$

$$f'(x) = \frac{15}{2}x^{3/2} - \frac{30}{2}x^{1/2} > 0$$

$$\Rightarrow \frac{15}{2}\sqrt{x}(x-2) \geq 0 \quad \Rightarrow x \geq 2$$

$$30. z^2 + z + 1 - a = 0$$

$\rightarrow z$ is imaginary $\Rightarrow D < 0$

$$1 - 4(1-a) < 0$$

$$4a < 3$$

$$a < \frac{3}{4}$$

Aliter : $a = z^2 + z + 1$

$\rightarrow a = \bar{a}$ (given a is real)

$$\therefore z^2 + z = \bar{z}^2 + \bar{z} \Rightarrow z^2 - \bar{z}^2 = \bar{z} - z$$

$$\Rightarrow z + \bar{z} = -1 \quad (\rightarrow \operatorname{Im}(z) \text{ is non zero})$$

$$\Rightarrow \operatorname{Re}(z) = -\frac{1}{2}$$

$\therefore z$ can be taken as $-\frac{1}{2} + iy$

where $y \in \mathbb{R}$

$$\therefore a = \left(-\frac{1}{2} + iy\right)^2 + \left(-\frac{1}{2} + iy\right) + 1$$

$$\Rightarrow a = \frac{1}{4} - \frac{1}{2} + 1 - iy + iy - y^2$$

$$\Rightarrow a = \frac{3}{4} - y^2 \Rightarrow a < \frac{3}{4}$$

$$\therefore a \neq \frac{3}{4}$$

31. Given : α satisfies $|z - z_0| = r$

$$\Rightarrow |\alpha - z_0| = r \quad \dots(i)$$

& $\frac{1}{\alpha}$ satisfies $|z - z_0| = 2r$

$$\Rightarrow \left|\frac{1}{\alpha} - z_0\right| = 2r \quad \dots(ii)$$

squaring (i) and (ii) we get $(\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$$\Rightarrow \alpha\bar{\alpha} - z_0\bar{\alpha} - \alpha\bar{z}_0 + z_0\bar{z}_0 = r^2 = 2|z_0|^2 - 2 \quad \dots(iii)$$

$$\& \left(\frac{1}{\alpha} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$$

$$\Rightarrow \frac{1}{\alpha\bar{\alpha}} - \frac{z_0}{\alpha} - \frac{\bar{z}_0}{\bar{\alpha}} + z_0\bar{z}_0 = 4r^2$$

$$\Rightarrow 1 - z_0\bar{\alpha} - \bar{z}_0\alpha + |z_0|^2|\alpha|^2 = 4(2|z_0|^2 - 2)|\alpha|^2$$

$$\Rightarrow 1 + 2|z_0|^2 - 2|\alpha|^2 - |z_0|^2|\alpha|^2 = 8|z_0|^2|\alpha|^2 - 8|\alpha|^2$$

$$\Rightarrow -1 + |z_0|^2 - 7|z_0|^2|\alpha|^2 + 7|\alpha|^2 = 0$$

$$\Rightarrow (|z_0|^2 - 1)(7|\alpha|^2 - 1) = 0$$

$$\Rightarrow |z_0| = 1 \text{ (rejected as } r = 0) \& |\alpha| = \frac{1}{\sqrt{7}}$$

$$32. P^2 = [\alpha_{ij}]_{n \times n}$$

$$\alpha_{ij} = \sum_{k=1}^n p_{ik} \cdot p_{kj}$$



MOCK TEST

1. (D) $e^{\frac{2\pi r}{p}i} = e^{\frac{2\pi m}{q}i}$

$r=0, 1, \dots, p-1$

$m=0, 1, \dots, q-1$

This is possible iff $r=m=0$

but for $r=m=0$ we get 1 which is not an imaginary number.

2. (D) $z^3 + \frac{3(\bar{z})^2}{|z|} = 0$ Let $z = re^{i\theta}$

$\Rightarrow r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$

Since 'r' cannot be zero

$\Rightarrow r^2 e^{i5\theta} = -3$ which will hold for

$r = \sqrt{3}$ and 5 distinct values of ' θ '

Thus there are five solution.

