TANGENT & NORMAL

EXERCISE # 1

Question based on **Equation of Tangent & Normal**

Q.1 If m be the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$, then -

(A) $m < 1$ (B) $|m| \le 1$

(C) $|m| > 1$ (D) None of these

Sol. **[B]**

$e^{2y} = 1 + 4x^2$

$2y = \log_e (1 + 4x^2)$

$y = \frac{1}{2} \log_e (1 + 4x^2)$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{1+4x^2} \times 4 \times 2x = \frac{4x}{1+4x^2}$

$\frac{dy}{dx} = \frac{4x}{1+4x^2} = m$

$1 + 4x^2$ will be positive for $x \in R$

Hence, $\frac{dy}{dx} = m = \frac{4x}{1+4x^2} < 1$, for all $x \in R$

$\Rightarrow |m| \le 1$

$\therefore$ option [B] is correct answer.

Q.2 The curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$, touches the line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b) for n =

(A) 1

(B) 2

(C) 3

(D) all non zero values of n

Sol. **[D]**

$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2,$

Differentiating above function w.r.t. x, we get

$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n} \times dy/dx = 0$

$\frac{ny^{n-1}}{b^n} \times (dy/dx) = -\frac{nx^{n-1}}{a^n}$

$\Rightarrow \left(\frac{dy}{dx}\right) = -\frac{nx^{n-1}}{a^n} \times \frac{b^n}{ny^{n-1}}$

$\left(\frac{dy}{dx}\right) = -\frac{b^n}{a^n} \times \left(\frac{x}{y}\right)^{n-1}$

$\left(\frac{dy}{dx}\right) = -\frac{b^n}{a^n} \times \left(\frac{a}{b}\right)^{n-1}$

$= -\frac{b^n}{a^n} \times \frac{a^n}{b^n} \times \frac{b}{a}$

$(dy/dx)|_{(x,\, b)} = -\frac{b}{a}$

compare slope of given tangent

slope = – b/a

Hence, it is valid for all values of n.

$\therefore$ option [D] is correct answer.

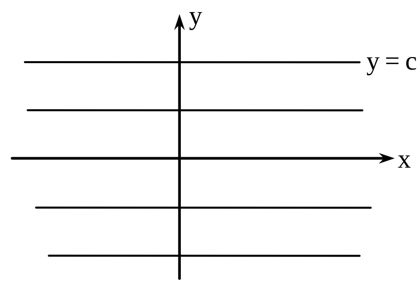
Q.3 All the points on the curve $y = \sqrt{x + \sin x}$ at which the tangents is || to x axis lie on

(A) straight line (B) circle

(C) parabola (D) ellipse

Sol. **[C]**

$y = \sqrt{[x + \sin x]}$



Let point of intersection of tangent y = c and given curve $y = \sqrt{[x + \sin x]}$ is (h, k).

Then k = c and $k = \sqrt{[h + \sin h]}$

$k^2 = h + \sin h$

use, $\sin h = h - h^3/3! + h^5/5! - \ldots\ldots\ldots\ldots\ldots\ldots$

$k^2 = h + (h - h^3/3! + h^5/5! \ldots\ldots\ldots\ldots\ldots\ldots)$

$k^2 \cong 2h \Rightarrow y^2 = 2x$

which is roughly parabola.

Hence option [C] is correct parabola.

Q.4 The normal of the curve given by the equation $x = a(\sin\theta + \cos\theta)$, $y = a(\sin\theta - \cos\theta)$ at the point θ is-

(A) $(x + y)\cos\theta + (x - y)\sin\theta = 0$

(B) $(x + y)\cos\theta + (x - y)\sin\theta = a$

(C) $(x + y)\cos\theta - (x - y)\sin\theta = 0$

(D) $(x + y)\cos\theta - (x - y)\sin\theta = a$

Sol. **[C]**

$x = a(\sin\theta + \cos\theta)$,

$y = a(\sin\theta - \cos\theta)$

Differentiating above functions w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(\cos\theta - \sin\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta + \sin\theta)$$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{dy}{dx} = \frac{a(\cos\theta + \sin\theta)}{a(\cos\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

slope of normal = $-1/(dy/dx)$

$$= -\frac{(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)} = \frac{(\sin\theta - \cos\theta)}{(\sin\theta + \cos\theta)}$$

Equation of normal

$$y - a(\sin\theta - \cos\theta) = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}[x - a(\sin\theta + \cos\theta)]$$

$$y - a(\sin\theta - \cos\theta) = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}[x - a(\sin\theta + \cos\theta)]$$

$y(\sin\theta + \cos\theta) - a(\sin^2\theta - \cos^2\theta) = x(\sin\theta - \cos\theta) - a(\sin^2\theta - \cos^2\theta)$

$\Rightarrow y(\sin\theta + \cos\theta) = x(\sin\theta - \cos\theta)$

$\Rightarrow (y - x)\sin\theta + (y + x)\cos\theta = 0$

$\Rightarrow (x + y)\cos\theta - (x - y)\sin\theta = 0$

$\therefore$ option [C] is correct answer.

Q.5 The normal to the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ at the point $\theta = \pi/4$ passes through the point-

(A) (2, –2) (B) (0, 0)

(C) (–1, 1) (D) None of these

Sol. **[B]**

$x = 3\cos\theta - \cos^3\theta$,

$y = 3\sin\theta - \sin^3\theta$

$$x|_{\theta = \pi/4} = 3|_{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{6-1}{2\sqrt{2}} = \frac{5}{2\sqrt{2}}$$

$$y|_{\theta = \pi/4} = 3|_{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{6-1}{2\sqrt{2}} = \frac{5}{2\sqrt{2}}$$

Differentiating above functions, w.r.t. θ, we get

$$\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta \times \sin\theta$$

$$\frac{dy}{d\theta} = -3\cos\theta + 3\cos^2\theta \times \cos\theta$$

$$\frac{dy}{d\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 3\sin^2\theta \times \cos\theta}{-3\sin\theta + 3\cos^2\theta \times \sin\theta}$$

$$= \frac{\cos\theta - \sin^2\theta \times \cos\theta}{-\sin\theta + \cos^2\theta \times \sin\theta}$$

$$= \frac{\cos\theta(1 - \sin^2\theta)}{-\sin\theta(1 - \cos^2\theta)}$$

$$= -\cot\theta \times \frac{\cos^2\theta}{\sin^2\theta} = -\cot^3\theta$$

$(dy/dx)|_{\theta = \pi/4} = -(\cot\pi/4)^3 = -1$

Slope of normal = 1

Equation of normal

$$y - \frac{5}{2\sqrt{2}} = 1\left(x - \frac{5}{2\sqrt{2}}\right)$$

$$\left(y - \frac{5}{2\sqrt{2}}\right) = \left(x - \frac{5}{2\sqrt{2}}\right)$$

It satisfies point (0, 0).

Hence, option [B] is correct answer.

Q.6 The normals to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the points $\theta = (2n + 1)\pi$, $n \in I$ are all -

(A) parallel to x-axis

(B) parallel to y-axis

(C) parallel to the line y = x

(D) None of these

Sol. **[A]**

$x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

Differentiating above functions, w.r.t. θ, we get

$$\left.\begin{array}{lcl} dx/d\theta & = & a(1+\cos\theta) \\ dy/d\theta & = & a(\sin\theta) \end{array}\right\}$$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ and $\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta}$$

$\theta = (2n + 1)\pi,\ n \in I$

$$(dy/dx)\Big|_{\theta=(2n+1)\pi} = \frac{\sin\theta}{1+\cos\theta} = 0$$

for all $\theta = (2n + 1)\pi$ in $n \in I$

Hence, equation of normal must be parallel to y- axis.

∴ option [B] is correct answer.

Q.7 The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is

(A) 0 (B) 1 (C) 2 (D) 4

Sol. **[B]**

$y' = \frac{4x^3}{2} - 1$

$2x^3 - 1 = -\frac{3}{4}$

$8x^3 = 1$

$x = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$

C has one value.

Q.8 The curve $y - e^{xy} + x = 0$ has a vertical tangent at

(A) (1, 1) (B) (0, 1)

(C) (1, 0) (D) no point

Sol. **[C]**

$y' = -\frac{ye^{xy}+1}{1-xe^{xy}}$

$1 = xe^{xy}$

$0 = \lambda nx + xy$

$y' \to \infty$ at (1, 0)

Question based on

Angle of Intersection of Curves

Q.9 If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$, cut each other at right angles then the value of b is -

(A) 2 (B) 4

(C) 9/2 (D) None of these

Sol. **[C]**

$y^2 = 6x$; $9x^2 + by^2 = 16$

Differentiating above two curves w.r.t.x, we get

$2y(dy/dx) = 6 \Rightarrow (dy/dx)_1 = m_1 = 3/y$

$\frac{d}{dx}(9x^2 + by^2 - 16) = 0$

$18x + 2by \times (dy/dx)_2 = 0$

$\Rightarrow (dy/dx)_2 = m_2 = -\frac{18x}{2by} = \frac{-9x}{by}$

$m_1 \times m_2 = -1$

$\frac{3}{y} \times \left(-\frac{9x}{by}\right) = -1$

$\frac{27x}{by^2} = 1 \Rightarrow 27x = b \times 6x$

$\Rightarrow b = \frac{27}{6} = \frac{9}{2}$

Hence, option [C] is correct answer.

Q.10 If the tangent at P of the curve $y^2 = x^3$ intersect the curve again at Q and the straight lines OP, OQ makes angles α, β with the x-axis where 'O' is the origin then tan α/tan β has the value equal to-

(A) –1 (B) –2

(C) 2 (D) $\sqrt{2}$

Sol. **[B]**

$P(t_1^2, t_1^3)$

$Q(t_2^2, t_2^3)$

$$\frac{\tan\alpha}{\tan\beta} = \frac{t_1}{t_2}$$

tangent at P

$$y - t_1^3 = \frac{3t_1^4}{2t_1^3}(x - t_1^2)$$

$$\Rightarrow y - t_1^3 = \frac{3}{2}t_1(x - t_1^2)$$

It passes through Q

$$\Rightarrow t_2^3 - t_1^3 = \frac{3}{2}t_1(t_2^2 - t_1^2)$$

$$2t_2^2 + 2t_1^2 + 2t_1t_2 = 3t_1t_2 + 3t_1^2$$

$$t_1^2 + t_1t - 2t_2^2 = 0$$

$$(t_1 - t_2)(t_1 + 2t_2) = 0$$

$$\frac{t_1}{t_2} = -2$$

Q.11 The abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts off equal intercepts from the axes is -

(A) 1 (B) $\frac{4a}{3}$ (C) 3 (D) $\frac{4a}{9}$

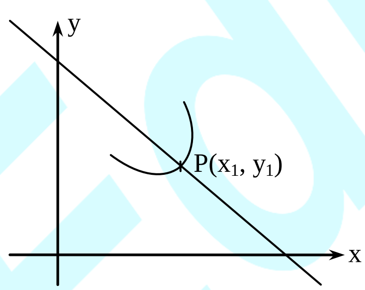
Sol. **[D]**

$ay^2 = x^3$

Differentiating above function w.r.t.x, we get

$$2ay \times \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$



$$(dy/dx)\big|_{P(x_1,y_1)} = 3x_1^2/2ay_1$$

$$\text{slope of normal} = -\frac{1}{(dy/dx)} = -2ay_1/3x_1^2$$

equation of normal is

$$y - y_1 = -\frac{2ay_1}{3x_1^2}(x - x_1)$$

$$y - y_1 = -\frac{2ay_1}{3x_1^2} \times x + \frac{2ay_1}{3x_1}$$

$$\frac{2ay_1}{3x_1^2}x + y = y_1 + \frac{2ay_1}{3x_1}$$

$$\frac{2ay_1.x + 3x_1^2.y}{3x_1^2} = \frac{y_1.3x_1 + 2ay_1}{3x_1}$$

$$2ay_1.x + 3x_1^2.y = y_1(3x_1^2 + 2ax_1)$$

$$\frac{x}{\frac{(3x_1^2 + 2ax_1)y_1}{2ay_1}} + \frac{y}{\frac{(3x_1^2 + 2ax_1).y_1}{2x_1^2}} = 1$$

Given condition, i.e. Normals equally intercepts

$$\frac{(3x_1^2 + 2ax_1).y_1}{2ay_1} = \frac{(3x_1^2 + 2ax_1).y_1}{3x_1^2}$$

$$\Rightarrow 2ay_1 = 3x_1^2 \Rightarrow y_1 = \frac{3x_1^2}{2a}$$

Also, point P (x_1, y_1) lie on the equation of the curve $ay_1^2 = x_1^3$

$$a\left(\frac{3x_1^2}{2a}\right)^2 = x_1^3$$

$$\Rightarrow \frac{9\times 9}{4a^2} \times x_1^4 = x_1^3$$

$$\Rightarrow \frac{9}{4a} \times x_1 = 1$$

$$\Rightarrow x_1 = 4a/9$$

Q.12 Tangent and normal to the curve

$y = 2\sin x + \sin 2x$ are drawn at p $\left(x = \frac{\pi}{3}\right)$.

The area of the quadrilateral formed by the tangent the normal and coordinate axes is

(A) $\frac{\pi\sqrt{3}}{2}$ sq. units (B) $\frac{\pi}{2}$ sq. units

(C) $\frac{\sqrt{3}}{2}$ sq. units (D) None of these

Sol. **[A]**

$$y = 2\sin x + \sin 2x \text{ at } p = \left(x = \frac{\pi}{3}\right)$$

$$y = 2 \times \sin \pi/3 + \sin 2 \times \pi/3$$

$y = 2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

Point p $\left[\frac{\pi}{3}, \frac{3\sqrt{3}}{2}\right]$

slope of tangent

$(dy/dx)|_p = (2\cos x + 2\cos 2x)|_p$

$= 2\cos \pi/3 + 2\cos 2\pi/3$

$= 2[\cos \pi/3 + \cos 2\pi/3]$

$= 2\left[\frac{1}{2} - \frac{1}{2}\right]$

$= 0$

Hence, equation of tangent

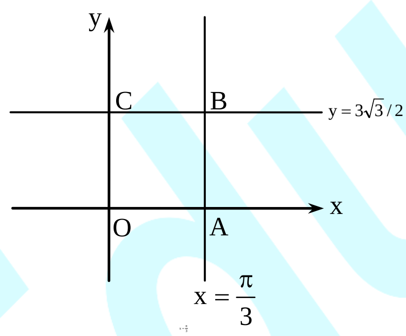
$y - 3\sqrt{3}/2 = 0\ (x - \pi/3)$

$\Rightarrow y = 3\sqrt{3}/2$

Equation of normal would be

$y - 3\sqrt{3}/2 = \frac{1}{0}(x - \pi/3)$

$\Rightarrow x = \pi/3$



Hence, area of quadrilateral OABC =

$\frac{\pi}{3} \times \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}\pi}{2}$ square unit.

