# Radii of Circle
# EXERCISE # 1

**Questions based on** **Circumcircle, Incircle, Circum radius & Inradius**

**Q.1** In an equilateral triangle of side $2\sqrt{3}$ cms, the circum radius is -

(A) 1 cm (B) $\sqrt{3}$ cm

(C) 2 cm (D) $2\sqrt{3}$ cm

**Sol.[C]** $\because$ triangle is equilateral so

$\Delta = \frac{\sqrt{3}}{4}a^2$ and $R = \frac{a^3}{4\Delta}$

$\Rightarrow R = \frac{a^3}{\frac{4\sqrt{3}}{4}a^2} = \frac{a}{\sqrt{3}}$ $\because a = 2\sqrt{3} \Rightarrow R = 2$

**Q.2** If $8R^2 = a^2 + b^2 + c^2$, then the $\Delta$ is -

(A) Right angled (B) Isosceles

(C) Equilateral (D) None of these

**Sol.[A]** $\because a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$

$\Rightarrow 8R^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C$

$\Rightarrow \cos^2 A + \cos^2 B - \sin^2 C = 0$

$\Rightarrow (\cos^2 A - \sin^2 C) + \cos^2 B = 0$

$\Rightarrow \cos(A + C)\cos(A - C) + \cos^2 B = 0$

$\Rightarrow -\cos B[\cos(A - C) + \cos(A + C)] = 0$

$\Rightarrow \cos A \cos B \cos C = 0$

$\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow A = \pi/2$ or $B = \pi/2$ or $C = \pi/2$

triangle is right angled.

**Q.3** In a $\Delta ABC$, $2R^2 \sin A \sin B \sin C =$

(A) $\Delta$ (B) $2\Delta$

(C) $3\Delta$ (D) $4\Delta$

**Sol.[A]** $\because 2R^2 \sin A \sin B \sin C$

$= \frac{1}{2}(2R \sin A)(2R \sin B)\sin C = \frac{1}{2}ab \sin C = \Delta$

**Q.4** If the sides of a triangle are 3 : 7 : 8 then R : r =

(A) 2 : 7 (B) 7 : 2

(C) 3 : 7 (D) 7 : 3

**Sol.[B]** Given $a = 3x,\ b = 7x,\ c = 8x$

$S = 9x \quad \Delta = 6\sqrt{3}x^2$

$R = \frac{abc}{4\Delta} = \frac{168x^3}{24\sqrt{3}x^2} = \frac{7x}{\sqrt{3}}$

$r = \frac{\Delta}{S} = \frac{6\sqrt{3}x^2}{9x} = \frac{2x}{\sqrt{3}} \Rightarrow \frac{R}{r} = \frac{7x}{\sqrt{3}}.\frac{\sqrt{3}}{2x} = \frac{7}{2}$

**Questions based on** **Escribed circle of a triangle and their radii**

**Q.5** If the sides be a, b, c then $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} =$

(A) $c/r_3$ (B) $c/r_2$

(C) $c/r$ (D) None of these

**Sol.[A]** $\because r_1 = \frac{\Delta}{s-a},\ r_2 = \frac{\Delta}{s-b},\ r = \frac{\Delta}{s}$

$\Rightarrow \frac{\frac{\Delta}{s-a} - \frac{\Delta}{s}}{a} + \frac{\frac{\Delta}{s-b} - \frac{\Delta}{s}}{b}$

$= \frac{\Delta(s-s+a)}{as(s-a)} + \frac{\Delta(s-s+b)}{bs(s-b)}$

$= \frac{\Delta}{s}\left(\frac{s-b+s-a}{(s-a)(s-b)}\right) = \frac{\Delta c}{s(s-a)(s-b)}$

$= \frac{\Delta c(s-c)}{\Delta^2} = \frac{c}{\frac{\Delta}{(s-c)}} = \frac{c}{r_3}$

**Q.6** If $r_1 = r_2 + r_3 + r$, then the $\Delta$ is -

(A) Equilateral (B) Isosceles

(C) Right angled (D) None of these

**Sol.[C]** $r_1 - r = r_2 + r_3$

$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$

$\Rightarrow \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \frac{(s-b)(s-c)}{s(s-a)} = 1$

$\Rightarrow \tan^2\frac{A}{2} = 1 \Rightarrow \tan\frac{A}{2} = 1$

$\Rightarrow \frac{A}{2} = 45° \Rightarrow A = 90°$

**Q.7** In an equilateral triangle, the in-radius, circum-radius and one of the ex-radii are in the ratio-

(A) 2 : 3 : 5 (B) 1 : 2 : 3

(C) 1 : 3 : 7 (D) 3 : 7 : 9

**Sol.[B]** In equilateral triangle

$\Delta = \frac{\sqrt{3}}{4}a^2$

$R = \frac{abc}{4\Delta} = \frac{a^3}{4.\frac{\sqrt{3}}{4}a^2} = \frac{a}{\sqrt{3}}$

$r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a} = \frac{a}{2\sqrt{3}}$

$r_1 = \frac{\Delta}{s-a} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{1}{2}a} = \frac{\sqrt{3}a}{2} = \frac{3a}{2\sqrt{3}}$

$\Rightarrow r : R : r_1 = 1 : 2 : 3$

**Q.8** If $\lambda_1$, $\lambda_2$, $\lambda_3$ are respectively the perpendicular from the vertices of a triangle on the opposite side, then $\lambda_1 \lambda_2 \lambda_3 =$

(A) $\frac{a^2b^2c^2}{8R^2}$ (B) $\frac{abc}{8R^2}$

(C) $\frac{a^2b^2c^2}{8R^3}$ (D) $\frac{abc}{8R^3}$

**Sol.[C]** $\because \Delta = \frac{1}{2}a\lambda_1 = \frac{1}{2}b\lambda_2 = \frac{1}{2}c\lambda_3$

$\Rightarrow \lambda_1 = \frac{2\Delta}{a},\ \lambda_2 = \frac{2\Delta}{b},\ \lambda_3 = \frac{2\Delta}{c}$

$\Rightarrow \lambda_1\lambda_2\lambda_3 = \frac{8\Delta^3}{abc} = \frac{8\Delta^3}{4\Delta R} = \frac{2\Delta^2}{R}$

$= \frac{2a^2b^2c^2}{(4R)^2R} = \frac{a^2b^2c^2}{8R^3}$

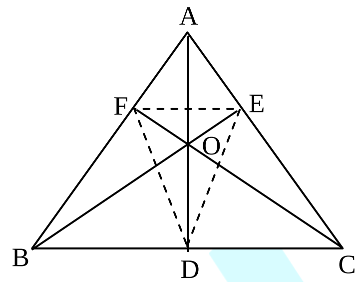
## Questions based on Geometrical distances, Orthocentre, Pedal Triangle & Regular Polygon

**Q.9** If in a triangle ABC; AD, BE and CF are the altitudes and R is the circum-radius, then the radius of the circle DEF is -

(A) R/2 (B) 2R

(C) R (D) None of these

**Sol.** **[A]**




We know that O is the orthocenter of ΔABC and O is in-center of pedal triangle DEF