3. (B)

$(x-1)(x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_8) \equiv x^9 - 1$

$\therefore (2-\alpha_1)(2-\alpha_2) \dots (2-\alpha_8) = 2^9 - 1$

Now since $2-\alpha_1$ and $2-\alpha_8$ are conjugates of each other

$\therefore |2-\alpha_1| = |2-\alpha_8|$

similarly

$|2-\alpha_2| = |2-\alpha_7|, |2-\alpha_3| = |2-\alpha_6|$

and $|2-\alpha_4| = |2-\alpha_5|$

$\therefore |(2-\alpha_1)(2-\alpha_3)(2-\alpha_5)(2-\alpha_7)| = \sqrt{2^9-1} = \sqrt{511}$

4. (D)

$\arg(z-i+2) = \frac{\pi}{6} \Rightarrow \tan \frac{\pi}{6} = \frac{y-1}{x+2}$

$\Rightarrow x - \sqrt{3}y = -(\sqrt{3}+2), x > -2, y > 1$

.....(i)

$\arg(z+4-3i) = -\frac{\pi}{4} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = \frac{y-3}{x+4}$

$\Rightarrow y+x = -1, x > -4, y < 3$

.....(ii)

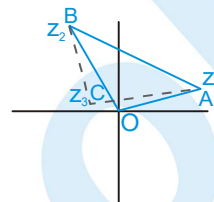
so, there is no point of intersection.

5. (D)

$|z_2 + iz_1| = |z_1| + |z_2|$

$\Rightarrow \arg(iz_1) = \arg(z_2)$

$\Rightarrow \arg(z_2) - \arg(z_1) = \frac{\pi}{2}$



Let $z_3 = \frac{z_2 - iz_1}{1-i}$

$(1-i)z_3 = z_2 - iz_1$

$\Rightarrow (z_3 - z_2) = i(z_3 - z_1) \Rightarrow (z_2 - z_3) = i(z_1 - z_3)$

$\therefore \angle ACB = \frac{\pi}{2}$ and $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow AC = BC$

$\rightarrow AB^2 = AC^2 + BC^2 \Rightarrow AC = \frac{5}{\sqrt{2}}$

$(\rightarrow AB = 5)$

$\therefore \Delta ABC = \frac{1}{2} AC \cdot BC = \frac{AC^2}{2} = \frac{25}{4}$ square unit

6. (C)

$\frac{(1+i)^5(1+\sqrt{3}i)^2}{-2i(-\sqrt{3}+i)}$

$= \frac{(\sqrt{2})^5 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^5 \cdot 2^2 \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^2}{2i \cdot 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}$

$\therefore \text{argument} = \frac{5\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{19\pi}{12}$

$\therefore \text{principal argument is } -\frac{5\pi}{12}$

7. (C)

Equation of the line is $2x - 2y = 0$ i.e. $y = x$

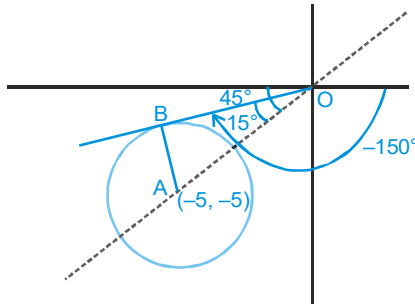
Now $\frac{2-i}{3+i} = \frac{(2-i)(3-i)}{10} = \frac{1-i}{2}$

$\therefore$ image is the point whose affix is $\frac{-1+i}{2}$



8. (C)

Point B has least principal argument



$$AB = \frac{5(\sqrt{3}-1)}{2}$$

$$OA = 5\sqrt{2}$$

$$\angle AOB = \frac{\pi}{12}$$

$$\therefore \text{Arg}(z) = -\frac{5\pi}{6}$$

9. (C)

$$z = \frac{at+b}{t-c} \Rightarrow t = \frac{b+cz}{z-a}$$

$$|t| = 1 \Rightarrow t\bar{t} = 1$$

$$\Rightarrow \frac{(b+cz)(\bar{b}+\bar{c}\bar{z})}{(\bar{z}-\bar{a})(z-a)} = 1$$

$$\Rightarrow z\bar{z}(c\bar{c}-1) + (c\bar{b}+\bar{a})z + (\bar{c}b+a)\bar{z} + b\bar{b}-a\bar{a} = 0$$

10. (B)

$$S_1. (z+1)^7 + z^7 = 0$$

$$z^7 + z^7 + {}^7C_1 z^6 + {}^7C_2 z^5 + \dots + 1 = 0$$

$$\therefore \sum_{k=0}^6 \text{Re}(z_k) = -\frac{7}{2}$$

$$S_2. \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 + i$$

squaring both the sides

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 2\left(\frac{\alpha\beta}{ab} + \frac{\beta\gamma}{bc} + \frac{\gamma\alpha}{ca}\right) = 2i$$

$$\text{i.e. } \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 2\left(\frac{c\alpha\beta + a\beta\gamma + b\gamma\alpha}{abc}\right) = 2i$$

$$\therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 2i \quad \rightarrow \quad \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0$$

$$S_3. z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = \prod_{i=1}^6 (z - z_i)$$

$$\text{Put } z = -1$$

$$\therefore 7 = \prod_{i=1}^6 (-1 - z_i) = \prod_{i=1}^6 (1 + z_i)$$

$$S_4. z^3 = \bar{z}i|z| \Rightarrow |z| = 1 \text{ or } |z| = 0$$

Thus $z = 0$ is a solution.