Question based on

Length of Tangent, Normal, Subtangent & Subnormal

Q.13 If the relationship between the subnormal SN and sub-tangents ST at any point of the curve $by^2 = (x + a)^3$ is of the form

$p\,(SN) = q\,(ST)^2$ then $\frac{p}{q} =$

(A) $\frac{8b}{27}$ (B) $\frac{b}{8}$

(C) $\frac{b}{27}$ (D) None of these

Sol. **[A]**

$by^2 = (x + a)^3 \Rightarrow y = (x + a)^{3/2} / \sqrt{b}$

$\frac{dy}{dx} = y' = \frac{3}{2\sqrt{b}} \times (x + a)^{1/2}$

Sub-tangent = ST

$= |y/y'|$

$= \left|\frac{(x+a)^{3/2}}{\sqrt{b}} \times \frac{2\sqrt{b}}{3(x+a)^{1/2}}\right| = \left|\frac{2}{3} \times (x+a)\right|$

subnormal = SN

$= |yy'| = \left|\frac{(x+a)^{3/2}}{\sqrt{b}} \times \frac{3}{2\sqrt{b}} \times (x+a)^{1/2}\right|$

$P(SN) = q(ST)^2$

$p \times \left|(x+a)^2 \times \frac{3}{2b}\right| = q \times \left|\frac{2}{3}(x+a)\right|^2$

$p \times \left|\frac{3}{2b} \times (x+a)^2\right| = q \times \left|\frac{4}{9} \times (x+a)^2\right|$

$\frac{p}{q} = \frac{4}{9} \times \frac{(x+a)^2}{(x+a)^2} \times \frac{2b}{3} = \frac{8b}{27}$

$\frac{p}{q} = \frac{8b}{27}$

Hence, option [A] is correct answer.

Q.14 For the curve $y = be^{x/a}$ -

(A) sub-tangent is constant

(B) sub-normal is constant

(C) Length of tangent is constant

(D) Length of normal is constant

Sol. **[A]**

$y = be^{x/a}$

Differentiate w.r.t.x, we get

$\frac{dy}{dx} = be^{x/a}.\ \frac{1}{a} = \frac{y}{a}$

$y' = \frac{y}{a}$

Subtangent $= \left|\frac{y}{y'}\right| = |a| = a =$ constant

Subnormal $= |y.y'| = \left|y.\frac{y}{a}\right| = |y^2/a|$

= not constant

Length of tangent = $\left|\frac{y}{y'}\times\sqrt{1+y'^2}\right|$

= $\left|a\times\sqrt{1+y^2/a}\right|$

= not constant

Length of normal = $\left|y\times\sqrt{1+y'^2}\right|$

= $\left|y\times\sqrt{1+y^2/a^2}\right|$

= Not constant

∴ option [A] is correct answer.

Q.15 The length of the tangent to the curve x = a (θ + sin θ), y = a (1 – cos θ) at θ points is

(A) $2a\sin\frac{\theta}{2}$ (B) a sin θ

(C) 2a sin θ (D) a cosθ

Sol. **[A]**

$y' = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$

length of tangent

$\Rightarrow \frac{y}{y'}\sqrt{1+(y')^2}$

$\Rightarrow \frac{a(1-\cos\theta)}{\tan\frac{\theta}{2}}\times\sec\frac{\theta}{2}$

$\Rightarrow a\times\frac{2\sin^2\theta/2}{\sin\theta/2}\times\frac{\cos\theta/2}{\cos\theta/2}$

$\Rightarrow 2a\sin\frac{\theta}{2}$

Question based on

Application of derivative as Rate Measure

Q.16 A particle is moving on a line, where its position **S** in meter is a function of time t in seconds given by **S** = $t^3 + at^2 + bt + c$ where a, b, c are constant. It is known that at t = 1 seconds, the position of the particle is given by **S** = 7m. Velocity is 7 m/s and acceleration is $12m/s^2$. The values of a, b, c are

(A) –3, 2, 7 (B) 3, –2, 5

(C) 3, 2, 1 (D) None of these

Sol. **[B]**

$S = t^3 + at^2 + bt + c$

At t = 1 second, position = 7m

velocity = 7m/sec.

acceleration = $12m/sec^2$

At t = 1 sec and S = 7m

$1 + a + b + c = 7 \Rightarrow a + b + c = 6$ …(1)

Differentiate equation, $S = t^3 + at^2 + bt + c$

w.r.t.t, we get

$\frac{ds}{dt}$ = velocity, $3t^2 + 2at + b$

At t = 1 sec, V = 7m/sec.

$3 + 2a + b = 7 \Rightarrow 2a + b = 4$ …(2)

Differentiating equation, $\frac{ds}{dt} = 3t^2 + 2at + b$

w.r.t. t, we get

$\frac{d^2s}{dt^2}$ = acceleration = 6t + 2a

At t = 1 sec, Acceleration = $12m/sec^2$

$6. (1) + 2a = 12 \Rightarrow 2a = 6$

$\Rightarrow a = 3$

b = 4 – 2a = 4 – 6 = –2

from a + b + c = 6

3 – 2 + c = 6

$\Rightarrow c = 5$

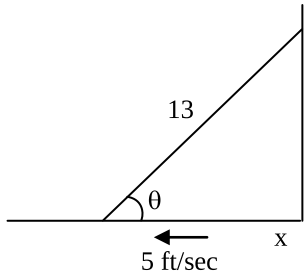
∴ option [B] is correct answer.

Q.17 A 13 ft. ladder is leaning against a wall when its base starts to slide away. At the instant when the base is 12ft. away from the wall, the base is moving away from the wall at the rate of 5ft./sec. The rate of which the angle θ between the ladder and the ground is changing is

(A) $-\frac{12}{13}$ rad/sec. (B) –1 rad/sec.

(C) $-\frac{13}{12}$ rad/sec. (D) $-\frac{10}{13}$ rad/sec.

Sol. **[B]**



$\cos\theta = \frac{x}{13}$

$-\sin\theta \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \frac{-1}{13\sin\theta} \times \frac{dx}{dt}$

$= \frac{-1\times 13}{13\times 5} \times 5$

$= -1$

Q.18 Water is poured into an inverted conical vessel of which the radius of the base is 2m and height 4m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is (use $\pi = 22/7$)

(A) 10 cm/min (B) 20 cm/min

(C) 40 cm/min (D) None of these

Sol. **[B]**

$\frac{h}{r} = \frac{H}{R}$

$v = \frac{1}{3}\pi\left(\frac{h}{H} - R\right)^2 h$

$v = \frac{\pi}{3}\frac{R^2}{H^2}h^3$

$\frac{dv}{dt} = \frac{\pi R^2}{H^2}h^2\frac{dh}{dt}$

$77000 = \frac{22}{7}\ \frac{1}{H}\ (70)^2\ \frac{dh}{dt}$

$\frac{dh}{dt}$ = 20 cm/mm

Q.19 A point is moving along the curve $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate, is

(A) (–3, –3) (B) $(-\infty, \infty)$

(C) (–1, 1) (D) $(-\infty, -3) \cup (3, \infty)$

Sol. **[C]**

$y^3 = 27x$

$3y^2 . y' = 27$

$y' = \frac{9}{y^2}$

$\frac{9}{y^2} > 1$

$\Rightarrow y^2 < 9$

$\Rightarrow -3 < y < 3$

$\Rightarrow -27 < y^3 < 27$

$\Rightarrow -1 < x < 1$

Question based on **Rolle's Theorem and Lagrange's Mean Value Theorem**

Q.20 Let f(x) = (x – 4) (x – 5) (x – 6) (x – 7) then-

(A) f′ (x) = 0 has four roots

(B) Three roots of f′(x) = 0 lie in $(4, 5) \cup (5, 6) \cup (6, 7)$

(C) The equation f′ (x) = 0 has one real root

(D) Three roots of f′(x) = 0 lie in $(3, 4) \cup (4, 5) \cup (5, 6)$

Sol. **[B]**

f(x) = (x – 4) (x –5) (x –6) (x –7)

$= x^4 - 22x^3 + 179x^2 - 638x + 84$

Differentiating f(x) w.r.t. x, we get

$f'(x) = 4x^3 - 66x^3 + 179 \times 2x - 638$

To make $f'(x) = 0$

$\Rightarrow 4x^3 - 66x^2 + 179 \times 2x - 638 = 0$

$\Rightarrow 2x^3 - 33x^2 + 179x - 319 = 0$

Put x = 11/2

$2x(11/2)^3 - 33(11/2)^2 + 179 \times 11/2 - 319$

$= \frac{11\times121 - 33\times121 + 179\times22 - 319\times4}{4}$

$= \frac{1331 - 3993 + 3938 - 1276}{4} = 0$

Hence (x– 11/2) is the root of above equation.

$2x^2 (x - 11/2) - 22x (x - 11/2) + 58 (x - 11/2) = 0$

$(x - 11/2)(2x^2 - 22x + 58) = 0$

$(x - 11/2)(x^2 - 11x + 29) = 0$

$x = \frac{11 \pm \sqrt{121 - 116}}{2\times1}$ & $x = \frac{11}{2}$

$x = \frac{11\pm\sqrt{5}}{2}$ and $x = \frac{11}{2}$

$x = \frac{11+\sqrt{5}}{2} = \frac{11+2.2}{2} \cong 6.5$

and $x = \frac{11-\sqrt{5}}{2} = \frac{11-2.2}{2} \cong 4.5$ & $x = 11/2$

Hence option [B] is correct answer.

Q.21 If $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+.....+\frac{a_{n-1}}{2}+a_n=0$, then the function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ has in (0, 1) is

(A) at least one zero (B) at most one zero

(C) only 3 zeros (D) only 2 zeros

Sol. **[A]**

$\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+...+\frac{a_{n-1}}{2}+a_n = 0$

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

Integrating w.r.t.x, we get

$$\int_0^1 f(x)\,dx = \int_0^1 [a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_n]\,dx$$

$$= \left[\frac{a_0x^{n+1}}{n+1}+\frac{a_1.x^n}{n}+\frac{a_2x^{n-1}}{n-1}+......+a_nx\right]_0^1$$

$$= \frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+......+a_n = 0$$

(Since, we have given $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+$ ....$+\frac{a_{n-1}}{2}+a_n = 0$)

$\int_0^1 f(x)\,dx = 0 \Rightarrow f(x)$ must be zero.

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+a_n = 0$

Hence, in (0, 1) if we put x = 0

$f(0) = 0 + 0 + 0 + ...+ 0 + a_n = 0$

Hence, $a_n = 0$

∴ at least one constant must be zero.

∴ option [A] is correct answer.

Q.22 Rolle's theorem in the indicated intervals will not be valid for which of the following function

(A) $f(x)=\begin{cases}\frac{\sin x}{x}; & x \neq 0\\ 1; & x=0\end{cases}$; $x\in[-1, 1]$

(B) $g(x)=\begin{cases}\frac{1-\cos x}{x}; & x \neq 0\\ 0; & x=0\end{cases}$; $x\in[-2\pi, 2\pi]$

(C) $h(x)=\begin{cases}\frac{1-\cos x}{x^2}; & x \neq 0\\ \frac{1}{2}; & x=0\end{cases}$; $x\in[-2\pi, 2\pi]$

(D) $k(x)=\begin{cases}x\sin\left(\frac{1}{x}\right); & x \neq 0\\ 0; & x=0\end{cases}$; $x\in\left[-\frac{1}{\pi},\frac{1}{2\pi}\right]$

Sol. **[D]**

Q.23 Consider the function for x = [–2, 3]

$f(x)=\begin{cases}\frac{x^3-2x^2-5x+6}{x-1} & ; x\neq 1\\ -6 & ; x=1\end{cases}$ then

(A) f is discontinuous at x = 1 ⇒ Rolle's theorem is not applicable n [–2, 3]

(B) f(–2) ≠ f(3) ⇒ Rolle's theorem is not applicable in [–2, 3]

(C) f is not derivable in (–2, 3)

⇒ Rolle's theorem is not applicable

(D) Rolle's theorem is applicable as f satisfies all the conditions and c of Rolle's theorem is 1/2

Sol. **[D]**

$f(x)=\begin{cases}\frac{(x-1)(x^2-x-6)}{x-1} & ; x\neq 1\\ -6 & ; x=1\end{cases}$

$2x - 1 = 0$

$\Rightarrow c = 1/2$

Q.24 If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at x = 2 on the closed interval [1, a] then the value of 'a' is equal to-

(A) 3 (B) 4 (C) 6 (D) 1

Sol. **[A]**

$f'(2) = \frac{f(a)-f(1)}{a-1}$

$\Rightarrow 11 = \frac{2a^2+3a+5-10}{a-1}$

$\Rightarrow 11a - 11 = 2a^2 + 3a - 5$

$\Rightarrow 2a^2 - 8a + 6 = 0$

$\Rightarrow a^2 - 4a + 3 = 0$

$\Rightarrow (a-1)(a-3) = 0$

$a = 1, 3$

$a = 3$ (Θ $a \neq 1$)

Q.25 Consider $f(x) = |1-x|$; $1 \le x \le 2$ and

$g(x) = f(x) + b\ \sin\dfrac{\pi}{2}x$, $1 \le x \le 2$ then which of the following is correct?

(A) Rolle's theorem is applicable to both f, g and $b = \dfrac{3}{2}$

(B) LMVT is not applicable to f and Rolle's theorem if applicable to g with $b = \dfrac{1}{2}$

(C) LMVT is applicable to f and Rolle's theorem is applicable to g with b = 1

(D) Rolle's theorem is not applicable to both f, g for any real b

Sol. **[C]**

$f(x) = x - 1$; $1 \le x \le 2$

$g(x) = x - 1 + b\sin\dfrac{\pi}{2}x$; $1 \le x \le 2$

LMVT applicable to f Rolle's applicable to g if b = 1

➤ True or false type questions

Q.26 Any tangent to $y = x^5 + 7x + 5$ makes an acute angle with x-axis.

Sol. **[True]**

$y = x^5 + 7x + 5$

Differentiate w.r.t.x, we get

$\dfrac{dy}{dx} = 5x^4 + 7 \ge 7$ for all $x \in R$

Hence, $\dfrac{dy}{dx} = m = \tan\theta$, θ will be acute with x- axis because θ either in first quadrant or third quadrant.

$\therefore$ option is true.

Q.27 If the tangent to the curve $2y^3 - ax^2 - x^3 = 0$ at the point (a, a) cuts off intercepts α and β on axes, where $\alpha^2 + \beta^2 = 162$, then value of a is 30.

Sol. **[False]**

$2y^3 - ax^2 - x^3 = 0$

Differentiating above curve w.r.t. x, we get

$2 \times 3y^2\dfrac{dy}{dx} - 2ax - 3x^2 = 0$

$\dfrac{dy}{dx} = \dfrac{2ax + 3x^2}{6y^2}$

$\left.\dfrac{dy}{dx}\right|_{(a,a)} = \dfrac{2a^2 + 3a^2}{6a^2} = \dfrac{5a^2}{6a^2} = \dfrac{5}{6}$

equation of tangent at (a, a)

$y - a = \dfrac{5}{6}(x - a)$

$6y - 6a = 5x - 5a$

$-5x + 6y = a \Rightarrow \dfrac{x}{(-a/5)} + \dfrac{y}{(a/6)} = 1$

Here, $\alpha = -a/5$; $\beta = a/6$

$\alpha^2 + \beta^2 = 162$

$\dfrac{a^2}{25} + \dfrac{a^2}{36} = 162$

$\dfrac{61a^2}{25 \times 36} = 162 \Rightarrow a^2 = \dfrac{162 \times 25 \times 36}{6}$

$a \neq 30$

$\therefore$ option is false.

➤ Fill in the blanks type questions

Q.28 The equation of tangent to the curve $y = \dfrac{x(x^2-1)}{x^2 - x - 12}$ at x = 0 is................