EF = a cos A or R sin 2A

DE = b cos B or R sin 2B

DF = c cos C or R sin 2C

$\angle$FDE = 180 – 2A, $\angle$DEF = 180 – 2B, $\angle$EFD = 180° – 2C

Circum-radius of Δ DEF

$= \frac{\text{side}}{2\sin(\text{angle opposite to side})}$

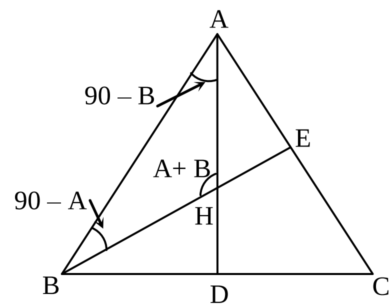
$= \frac{EF}{2\sin(180-2A)} = \frac{R\sin 2A}{2\sin 2A} = \frac{R}{2}$

**Q.10** If H is the orthocentre of the triangle ABC, then AH is equal to -

(A) a cot A (B) a cot B

(C) b cot A (D) c cot A

**Sol.** **[A]**




From ΔABH, we have

$\frac{AH}{\sin(90-A)} = \frac{AB}{\sin(A+B)} = \frac{BH}{\sin(90-B)}$

$\Rightarrow AH = \frac{c\cos A}{\sin C}$

$\Rightarrow AH = \frac{a\cos A}{\sin A}$ $\because \frac{c}{\sin C} = \frac{a}{\sin A}$

$\Rightarrow AH = a \cot A$

**Q.11** The radius of the circumscribing circle of a regular polygon of n sides each of length a is -

(A) 2a cosec $\left(\frac{\pi}{n}\right)$ (B) a cosec $\left(\frac{2\pi}{n}\right)$

(C) a cosec $\left(\frac{\pi}{n}\right)$ (D) none of these

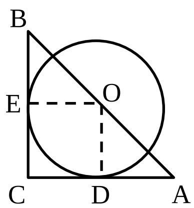
**Sol.[D]** $\because$ $R = \frac{a}{2}$ cosec$\left(\frac{\pi}{n}\right)$

**Q.12** A circle touches two of the smaller sides of a $\Delta ABC$ $(a < b < c)$ and has its centre on the greatest side. Then the radius of the circle is -

(A) $\frac{a-b-c}{2}$ (B) $\frac{abc}{2}$

(C) $\frac{2\Delta}{a+b}$ (D) none of these

**Sol.[C]**




Let OD = OE = r

in $\Delta ODA$

$$\frac{OD}{OA} = \sin A \Rightarrow OA = \frac{r}{\sin A} \quad \text{......(1)}$$

in $\Delta OEB$

$$\frac{OE}{OB} = \sin B \Rightarrow OB = \frac{r}{\sin B} \quad \text{......(2)}$$

adding (1) & (2)

$$OA + OB = c = \frac{r}{\sin A} + \frac{r}{\sin B}$$

$$\Rightarrow r\left(\frac{bc}{2\Delta} + \frac{ac}{2\Delta}\right) = c \qquad [\because \Delta = \frac{1}{2} bc \sin A]$$

$$\Rightarrow r = \frac{2\Delta}{b+a}$$

## ➤ Fill in the Blanks type Questions

**Q.13** $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} =$ ............

**Sol.** $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$

$$= (b-c).\frac{(s-a)}{5} + (c-a).\frac{(s-b)}{5} + (a-b).\frac{(s-c)}{5}$$

$$= \frac{s}{5} [b - c + c - a + a - b] - \frac{1}{5}[a(b - c) + b (c - a) + c(a - b)]$$

$$= \frac{s}{5}[0] - \frac{1}{5}[0] = 0$$

**Q.14** $r_1 \cot\frac{A}{2} + r_2 \cot\frac{B}{2} + r_3 \cot\frac{C}{2} =$ .....

**Sol.** $r_1 \cot\frac{A}{2} + r_2 \cot\frac{B}{2} + r_3 \cot\frac{C}{2}$

$$= \frac{\Delta}{(s-a)}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \frac{\Delta}{(s-b)}\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \frac{\Delta}{(s-c)}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta\left(\frac{s}{\Delta} + \frac{s}{\Delta} + \frac{s}{\Delta}\right) = 3s$$

## ➤ True or False type Questions

**Q.15** $\frac{\text{Area of the incircle}}{\text{Area of triangle}} = \frac{\pi}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$

**Sol.** $\frac{\text{Area of the incircle}}{\text{Area of triangle}}$

$$= \frac{\pi r^2}{\frac{1}{2} bc \sin A}$$

$$= \frac{16\pi R^2 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{2R^2 \sin A \sin B \sin C}$$

$$= \frac{\pi}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} \qquad \text{(True)}$$

**Q.16** $a(rr_1 + r_2r_3) = b\,(rr_2 + r_3r_1) = c(rr_3 + r_1r_2)$.

**Sol.** $a\,(rr_1 + r_2r_3)$

$$= a\Delta^2\left(\frac{1}{s(s-a)} + \frac{1}{(s-b)(s-c)}\right)$$

$$= \frac{a\Delta^2}{4}\frac{[(a+c-b)(a+b-c)+(a+b+c)(b+c-a)]}{s(s-a)(s-b)(s-c)}$$

$$= \frac{a}{4}[a^2 - (b-c)^2 + (b+c)^2 - a^2]$$

$$= \frac{a}{4}.4bc = abc$$

Similarly $b(rr_2 + r_3r_1) = c\,(rr_3 + r_1r_2) = abc$ (True)

# EXERCISE # 2

**Part-A Only Single Correct answer type Question**

**Q.1** If the angles of a triangle are in the ratio 1: 2: 3, then the sides opposite to the respective angles are in the ratio -

(A) $1 : \sqrt{2} : \sqrt{3}$ (B) $1 : \sqrt{3} : 2$

(C) $1 : \sqrt{2} : 3$ (D) $1 : 2 : 3$

**Sol.** **[B]**

$A : B : C = K : 2K : 3K$

$\Theta$ $A + B + C = K + 2K + 3K = 180^\circ \Rightarrow K = 30^\circ$

$\Rightarrow A = 30^\circ, B = 60^\circ, C = 90^\circ$

$\Theta$ $a : b : c = \sin A : \sin B : \sin C$

$= \sin 30^\circ : \sin 60^\circ : \sin 90^\circ = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$a : b : c = 1 : \sqrt{3} : 2$

**Q.2** In $\Delta ABC$, $a : b : c = (1 + x) : 1 : (1 - x)$ where $x \in (0, 1)$. If $\angle A = \frac{\pi}{2} + \angle C$, then $x =$

(A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{2\sqrt{7}}$

**Sol.** **[C]**

Sides are in A.P. so

$2b = a + c$

$\Rightarrow 2 \sin B = \sin A + \sin C$

$\Rightarrow 4 \sin \frac{B}{2} \cos \frac{B}{2} = 2 \cos \frac{A+C}{2} \cos \frac{A-C}{2}$

$\Rightarrow 2 \sin \frac{B}{2} = \cos \frac{A-C}{2}$

$\Rightarrow 2 \sin \frac{B}{2} = \frac{1}{2\sqrt{2}}$ $[\Theta\ A - C = \frac{\pi}{2}]$