$$\text{If } |z| = 1, \quad \text{Let } z = e^{i\theta} \text{ then } e^{i3\theta} = e^{-i\theta}i$$

$$\Rightarrow e^{i4\theta} = i$$

$$\Rightarrow 4\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$$

$$\therefore \theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8} \text{ are solutions.}$$

$\therefore$ In all there are 5 solutions.

11. (A, B)

$$n = 1 \Rightarrow b = 0 \quad \text{not possible}$$

$$n = 2 \Rightarrow \text{either } a = 0 \text{ or } b = 0 \quad \text{not possible}$$

$$n = 3 \Rightarrow a^3 - ib^3 + 3a^2bi - 3ab^2 = a^3 + ib^3 - 3a^2bi - 3ab^2$$

$$\Rightarrow 2ib^3 = 3a^2bi$$

$$\Rightarrow \frac{b^2}{a^2} = 3 \Rightarrow \frac{b}{a} = \sqrt{3} = \tan \frac{\pi}{3}$$

12. (A, B)

$$|z - i \text{Re}(z)| = |z - i \text{Im}(z)|$$

Let $z = x + iy$, then

$$|x + iy - ix| = |x + iy - y|$$

$$\text{i.e. } x^2 + (y-x)^2 = (x-y)^2 + y^2$$

$$\text{i.e. } x^2 = y^2 \quad \text{i.e. } y = \pm x$$

13. (A, D)

$$z_1 = 5 + 12i, |z_2| = 4$$

$$|z_1 + iz_2| \leq |z_1| + |z_2| = 13 + 4 = 17$$

$$\therefore |z_1 + (1+i)z_2| \geq ||z_1| - |1+i||z_2|| = 13 - 4\sqrt{2}$$

$$\therefore \min(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$$

$$\left| z_2 + \frac{4}{z_2} \right| \leq |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left| z_2 + \frac{4}{z_2} \right| \geq |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \max \left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{3} \quad \text{and} \quad \min \left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{5}$$

14. (A, D)

$$\mu^2 - 2\mu + 2 = 0$$

$$\therefore \alpha = 1 + i \text{ and } \beta = 1 - i$$

$$x = \cot \theta - 1$$

$$\therefore x + \alpha = \cot \theta + i = \operatorname{cosec} \theta \cdot e^{i\theta}$$

$$\text{and } x + \beta = \cot \theta - i = \operatorname{cosec} \theta \cdot e^{-i\theta}$$

$$\alpha - \beta = 2i$$

$$\frac{(x + \alpha)^n - (x + \beta)^n}{\alpha - \beta} = \frac{\operatorname{cosec}^n \theta \cdot [e^{in\theta} - e^{-in\theta}]}{2i}$$

$$= \operatorname{cosec}^n \theta \cdot \frac{2i \sin n\theta}{2i} = \frac{\sin n\theta}{\sin^n \theta}$$

15. (A, B, C, D)

$$|z_1| = 1, |z_2| = 2$$

$$(A) \quad ||z_1| - 2||z_2| \leq |z_1 - 2z_2| \leq |z_1| + 2|z_2|$$

$$|1 - 2(2)| \leq |z_1 - 2z_2| \leq 1 + 2(2)$$

$$3 \leq |z_1 - 2z_2| \leq 5$$

$$(B) \quad ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$|1 - 2| \leq |z_1 + z_2| \leq 1 + 2$$

$$1 \leq |z_1 + z_2| \leq 3$$

$$(C) \quad ||z_1| - 3|z_2|| \leq |z_1 - 3z_2| \leq |z_1| + 3|z_2|$$

$$5 \leq |z_1 - 3z_2| \leq 7$$

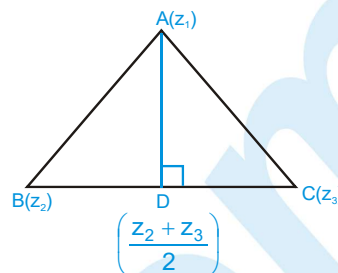
$$(D) \quad ||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$$

$$1 \leq |z_1 - z_2| \leq 3$$

16. (D)