Sol. $y = \dfrac{x(x^2-1)}{x^2 - x - 12} = \dfrac{(x^3 - x)}{(x^2 - x - 12)}$

Differentiating above function w.r.t.x, we get

$\dfrac{dy}{dx} = \dfrac{(3x^2-1)(x^2-x-12) - (x^3-x)(2x-1)}{(x^2-x-12)^2}$

$\left.\left(\dfrac{dy}{dx}\right)\right|_{a+x=0} = \dfrac{(-1)(-12) - 0}{(-12)^2} = 1/12$

Also, $y = \dfrac{x^3 - x}{x^2 - x - 12}$ at x = 0 = 0

Equation of tangent at (0, 0) is $y - 0 = \frac{1}{12}(x - 0)$

$\Rightarrow x = 12y$

Q.29 If the normal to the curve y = f(x) at x = 0 be given by the equation 3x – y + 3 = 0, then the value of

$\lim_{x \to 0} x^2[f(x^2) - 5f(4x^2) + 4f(7x^2)]^{-1}$ is............

Sol. y = f(x)

3x – y + 3 = 0

$\Rightarrow y = 3x + 3 = f(x)$

$\lim_{x \to 0} x^2 [f(x^2) - 5f(4x^2) + 4f(7x^2)]^{-1}$

$= \lim_{x \to 0} x^2[(3x^2 + 3) - 5(12x^2 + 3) + 4(21x^2 + 3)]^{-1}$

$= \lim_{x \to 0} x^2[3x^2 + 3 - 60x^2 - 15 + 84x^2 + 12]^{-1}$

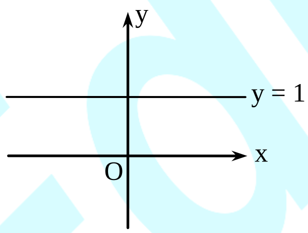
$= \lim_{x \to 0} x^2[27x^2]^{-1} = \frac{1}{27}$

Q.30 The value of a for which the area of the triangle included between the axes and any tangent to the curve $x^a y = k^a$ is constant is.............

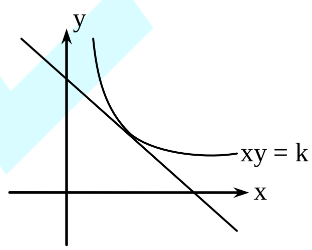
Sol. $x^a y = k^a$

put a = 0 $\Rightarrow$ y = 1

Area of triangle not possible.



Put a = 1 $\Rightarrow$ xy = k



Here, area of triangle is possible

Hence, a must be 1.

EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 The tangent at $(t, t^2 - t^3)$ on the curve $y = x^2 - x^3$ meets the curve again at Q, then abscissa of Q must be -

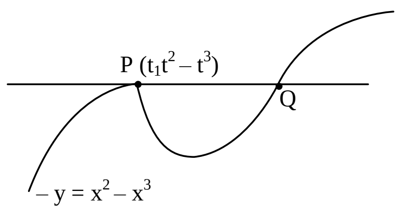
(A) $1 + 2t$ (B) $1 - 2t$

(C) $-1 -2t$ (D) $2t - 1$

Sol. **[B]**

$y = x^2 - x^3$

Let graph of this curve as follows:



slope of tangent, $\left.\frac{dy}{dx}\right|_P$

$= 2x - 3x^2$

$= 2t - 3t^2$

Equation of tangent is

$y - (t^2 - t^3) = (2t - 3t^2)(x - t)$

$y - (t^2 - t^3) = (2t - 3t^2)x - (2t - 3t^2)t$

$y = (2t - 3t^2)x - 2t^2 + 3t^3 + t^2 - t^3$

$y = (2t - 3t^2)x + 2t^3 - t^2$ .....(1)

Solving equation (1) and $y = x^2 - x^3$

$(2t - 3t^2)x + 2t^3 - t^2 = x^2 - x^3$

$\Rightarrow x^3 - x^2 + (2t - 3t^2)x + (2t^3 - t^2) = 0$

Put x = t,

$t^3 - t^2 + (2t - 3t^2)t + (2t^3 - t^2) = 0$

$t^3 - t^2 + 2t^2 - 3t^3 + 2t^3 - t^2 = 0$

$3t^3 - 3t^3 + 2t^2 - 2t^2 = 0$

Hence, x = t is the factor of above equation

$x^2(x - t) + (x - x(x - t) - (2t^2 - t)(x - t) = 0$

$[x^2 + x(t - 1) - (2t^2 - t)](x - t) = 0$

$$x = \frac{(1-t) \pm \sqrt{(1-t)^2 + 4(2t^2 - t)}}{2 \times 1}$$

$$x = \frac{1 - t \pm \sqrt{1 + t^2 - 2t + 8t^2 - 4t}}{2 \times 1}$$

$$x = \frac{1 - t \pm \sqrt{9t^2 = 6t + 1}}{2 \times 1}$$

$$x = \frac{(1-t) \pm (1-3t)}{2}$$

$x = \frac{1 - t + 1 - 3t}{2}$ and $x = \frac{1 - t - 1 + 3t}{2}$

$x = \frac{2 - 4t}{2}$ and $x = \frac{2t}{2}$

$\Rightarrow x = (1 - 2t)$ & $x = t \Rightarrow x = 1 - 2t$

Q.2 If the tangent at 't' on the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$ meets the curve again at t′ and is normal to the curve at that point, then value of t must be -

(A) $\pm \frac{1}{\sqrt{3}}$ (B) $\pm \frac{1}{\sqrt{2}}$

(C) $\pm \frac{\sqrt{2}}{3}$ (D) None of these

Sol. **[C]**

$y = 8t^3 - 1$ and $x = 4t^2 + 3$

$y + 1 = 8t^3$ and $(x - 3) = 4t^2$

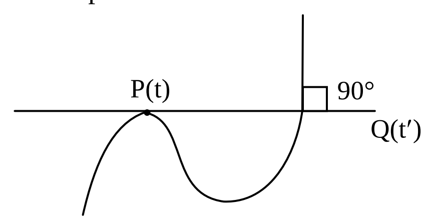
$(y + 1)^2 = 64t^6$ and $(x - 3) = 4t^2$

$(y + 1)^2 = (4t^2)^3 = (x - 3)^3$

$\Rightarrow (y + 1)^2 = (x - 3)^3$

$\Rightarrow y = (x - 3)^{3/2} - 1$ ...(1)

Let shape of curve is



Slope at p(t)

$\frac{dy}{dt} = 24t^2$ and $\frac{dx}{dt} = 8t$

$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = 24t^2/8t = 3t$

equation of tangent at p(t)

$y - (8t^3 - 1) = 3t(x - (4t^2 + 3))$ ...(2)

point Q $[t', (t' - 3)^{3/2} - 1]$

Point Q satisfies equation (2), we get

$(t' - 3)^{3/2} - 1 - 8t^3 + 1 = 3t(t' - 4t^2 - 3)$

$(t' - 3)^{3/2} - 8t^3 = 3tt' - 12t^3 - 9t$

$(t' - 3)^{3/2} - 8t^3 = 3tt' - 12t^3 - 9t$

$(t'-3)^{3/2} = 3tt' - 9t - 4t^3$

$(t'-3)^{3/2} = 3t(t'-3) - 4t^3$ ….(3)

from (1),

$\frac{dy}{dx} = \frac{3}{2}.(x-3)^{1/2} - 0$

$\frac{dy}{dx} = \frac{3}{2}(x-3)^{1/2}$

$\frac{dy}{dx} = \frac{3}{2}(t'-3)^{1/2}$ = slope of tangent at Q(t′)

Also, $3t \times \frac{3}{2}(t'-3)^{1/2} = -1$

$(t'-3)^{1/2} = -2/9 \times \frac{1}{t}$

$(t'-3)^{3/2} = (-2/9)^3 = -8/9 \times 81 \times \frac{1}{t^3}$

$(t'-3) = 4/81 \times \frac{1}{t^2}$

from (3), we get

$\frac{-8}{9\times 81} \times \frac{1}{t^3} = 3t \times \frac{4}{81} \times \frac{1}{t^2} - 4t^3$

$= \frac{-8}{9\times 81} \times \frac{1}{t^3} = \frac{12 - 4\times 81 \times t^4}{81 \times t}$

$= \frac{4\times 3(1-27t^4)}{81\times t}$

$= \frac{-8}{9\times 81} \times \frac{1}{t^3} = \frac{4\times 3(1-27t^4)}{81\times t}$

$-\frac{2}{27} \times \frac{1}{t^2} = 1 - 27t^4$

$\Rightarrow 27t^4 - \frac{2}{27} \times \frac{1}{t^2} - 1 = 0$

If we put, $t = \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{2}}{3}$

as we given in options only $t = \pm\frac{\sqrt{2}}{3}$ holds good.

Hence, option [C] is correct answer.

Q.3 The point of intersection of the tangents drawn to the curve $x^2y = 1-y$ at the points where it is meet by the curve $xy = 1-y$, is given by

(A) (0, –1) (B) (1, 1)

(C) (0, 1) (D) None of these

Sol. **[C]**

$x^2y = 1-y$ …(1)

$xy = 1-y$ … (2)

Solving (1) and (2), we get

$x(xy) = 1-y$

$\Rightarrow x(1-y) = (1-y)$

$\Rightarrow (1-y)(x-1) = 0$

$\Rightarrow x = 1, y = 1$

It must satisfy equation of the curve

$xy = 1-y$

$1.\ 1 = 1 - \phi \neq 0$

If take x = 1, y = 0

then $1.0 \neq 1 - 0$

If take x = 0 and y = 1

$0.1 = 1 - 1 = 0$

Hence, option [C] is correct answer.

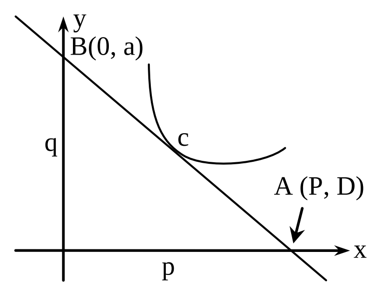
Q.4 The tangent at any point of the curve $x^3 + y^3 = 2$ cuts off length p, q on the coordinate axes, the value of $p^{-3/2} + q^{-3/2} =$

(A) $2^{-3/2}$ (B) $\frac{1}{\sqrt{2}}$

(C) $\sqrt{2}$ (D) None of these

Sol. **[B]**

$x^3 + y^3 = 2$



Differentiating w.r.t. x, we get

$3x^2 + 3y^2.\ dy/dx = 0$

$dy/dx = -\frac{x^2}{y^2} = -\left(\frac{x}{y}\right)^2$

Also, slope of tangent $= \frac{q-0}{0-p} = -\frac{q}{p}$

Hence, $\frac{x^2}{y^2} = \frac{q}{p} \Rightarrow x = y \times \sqrt{q/p}$

Also equation of tangent is

$\frac{x}{p} + y/q = 1 \Rightarrow qx + py = pq$

$q \times y\sqrt{q/p} + py = pq$

$y\left(\frac{q^{3/2}}{\sqrt{p}} + p\right) = pq$

$y(q^{3/2} + p^{3/2}) = qp^{3/2}$

$y = q.\,p^{3/2}/(p^{3/2} + q^{3/2})$

$x = \frac{q.p^{3/2}}{(p^{3/2} + q^{3/2})} \times \sqrt{q/p}$

$x = pq^{3/2}/(p^{3/2} + q^{3/2})$

Put values of x, y in equation of curve

$x^3 + y^3 = 2$

$\frac{p^3.q^{9/2}}{(p^{3/2} + q^{3/2})^3} + \frac{q^3.p^{9/2}}{(p^{3/2} + q^{3/2})^3} = 2$

$\frac{p^3.q^3[p^{3/2} + q^{3/2}]}{(p^{3/2} + q^{3/2})^3} = 2$

$= \frac{p^3q^3[p^{3/2} + q^{3/2}]}{(p^{3/2} + q^{3/2})^3} = 2$

$\frac{p^3q^3}{(p^{3/2} + q^{3/2})^2} = 2$

$p^3q^3/(p^{3/2} + q^{3/2})^2 = 2$

$\Rightarrow \frac{(p^{3/2} + q^{3/2})^2}{(p^{3/2}q^{3/2})^2} = \frac{1}{2}$

$\Rightarrow \frac{p^{3/2} + q^{3/2}}{p^{3/2}q^{3/2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1}{q^{3/2}} + \frac{1}{p^{3/2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1}{p^{3/2}} + \frac{1}{q^{3/2}} = \frac{1}{\sqrt{2}}$

$= p^{-3/2} + q^{-3/2}$

Hence, option [B] is correct answer.

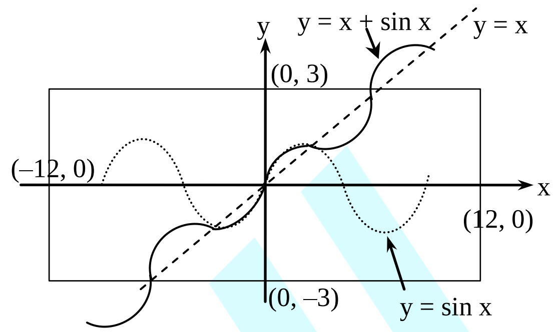
Q.5 The number of points in the rectangle $\{(x, y) | -12 \le x \le 12, -3 \le y \le 3\}$ which lie on the curve $y = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis is-

(A) 0 (B) 2

(C) 4 (D) 8

Sol. **[A]**

$y = x + \sin x$



Points on the curve at which tangents are parallel to x- axis i.e. slope of tangents at that points must be zero.

Differentiating above curve w.r.t. x, we get

$\frac{dy}{dx} = 1 + \cos x \quad \Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 1 + \cos x = 0$

$\cos x = -1 = \cos \pi$

$\Rightarrow x = 2n\pi + \pi = (2n + 1)\pi;\ n \in I$

put $n = 0$; $x = \pi$

$n = -1$; $x = -\pi$

When $x = \pi \Rightarrow y = \pi + \sin \pi = \pi$

(outside rectangle)

When $x = -\pi \Rightarrow y = -\pi + \sin(-\pi) = -\pi$

(outside rectangle)

Hence, there is no points on the curve in the rectangle at which tangents are parallel to x- axis.

$\therefore$ option [A] is correct answer.

Q.6 A curve with equation of the form $y = ax^4 + bx^3 + cx + d$ has zero gradient at the point (0, 1) and also touches the x-axis at the point (–1, 0) then the value of x for which the curve has a negative gradient are

(A) $x > -1$ (B) $x < 1$

(C) $x < -1$ (D) $-1 \le x \le 1$

Sol. **[C]**

$y' = 4ax^3 + 3bx^2 + c$

$c = 0$

$3b - 4a = 0$

$y' < 0$

$4ax^3 + 3bx^2 + c < 0$

$c = 0$

$x^2(4ax + 3b) < 0$

$x^2(x + 1) < 0$

$\Rightarrow x + 1 < 0$

$\Rightarrow x < -1$

Q.7 The tangent to the curve $3xy^2 - 2x^2y = 1$ at (1, 1) meets the curve again at the point.