Now $\frac{a}{\sin A} = \frac{c}{\sin C}$

$\Rightarrow \frac{1+x}{1-x} = \frac{\sin A}{\sin C}$

using componendo & devidendo

$\Rightarrow \frac{1}{x} = \frac{\sin A + \sin C}{\sin A - \sin C}$

$\Rightarrow \frac{1}{x} = \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{2\cos\frac{A+C}{2}\sin\frac{A-C}{2}}$

$\Rightarrow \frac{1}{x} = \tan\frac{A+C}{2}$ $[\Theta\ A - C = \frac{\pi}{2}]$

$\Rightarrow \frac{1}{x} = \cot\frac{B}{2} = \sqrt{7}$ $[\Theta\ \sin\frac{B}{2} = \frac{1}{2\sqrt{2}}]$

$\Rightarrow x = \frac{1}{\sqrt{7}}$

**Q.3** If the area of a triangle is 81 square cm and its perimeter is 27cm then its in-radius in centi-metres is -

(A) 6 (B) 3 (C) 1.5 (D) None

**Sol.** **[A]**

$\Theta$ $r = \frac{\Delta}{s}$ given $\Delta = 81$; $2s = 27$

$\Rightarrow r = 81 \times \frac{2}{27} = 6$ cm

**Q.4** abc =

(A) $Rrs$ (B) $4Rr\Delta$ (C) $4Rrs$ (D) $4Rr\Delta s$

**Sol.** **[C]**

$\Theta$ $R = \frac{abc}{4\Delta} \Rightarrow abc = 4R\Delta$

$\therefore r = \frac{\Delta}{s} \Rightarrow \Delta = rs$

$\Rightarrow abc = 4Rrs$

**Q.5** In a $\Delta ABC$, if $\frac{R}{r} \le 2$, then the triangle is -

(A) scalene (B) isosceles

(C) right angled (D) equilateral

**Sol.** **[D]**

We know that

$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$\Rightarrow \frac{r}{R} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$\Rightarrow \frac{r}{R} = 2\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\} \sin\frac{C}{2}$

$\Rightarrow \frac{r}{R} = 2\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}\sin\frac{C}{2}$

$\Rightarrow \frac{r}{R} \le 2\left(1-\sin\frac{C}{2}\right)\sin\frac{C}{2}$

$\Rightarrow \frac{r}{R} \le 2\left(\sin\frac{C}{2} - \sin^2\frac{C}{2}\right)$

$\Rightarrow \frac{r}{R} \le 2\left(\frac{1}{4} - \left(\frac{1}{2} - \sin\frac{C}{2}\right)^2\right)$

$\Rightarrow \frac{r}{R} \le \frac{2}{4} \Rightarrow \frac{R}{r} \ge 2$

But given $\frac{R}{r} \le 2$

This means

$\frac{R}{r} = 2$

It is possible only when

$\frac{A-B}{2} = 0$ and $\frac{1}{2} - \frac{C}{2} = 0$

$\Rightarrow$ A = B and C = 60

$\Rightarrow$ A = B = C = 60°

Triangle is equilateral.

**Q.6** If in a triangle $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then the triangle is-

(A) Right angled (B) Isosceles

(C) Equilateral (D) None of these

**Sol.** **[A]**

$\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right) = 2$ $\quad \because r_1 = \frac{\Delta}{s-a}$

$\Rightarrow \frac{(b-a)(c-a)}{(s-a)^2} = 2 \Rightarrow 2(b-a)(c-a) = 4(s-a)^2$

$\Rightarrow 2(bc - ac - ab + a^2) = (b + c - a)^2$

$\Rightarrow 2bc - 2ac - 2ab + 2a^2 = a^2 + b^2 + c^2 + 2bc - 2ab - 2ac$

$\Rightarrow a^2 = b^2 + c^2$ triangle is right angled.

**Q.7** If the sides be a, b, c then $(r + r_1)\tan\frac{B-C}{2} + (r + r_2)\tan\frac{C-A}{2} + (r + r_3)\tan\frac{A-B}{2} =$

(A) 1 (B) 0 (C) 2 (D) 4

**Sol.** **[B]**

Taking $(r + r_1)\tan\left(\frac{B-C}{2}\right)$

$= \Delta\left(\frac{s-a+s}{s(s-a)}\right)\frac{b-c}{b+c}\cot\frac{A}{2}$ $\quad \because r_1 = \frac{\Delta}{s-a}$

$= \Delta\left(\frac{b+c}{s(s-a)}\right)\frac{b-c}{b+c}\cot\frac{A}{2}$

$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\,.\,(b-c)\cot\frac{A}{2}$

$= (b-c)\tan\frac{A}{2}\cot\frac{A}{2} = b - c$

similarly $(r + r_2)\tan\frac{C-A}{2} = c - a$

and $(r + r_3)\tan\frac{A-B}{2} = a - b$ adding, we get

$(r + r_1)\tan\frac{B-C}{2} + (r + r_2)\tan\frac{C-A}{2} + (r + r_3)\tan\frac{A-B}{2}$

$= b - c + c - a + a - b = 0$

**Q.8** If A, $A_1$, $A_2$, $A_3$ be the area of the in-circle and ex-circles, then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$ is equal to

(A) $\frac{1}{\sqrt{A}}$ (B) $\frac{2}{\sqrt{A}}$ (C) $\frac{3}{\sqrt{A}}$ (D) None

**Sol.** **[A]**

$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_2^2}} + \frac{1}{\sqrt{\pi r_3^2}}$

$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$

$= \frac{1}{\sqrt{\pi}}\,\frac{s-a+s-b+s-c}{\Delta} = \frac{1}{\sqrt{\pi}}\,.\,\frac{s}{\Delta} = \frac{1}{\sqrt{\pi}}\,.\,\frac{1}{r}$

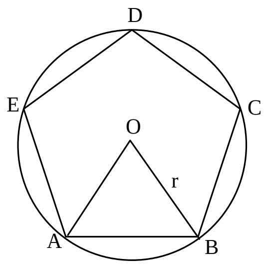
$= \frac{1}{\sqrt{\pi r^2}} = \frac{1}{\sqrt{A}}$

**Q.9** The area of a circle is $A_1$ and the area of a regular pentagon inscribed in the circle is $A_2$. Then $A_1 : A_2$ is -

(A) $\frac{\pi}{5}\cos\frac{\pi}{10}$  (B) $\frac{2\pi}{5}\sec\frac{\pi}{10}$

(C) $\frac{2\pi}{5}\text{cosec}\frac{\pi}{10}$  (D) None of these

**Sol.** **[B]**




In ΔOAB, OA = OB = r and ∠AOB

$$= \frac{360°}{5} = 72°$$

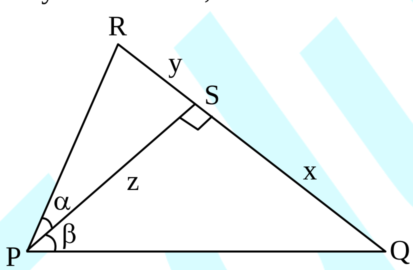
$$\text{ar}(\Delta AOB) = \frac{1}{r}\text{r. r. sin } 72° = \frac{1}{r}r^2\cos 18°$$

$$\text{ar (Pentagon)} = \frac{5}{2}r^2\cos 18°$$

Area of circle = $\pi r^2$

$$\frac{A_1}{A_2} = \frac{2\pi r^2}{5r^2\cos 18°} = \frac{2\pi}{5}\sec\frac{\pi}{10}$$