$$\rightarrow \arg \left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2} \right) = \arg \left(\frac{\frac{z_2 + z_3}{2} - z_1}{\frac{z_3 - z_2}{2}} \right)$$

$$= \frac{\pi}{2} \quad (\text{as } AD \perp BC)$$



17. (D)

$$x + \frac{1}{x} = 1$$

$$\Rightarrow x^2 - x + 1 = 0$$

$$\therefore x = -\omega, -\omega^2$$

$$\text{Now for } x = -\omega, P = \omega^{4000} + \frac{1}{\omega^{4000}} = \omega + \frac{1}{\omega} = -1$$

$$\text{Similarly for } x = -\omega^2 \text{ also } p = -1$$

$$\text{for } n > 1, 2^n = 4k$$

$$\therefore 2^{2^n} = 2^{4k} = (16)^k = \text{a number with last digit} = 6$$

$$\Rightarrow q = 6 + 1 = 7$$

$$\text{Hence } p + q = -1 + 7 = 6$$

18. (B)

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$

$$\Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$$

$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3} \Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \arg \left(-\frac{z_2}{z_3} \right)$$

$$\Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \pm \pi + \arg \left(\frac{z_2}{z_3} \right) = \pm \pi + \arg \left(\frac{z_2 - O}{z_3 - O} \right)$$

(+ or - as applicable)

$\Rightarrow$ Points O, A, B, C are concyclic.

19. (D)

If $3 + ix^2y$ and $x^2 + y + 4i$ are conjugate

then $x^2y = -4$ and $x^2 + y = 3$

$$\Rightarrow x^2 = 4, y = -1$$

$$\Rightarrow x^2 + y^2 = 5$$

20. (A)

$$|z^2 + 2z \cos \alpha| \leq |z|^2 + |2z \cos \alpha| < |z|^2 + 2|z| |\cos \alpha| < (\sqrt{2} - 1)^2$$

$$+ 2(\sqrt{2} - 1) = 1$$

$$(\rightarrow |\cos \alpha| \leq 1)$$

21. (A) $\rightarrow s$; (B) $\rightarrow q, r$; (C) $\rightarrow q, t$; (D) $\rightarrow t$

(A) Put $z = x + iy$

$$\therefore \operatorname{Re}(x + iy)^2 = \operatorname{Re}(x + iy + x - iy)$$

$$x^2 - y^2 = 2x$$

$$\text{or } x^2 - y^2 - 2x = 0$$

Rectangular hyperbola, eccentricity = $\sqrt{2}$

(B) For ellipse $\lambda > |z_1 - z_2|$ and for straight line

$$\lambda = |z_1 - z_2|$$

$$(C) \left| \frac{2z - i}{z + i} \right| = m \Rightarrow \left| \frac{z - \frac{i}{2}}{z + i} \right| = \frac{m}{2}$$

$$\text{for } m = 2, \left| \frac{z - \frac{i}{2}}{z + i} \right| = 1 \Rightarrow \left| z - \frac{i}{2} \right| = |z + i|$$

ie, a straight line and for $m \neq 2$, locus is circle

(D) Let $z = x + iy$

$$\Rightarrow x^2 + y^2 = 25^2$$

$$-1 + 75\bar{z} = 75x - 1 + i75y = h + ik$$

$$\Rightarrow \left(\frac{h+1}{75} \right)^2 + \left(\frac{k}{75} \right)^2 = 25^2$$

$\Rightarrow$ Locus of (h, k) is a circle

22. (A) $\rightarrow (r)$, (B) $\rightarrow (q, t)$, (C) $\rightarrow (s)$, (D) $\rightarrow (p)$

The given equation is $\frac{z^5 - 1}{z - 1} = 0$ which means that $z_1, z_2,$

z_3, z_4 are four out of five roots of unity except 1.

$$(A) z_1^4 + z_2^4 + z_3^4 + z_4^4 + 1^4 = 0 \Rightarrow \left| \sum_{i=1}^4 z_i^4 \right| = 1$$

$$(B) z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1^5 = 5 \Rightarrow \sum_{i=1}^4 z_i^5 = 4$$

$$(C) z^4 + z^3 + z^2 + z + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4).$$

Putting $z = -2$ both the sides and we get $\prod_{i=1}^4 (z_i + 2) = 11$

$$(D) |z_1 + z_2| = \sqrt{2 + 2\cos 144^\circ} \text{ for minimum}$$

$$= 2 \cos 72^\circ = \frac{\sqrt{5} - 1}{2} \text{ whose greatest integer is } 0.$$

23.