(A) $\left(\frac{16}{5}, \frac{1}{20}\right)$ (B) $\left(-\frac{16}{5}, -\frac{1}{20}\right)$

(C) $\left(\frac{1}{20}, \frac{16}{5}\right)$ (D) $\left(-\frac{1}{20}, \frac{16}{1}\right)$

Sol. **[B]**

$y' = -\frac{3y^2 - 4xy}{6xy - 2x^2}$

$(y')_{(1,\,1)} = \frac{1}{4}$

$y - 1 = \frac{1}{4}(x - 1)$

check option

$\left(-\frac{16}{5}, -\frac{1}{20}\right)$

Q.8 The point (s) at each of which the tangents to the curve $y = x^3 - 3x^2 - 7x + 6$ cut off on the positive semi axis OX a line segment half the negative semi axis OY then the coordinates the point (s) is/are given by

(A) (–1, 9) (B) (3, –15)

(C) (1, –3) (D) None of these

Sol. **[B]**

Slope = 2

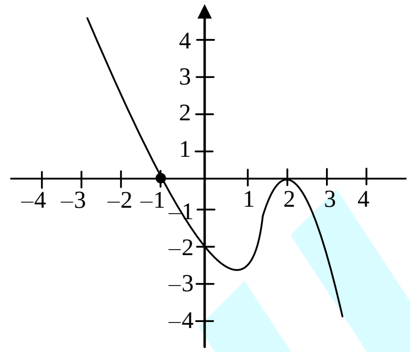
$y' = 3x^2 - 6x - 7 = 2$

$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$

$x = 3, -1 \Rightarrow (-1, 9)$ not possible

$y = -15, 9$

Q.9 A cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has a graph which touches the x-axis at 2, has another x-intercept at –1 and has y-intercept at –2 as shown. The value of, a + b + c + d equals



(A) –2 (B) –1 (C) 0 (D) 1

Sol. **[B]**

$x = 0,\ f(x) = -2,\ d = -2$

$x = -1\ ;\ f(x) = 0$

$-a + b - c - 2 = 0$

$-a + b - c = 2$

$x = 2;\ f(x) = 0$

$8a + 4b + 2c = 2$

$4a + 2b + c = 1$

$3ax^2 + 2bx + c = 0$

at x = 2

$12a + 4b + c = 0$

$2a + 2b = 2$

$8a + 2b = -1$

$a = -1/2,\ b = 3/2$

$\Rightarrow c = 0$

$a + b + c + d$

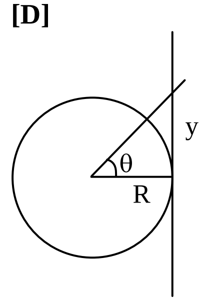
$\Rightarrow -\frac{1}{2} + \frac{3}{2} + 0 + (-2)$

$\Rightarrow -1$

Q.10 A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse move along the fence at the moment when it covers 1/8 of the circle in km/hr is.

(A) 20 (B) 60 (C) 30 (D) 40

Sol. **[D]**



$R \tan\theta = y$

$\Rightarrow R\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{dy}{dt}$

$\Rightarrow (R\omega)\sec^2\theta = \frac{dy}{dt}$

$\Rightarrow \frac{dy}{dt} = v\sec^2\theta$

$= 20 \times \sec^2\frac{\pi}{4}$

$= 20 \times 2 = 40$ km/hr.

Q.11 If the tangent at P on the curve $x^m y^n = a^{m+n}$ meets the coordinate axes at A and B then AP: PB =

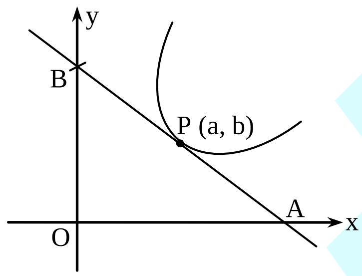
(A) m : n (B) n : m

(C) –m : n (D) –n : m

Sol. **[B]**

$x^m y^n = a^{m+n}$

Taking log both sides, we get



m log x + n log y = (m + n) log a

Differentiating above equation w.r.t. x, we get

$\frac{m}{x} \times 1 + \frac{n}{y} \times \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-m}{x} \times \frac{y}{n} = \frac{-m}{n} \times \left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{-m}{n} \times \frac{b}{a}$

Hence, equation of tangent at p (a, b) is

$y - b = \frac{-m}{n} \times \frac{b}{a}(x - a)$

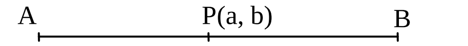
nay – abn = – mbx + abm

$\Rightarrow$ mbx + nay = ab (m +n)

$\Rightarrow \frac{x}{\frac{ab(m+n)}{mb}} + \frac{y}{\frac{ab(m+n)}{na}} = 1$

$\Rightarrow \frac{x}{\frac{ab(m+n)}{mb}} + \frac{y}{\frac{ab(m+n)}{na}} = 1$

$\Rightarrow \frac{x}{\frac{a(m+n)}{m}} + \frac{y}{\frac{b(m+n)}{n}} = 1$



$\left[\frac{a(m+n)}{m}, 0\right]$ $\left[0, \frac{b(m+n)}{n}\right]$

$a = \frac{\frac{a(m+n)}{m} \times PB + AP \times 0}{(AP + BP)}$

$\Rightarrow AP + BP = PB \times \frac{(m+n)}{m}$

$b = \frac{O \times PB + \frac{b(m+n)}{n} \times AP}{AP + BP}$

$\Rightarrow AP + BP = \left(\frac{m+n}{n}\right) \times AP$

Hence, $AP \times \left(\frac{m+n}{n}\right) = PB \times \left(\frac{m+n}{m}\right)$

$\Rightarrow \frac{AP}{BP} = \frac{n}{m}$

AP : BP = n : m

$\therefore$ option [B] is correct answer.

Q.12 The curve x + y – λn(x + y) = 2x + 5 has a vertical tangent at the point (α, β). Then α + β is equal to-

(A) –1 (B) 1 (C) 2 (D) –2

Sol. **[B]**

$y' = -\frac{1 - \frac{1}{x+y} - 2}{1 - \frac{1}{x+y}}$

$1 = \frac{1}{x+y}$

$\alpha + \beta = 1$

Q.13 The curves $4x^2 + 9y^2 = 72$ & $x^2 - y^2 = 5$ at (3, 2)

(A) touch each other (B) cut orthogonally

(C) intersect at 45° (D) intersect at 60°

Sol. **[B]**

$8x + 18yy' = 0$

$y' = \frac{-8x}{18y}$

$y'_{(3,\,2)} = \frac{-8(3)}{18(2)} = -\frac{2}{3}$

$2x - 2y.y' = 0$

$y' = \frac{x}{y}$

$y'_{(3,\,2)} = \frac{3}{2}$

$m_1 m_2 = -1$

Q.14 A monkey is attempting to climb a rope that is not securely fastened. If he pulls himself up x meters at once then the rope slips x^3 meters down. How many meter at a time must he pull himself up to climb with as few pulls as possible?

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Sol. **[B]**

$d = x - x^3$

$\frac{d(d)}{dx} = 1 - 3x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{3}}$

Q.15 A curve is represented by the equations, $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then |PQ| is equal to-

(A) $\frac{5\sqrt{3}}{2}$ (B) $\frac{5\sqrt{5}}{2}$ (C) $\frac{2\sqrt{5}}{3}$ (D) $\frac{3\sqrt{5}}{2}$

Sol. **[D]**

$y' = \frac{-\operatorname{cosec}^2\theta}{2\sec^2 t\,\tan t} = -\frac{\cot^3\theta}{2}$

$y'_{(\pi/4)} = -\frac{1}{2}$

$y - 1 = -\frac{1}{2}(x - 2)$

$\Rightarrow 2y - 2 = -x + 2$

$\Rightarrow 2y + x - 4 = 0$

$2\cot t + \sec^2 t - 4 = 0$

$\frac{2}{\tan t} + 1 + \tan^2 t - 4 = 0$

$\Rightarrow 2 + \tan^3 t - 3\tan t = 0$

$\Rightarrow \tan^3 t - 3\tan t + 2 = 0$

$\Rightarrow (\tan t - 1)(\tan^2 t + \tan t - 2) = 0$

$\Rightarrow (\tan t - 1)(\tan^2 t + 2\tan t - \tan t - 2) = 0$

$\Rightarrow (\tan t - 1)^2(\tan t + 2) = 0$

$\Rightarrow \tan t = -2$

$1 + \tan^2 t = 5$

$\sec^2 t = x$

$y = \cos t = -\frac{1}{2}$

$P(2, 1),\ Q\left(5, -\frac{1}{2}\right)$

$PQ = \sqrt{9 + \frac{9}{4}} = \frac{\sqrt{45}}{2}$

$= \frac{3\sqrt{5}}{2}$

Q.16 The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to-

(A) square of the abscissa of the point of tangency

(B) square root of the abscissa of the point of tangency

(C) cube of the abscissa of the point of tangency

(D) cube root of the abscissa of the point of tangency

Sol. **[C]**

$ax^{-2} + by^{-2} = 1$

equation of tangent

$ax.x_1^{-3} + by.y_1^{-3} = 1$

x intercept $= \frac{x_1^3}{a}$

Q.17 At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$; $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by-

(A) $2\pi/3$ (B) $3\pi/4$ (C) $\pi/2$ (D) $\pi/3$

Sol. **[A]**

$y' = \frac{a(2\cos t - 2\cos 2t)}{a(-2\sin t + 2\sin 2t)}$

$\cos t - \cos 2t = 0$

$\cos 2t - \cos t = 0$

$\Rightarrow 2\cos^2 t - \cos t - 1 = 0$

$\Rightarrow 2\cos^2 t - 2\cos t + \cos t - 1 = 0$

$\Rightarrow (2\cos t + 1)(\cos t - 1) = 0$

$\cos t = 1, \quad \cos t = -\frac{1}{2}$

$t = 0 \qquad t = \frac{2\pi}{3}$

exclude the point where denominator of y' becomes zero

Q.18 Consider the function $f(x) = \begin{cases} 2+x^3, & \text{if } x \le 1 \\ 3x, & \text{if } x > 1 \end{cases}$, then

(A) f is continuous on [–1, 2] but is not differentiable on (–1, 2)

(B) Mean value theorem is not applicable for the function on [–1, 2]

(C) Mean value theorem is applicable on [–1, 2] and the value of c = 1

(D) Mean value theorem is applicable on [–1, 2] and the value of c is $\pm\frac{\sqrt{5}}{3}$

Sol. **[D]**

Continuous & diff. $\forall\ x \in R$

Rolle's theorem not applicable

$f(-1) \ne (-2)$

LMVT

$3x^2 = \frac{5}{3}$

$c = \pm\frac{\sqrt{5}}{3}$

Q.19 Let $f(x) = 1 + x^m (x-1)^n$ where $m, n \in N$. Then in (0, 1) the equation $f'(x) = 0$ has

(A) no root (B) at least one root

(C) at most one root (D) exactly one root

Sol. **[D]**

$f(0) = 1,\ f(1) = 1$

$f'(x) = 0$ must have at least one root in (0, 1)

$f'(x) = x^{m-1}.\ m(x-1)^n + nx^m(x-1)^{n-1}$

$= x^{m-1}(x-1)^{n-1}[(m+n)x - n]$

$x = 0, 1, \left(\frac{m}{m+n}\right)$

Part-B One or more than one correct answer type questions

Q.20 The angle at which the curve $y = Ke^{kx}$ intersects the y-axis is-

(A) $\tan^{-1} k^2$ (B) $\cot^{-1}(k^2)$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{1+k^4}}\right)$ (D) $\sec^{-1}\left(\sqrt{1+k^4}\right)$

Sol. **[C]**

$y' = k^2.2e^{kx}$

$y'_{(0,4)} = k^2$

angle with y-axis

$\frac{\pi}{2} - \theta = \frac{\pi}{2} - \tan^{-1} k^2$

$= \cot^{-1} k^2 = \sin^{-1}\frac{1}{\sqrt{1+k^4}}$

Q.21 Which of the following pairs(s) of curve is/are orthogonal.

(A) $xy = a^2\ ;\ x^2 - y^2 = b^2$

(B) $y = ax\ ;\ x^2 + y^2 = c^2$

(C) both

(D) None

Sol. **[A, B, C]**

(A) $xy' + y = 0$

$y' = -\frac{y}{x}$

$2x - 2y.y' = 0$

$y' = \frac{x}{y}$

$\theta = 90^o$

(B) curve passes through centre of circle (becomes diameter)

Q.22 If $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to the curve $x = Kt$, $y = \frac{K}{t}$, $K > 0$ then:

(A) a > 0 , b > 0 (B) a > 0, b < 0

(C) a < 0, b > 0 (D) a < 0, b < 0

Sol. **[D]**

$xy = k^2$

$y + xy' = 0$

$y' = -\frac{y}{x}$

$Y - y = -\frac{y}{x}(X - x)$

$xY - xy = -yX + xy$

$xY + yX = 2xy$

$\frac{X}{2x} + \frac{Y}{2y} = 1$

$xy = k^2$

$k > 0$

x, y same sign
$x > 0,\ y > 0$
or $x < 0,\ y < 0$

Q.23 Equation of a tangent to the curve $y \cot x = y^3 \tan x$ at the point where the abscissa is $\frac{\pi}{4}$ is-

(A) $4x + 2y = \pi + 2$ (B) $4x - 2y = \pi + 2$
(C) $x = 0$ (D) $y = 0$

Sol. **[]**

$x = \frac{\pi}{4},\ y = 0, \pm 1$

$y = y^3 \tan^2 x$

$y^2 = \cot^2 x$

$2yy' = 2 \cot x\,(-\operatorname{cosec}^2 x)$

$y' = \frac{-\cot x \operatorname{cosec}^2 x}{y}$

$y'_{(\pi/4)} = \pm 2, \infty$

Q.24 For the curve represented parametrically by the equations, $x = 2\,\lambda n \cot t + 1$ & $y = \tan t + \cot t$

(A) tangent at $t = \pi/4$ is parallel to x-axis
(B) normal at $t = \pi/4$ is parallel to y-axis
(C) tangent at $t = \pi/4$ is parallel to the line $y = x$
(D) tangent and normal intersect at the point (2, 1)

Sol. **[A, B]**

$y' = \sec^2 t - \operatorname{cosec}^2 t$

$= \frac{-2 \operatorname{cosec}^2 t}{\cot\ t}$

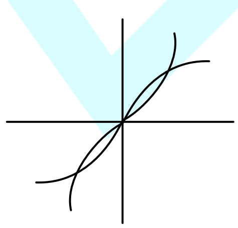
$(y')_{(\pi/4)} = 0$

Q.25 Consider the curve $f(x) = x^{1/3}$, then-

(A) the equation of tangent at (0, 0) is $x = 0$
(B) The equation of normal (0, 0) is $y = 0$
(C) normal to the curve does not exist at (0, 0)
(D) f(x) and its inverse meet at exactly 3 points

Sol. **[D]**

$f'(x) = \frac{1}{3x^{2/3}}$



f, f^{-1} meet at three points.

Q.26 If $y = f(x)$ be the equation of a parabola (axis parallel to the y-axis) which is touched by the line $y = x$ at the point where $x = 1$, then-

(A) $f''(0) = 2 f(0)$
(B) $f'(1) = 1$
(C) $f(0) + f'(0) + f''(0)/2 = 1$
(D) $2f(0) = 1 - f'(0)$

Sol. **[A, B, C, D]**

$f(x) = y = ax^2 + bx + c$

$y'_p = 2a + b = 1 = f'(1)$

$a + b + c = 1$

$f(0) = c$

$f'(0) = b$

$f''(c) = 2a$

$\frac{f''(0)}{2} + f'(0) + f(0) = 1$

$2a + b = 1$

$f'(0) + f(0) = 1$

$a = c$

$\frac{f''(0)}{2} = f(0)$

Q.27 The families of curves defined by the equations $y = ax$, $x^2 + y^2 = b^2$ are perpendicular for -

(A) $a = 3, b = 4$ (B) $a = -2, b = 5$
(C) $a = 3, b = 3$ (D) $a = 3, b = 2$

Sol. **[A,B,C,D]**

$y = ax,\ x^2 + y^2 = b^2$

$y = ax \Rightarrow (dy/dx)_1 = a$

$x^2 + y^2 = b^2 \Rightarrow 2x \times 1 + 2y \times \frac{dy}{dx} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)_2 = -\frac{x}{y}$

$\left(\frac{dy}{dx}\right)_1 \times \left(\frac{dy}{dx}\right)_2 = \frac{-ax}{y} = \frac{-ax}{ax} = -1$

For all $a, b \in R$

Hence, all options are correct answers.