**Q.10** In a triangle PQR as shown in figure given that x : y : z :: 2 : 3 : 6, then the value of ∠QPR is -




(A) π/6  (B) π/4  (C) π/3  (D) none

**Sol.** **[B]**

Θ x : y : z : : 2 : 3 : 6

⇒ x = 2k, y = 3k, z = 6k

from figure, we have

$$\tan\alpha = \frac{RS}{PS} = \frac{y}{z} = \frac{3k}{6k} = \frac{1}{2}$$

$$\text{and } \tan\beta = \frac{QS}{PS} = \frac{x}{z} = \frac{2k}{6k} = \frac{1}{3}$$

$$\Theta \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}} = \frac{5}{5} = 1$$

$\Rightarrow \alpha + \beta = \frac{\pi}{4}$  $\Rightarrow \angle QPR = \frac{\pi}{4}$

## Part-B One or More Than one Correct Answer type Questions

**Q.11** If the lengths of the sides of a ΔABC are 3, 5 and 7, then-

(A) largest angle is 2π/3

(B) area of $\Delta = \frac{15\sqrt{3}}{4}$

(C) $R = \frac{7\sqrt{3}}{3}$

(D) $r = \frac{\sqrt{3}}{4}$

**Sol.** **[A,B,C,D]**

Let a = 3, b = 5, c = 7 then largest angle is ∠ C

$$\Rightarrow \cos C = \frac{9+25-49}{2.3.5}$$

$$\Rightarrow \cos C = -\frac{1}{2}$$

$\Rightarrow C = \frac{2\pi}{3}$ hence A is correct.

$$\sin C = \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$\Delta = \frac{15\sqrt{3}}{4}$  B is correct.

$\Theta\ R = \frac{abc}{4\Delta} = \frac{3.5.7}{4.\frac{15\sqrt{3}}{4}} = \frac{7\sqrt{3}}{3}$  C is correct

$$r = \frac{\Delta}{s}$$

$= \frac{15\sqrt{3}}{4}.\frac{2}{15}$  $\Theta\ s = \frac{3.5.7}{2} = \frac{15}{2}$

$r = \frac{\sqrt{3}}{2}$  D is wrong.

**Q.12** If A = 30º and the area of triangle ABC is $\frac{\sqrt{3}}{4}a^2$, then the triangle ABC is -

(A) Obtuse angled triangle
(B) $\angle B = 120^o$
(C) $\angle C = 30^o$
(D) Acute angled triangle

**Sol.** **[A, B, C]**

$A = 30°$, $\Delta = \frac{\sqrt{3}}{4} a^2$

Θ $\Delta = \frac{1}{2} bc \sin A = \frac{1}{4} bc$ Θ $A = 30°$

$\frac{1}{4} bc = \frac{\sqrt{3}}{4} a^2$

$\sin B \sin C = \frac{\sqrt{3}}{4}$

Θ $\cos (B + C)$

$= \cos B \cos C - \sin B \sin C = -\cos A$

$\Rightarrow \cos B \cos C = -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} = -\frac{\sqrt{3}}{4}$

$\Rightarrow \tan B \tan C = -1$

$\Rightarrow B - C = 90°$ and $B + C = 150°$

$\Rightarrow B = 120°$ ; $C = 30°$

option A, B, C are correct.

**Q.13** If for a $\Delta ABC$, cot A. cot B. cot C > 0 then the triangle is-
(A) right angled (B) acute angled
(C) obtuse angled
(D) all the options are possible

**Sol.** **[B]**

cot A. cot B. cot C > 0

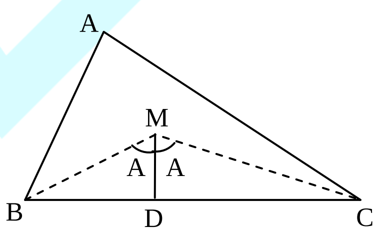
$\Rightarrow$ cot A > 0, cot B > 0, cot C > 0

Because in any triangle two or more cot θ negative are impossible.

$\Rightarrow$ triangle is acute angled triangle.

**Q.14** The distances of the circumcentre of the acute-angled $\Delta ABC$ from the sides BC, CA and AB are in the ratio-
(A) a sin A : b sin B : c sin C
(B) cos A : cos B : cos C
(C) a cot A : b cot B : c cot C
(D) none of these

**Sol.** **[A, C]**




MD = MB cos A = R cos A etc.

then required ratio

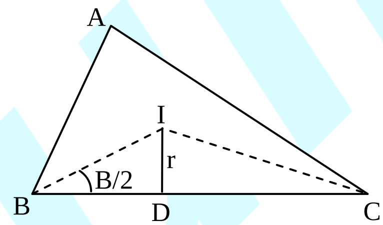
R cos A : R cos B : R cos C

$\Rightarrow \frac{a}{2} \cot A : \frac{b}{2} \cot B : \frac{c}{2} \cot C$

A, C are correct.

**Q.15** If I be the incentre of the $\Delta ABC$, then AI.BI.CI is equal to
(A) abc tan (A/2) tan (B/2) tan (C/2)
(B) $\frac{r^3}{\sin(A/2)\sin(B/2)\sin(C/2)}$
(C) $64R^3 \sin^2 (A/2) \sin^2 (B/2) \sin^2 (C/2)$
(D) None of these

**Sol.** **[A, B, C]**




In $\Delta ABC$ we have

$\frac{DI}{BI} = \sin\frac{B}{2} \Rightarrow BI = \frac{r}{\sin\frac{B}{2}}$ etc.

$\Rightarrow AI . BI . CI = \frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$ ... (i)

Θ $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$\Rightarrow AI. BI. CI = 64 R^3 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$

and we have

$r = (s-a) \tan\frac{A}{2} = (s-b) \tan\frac{B}{2} = (s-c) \tan\frac{C}{2}$

so from (i), we get

AI. BI. CI =

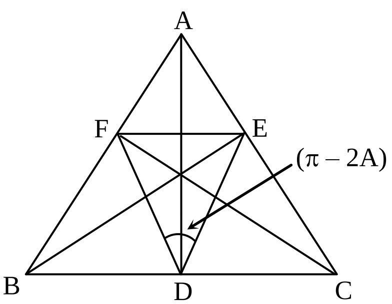
$$\frac{(s-a)(s-b)(s-c)\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{\sqrt{\frac{(s-b)(s-c)}{bc}\frac{(s-a)(s-c)}{ac}\frac{(s-a)(s-b)}{ab}}}$$

$\Rightarrow AI. BI. CI = abc \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$

option (A), (B) and (C) are correct.

**Q.16** In a $\Delta ABC$, the line segments AD, BE and CF are three altitudes. If R is the circum-radius of the $\Delta ABC$, a side of the $\Delta DEF$ will be-
(A) Rsin2A (B) c cos B
(C) a sin A (D) b cos B

**Sol.** **[A, D]**




We know that sides of pedal triangle are

EF = a cos A = R sin 2A

FD = b cos B = R sin 2B

ED = c cos C = R sin 2C

Clearly option (A) and (D) are correct

## Part-C Assertion- Reason type Questions

**The following questions 17 to 18 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.**

**(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.**

**(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.**

**(C) If Assertion is true but the Reason is false.**

**(D) If Assertion is false but Reason is true**

**Q.17** **Assertion (A) :** In any triangle ABC, $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2rR}$, where r is inradius and R is circum radius.