1. (B)

$$\omega_1 = \omega_2 e^{i\frac{4\pi}{3}}$$

$$\Rightarrow \omega_1^3 = \omega_2^3$$

$$\Rightarrow \omega_1^3 \bar{\omega}_1^2 \bar{\omega}_2^2 = \omega_2^3 \bar{\omega}_1^2 \bar{\omega}_2^2$$

$$\Rightarrow \omega_2 \bar{\omega}_1^2 = \omega_1 \bar{\omega}_2^2$$

2. (B)

Since $i\beta$ is real

$\therefore \beta$ is pure imaginary

3. (C)

$$-\frac{\alpha}{\bar{\alpha}} = e^{\pm i\frac{\pi}{2}} = \pm i$$

$$\therefore (1+i) \left(-\frac{2\alpha}{\bar{\alpha}} \right) = \pm 2(-1+i)$$

25. 1. (B) 2. (C) 3. (D)

25. (Q. 1. to 3.)

$$\text{BM} \equiv y - 0 = -1(x - 1)$$

$$x + y = 1$$

$$\therefore \sqrt{u-1} = t + i(1-t)$$

$$u = 2t + 2it(1-t)$$

$$x = 2t \text{ and } y = 2t(1-t), \text{ where } u = x + iy$$

$$\Rightarrow (x-1)^2 = -2(y - \frac{1}{2}) \text{ which is a parabola}$$

$$\text{axis is } x = 1 \quad \text{i.e. } z + \bar{z} = 2$$

$$\text{directrix is } y = 1 \quad \text{i.e. } z - \bar{z} = 2i$$

26. (4)

$$\left(\frac{1+i}{1-i} \right)^n = \frac{2}{\pi} \times \frac{\pi}{2} = 1$$

$$\text{Now } \left(\frac{1+i}{1-i} \right)^n = 1$$

$$\Rightarrow i^n = 1 \Rightarrow n = 4, 8, 12, 16$$



27. (1)

$$f_p(\alpha) = e^{\frac{i\alpha}{p^2}(1+2+\dots+p)} = e^{\frac{i\alpha}{2}\left(1+\frac{1}{p}\right)}$$

$$\lim_{n \rightarrow \infty} f_n(\pi) = \lim_{n \rightarrow \infty} e^{\frac{i\alpha}{2}\left(1+\frac{1}{n}\right)} = e^{\frac{i\alpha}{2}}$$

$$\left| \lim_{n \rightarrow \infty} f_n(\pi) \right| = \left| e^{\frac{i\alpha}{2}} \right| = 1$$

28. (1)

$$\text{If } |z| = |z-1| \Rightarrow |z|^2 = |z-1|^2$$

$$\Rightarrow z\bar{z} = (z-1)(\bar{z}-1)$$

$$\Rightarrow z + \bar{z} = 1$$

$$\text{again if } |z| = |z+1|$$

$$\Rightarrow |z|^2 = |z+1|^2 \Rightarrow z + \bar{z} = -1 \Rightarrow |z + \bar{z}| = 1$$

29. (30)

$$|z| + |z-1| + |2z-3| = |z| + |z-1| + |3-2z| \geq |z+z-1+3-2z| = 2$$

$$\therefore |z| + |z-1| + |2z-3| \geq 2$$

$$\therefore \lambda = 2$$

$$\text{then } 2[x] + 3 = 3[x - \lambda] \\ = 3[x - 2]$$

$$\Rightarrow 2[x] + 3 = 3([x] - 2)$$

$$\text{or } [x] = 9, \text{ then } y = 2.9 + 3 = 21$$

$$\therefore [x+y] = [x+21] = [x] + 21 = 9 + 21 = 30$$

30. (7) $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$

$$= \sum_{k=0}^{20} A_k x^k$$

$$\rightarrow \sum_{r=0}^6 f(\alpha^r x) = f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + f(\alpha^4 x) \\ + f(\alpha^5 x) + f(\alpha^6 x)$$

$$= \sum_{k=0}^{20} \{A_k x^k + A_k (\alpha x)^k + A_k (\alpha^2 x)^k + A_k (\alpha^3 x)^k \\ + A_k (\alpha^4 x)^k + A_k (\alpha^5 x)^k + A_k (\alpha^6 x)^k\}$$

$$= \sum_{k=0}^{20} A_k x^k (1 + (\alpha)^k + (\alpha^2)^k + (\alpha^3)^k + (\alpha^4)^k + (\alpha^5)^k + (\alpha^6)^k)$$

$$= A_0 x^0 (7) + A_7 x^7 (7) + A_{14} x^{14} (7)$$

$$= 7(A_0 + A_7 x^7 + A_{14} x^{14})$$

$$\therefore n = 7$$