Q.28 Which of the following functions do not satisfy conditions of Rolle's Theorem-

(A) $e^x \sin x,\ \left[0, \frac{\pi}{2}\right]$

(B) $(x + 1)^2 (2x - 3)^5,\ \left[-1, \frac{3}{2}\right]$

(C) $\sin |x|,\ [\pi, 2\pi]$

(D) $\sin\frac{1}{x}, \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Sol. **[A, D]**

We have to check for every option

For A : $f(x) = e^x \sin x$ in $[0, \pi/2]$

$f(b) = f(\pi/2) = e^{\pi/2} \sin \pi/2 = e^{\pi/2}. 1 = e^{\pi/2}$

$f(a) = f(0) = e^0 \cdot \sin 0 = 1. 0 = 0$

since, $f(b) \neq f(a)$

$\therefore$ [A] is correct answer.

Part-C Assertion-Reason type questions

The following questions 29 to 35 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.29 **Assertion**: If the curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then a^2 is equal to 4/3.

Reason: If two curves cut each other orthogonally, then product of slopes of tangent at point of intersection is equal to –1.

Sol. **[C]**

$\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$

Differentiating above two curves w.r.t.x , we get

$\frac{2x}{a^2} + \frac{2y}{4} \times \frac{dy}{dx} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)_1 = m_1 = \frac{-2x}{a^2} \times \frac{4}{2y}$

$\left(\frac{dy}{dx}\right)_1 = \frac{-4x}{a^2 y}$

$y^3 = 16x \Rightarrow 3y^2 \frac{dy}{dx} = 16 \Rightarrow \left(\frac{dy}{dx}\right)_2 = \frac{16}{3y^2}$

Since, above two curves intersect orthogonally, we follow as $\left(\frac{dy}{dx}\right)_1 \times \left(\frac{dy}{dx}\right)_2 = -1$

$\frac{-4x}{a^2 y} \times \frac{16}{3y^2} = -1$

$\Rightarrow \frac{64x}{3y^2} \times \frac{1}{a^2} = 1 \Rightarrow a^2 = \frac{64x}{3 \times 16x} \Rightarrow a^2 = 4/3$

$\Rightarrow a = \pm 2/\sqrt{3}$

$\therefore$ assertion is not true. Reason is true.

$\therefore$ option [D] is correct answer.

Q.30 **Assertion**: The slope of the normal at the point with abscissa x = –2 of the graph of the function $f(x) = |x^2 - |x||$ is 1/3.

Reason : At x = –2, the slope of tangent $\left(\frac{dy}{dx}\right)$ of the curve is –3 and a normal is perpendicular to the tangent.

Sol. **[A]**

$f(x) = \left|x^2 - |x|\right|$

$= \begin{cases} x^2 - |x| & ; \quad x^2 - |x| \geq 0 \\ -(x^2 - |x|) & ; \quad x^2 < |x| \text{ not possible} \end{cases}$

$f(x) = x^2 - |x|; x^2 \geq |x|$

$= \begin{cases} x^2 - x & ; \quad x^2 \leq 0 \text{ or } x \geq 1 \text{ but } x > 0 \\ x^2 + x & ; \quad x^2 < 0 \end{cases}$

We take only $y = f(x) = x^2 + x; x < 0$

$\frac{dy}{dx} = 2x + 1$

$\left.\frac{dy}{dx}\right|_{x=-2} = -4 + 1 = -3$

when x = –2, y = 4 – 2 = 2

Slope of normal is 1/3.

Assertion is true. Reason also true.

Hence, option [A] is correct answer.

Q.31 **Assertion**: The ratio of length of tangent to length of normal is proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Reason: Length of normal and tangent to a curve $y = f(x)$ is $\left|y\sqrt{1+m^2}\right|$ and $\left|\frac{y\sqrt{1+m^2}}{m}\right|$,

where $m = dy/dx$.

Sol. **[A]**

Q.32 **Assertion** : The product of the ordinates of the point of tangency to the curve $(1 + x^2)\, y = 2 - x$, where the tangent makes equal intercept with coordinate axes is equal to 1

Reason : Slope of straight line making equal intercept with coordinate axis is equal to 1

Sol. **[C]**

Q.33 **Assertion** : The quadratic equation $10x^2 - 28x + 17 = 0$ has at least one root in [1, 2]

Reason : $f(x) = e^{10x}(x - 1)(x - 2)$ satisfies all the conditions for Rolle's theorem in [1, 2]

Sol. **[A]**

Reason :

$f(x) = e^{10x}(x - 1)(x - 2)$

$f(1) = f(2)$

$1 < \alpha < 2$

$f(\alpha) = e^{10x}\, 10(x^2 - 3x + 2) + e^{10x}(2x - 3) = 0$

$\Rightarrow e^{10x}[10x^2 - 28x + 17] = 0$

$10x^2 - 28x + 17 = 0$

Q.34 Consider function $f(x) = \begin{cases} x\{x\}+1, & 0 \le x < 1 \\ 2-\{x\}, & 1 \le x \le 2 \end{cases}$

where $\{x\}$: fractional part function of x.

Assertion : Rolle's Theorem is not applicable f(x) in [0, 2]

Reason : $f(0) \ne f(2)$

Sol. **[A]**

If both Assertion and Reason are true and the Reason is correct explanation of the Assertion

Q.35 Consider the function $f(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$

Assertion : x = 0 is not the critical point on the graph of y = f(x).

Reason : f(x) is discontinuous at x = 0.

Sol. **[A]**

Part-D Column Matching type questions

Q.36

Column 1		Column II
(A)	The angle of intersection of $y^2 = 4x$ and $x^2 = 4y$ is 90° & $\tan^{-1}\left(\frac{m}{n}\right)$ then $\|m + n\|$ is equal to (m and n are co-prime)	(P) 0
(B)	The area of triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is	(Q) 1/2
(C)	The angle between curve $x^2 y = 1$ and $y = e^{2(1-x)}$	(R) 7
(D)	The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to	(S) 3

Sol. (A) → R; (B) → Q; (C) → P; (D) → S

(A) $x^4 = 64x$

$x(x^3 - 64) = 0$

$x = 0; 4$

$y = 0, 4$

$y'_1 = \frac{2}{y}$; $y'_2 = \frac{x}{2}$

$\tan^{-1}\left|\frac{2-\frac{1}{2}}{1+1}\right| \Rightarrow \tan^{-1}\left(\frac{3}{4}\right)$

$(m + n) = 7$

(B) $1 = e^{\sin y}\cos y . y'$

$y' = \frac{1}{e^{\sin y}\cos y}$

$y'_{(1,\,0)} = 1$

$y = -x + 1$

$y + x = 1$

Area $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

(C) (1, 1)

$2xy + x^2y' = 0$

$y' = -\frac{2y}{x} = -2$

$y' = e^{2(1-x)}(-2) = (-2)$

$\theta = 0°$

(D) $y' = \frac{b}{3}e^{x/3}$

$\frac{y}{y'} = \frac{be^{x/3}}{\frac{b}{3}e^{x/3}} = 3$

Q.37

Column 1	Column II
(A) The slope of the curve $2y^2 = ax^2 + b$ at (1, 1) is –1, then	(P) $a - b = -6$
(B) If (a, b) be the point on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes, then	(Q) $a - b = 7/2$
(C) If the tangent at a point (1, 2) on the curve $y = ax^2 + bx + \frac{7}{2}$ be parallel to the normal at (–2, 2) on the curve $y = x^2 + 6x + 10$, then	(R) $a - b = 4/3$
(D) If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with x-axis, then	(S) $a - b = 3$

Sol. (A) → P; (B) → R; (C) → Q; (D) → S

(A) $4yy' = 2ax$

$\frac{a}{2} = -1$

$a = -2$

$a + b = 2$

$b = 4$

$a - b = -6$

(B) $18y.y' = 3x^2$

$y' = \frac{x^2}{6y}$

$-\frac{6y}{x^2} = \pm 1$

$6y = \pm x^2$

$6b = \pm a^2$

(C) $y' = 2x + 6$

$y'_{(-2,\,2)} = -2$

normal = –1/2

$y' = 2ax + b$

$2a + b = -1/2$ ….(1)

$a + b = -\frac{3}{2}$ ….(2)

Solving (1) & (2)

$a = 1,\ b = -\frac{5}{2}$

$a - b = \frac{7}{2}$

(D) $xy' + y + a + by' = 0$

$y' = \frac{-y-a}{b+x}$

$\frac{-1-a}{1+b} = 2$

$-1 - a = 2 + 2b$

$a + 2b = -3$ ….(1)

$a + b = -1$ …(2)

Solving (1) & (2)

$b = -2,\ a = 1$

$a - b = 3$

Q.38

Column 1	Column II
(A) The equation $x \log x = 3 - x$ has at least one root in	(P) (0, 1)
(B) If $27a + 9b + 3c + d = 0$, Then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(Q) (1, 3)
(C) If $c = \sqrt{3}$ & $f(x) = x + \frac{1}{x}$ then interval in which LMVT is applicable for f(x) is	(R) (0, 3)
(D) If $c = \frac{1}{2}$ & $f(x) = 2x - x^2$, then interval in which LMVT is applicable for f(x) is	(S) (–1, 1)

Sol. (A) → Q, R; (B) → R; (C) → Q; (D) → P, S

(B) $\phi(x) = ax^4 + bx^3 + cx^2 + dx$

$= x(ax^3 + bx^2 + cx + d)$

(C) $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

in interval (1, 3)

$\frac{2}{3} = \frac{3 + \frac{1}{3} - 1 - 1}{2} = \frac{2}{3}$

(D) $f'(x) = 2 - 2x$

$f'(c) = 1$

(1, 3)

⇒ 1/1

(–1, 1)

$\frac{1+2+1}{2} = 2$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Show that the normal at the point (3t, 4/t) of the curve xy = 12 cuts the curve again at the point whose parameter t_1 is given by $t_1 = -\frac{16}{9t^3}$

Sol. (3t, 4/t)

xy = 12

$x.\frac{dy}{dx} + y.1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\left.\frac{dy}{dx}\right|_{(3t,4/t)} = \frac{-4/t}{3t} = -\frac{4}{3t^2}$

slope of normal = $3t^2/4$

Equation of normal

$y - \frac{4}{t} = \frac{3t^2}{4}(x - 3t)$

$y - \frac{4}{t} = \frac{3t^2}{4}x - \frac{9t^3}{4}$

$y = \frac{3t^2}{4}x - \frac{9t^3}{4} + \frac{4}{t}$

$y = \left(\frac{3t^2}{4}\right)x + (4/t - 9t^3/4)$ … (1)

solving equation (1) and xy = 12, we get

$\frac{12}{x} = \left(\frac{3t^2}{4}\right)x + (4/t - 9t^3/4)$

$\frac{12}{x} = \frac{3t^2}{4}.x + \frac{4}{t} - \frac{9t^3}{4}$

$\frac{12}{x} = \frac{3t^3x + 16 - 9t^4}{4t}$

$12 \times 4t = 3t^3x^2 + 16x - 9t^4.x$

$12 \times 4t = 3t^3x^2 + x(16 - 9t^4)$

$\Rightarrow 3t^3x^2 + (16 - 9 + 4)x - 48t = 0$

$x = \frac{-(16-9+4) \pm \sqrt{(16-9t^4) + 12 \times 48t^4}}{6t^3}$

$x = \frac{(9t^4 - 16) \pm \sqrt{256 + 81t^8 - 18 \times 16t^4 + 12 \times 48t^4}}{6t^3}$

$x = \frac{(9t^4 - 16) \pm \sqrt{81t^8 - 256 + 18 \times 16t^4}}{6t^3}$

$x = \frac{(9t^4 - 16) \pm \sqrt{81t^8 + 256 + 18 \times 16t^4}}{6t^3}$

$x = \frac{(9t^4 - 16) \pm \sqrt{(9t^4 + 16)^2}}{6t^3}$

$x = \frac{(9t^4 - 16) \pm (9t^4 + 16)}{6t^3}$

positive sign

$x = \frac{9t^4 - 16 + 9t^4 + 16}{6t^3} = 3t$

negative sign

$x = \frac{(9t^4 - 16) - (9t^4 + 16)}{6t^3}$

$x = \frac{-32}{6t^3} = -\frac{16}{3t^3}$

put $x = 3t_1 = -\frac{16}{3t^3}$

$\Rightarrow t_1 = -\frac{16}{9t^3}$ Hence proved.

Q.2 Prove that the tangents to the curve $y = \frac{(1+3x^2)}{(3+x^2)}$ drawn at the points for which y = 1, intersect at the origin.

Sol. $y = \frac{1+3x^2}{3+x^2}$

$y = 1 = \frac{1+3x^2}{3+x^2}$

$3 + x^2 = 1 + 3x^2$

$2 = 2x^2$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

Hence, points (+1, 1) and (–1, 1) would be points of tangent

$\frac{dy}{dx} = \frac{6x(3+x^2) - (1+3x^2)(2x)}{(3+x^2)^2}$

$\frac{dy}{dx} = \frac{18x + 6x^3 - 2x - 6x^3}{(3+x^2)^2}$

$\frac{dy}{dx} = \frac{16x}{(3+x^2)^2}$

$\left.\left(\frac{dy}{dx}\right)\right|_{(1,1)} = \frac{16}{16} = 1$

Equation of tangent

$y - 1 = 1.(x - 1)$

$\Rightarrow y = x$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = -\frac{16}{16} = -1$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = -1$$

Equation of tangent at (–1, 1)

$y - 1 = -1\,(x + 1)$

$= -x - 1$

$y = -x$

Hence, solving $y = x$ and $y = -x$, we will get point of intersection (0, 0).

Q.3 Find the angle of intersection of the curves,

(i) $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$.

(ii) $y = \sin x$ and $y = \cos x$ $(0 \le x \le \pi)$.