**Reason (R):** $R \ge 2r$.

**Sol.** **[B]**

$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$

$= \frac{a+b+c}{abc} = \frac{2S}{4\Delta R} = \frac{1}{2rR}$

$\Rightarrow$ A is true

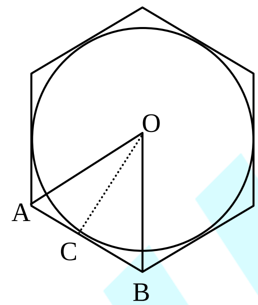
$\because R \ge 2r \Rightarrow$ R is true

But R is not a correct explanation of A

**Q.18** **Assertion (A) :** The side of regular hexagon is 5 cm whose radius of inscribed circle is $5\sqrt{3}$ cm.

**Reason (R) :** The radius of inscribed circle of a regular polygon of side a is $\frac{a}{2}\cot\left(\frac{\pi}{n}\right)$.

**Sol.** **[D]**




$\angle AOB = \frac{2\pi}{6} = \frac{\pi}{3} = 60°$

$\angle AOC = 30°$

Given AB = 5 cm $\Rightarrow$ AC = 2.5 cm

From $\Delta AOC$, we have $\tan 30° = \frac{AC}{OC} = \frac{2.5}{r}$

$\Rightarrow r = 2.5\sqrt{3}$ cm assertion is false.

radius $= \frac{a}{2}\cot\frac{\pi}{n}$ Reason is true.

## Part-D Column Matching Questions

**Q.19** Match the following

| **Column I** | **Column II** |
|---|---|
| In a triangle ABC | |
| (A) If a,b,c are 13,14,15 respectively then $r_1$ = | (P) 8 |
| (B) The inradius of triangle whose sides are 3, 5, 6 is r then $7r^2$ = | (Q) 10.5 |
| (C) If the radius of circumcircle of an isosceles triangle ABC is equal to AB = AC then angle A = $\frac{p\pi}{3}$ then 4p = | (R) 9.5 |
| (D) In an equilateral triangle for inradius and circumradius $\frac{4R}{r}$ = | (S) 7 |

**Sol.** **$A \to Q$, $B \to P$, $C \to P$, $D \to P$**

(A) $r_1 = \frac{\Delta}{(s-a)} = \frac{84}{8} = 10.5$

(B) $r = \frac{\Delta}{S} = \frac{2\sqrt{2}}{\sqrt{7}}$

$\Rightarrow 7r^2 = 7.\ \frac{8}{7} = 8$

(C) R = c

$\Rightarrow \frac{abc}{4\Delta} = c \Rightarrow ab = 4\Delta$

$\Rightarrow \sin C = \frac{1}{2} \Rightarrow C = 30°$

$\angle B = \angle C = 30°$

$\angle A = 120° = \frac{2\pi}{3} \Rightarrow p = 2$

4p = 8

(D) $R = \frac{abc}{4\Delta}$, $r = \frac{\Delta}{s}$

$\Rightarrow \frac{4R}{r} = \frac{abc}{\Delta}.\frac{s}{\Delta} = \frac{a^3 s}{\Delta^2}$

In equilateral triangle $\Delta = \frac{\sqrt{3}}{4}a^2$, $s = \frac{3a}{2}$

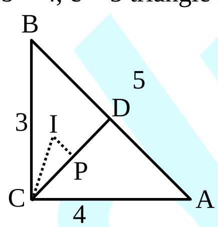
$\Rightarrow \frac{4R}{r} = a^3.\frac{3a}{2}.\frac{16}{3a^4} = 8$

**Q.20** If ABC is a triangle with a = 3, b = 4 and c = 5 then :

| **Column-I** | **Column-II** |
|---|---|
| (A) Distance between circumcentre & orthocentre | (P) 1/3 |
| (B) Distance between centroid & circumcentre | (Q) 5/2 |
| (C) Distance between centroid & incentre | (R) 5/6 |
| (D) Distance between centroid & orthocentre | (S) 5/3 |

**Sol.** $\mathbf{A \to Q,\ B \to R,\ C \to P,\ D \to S}$

Θ a = 3, b = 4, c = 5 triangle is right angled




(A) orthocenter is C and circum centre is mid point of AB is D

Θ $AD = CD = \frac{5}{2}$

(B) centroid is at P then we know that

$\frac{CP}{PD} = \frac{2}{1} \Rightarrow \frac{CD - PD}{PD} = \frac{2}{1}$

$\Rightarrow 3PD = CD \Rightarrow 3PD = \frac{5}{2}$

$\Rightarrow PD = \frac{5}{6}$

(C) In $\Delta CDA$

$\cos C = \frac{16}{2.\frac{5}{2}.4} = \frac{4}{5}$

$\angle DCA = \cos^{-1}\frac{4}{5}$

$\angle ICP = 45° - \cos^{-1}\frac{4}{5}$

in $\Delta IPC$, Let IP = a

$IC = \sqrt{2}$, $CP = \frac{5}{3}$, $\angle ICP = 45° - \cos^{-1}\frac{4}{5}$

$\cos\left(45° - \cos^{-1}\frac{4}{5}\right) = \frac{2 + \frac{25}{9} - a^2}{2.\sqrt{2}.\frac{5}{3}}$

solving we get

$a = \frac{1}{3} \Rightarrow IP = \frac{1}{3}$

(D) Θ $\frac{CP}{PD} = \frac{2}{1} = \frac{CP}{CD - CP} = \frac{2}{1}$

$\Rightarrow 2CD = 3CP$

$\Rightarrow CP = \frac{5}{3}$

**Q.21**

| **Column-I** | **Column-II** |
|---|---|
| (A) If $\angle B = 90°$ in $\Delta ABC$ then inradius is | (P) sin (B – C) |
| (B) If R denotes circumradius then in $\Delta ABC$, $\frac{b^2 - c^2}{2aR}$ equals | (Q) s |
| (C) In $\Delta ABC$, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ equals | (R) $\frac{a-b+c}{2}$ |
| (D) In a right angled triangle r + 2R equals | (S) 0 |

**Sol.** $\mathbf{A \to R,\ B \to P,\ C \to S,\ D \to Q}$

(A) $\angle B = 90°$

$r = (s - b)\tan\frac{B}{2} = \frac{a-b+c}{2}$

(B) $\frac{b^2 - c^2}{2aR} = \frac{4R^2(\sin^2 B - \sin^2 C)}{4R^2 \sin A}$

$= \frac{\sin(B+C)\sin(B-C)}{\sin(B+C)} = \sin(B - C)$

(C) Θ $r_1 = \frac{\Delta}{s-a}$ etc. then

$\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$

$= \frac{1}{\Delta}((s-a)(b-c) + (s-b)(c-a) + (s-c)(a-b))$

$= 0$

(D) Let $\angle B = 90°$

$r = (s-b)\tan\frac{B}{2} = s-b$

$2R = b$

$r + 2R = s - b + b = s$

# EXERCISE # 3

## Part-A Subjective Type Questions

**Q.1** In a triangle, prove that

$a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$

**Sol.** A cos B cos C + b cos C cos A + c cos A cos B

= 2R(sin A cos B cos C + sin B cos A cos C + sin C cos A cos B)