(iii) $y^2 = \frac{2x}{\pi}$ and $y = \sin x$

(iv) $y^2 = 4x$ and $x^2 + y^2 = 5$

Sol. (i) $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$

We have to first find point of intersection of above two curves

$x^2 = y^2 + 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$

$$\frac{y^2+5}{18} + \frac{y^2}{8} = 1$$

$$\frac{4y^2+20+9y^2}{72} = 1$$

$\Rightarrow 13y^2 + 20 = 72$

$\Rightarrow 13y^2 = 52$

$\Rightarrow y = \pm 2$

Put $y = +2 \Rightarrow x^2 = 4 + 5 = 9$

$x^2 = 9$

$\Rightarrow x = \pm 3$

Put $y = -2 \Rightarrow x^2 = 4 + 5 = 9$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

Hence, point of intersection would be as follows: (3, 2), (–3, 2) (3, –2), (–3, –2)

Differentiate above two equations w.r.t. x, we get

$$2x - 2y \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = 3/2 = m_1$$

$$\frac{2x}{18} + \frac{2y}{8} \times \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} \times \frac{2y}{8} = -\frac{2x}{18}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{18} \times \frac{8}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-8}{18} \times \frac{x}{y}$$

$$\left.\left(\frac{dy}{dx}\right)\right|_{(3,2)} = \frac{-8}{18} \times \frac{3}{2}$$

$$= \frac{-2}{3} = m_2$$

Hence, angle of intersection

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{\frac{3}{2} + \frac{2}{3}}{1 - \frac{3}{2} \times \frac{2}{3}}\right| = \infty$$

$\Rightarrow \theta = \pi/2$

(ii) $y = \sin x$ and $y = \cos x$ $(0 \le x \le \pi)$

We have to first find out point of intersection,

$y = \sin x = \cos x$

$\tan x = 1$, $\tan \pi/4$

$x = n\pi + \pi/4$; $n \in$ Integer in $0 \le x \le \pi$,

only $x = \pi/4$

Differentiating above two equations w.r.t.x, we get

$$\frac{dy}{dx} = \cos x \Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi/4} = \left.\cos x\right|_{(x=\pi/4)} = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = -\sin x \Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi/4} = -\frac{1}{\sqrt{2}}$$

Hence, angle of intersection can be found out as follows:

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = \left|\dfrac{\frac{2}{\sqrt{2}}}{\frac{1}{2}}\right| = |4/\sqrt{2}\,|$

$\theta = \tan^{-1}(2\sqrt{2})$

Q.4 If p and q be the intercepts on the axes by the tangent at any point on the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$ show that $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} = 1$

Sol. $(x/a)^{2/3} + (y/b)^{2/3} = 1$

Differentiating above curve w.r.t. x, we get

$$\frac{1}{a^{2/3}}\,\frac{2}{3}x^{2/3-1} + \frac{1}{b^{2/3}}\,\frac{2}{3}y^{2/3-1}\,.\,\frac{dy}{dx} = 0$$

$$\frac{2}{3} \times \frac{1}{b^{2/3}}\,\frac{1}{y^{1/3}}(dy/dx) = -\,\frac{2}{3} \times \frac{1}{a^{2/3}}\,\frac{1}{x^{1/3}}$$

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -(b/a)^{2/3}\,(y_1/x_1)^{1/3}$$

Hence, equation of tangent at point (x_1, y_1)

$y - y_1 = -(b/a)^{2/3}\,(y_1/\,x_1)^{1/3}\,(x - x_1)$

$a^{2/3}\;x_1^{1/3} y - a^{2/3} y_1\, x_1^{1/3} = -\,b^{2/3}\;y_1^{1/3} x + b^{2/3} x_1\, y_1^{1/3}$

$b^{2/3}\,y_1^{1/3} x + a^{2/3}\;x_1^{1/3} y = a^{2/3} y_1\, x_1^{1/3} + b^{2/3} x_1\, y_1^{1/3}$

$= x_1^{1/3}\;y_1^{1/3}\,(a^{2/3}\;y_1^{2/3} + b^{2/3}\;x_1^{2/3})$

$$\frac{x}{\dfrac{x_1^{1/3}y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{b^{2/3}y_1^{1/3}}} + \frac{y}{\dfrac{x_1^{1/3}y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{a^{2/3}x_1^{1/3}}} = 1$$

$$\frac{x}{\dfrac{x_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{b^{2/3}}} + \frac{y}{\dfrac{y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{a^{2/3}}} = 1$$

Hence,

$$p = \frac{x_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{b^{2/3}}$$

$$q = \frac{y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{a^{2/3}}$$

$$\frac{p^2}{a^2} = \frac{x_1^{2/3}}{a^2b^{4/3}}\,(a^{2/3}\,y_1^{2/3} + b^{2/3}x_1^{2/3})^2$$

$$\frac{q^2}{b^2} = \frac{y_1^{2/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})^2}{b^4a^{4/3}}$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = (a^{2/3}y_1^{2/3}\;x_1^{1/3})^2\left[\frac{x_1^{2/3}}{a^2b^{4/3}} + \frac{y_1^{2/3}}{a^2b^{4/3}}\right]$$

$$= (a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})^2\;\frac{\left[b^{2/3}x_1^{2/3} + a^{2/3}y_1^{2/3}\right]}{a^2b^2}$$

$$= \frac{(a^{2/3}b^{2/3})^3}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

Hence, $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} = 1$ proved.

Q.5 Prove that If the tangent at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ meets the axes of coordinates in A and B then locus of mid point of AB is a circle.

Sol. equation of tangent

$x x_1^{-1/3} + y y_1^{-1/3} = a^{2/3}$

$A \equiv (a^{2/3}x^{1/3}, 0)$

$B \equiv (0, a^{2/3}y_1^{1/3})$

$h = \dfrac{a^{2/3}x_1^{1/3}}{2}$

$k = \dfrac{a^{2/3}xy_1^{1/3}}{2}$

$\Rightarrow \dfrac{4h^2}{a^2} + \dfrac{4k^2}{a^2} = 1$

$\Rightarrow a.x^2 + y^2 = \left(\dfrac{a^2}{4}\right)$

circle.

Q.6 Show that in the curve $y = a\,\lambda n\ (x^2 - a^2)$, sum of the length of tangent & sub-tangent varies as the product of coordinates of the point of contact.

Sol. $y = a\lambda n\,(x^2 - a^2)$

Differentiate above curve w.r.t.x, we get

$$\frac{dy}{dx} = \frac{a}{(x^2 - a^2)} \times 2x = \frac{2ax}{(x^2 - a^2)} = y'$$

$$1 + y'^2 = 1 + \left(\frac{2ax}{x^2 - a^2}\right)^2$$

$$= 1 + \frac{4a^2x^2}{(x^2 - a^2)^2}$$

$$= \frac{x^4 + a^4 - 2a^2x^2 + 4a^2x^2}{(x^2 - a^2)^2}$$

$$= \frac{x^4 + a^4 + 2a^2x^2}{(x^2 - a^2)^2}$$

$$= \left(\frac{x^2 + a^2}{x^2 - a^2}\right)^2$$

$$\sqrt{1 + y'^2} = \frac{x^2 + a^2}{x^2 - a^2}$$

Sum of tangent and sub-tangent

$$= \left|\frac{y}{y'} \times \sqrt{1 + y'^2}\right| + \left|\frac{y}{y'}\right|$$

$$= \left|\frac{y}{y'}\right| \left|\sqrt{1 + y'^2} + 1\right|$$

$$= \left|\frac{y}{\frac{2ax}{x^2 - a^2}}\right| \left|\frac{x^2 + a^2}{x^2 - a^2} + 1\right|$$

$$= \left|\frac{y \times (x^2 - a^2)}{2ax}\right| \left|\frac{x^2 + a^2 + x^2 - a^2}{x^2 - a^2}\right|$$

$$= \left|\frac{y}{2ax} \times 2x^2\right|$$

$$= \left|\frac{xy}{a}\right|$$

Hence, length of tangent and sub-tangent

$\propto |xy|$. proved.

Q.7 An edge of variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long.

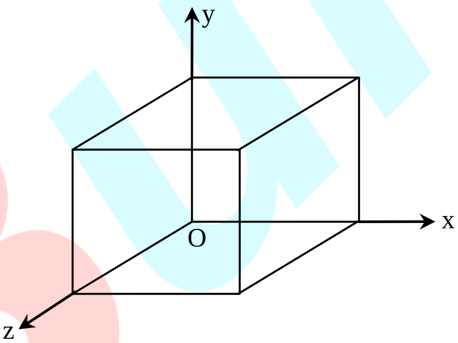
Sol. Volume of cube,

$V = xyz$

Put $x = y = z$

$V = x^3$

Differentiate above equation



w.r.t. time, we get

$$\frac{dv}{dt} = 3x^2 \times \frac{dx}{dt}$$

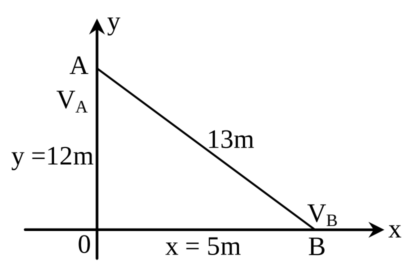
Given, $\frac{dx}{dt} = \frac{3\,cm}{sec}$, $x = 10$ cm

$$\frac{dv}{dt} = 3 \times 100\ cm^2 \times \frac{3\,cm}{sec}$$

$\frac{dv}{dt} = 900\ cm^3/sec.$ **Ans.**

Q.8 The top of a ladder 13 m long is resting against a vertical wall, when a ladder begins to slide. When the foot of the ladder is 5m from the wall, it is sliding at the rate of 2 m/s. How fast then the top sliding downwards.

Sol. $x^2 + y^2 = (13)^2 = 169$



Differentiating above curve w.r.t. time, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$y.\ V_A = -x\ V_B$$

$$V_A = -\frac{x}{y} \times V_B$$

$$V_A = -\frac{5}{12} \times 2 \text{ m/sec.}$$

$$V_A = \frac{-5}{6} \text{ m/sec.}$$

Q.9 Sand is pouring from a pipe at the rate of 12 cc/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6 th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm .

Sol. $\frac{dy}{dt} = 12$ cc/sec.

$$h = \frac{1}{6}r \Rightarrow r = 6h$$

V = volume of cone

$$= \frac{1}{3}\pi(6h)^2 \times h$$

$$= 12\pi h^3$$

Differentiating equation of $V = 12\pi h^3$

w.r.t. time, we get

$$\frac{dv}{dt} = 12\pi \times 3h^2 \times \frac{dh}{dt}$$

$$\Rightarrow (dh/dt) = \frac{dv}{dt} \times \frac{1}{36\pi \times h^2}$$

$$\frac{dh}{dt} = \frac{12}{36\pi \times 4 \times 4}$$

$$\frac{dh}{dt} = \frac{1}{48\pi} \text{ per sec.}$$

Q.10 At what point of (1st quadrant) the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease at the same rate at which the abscissa increases?

Sol. $16x^2 + 9y^2 = 400$

Differentiate above curve w.r.t. x, we get

$$16 \times 2x + 9 \times 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-16 \times 2x}{9 \times 2y} = \frac{-16}{9} \times \frac{x}{y}$$

As given condition $\frac{16}{9}\frac{x}{y} = \frac{9y}{16x}$

$$\Rightarrow (16x)^2 = (9y)^2$$

Now, solving $16x^2 + 9y^2 = 400$

$$\frac{81y^2}{16} + 9y^2 = 400$$

$$\Rightarrow 81y^2 + 16 \times 9y^2 = 400 \times 16$$

$$\Rightarrow 9y^2 (9 + 16) = 400 \times 16$$

$$\Rightarrow y^2 = \frac{400 \times 16}{9 \times 25}$$

$$\Rightarrow y = \frac{20 \times 4}{3 \times 5} = \frac{16}{3}$$

$$16x^2 + 9 \times \left(\frac{16}{3}\right)^2 = 400$$

$$16x^2 = 400 - 256 = 144$$

$$x^2 = 9 \Rightarrow x = \pm 3$$

Hence, required point $\left(3, \frac{16}{3}\right)$

Q.11 If a, b are two real number with a < b, show that a real number c can be found between a and b such that $3c^2 = b^2 + ab + a^2$. a, b & c satisfy the function $f(x) = x^3$.

Sol. Given a, b $\in$ R; a < b

$f(a) = a^3$

$f(c) = c^3$

$f(b) = b^3$

From Lagrange's Mean Value Theorem, we get

$$F'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{b^3 - a^3}{b - a} = (a^2 + ab + b^2)$$

Also, $f(x) = x^3$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2$

$f'(c) = 3c^2 = a^2 + b^2 + ab$

Hence, c must be between a and b

i.e. a < c, < b; proved.

Q.12 Verify Rolle's Theorem for $f(x) = (x - a)^m \times (x - b)^n$ on [a, b]; m , n being positive integer.

Sol. $f(x) = (x - a)^m (x - b)^n$ on [a, b];

m, n being positive integer

$f(a) = (a - a)^m (a - b)^n = 0$

$f(b) = (b - a)^m (b - b)^n = 0$

According to Rolle's theorem

(i) f(x) must be continuous in $a \le x \le b$

(ii) f(x) must be differentiable in $a < x < b$

(iii) $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0$

Then there exist a point x = c such that a < c < b.

Differentiating above function w.r.t x, we get

$f'(x) = m(x - a)^{m-1}(x - b)^n + (x - a)^m . n(x - b)^{n-1}$

$= (x - a)^m (x - b)^n \left[\frac{m}{x - a} + \frac{n}{x - b}\right]$

$f'(c) = f(c)\left[\frac{m}{c - a} + \frac{n}{c - b}\right] = 0$

$\Rightarrow \frac{m}{c - a} + \frac{n}{c - b} = 0$ (because $f(c) \neq 0$)

$\Rightarrow \frac{m}{c - a} = -\frac{n}{c - b}$

$\Rightarrow \frac{m}{c - a} = \frac{n}{b - c}$

$\Rightarrow \frac{m}{n} = \frac{c - a}{b - c}$

Since, m and n, both are +ve integers then (c – a) and (b – c) must be +ve or –ve at a time.

Since, a < c < b.

It verify Rolle's mean value Theorem.

Q.13 f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that f(0) = 5, g(0) = 0, f(2) = 8, g(2) = 1. Show that there exists a number c satisfying 0 < c < 2 and $f'(c) = 3g'(c)$

Q.14 Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

Part-B Passage based objective questions

Passage I (Question 15 to 17)

Consider the function

$f(x) = x^2 f(1) - xf'(2) + f''(3)$ such that f(0) = 2

Q.15 The values of $f'(1)$ is-

(A) 0 (B) 1

(C) 2 (D) –1

Sol. **[A]**

$f(x) = x^2 f(1) - xf'(2) + f'')3)$

$f''(3) = 2$

Now, $f'(x) = 2x f(1) - f'(2)$

$f''(x) = 2f(1)$

$f(1) = 1$

$f'(2) = 2$

$f(x) = x^2 - 2x + 2$

$f'(1) = 0$

Q.16 Equation of tangent to y= f(x) at x = 3 is-

(A) y = x – 7 (B) $y = \frac{x}{4} - 7$

(C) y = 4x – 7 (D) none of these

Sol. **[C]**

$f'(x) = 2x - 2$

$= 4$

$f(3) = 5$

(3, 5)

$y - 5 = 4(x - 3)$

$y = 4x - 7$

Q.17 The angle of intersection of y = f(x) and $y = 2e^{2x}$ is-

(A) $\tan^{-1}\left(\frac{3}{4}\right)$ (B) $\tan^{-1}\left(\frac{4}{3}\right)$

(C) 0 (D) none of these

Sol. **[D]**

$x^2 - 2x + 2 = 2e^{2x}$

x = 0, y = 2, (0, 2)

$(y_1')_{(0, 2)} = -2$

$(y_2')_{(2, 2)} = 4e^{2x} = 4$

$\tan\theta = \left|\frac{4 + 2}{1 - 8}\right|$

$\theta = \tan^{-1}\left(\frac{6}{7}\right)$

Passage II (Question 18 to 20)

Let $y = f(x)$ be a differentiable function which satisfies $f'(x) = f^2(x)$ and $f(0) = -\frac{1}{2}$. The graph of the differentiable function $y = g(x)$ contains the point (0, 2) and has the property that for each number 'P', the line tangent to $y = g(x)$ at (P, g(P)) intersects x-axis at P + 2.

Q.18 If the tangent is drawn to the curve $y = f(x)$ at a point where it cross the y-axis then its equation is-

(A) $x - 4y = 2$ (B) $x + 7y = 2$

(C) $x + 4y + 2 = 0$ (D) none of these

Sol. **[A]**

$f'(x) = f''(x)$

$y = f(x)$

$$\frac{dy}{dx} = y^2$$

$$\Rightarrow \int \frac{dy}{y^2} = \int dx$$

$$\Rightarrow -\frac{1}{y} = x + c$$

$c = 2$

$$\Rightarrow x + \frac{1}{y} + 2 = 0$$

at $x = 0$, $y = -1/2$

$$y = -\frac{1}{x+2}$$

$$y' = \frac{1}{(x+2)^2}$$

$$y'_{(0,-1/2)} = \frac{1}{4}$$

$$y + \frac{1}{2} = \frac{1}{4}(x)$$

$4y + 2 = x$

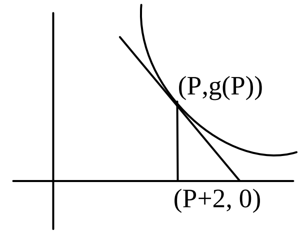
$x - 4y = 2$

Q.19 The function $y = g(x)$ is given by-

(A) $\frac{e^{-x/2}}{2}$ (B) $e^{-x/2}$

(C) $2.e^{-x/2}$ (D) $e^{-x/2} + 2$

Sol. **[C]**



$y - g(p) = g'(p)(x - p)$

$y = 0$

$$\frac{-g(p)}{g(p)} + p = p + 2$$

$-g(p) = 2g'(p)$

$$-y = 2\frac{dy}{dx}$$

$$-\int dx = 2\int \frac{dy}{dx}$$

$\Rightarrow x + c = 2\,\lambda ny$

$c = 2\lambda n\, 2$

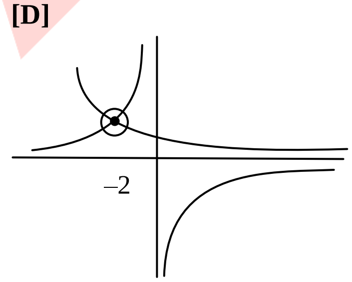
$$\Rightarrow -x = 2\,\lambda n \frac{y}{2}$$

$y = 2e^{-x/2}$

Q.20 The number of point intersection of $y = f(x)$ and $y = g(x)$

(A) 4 (B) 0 (C) 2 (D) 1

Sol. **[D]**



one intersection point

Passage III (Question 21 to 23)

Let f(x) be differentiable at every value of x and suppose that $f(1) = 1$, that $f'(x) < 0$ on $x \in (-\infty, 1)$ and that $f'(x) > 0$ on $x \in (1, \infty)$. Suppose g(x) be differentiable on [a, b] and that $f(a) = g(a)$ and $f(b) = g(b)$.

Q.21 Tangents to f(x) & g(x) are parallel at -

(A) least one point between a & b

(B) most one point between a & b

(C) exactly one point between a & b

(D) no point between a & b

Sol. **[A]**

From Rolle's theorem

i.e. $f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0 \Rightarrow f(b) = f(a)$

Also $g'(c) = \dfrac{g(b)-g(a)}{b-a} = 0 \Rightarrow g(b) = g(a)$

Hence, tangents to f(x) and g(x) must be parallel at least one point between a and b.

$\therefore$ option [A] is correct answer.

Q.22 If it is known that $f''(x) > g''(x)$ throughout the interval [a, b], then the tangents to f(x) & g(x) are parallel at -

(A) least once in between a & b
(B) most once in between a & b
(C) exactly once in between a & b
(D) no where in between a & b

Sol. **[C]**

$f''(x) > g''(x)$

then tangent to f(x) and g(x) must be parallel at exactly once in between a and b.

$\therefore$ option [C] is correct answer.

Q.23 The equation f (x) = 0 has-

(A) at least one real root
(B) at most one real root
(C) exactly one real root
(D) no real root

Sol. **[B]**

f(x) = 0

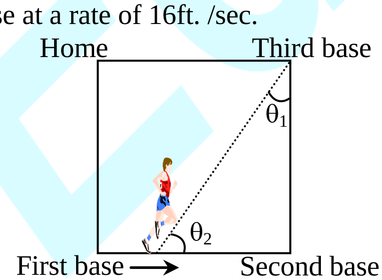
i.e. f(x) must have at most real roots

i.e. at least two real roots.

$\therefore$ option [B] is correct answer.

Passage IV (Question 24 to 26)

The baseball diamond is a square 90ft. on a side. A player runs from first base to second base at a rate of 16ft. /sec.

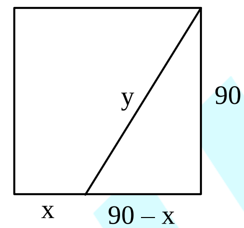


Q.24 At what rate is the player's distance from third base changing when the player is 30ft from first base?

(A) $\dfrac{30}{\sqrt{13}}$ ft/sec (B) $-\dfrac{32}{\sqrt{13}}$ ft/sec

(C) $-\dfrac{30}{\sqrt{13}}$ ft/sec (D) $\dfrac{32}{\sqrt{13}}$ ft/sec

Sol. **[B]**



$y = \sqrt{90^2 + (90-x)^2}$

$\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{90^2 + (90-x)^2}}$

$x - 2\,(90-x)\,\dfrac{dx}{dt}$

$y'_{(30)} = \dfrac{-60}{\sqrt{90^2 + 60^2}} \times 16$

$= -\dfrac{32}{\sqrt{13}}$ ft/sec

Q.25 At what rate the angle θ_1 is changing when the player is at 30ft from first base?

(A) $-\dfrac{8}{65}$ rad/sec (B) $\dfrac{-4}{65\sqrt{13}}$ rad/sec

(C) $\dfrac{8}{65}$ rad/sec (D) $\dfrac{4}{65\sqrt{13}}$ rad/sec

Sol. **[A]**

$\tan\theta_1 = \dfrac{90-x}{90}$

$\Rightarrow \sec^2\theta_1 . \dfrac{d\theta_1}{dt} = -\dfrac{1}{90}\left(\dfrac{dx}{dt}\right)$

$\Rightarrow \dfrac{d\theta_1}{dt} = -\dfrac{\cos^2\theta_1}{90} \times 16$

$= \dfrac{-90 \times 90 \times 10}{90 \times 11700} = -\dfrac{8}{65}$ rad/sec

Q.26 If the player slides into second base at the rate of 15 ft/sec. then at what rate the angle θ_2 changing as the player touches second base?

(A) $-\dfrac{1}{6}$ rad/sec (B) $\dfrac{8}{65}$ rad/sec

(C) $-\dfrac{8}{65}$ rad/sec (D) $\dfrac{1}{6}$ rad/sec

Sol. **[D]**

EXERCISE # 4

Old IIT-JEE Questions

Q.1 The point (s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are)

[IIT Sc 2002]

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$

(C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Sol. **[D]**

$y^3 + 3x^2 = 12y$

Differentiating above curve w.r.t. x, we get

$3y^2 \times \frac{dy}{dx} + 6x = 12\frac{dy}{dx}$

$6x = \frac{dy}{dx} = (12 - 3y^2)$

$\frac{dy}{dx} = \frac{6x}{3(4-y^2)}$ … (1)

Use, given condition, slope of tangent is ∞

Hence, $\frac{dy}{dx} = \frac{6x}{3(4-y^2)} = \frac{1}{0}$

$\Rightarrow 4 - y^2 = 0$

$\Rightarrow y = \pm 2$

When $y = 2 \Rightarrow 3x^2 = 12y - y^3$

$3x^2 = 24 - 8$

$= 16$

$x = \pm\frac{4}{\sqrt{3}}$

when $y = -2 \Rightarrow 3x^2 = -24 - 8 = -$ve (rejected)

Hence, required points $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

∴ option (D) is correct answer.

Q.2 According to mean value theorem in the interval $x \in [0, 1]$ which of the following does not follow- **[IIT Sc 2003]**

(A) $f(x) = \frac{1}{2} - x \quad ; x < \frac{1}{2}$

$= \left(\frac{1}{2} - x\right)^2 ; x \ge \frac{1}{2}$

(B) $f(x) = \frac{\sin x}{x} \quad ; x \ne 0$

$= 1 \quad ; x = 0$

(C) $f(x) = x\,|x|$

(D) $f(x) = |x|$

Sol. **[A]**

There is only one function in option (a) whose critical pt. $\frac{1}{2} \in (0, 1)$ for the rest of the parts critical pt. $0 \notin (0, 1)$. It can be easily seen that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1).

Now for $f(x) = \begin{cases} \left(\frac{1}{2} - x\right) & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \ge \frac{1}{2} \end{cases}$

Here $f'\left(\frac{1^-}{2}\right) = -1$ and $f'(1/2^+) = -2\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

Θ $f'\left(\frac{1^-}{2}\right) \ne (1/2^+)$

∴ f is not differentiable at $1/2 \in (0, 1)$

∴ LMV is not applicable for this function is [0, 1]

Q.3 Let $f(x) = x^\alpha \log x$ for $x > 0$ & $f(0) = 0$, follows roll's theorem for [0, 1] then α is-

[IIT Sc 2004]

(A) – 2 (B) –1

(C) 0 (D) 2

Sol. **[C]**

$f(x) = x^2 \log x$ for $x > 0$

$f(0) = 0$

$f(1) = (1)^\alpha . \log 1 = 0$

$f(1) = f(0) = 0$

According to Rolle's theorem

$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 0}{1} = 0$

Differentiate $f(x) = x^\alpha . \log x$ w.r.t. x, we get

$f'(x) = \alpha x^{\alpha-1} \log x + x^{\alpha} . \frac{1}{x}$

$= \alpha . x^{2-1} \log x + x^{2-1}$

$f'(c) = 2c^{\alpha-1} \log c + c^{\alpha-1} = 0$

$= c^{\alpha-1} [\alpha . \log c + 1] = 0$

$\Rightarrow \alpha . \log c + 1 = 0$

$\Rightarrow \alpha = -\frac{1}{\log c}$

Put $\alpha = -2 \Rightarrow \log c = -1/\alpha = \frac{1}{2}$

$\Rightarrow c = e^{1/2} > 1$ (rejected)

Put $\alpha = -1 \Rightarrow \log c = 1 \Rightarrow c = e >$ (rejected)

Put $\alpha = 0 \Rightarrow \log c = 0 \Rightarrow c = e^{0} = 1$

Put $\alpha = \frac{1}{2} \Rightarrow \log c = -2 \Rightarrow c = e^{-2} < 1$

Q.4 For any two distinct real numbers x_1 & x_2, $y = f(x)$ is satisfying the condition, $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$. Find the equation of the tangent at the point (1, 2) to the curve $y = f(x)$. **[IIT 2005]**

Sol. Given that, $| f(x_1) - f(x_2) | < (x_1 - x_2)^2$, $\forall\ x_1, x_2 \in R$

Let $x_1 = x + h$ and $x_2 = x$ then we get

$| f(x + h) - f(x) | < h^2$

$\Rightarrow\ | f(x + h) - f(x) | < | h |^2$

$\Rightarrow\ \left| \frac{f(x+h) - f(x)}{h} \right| < | h |$

Taking limit as $h \to 0$ on both sides we get

$\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| < \delta$ (a small +ve number)

$\Rightarrow\ | f'(x) | < \delta \Rightarrow f'(x) = 0$

$\Rightarrow$ f(x) is a constant function. Let $f(x) = k$ i.e., $y = k$

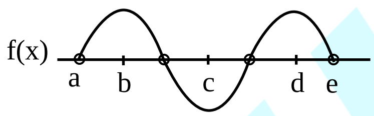
As f(x) passes through $(1, 2) \Rightarrow y = 2$

$\therefore$ Equation of tangent at (1, 2) is

$y - 2 = 0 (x - 1)$ i.e., $y = 2$

Q.5 If f(x) is twice differentiable and $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$, where $a < b < c < d < e$ then find the minimum number of zeroes of $g(x) = \{f'(x)\}^2 + \{f''(x) f(x)\}$ in [a, e] is? **[IIT 2006]**

Sol. $g(x) = \frac{d}{dx} (f'(x) f(x)) = \frac{d}{dx} h(x)$



$f(x) = 0 \Rightarrow 4$ roots

$f'(x) = 0 \Rightarrow 3$ roots

$\Rightarrow h(x) = 0 \Rightarrow 7$ roots

$\Rightarrow g(x) = 0 \Rightarrow 6$ roots (min)

EXERCISE # 5

Q.1 Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$. **[IIT-1985]**

Sol. Equation of given curve is

$y = \cos(x + y),\ -2\pi \le x \le 2\pi$

Differentiating with respect to x,

$$\frac{dy}{dx} = -\sin(x+y)\cdot\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow [1+\sin(x+y)]\frac{dy}{dx} = -\sin(x+y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)} \qquad \ldots\ldots(1)$$

Since the tangent to given curve is parallel to $x + 2y = 0$

$$\therefore \frac{-\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$$

[For parallel line $m_1 = m_2$]

$\Rightarrow$ $2\sin(x + y) = 1 + \sin(x + y)$

$\Rightarrow$ $\sin(x + y) = 1$

Thus, $\cos(x + y) = 0$

Using eq. of curve and above result,

we get, $y = 0$

$\Rightarrow \sin x = 1 \Rightarrow x = n\pi + (-1)^n \pi/2,\ n \in Z$

$\Rightarrow x = \pi/2, -3\pi/2$

which belong to the interval $[-2\pi, 2\pi]$

Thus the pts on curve at which tangents are parallel to given line are $(\pi/2, 0)$ and $(-3\pi/2, 0)$

The equation of tangent at $(\pi/2, 0)$ is

$$y - 0 = -\frac{1}{2}(x - \pi/2) \Rightarrow 2y = -x + \pi/2$$

$\Rightarrow 2x + 4y - \pi = 0$

The equation of tangent at $(-3\pi/2, 0)$ is

$$y - 0 = -\frac{1}{2}(x + 3\pi/2)$$

$\Rightarrow 2y = -x - 3\pi/2 \Rightarrow 2x + 4y - 3\pi = 0$

Thus the required equations of tangent are

$2x + 4y - \pi = 0$ and $2x + 4y + 3\pi = 0$.

Q.2 If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then **[IIT-1986]**

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$

(C) $a < 0, b > 0$ (D) $a < 0, b < 0$

Sol. **[B, C]**

Here, $xy = 1 \Rightarrow y = \frac{1}{x}$ or $\frac{dy}{dx} = -\frac{1}{x^2}$.

Thus, slope of normal $= x^2$ (which is always positive) and it is given $ax + by + c = 0$ is normal whose slope $= -\frac{a}{b}$.

$\Rightarrow -\frac{a}{b} > 0$ or $\frac{a}{b} < 0$,

$\therefore$ a and b are of opposite sign.

Q.3 Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$. **[IIT-1993]**

Sol. The given curve is $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

Here at $x = 0$, $y = (1 + 0)^y + \sin^{-1}(0)$

$\Rightarrow y = 1$

$\therefore$ Pt. at which normal has been drawn is (0, 1).