Θ sin (A + B + C) = sin A cos B cos C + cos A sin B cos C + cos A cos B sin C – sin A sin B sin C

= 2R [sin (A + B + C) + sin A sin B sin C]

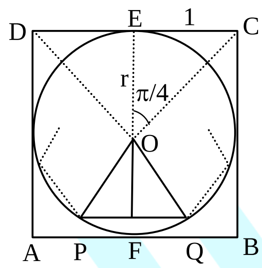
But A + B + C = π ⇒ sin π = 0

= 2R sin A sin B sin C

$= \frac{4R^2 \sin A \sin B \sin C}{2R} = \frac{ab \sin C}{2R} = \frac{\Delta}{R}$

**Q.2** A square, whose side is 2 cm, has its corners cut away so as to form a regular octagon; find its area.

**Sol.**




In Δ ODC, ∠OCE = $\frac{\pi}{4}$, EC = 1

$r = \tan \frac{\pi}{4} = 1$

Area of polygon = $nr^2 \tan \frac{\pi}{n}$

Then area of octagon = $10 \tan \frac{\pi}{10}$

= 10 tan 18° = 10 × .32492 = 3.25 sq.cm.

**Q.3** In a ΔABC, if ∠C = 90°, then prove that

$\frac{c}{r} = \frac{c+a}{b} + \frac{c+b}{a}$.

**Sol.** Taking R.H.S., we have

$= \frac{ac + a^2 + bc + b^2}{ab}$

$= \frac{ac + bc + 2ab \cos C + c^2}{ab}$

$= \frac{c(a+b+c)}{ab} = \frac{2cs}{ab}$ s

$= \frac{cs}{\frac{1}{2} ab \sin C}$

Θ sin C = 1

$= \frac{cs}{\Delta} = \frac{c}{r}$ L.H.S.

**Q.4** The area of a triangle is 6 $cm^2$. If the radii of its ex-circles are 2, 3 & 6 cms respectively then compute the length of its sides and the largest angle.

**Sol.** Given Δ = 6, $r_1 = 2$, $r_2 = 3$, & $r_3 = 6$

Θ $r_1 = \frac{\Delta}{s-a} \Rightarrow s - a = \frac{\Delta}{r_1} = \frac{6}{2}$

⇒ b + c – a = 6 … (i)

Similarly $r_2 = \frac{\Delta}{s-b} \Rightarrow s - b = \frac{\Delta}{r_2} = \frac{6}{3}$

⇒ a – b + c = 4 … (ii)

And $r_3 = \frac{\Delta}{s-c} \Rightarrow s - c = \frac{\Delta}{r_3} = \frac{6}{6}$

⇒ a + b – c = 2 … (iii)

From (i), (ii), (iii), we get

a = 3, b = 4, c = 5

Largest angle C so

$\cos C = \frac{9+16-25}{2.3.4} = 0$ ; C = 90°

**Q.5** In a ΔABC, prove that

$r_1 (r_2 + r_3) = a \sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}$

**Sol.** Taking L.H.S., we have

$\frac{\Delta^2}{(s-a)} \left( \frac{1}{s-b} + \frac{1}{s-c} \right) = \frac{\Delta^2 a}{(s-a)(s-b)(s-c)}$

$= as$ …(i)

Taking R.H.S., we have

$$a\sqrt{r_2\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-c}\right)+\frac{\Delta^2}{(s-c)(s-a)}}$$

$$= a\sqrt{\frac{\Delta^2 b}{(s-a)(s-b)(s-c)}+\frac{\Delta^2}{(s-c)(s-a)}}$$

$$= a\sqrt{\frac{\Delta^2 b+\Delta^2 s-\Delta^2 b}{(s-a)(s-b)(s-c)}} = \sqrt{\frac{\Delta^2 s^2}{\Delta^2}} = as \text{ … (ii)}$$

From (i) and (ii), we get

L.H.S. = R.H.S.

**Q.6** In a triangle, prove that $r_1 = r\cot\frac{B}{2}\cot\frac{C}{2}$

**Sol.** Taking R.H.S., we have

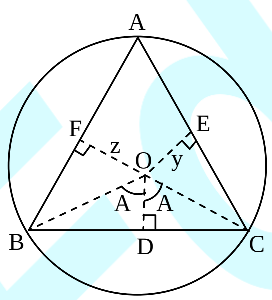
$$r\cot\frac{B}{2}\cot\frac{C}{2} = r\sqrt{\frac{s(s-b)}{(s-a)(s-c)}\cdot\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{rs}{s-a} = \frac{\Delta}{s-a} = r_1$$

L.H.S.

**Q.7** If x, y, z are respectively perpendicular from the circumcentre on the sides of the $\Delta ABC$, prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

**Sol.**




In $\Delta BOD$ we have

Let OD = x, OE = y, OF = z

$$\cos A = \frac{OD}{OB} = \frac{x}{R}$$

$$\Rightarrow x = R\cos A = \frac{a}{2\sin A}\cos A = \frac{a}{2}\cot A$$

$$\Rightarrow \frac{a}{x} = 2\tan A$$

similarly, we have

$$\frac{b}{y} = 2\tan B \text{ and } \frac{c}{z} = 2\tan C$$

In $\Delta ABC$, we have

tan A + tan B + tan C = tan A tan B tan C

put the values of tan A, tan B, tan C, we get

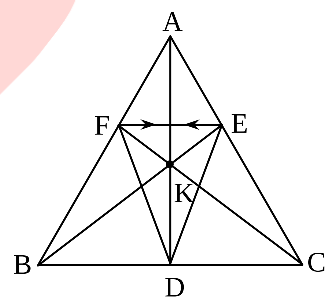
$$\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=\frac{abc}{8xyz}$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

## Part-B Passage based objective questions

**Passage-I (Q. 8 to 10)**

In a $\Delta ABC$, draw the perpendicular's from the vertices A, B and C to the opposite side meet at the point D, E and F respectively. Join these feet of the perpendiculars and make a triangle. This triangle is called pedal triangle of a given triangle, where R is the circum-radius $\left(\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{1}{2R}\right)$.




**On the basis of above passage, answer the following questions:**

**Q.8** The distance between A and K is-

(A) 2R cos A (B) 2R cos B

(C) 2R cos C (D) 2R cos B cos C

**Sol.** **[A]**

From $\Delta ABE$

$$\sin(90-A) = \frac{AE}{AB} = \frac{AE}{C}$$

$\Rightarrow$ AE = c cos A ... (i)

and from $\Delta AKE$, we have

$\angle AKE = 180 - (90 + 90 - C) = C$

$$\frac{AE}{AK} = \sin C$$

$\Rightarrow$ AK = AE cosec C ... (ii)

from (i) and (ii)

AK = c cos A. cosec C

= 2R cos A

**Q.9** Distance between K and D is-

(A) 2R cos A cos B (B) 2R cos B cos C

(C) 2R cos A cos C (D) none of these

**Sol.** **[B]**

See figure

from $\Delta$CKD, we have

$\tan(90-B) = \frac{KD}{DC}$

$\Rightarrow$ KD = DC cot B ... (i)

from $\Delta$ADC, we get

$\sin(90-C) = \frac{DC}{AC}$

$\Rightarrow$ DC = b cos C ... (ii)

from (i) and (ii), we get

$KD = b \cot B \cos C = \frac{b}{\sin B} \cos B \cos C$

KD = 2R cos B cos C

**Q.10** The $\angle$FDE is-

(A) 180º – 2$\angle$B (B) 180º – 2$\angle$A

(C) 180º – 2$\angle$C (D) $\angle$A

**Sol.** **[B]**

From fig. consider the quadrilateral KDCE

since $\angle$KDC = $\angle$OEC = 90°

$\because$ $\angle$KDC + $\angle$OEC = 180°

so KDCE is a cyclic quadrilateral

since angles in the same segment of a circle are equal.

Therefore $\angle$KDE = $\angle$KCE

But in $\Delta$AFC, we have

$\angle$FCE = 90° – A $\Rightarrow$ $\angle$KCE = 90° –A = $\angle$KDE

Similarly KDBF is a cyclic quadrilateral

so $\angle$KDF = $\angle$KBF = 90° –A

$\Rightarrow$ $\angle$FDE = $\angle$KDF + $\angle$KDE = 90° – A + 90° – A

$\angle$FDE = 180° – 2A

# EXERCISE # 4

## Old IIT-JEE Questions

**Q.1** In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then 2(r + R) is equal to - **[IIT 2000]**

(A) a + b (B) b + c
(C) c + a (D) a + b + c

**Sol.** **[A]**

$\angle C = \frac{\pi}{2}$

We know that

$r = (s - c) \tan \frac{C}{2} = (s - c) \tan 45° = s - c$

$2r = 2s - 2c = a + b - c$

and $\frac{C}{\sin C}$ = 2R sine formula

$\Rightarrow 2R = C$

$\Rightarrow 2(r + R) = 2r + 2R = a + b - c + c = a + b$

**Q.2** Which of the following pieces of data does NOT uniquely determine an acute angled triangle ABC (R being the radius of the circumcircle) - **[IIT 2002]**

(A) a, sin A, sin B (B) a, b, c
(C) a, sin B, R (D) a, sin A, R

**Sol.** **[D]**

We know by sine law

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A+B)} = 2R$

$\Theta$ A + B + C = $\pi$

(A) If we know a, sin A, sin B then we can find b, c, $\angle C$ and all values.

(B) If we know a, b, c then we can find all the values by cosine rule.

(C) If a, sin B, R are given then sin A, b , $\angle C$ we can find.

(D) If we know a, sin A, R then we know only the ratio $\frac{b}{\sin B}$ or $\frac{c}{\sin(A+B)}$, we can not determine the values of b, c, sin B and other.

$\Rightarrow$ (D) is correct choice

**Q.3** If the angles of a triangle are in ratio 4 : 1 : 1 then the ratio of the longest side and perimeter of triangle is - **[IIT 2003]**

(A) $\frac{1}{2+\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}-2}$

(C) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (D) None of these

**Sol.** **[C]**

Given that A : B : C = 4 : 1 : 1

Let A = 4x, B = x, C = x

$\Rightarrow$ A + B + C = 180

$\Rightarrow$ x = 30°

$\Rightarrow$ A = 120°, B = 30°, C = 30°

By sine law

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$

$\Rightarrow a : b : c = \sqrt{3} : 1 : 1$

Ratio of longest side to the perimeter

$= \frac{\sqrt{3}}{\sqrt{3}+1+1} = \frac{\sqrt{3}}{2+\sqrt{3}}$

**Q.4** If the sides a, b, c of a triangle are such that a : b : c : : 1 : $\sqrt{3}$ : 2, then the A : B : C is - **[IIT Scr.2004]**

(A) 3 : 2 : 1 (B) 3 : 1 : 2
(C) 1 : 3 : 2 (D) 1 : 2 : 3

**Sol.** **[D]**

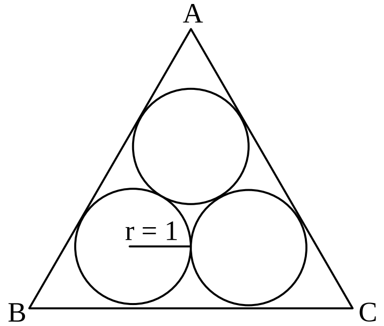
$a : b : c : : 1 : \sqrt{3} : 2 = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$\Rightarrow$ sin A : sin B : sin C = $\frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

A = 30°, B = 60°, C = 90°

A : B : C = 1 : 2 : 3

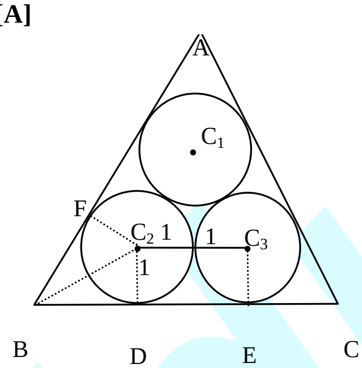
**Q.5** In any equilateral Δ, three circles of radii one are touching to the sides given as in the figure then area of the Δ is **[IIT 2005]**




(A) $6 + 4\sqrt{3}$ (B) $12 + 8\sqrt{3}$

(C) $7 + 4\sqrt{3}$ (D) $4 + \frac{7}{2}\sqrt{3}$

**Sol.** **[A]**




Let BD = x

But BD and BF are two tangents from B to circle, therefore $BC_2$ must be angle bisector of ∠ FBD.

But ∠ B = 60° ⇒ $\angle C_2BD = 30°$

from $\Delta BC_2D$, we have

$\tan 30° = \frac{1}{x}$

$\Rightarrow x = \frac{1}{\tan 30°} = \sqrt{3}$

so BC = x + 1 + 1 + x = 2x + 2

$= 2(1 + \sqrt{3})$

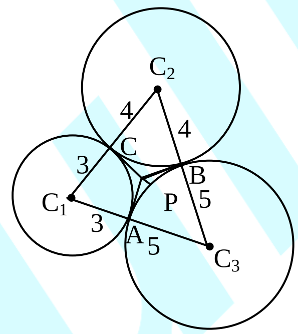
ΔABC is equilateral so

Area of ΔABC = $\frac{\sqrt{3}}{4}(BC)^2$

$= \frac{\sqrt{3}}{4} . 4(1 + \sqrt{3})^2 = \sqrt{3}(4 + 2\sqrt{3}) = 6 + 4\sqrt{3}$

**Q.6** 3, 4, 5 are radii of three circles touch each other externally if P is the point of intersection of tangents of these circles at their points of contact, find the distances of P from the points of contact. **[IIT 2005]**

**Sol.**




Θ P is radical center of three circles

so AP perpendicular $C_1C_3$, BP perpendicular $C_2C_3$ and CP perpendicular $C_1C_2$