For slope of normal we need to find $\frac{dy}{dx}$, and for that we consider the curve as

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

where $u = (1 + x)^y$ ....(i)

and $v = \sin^{-1}(\sin^2 x)$ ....(ii)

Taking log on both sides of equation (i) we get

$\log u = y \log(1 + x)$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{1+x} + \log(1+x).\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = (1+x)^y\left[\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right]$$

Also $v = \sin^{-1}(\sin^2 x)$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-\sin^4 x}}\ .\ 2\sin x\cos x$$

$$\Rightarrow \frac{dv}{dx} = \frac{2\sin x}{\sqrt{1+\sin^2 x}}$$

Thus we get, $$\frac{dy}{dx} = (1+x)^2\left[\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right] + \frac{2\sin x}{\sqrt{1+\sin^2 x}}$$

$$\Rightarrow [1-(1+x)^y \log(1+x)]\frac{dy}{dx} = y(1+x)^{y-1} + \frac{2\sin x}{\sqrt{1+\sin^2 x}}$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{y(1+x)^{y-1} + \dfrac{2\sin x}{\sqrt{1+\sin^2 x}}}{1-(1+x)^y \log(1+x)}\right]$$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = 1 \quad \therefore \text{ Slope of normal} = -1$$

$\therefore$ Equation of normal to given curve at (0, 1) is

$y - 1 = -1(x - 0)$

$\Rightarrow x + y = 1.$

Q.4 Tangent at a point P_1 {other than (0, 0)} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissa of $P_1, P_2, P_3, \ldots, P_n$, form a GP. Also find the ratio [area $(\Delta P_1P_2P_3)$]/[area $(\Delta P_2P_3P_4)$] **[IIT-1993]**

Sol. The given curve is $y = x^3$ …..(i)

Let the pt. P_1 be (t, t^3), $t \neq 0$

Then slope of tangent at P_1 is $= \frac{dy}{dx} = (3x^2)_{x=t} = 3t^2$

$\therefore$ Equation of tangent at P_1 is

$y - t^3 = 3t^2(x - t)$

$\Rightarrow y = 3t^2x - 2t^3$

$\Rightarrow 3t^2x - y - 2t^3 = 0$ …..(2)

Now this tangent meets the curve again at P_2 which can be obtained by solving (1) and (2)

i.e., $3t^2x - x^3 - 2t^3 = 0$

or $x^3 - 3t^2x + 2t^3 = 0$

$(x - t)^2(x + 2t) = 0 \Rightarrow x = -2t$

as $x = t$ is for P_1

$\therefore$ $y = -8t^3$

Hence pt P_2 is $(-2t, -8t^2) = (t_1, t_1^3)$ say.

Similarly we can find that tangent at P_2 meets the curve again at P_3 $(2t_1, 8t_1^3)$ i.e., $(4t, 64t^3)$.

Similarly $P_4 \equiv (-8t, -512t^3)$ and so on.

We observe that abscissa of pts. $P_1, P_2, P_3, \ldots$ are $t, -2t, 4t, \ldots$ which form a GP with common ratio -2. Also ordinates of these pts. $r^3, -8t^3, 64t^3, \ldots$ also form a GP with common ratio -8.

Now, $$\frac{Ar(\Delta P_1P_2P_3)}{Ar(\Delta P_2P_3P_4)} = \frac{\begin{vmatrix} 1 & t & t^3 \\ 1 & -2t & -8t^3 \\ 1 & 4t & 64t^3 \end{vmatrix}}{\begin{vmatrix} 1 & -2t & -8t^3 \\ 1 & 4t & -64t^3 \\ 1 & -8t & -512t^3 \end{vmatrix}}$$

$$= \frac{t^4\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -8 \\ 1 & 4 & 64 \end{vmatrix}}{(-2)(-8)t^4\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -8 \\ 1 & 4 & -64 \end{vmatrix}} = \frac{1}{64} \text{ sq. units}$$

Q.5 The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at P(–2, 0) and cuts the y-axis at a point Q, where its gradient is 3. Find a, b, c. **[IIT-1994]**

Sol. Given that $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at P (– 2, 0)

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=-2} = 0$ and P (– 2, 0) lies on curve

$\Rightarrow 3ax^2 + 2bx + c]_{x=-2} = 0$

$\Rightarrow 12a - 4b + c = 0$ ……(1)

and $-8a + 4b - 2c + 5 = 0$ ……(2)

[Θ (– 2, 0) lies on curve]

Also the curve cuts the y-axis at Q

$\therefore$ for $x = 0, y = 5$

$\therefore$ Q (0, 5)

At Q gradient of the curve is 3

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 3 \Rightarrow 3ax^2 + 2bx + c]_{x=0} = 3$

$\Rightarrow c = 3$ ……(3)

Solving (1), (2) and (3), we get

$a = -\frac{1}{2}$, $b = -\frac{3}{4}$ and $c = 3$.

Q.6 Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H = ...and V = ... **[IIT-1994]**

Sol. The given curve is C : $y^3 - 3xy + 2 = 0$

Differentiating it with respect to x, we get

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{-x+y^2}$$

$\therefore$ slope of tangent to C at pt (x_1, y_1) is

$$\frac{dy}{dx} = \frac{y_1}{-x_1+y_1^2}$$

For horizontal tangent $\frac{dy}{dx} = 0 \Rightarrow y_1 = 0$

For $y_1 = 0$ in C, we get no value of x_1

$\therefore$ there is no pt. on C at which tangent is horizontal $\therefore H = \phi$

For vertical tangent $\frac{dy}{dx} = \frac{1}{0} \Rightarrow -x_1 + y_1^2 = 0$

$\Rightarrow x_1 = y_1^2$

From C, $y_1^3 - 3y_1^3 + 2 = 0 \Rightarrow y_1^3 = 1 \Rightarrow y_1 = 1$

$\Rightarrow x_1 = 1$

$\therefore$ There is only one pt. (1, 1) at which vertical can be drawn

$\therefore V = \{(1, 1)\}$

Q.7 If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $3\pi/4$ with the positive x-axis, then $f'(3) =$ **[IIT Sc 2000]**

(A) –1 (B) $-\frac{3}{4}$

(C) $\frac{4}{3}$ (D) 1

Sol. **[D]**

$\tan(3\pi/4) = m = -1$

Let $y = ax^2 + bx$

$$\left.\frac{dy}{dx}\right|_{(3,4)} = 2ax + b\big|_{(3,4)} = 1 = 6a + b$$

Also, $4 = 9a + 3b$

$$\begin{array}{rcrcr} 9a & + & 3b & = & 4 \\ _{-}18a & + & _{-}3b & = & _{-}3 \\ \hline -9a & & & = & 1 \end{array} \Rightarrow a = -\frac{1}{9}$$

$$b = 1 - 6a = 1 - 6\left(-\frac{1}{9}\right)$$

$$b = 1 + \frac{2}{3} = \frac{5}{3}$$

$$f(x) = y = -\frac{1}{9}x^2 + \frac{5}{3}x$$

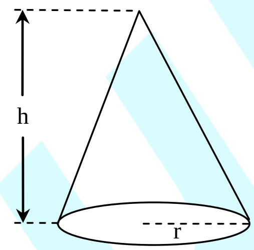
$$f'(x) = -\frac{2}{9}x + \frac{5}{3}$$

$$f'(3) = -\frac{2}{3} + \frac{5}{3} = \frac{3}{3} = 1$$

$\therefore$ option (D) is correct answer.

Q.8 A filter paper in the form of a right circular cone has diameter of its base equal to 8cm. and depth 6 cm. Water is flowing out through the bottom of the cone at the rate of 50 cc/min. Find the rate at which the level of water is falling when the height of the water is 3cm.

Sol. Given



$$\left.\begin{array}{l} r = 4\text{ cm} \\ h = 6\text{ cm} \end{array}\right\} \frac{r}{h} = \frac{4}{6} = \frac{2}{3}$$

$$r = \frac{2}{3} \times h$$

$$\frac{dv}{dt} = 50\text{cc/min.}$$

Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 \times h$$

$$V = \frac{1}{3}\pi \times \frac{4}{9}h^3$$

$$V = \frac{4\pi}{27} \times h^3$$

Differentiating above expression w.r.t, time, we get

$$\frac{dv}{dt} = \frac{4\pi}{27} \times 3h^2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{dv}{dt} \times \frac{27}{4\pi \times 3h^2}$$

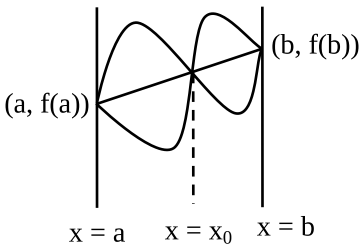
$$\frac{dh}{dt} = \frac{50 \times 27}{4\pi \times 3 \times (3)^2}$$

$$= \frac{50 \times 27}{4\pi \times 3 \times 9}$$

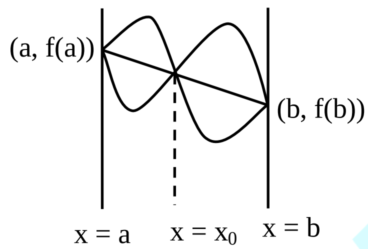
$$\frac{dh}{dt} = \frac{25}{2\pi}\text{ cm/min.}$$

Q.9 Let f be continuous on [a, b] and assume the second derivative f'' exists on (a, b). Suppose that the graph of f and the segment joining the point (a, f(a)) and (b, f(b)) intersect at a point $\{x_0, f(x_0)\}$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$

Sol.



or



Let $g(x) = f(x) - f(x_0)$

$g'(x) = 0$ has two roots (at least)

Hence $g''(x) = 0$ has at least 1 root

so $f''(c)$ for $c \in (a, b)$

Hence proved

Q.10 For what value of a, m and b does the function $f(x) = \begin{cases} 3; & x = 0 \\ -x^2 + 3x + a; & 0 < x < 1 \\ mx + b; & 1 \le x \le 2 \end{cases}$ satisfy the hypothesis of the mean value theorem for the interval [0, 2].

Sol. For mean value theorem function should be continuous and differentiable in the given interval

for continuity at x = 0 at x = 1

$f(0^+) = a = 3$ $f(1^-) = a + 2 = f(1^+) = b + m$

$b + m = 5$

for differentiability at x = 1

$f'(1^-) = -2 + 3$ & $f'(1^+) = m$

$m = 1 \Rightarrow b = 4$

so a = 3, m = 1, b = 4

Q.11 Assume that f is continuous on [a, b], a > b and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

Sol. Let $g(x) = \frac{f(x)}{x}$

g(x) is continuous in [a, b]

differentiable in (a, b)

g(a) = g(b) hence Rolle's theorem can be applicable

so $g'(x_0) = 0$ for $x_0 \in (a, b)$

$\frac{x_0 f'(x_0) - f(x_0)}{x^2} = 0 \Rightarrow x_0 f'(x_0) - f(x_0) = 0$

Hence proved

Q.12 Let f be continuous on [a, b] and differentiable on (a, b). If f(a) = a and f(b) = b, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Sol. Let g(x) = f(x) – x Rolle's theorm can be applicable to g(x)

g(a) = 0, g(b) = 0

$g'(c_1) = f'(c_1) - 1 = 0$ $c_1 \in (a, b)$

$g'(c_2) = f'(c_2) - 1 = 0$ $c_2 \in (a, b)$

so $f'(c_1) + f'(c_2) = 1$

Hence proved

Q.13 Let a > 0 and f be continuous in [–a, a]. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If f(a) = a and f(–a) = –a, show that f(0) = 0.

Sol. g(x) = f(x) – x

$g'(x) = f'(x) - 1 \Rightarrow g'(x) < 0$ or $= 0$

$g(x) \downarrow$ or continuous

also $g(a) = g(-a) \Rightarrow \downarrow$ not applicable

so g(x) is continuous $\Rightarrow$ also g(a) = 0

so $g(0) = 0 \Rightarrow f(0) = 0$

Hence prove

Passage (Question 14 to 16)

Given the continuous function

$$y = f(x) = \begin{cases} x^2 + 10x + 8, & x \le -2 \\ ax^2 + bx + c, & -2 < x < 0, \quad a \ne 0 \\ x^2 + 2x, & x \ge 0 \end{cases}$$

If a line L touches the graph of y = f(x) at three points then-

Q.14 The gradient of the line 'L' is equal to-
(A) 1 (B) 2 (C) 4 (D) 6

Sol.[C] Function is continuous

at x = –2

at x = 0

c = 0

–8 = 4a – 2b + c

so 2a = b – 4 & c = 0

Slope of line = $\frac{f(0) - f(-2)}{2}$ (by LMVT)

$= \frac{8}{2} = 4$

Q.15 The value of (a + b + c) is equal to-
(A) $5\sqrt{2}$ (B) 5 (C) 6 (D) 7

Sol.[D] Function is continuous

at x = –2

at x = 0

c = 0

–8 = 4a – 2b + c

so 2a = b – 4 & c = 0

Equation of line

for curve $y = x^2 + 2x$

$y' = 2x + 2 = 4 \Rightarrow x = 1 \Rightarrow y = 3$

tangent touches at (1, 3)

equation of tangent y – 3 = 4 (x – 1)

$\Rightarrow y = 4x - 1$

also for II^nd curve

$y = ax^2 + bx + 0$

$4x - 1 = ax^2 + bx$

$ax^2 + (b - 4)x + 1 = 0$

$\Rightarrow D = 0 \Rightarrow (b - 4)^2 = 4a \Rightarrow 4a^2 = 4a$

$\Rightarrow a \ne 0 \Rightarrow a = 1$, so b = 6

a + b + c = 7

Q.16 If y = f(x) is differentiable at x = 0 then the value of b
(A) is –1 (B) is 2
(C) is 4 (D) can't be determined

Sol.[B] Function is continuous

at x = –2

at x = 0

c = 0

–8 = 4a – 2b + c

so 2a = b – 4 & c = 0

Differentiability at x = 0

$f'(0^-) = 2a(0) + b$

$f'(0^+) = 2(0) + 2$

$\Rightarrow b = 2$

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	B	D	C	C	B	A	B	C	C	B	D	A	A
Q.No.	14	15	16	17	18	19	20	21	22	23	24	25	
Ans.	A	A	B	B	B	C	B	A	D	D	A	C	

26. True **27.** False **28.** x = 12y **29.** – 1/3 **30.** a = 1

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8	9	10
Ans.	B	C	C	B	A	C	B	B	B	D
Q.No.	11	12	13	14	15	16	17	18	19	
Ans.	B	B	B	B	D	C	A	B	D	

(PART-B)

Q.No.	20	21	22	23	24	25	26	27	28
Ans.	B,C	A,B,C	A,D	A,B,D	A,B	A,B,D	A,B,C,D	A,B,C,D	A,D

(PART-C)

Q.No.	29	30	31	32	33	34	35
Ans.	C	A	A	C	A	A	A

(PART-D)

36. (A) $\rightarrow$ R; (B) $\rightarrow$ Q; (C) $\rightarrow$ P; (D) $\rightarrow$ S

37. (A) $\rightarrow$ P; (B) $\rightarrow$ R; (C) $\rightarrow$ Q; (D) $\rightarrow$ S

38. (A) $\rightarrow$ Q, R; (B) $\rightarrow$ R; (C) $\rightarrow$ Q; (D) $\rightarrow$ P, S

EXERCISE # 3

(3) (i) $\theta = 90^\circ$ **(ii)** $\theta = \tan^{-1}(2\sqrt{2})$ **(iii)** $\pi/4$, $\cot^{-1}(\pi)$ **(iv)** $\tan^{-1}3$

(7) 900 cc/sec. **(8)** $-\frac{5}{6}$ m/sec **(9)** $\frac{1}{(48\pi)}$ cm/sec **(10)** (3, 16/3)

(12) $c = \frac{mb + na}{m + n}$

Q.No.	15	16	17	18	19	20	21	22	23	24	25	26
Ans.	A	C	D	A	C	D	A	C	D	B	A	D

EXERCISE # 4

(1) D **(2)** A **(3)** D **(4)** y = 2 **(5)** 6

EXERCISE # 5

(1) $x + 2y = \frac{\pi}{2}$ and $x + 2y = -\frac{3\pi}{2}$ **(2)** B, C **(3)** x + y = 1 **(4)** 1 : 16

(5) $a = -\frac{1}{2}$, $b = -\frac{3}{4}$, c = 3 **(6)** ϕ, {1, 1} **(7)** D **(8)** $\frac{25}{2\pi}$ cm/min

(10) a = 3, b = 4, m = 1 **(14)** C **(15)** D **(16)** B