Θ AP = BP = CP = in-radius of $\Delta C_1C_2C_3$ = r

perimeter of $\Delta C_1C_2C_3$

2s = 2(4 + 3 + 5) = 24

s = 12

$\Delta = \sqrt{s(s-8)(s-9)(s-7)}$

$\Delta = \sqrt{12.4.3.2} = 12\sqrt{2}$

$r = \frac{\Delta}{s} = \frac{12\sqrt{2}}{12} = \sqrt{2}$

$\Rightarrow AP = BP = CP = \sqrt{2}$

**Q.7** If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is - **[IIT 2010]**

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

**Sol.[D]** $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$

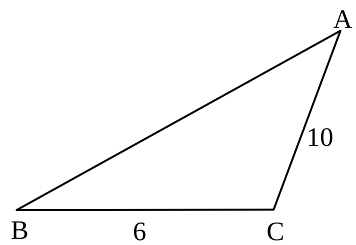
$\frac{a}{c}2\sin C\cos C + \frac{c}{a}2\sin A\cos A$

$= \dfrac{a\cos C + c\cos A}{R} = \dfrac{b}{R} = \dfrac{2R\sin B}{R} = \sqrt{3}$

**Q.8** Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then $r^2$ is equal to

**[IIT 2010]**

**Sol.[3]**




$\Delta = \dfrac{1}{2}$ ab sin C

$15\sqrt{3} = \dfrac{1}{2}6(10)\sin C \Rightarrow \sin C = \sqrt{3}/2$

$\Rightarrow C = 120°$

$\cos C = \dfrac{100 + 36 - c^2}{2.10.6} \Rightarrow C^2 = 136 + 120° (1/2)$

$\Rightarrow C^2 = 196 \Rightarrow C = 14$

s = 15

$r = \dfrac{\Delta}{s} = \sqrt{3}$

$r^2 = 3$

# EXERCISE # 5

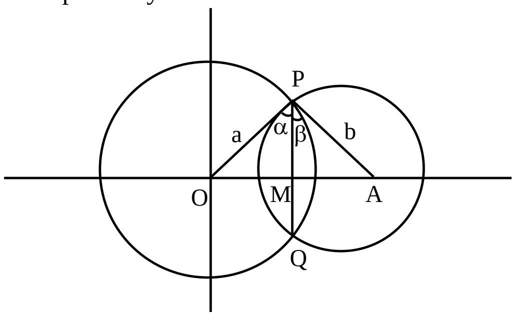
**Q.1** Two circles of radii a and b, cut each other at an angle θ. Prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$.

**Sol.** Let the equation of two circles are

$x^2 + y^2 = a^2$ ………..(1)

and $(x - c)^2 + y^2 = b^2$ ……….(2)

Since the radius of (1) & (2) circles are a & b respectively




Let ∠OPM = α, ∠APM = β

∴ ∠OPA = α + β = Q

Let PQ = x = length of common chord

∴ $\cos\alpha = \frac{PM}{a} = \frac{x}{2a}$, $\cos\beta = \frac{x}{2b}$

Now $\cos\theta = \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

∴ $\sin\alpha\sin\beta = \cos\alpha\cos\beta - \cos\theta$

on squaring both sides, we get

$\sin^2\alpha\sin^2\beta = \cos^2\alpha\cos^2\beta + \cos^2\theta - 2\cos\theta\cos\alpha\cos\beta$

$1 - \cos^2\alpha - \cos^2\beta + \cos^2\alpha\cos^2\beta = \cos^2\alpha\cos^2\beta + \cos^2\theta - 2\cos\theta\cos\alpha\cos\beta$

∴ $\sin^2\theta = \cos^2\alpha + \cos^2\beta - 2\cos\theta\cos\alpha\cos\beta$

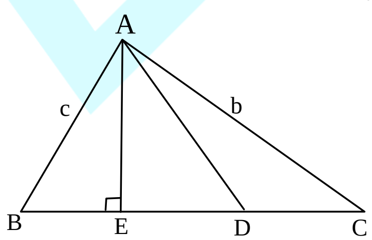
$\Rightarrow \sin^2\theta = \frac{x^2}{4a^2} + \frac{x^2}{4b^2} - \frac{2x^2}{4ab}.\cos\theta$

$\Rightarrow 4a^2b^2\sin^2\theta = x^2(b^2 + a^2 - 2ab\cos\theta)$

$\Rightarrow x = \frac{2ab\sin\theta}{\sqrt{a^2+b^2-2ab\cos\theta}}$ (Hence proved)

**Q.2** In the triangle ABC, prove that the distance between the middle point of BC and the foot of the perpendicular from A is $\frac{b^2 \sim c^2}{2a}$.

**Sol.**




D is the mid point and E is the foot of perpendicular from A so ED = BD – BE

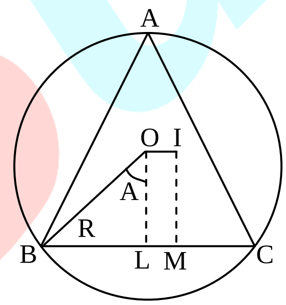
∵ $BD = \frac{a}{2}$ and BE = c cos B

$\Rightarrow ED = \frac{a}{2} - c\cos B = \frac{a}{2} - \frac{c(a^2+c^2-b^2)}{2ac}$

$= \frac{a^2-a^2-c^2+b^2}{2a} = \frac{b^2 \sim c^2}{2a}$

**Q.3** If in a triangle ABC, the line joining the circumcentre O and incentre I is parallel to BC, then prove that cosB + cosC = 1.

**Sol.** Since OI is parallel to BC, O and I are equidistant from BC (see figure). Now, ∠BOL = A, OB = R and IM = r.




Also, IM = OL = R cosA, so r = R cosA.

$4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = R\cos A$

From the identities, we know that

if A + B + C = π

$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{C}{2}$

= 1 + cosA

⇒ cosB + cosC = 1

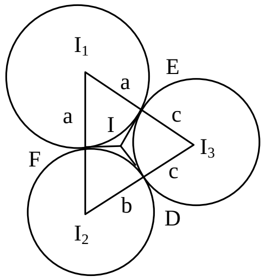
**Q.4** Three circles whose radii are a, b, c touch one another externally, and the tangents at their point of contact meet in a point. Prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{1/2}$.

**Sol.** Let $I_1$, $I_2$, $I_3$ be the centres of the circles of radii a, b, c respectively which touch externally at points D, E and F. If the tangents at D, E ,F meet at I, then ID = IE = IF also

$ID \perp I_2 I_3$, $IE \perp I_3 I_1$ and $IF \perp I_1 I_2$

Hence I is the incentre of the $\Delta\, I_1 I_2 I_3$, whose sides are of lengths

$I_2 I_3 = b + c$, $\quad I_3 I_1 = c + a$




and $I_1 I_2 = a + b$

Perimeter of $\Delta\, I_1 I_2 I_3 = I_2 I_3 + I_3 I_1 + I_1 I_2$

$2s = 2(a + b + c)$

$\therefore\ s = a + b + c$

$\therefore$ ID = IE = IF = radius of inscribed circle of $\Delta\, I_1 I_2 I_3$

$$= \frac{\Delta}{s} = \frac{\sqrt{s(s - I_2 I_3)(s - I_3 I_1)(s - I_1 I_2)}}{s}$$

$$= \frac{\sqrt{(a+b+c)abc}}{(a+b+c)}$$

$$= \sqrt{\left(\frac{abc}{(a+b+c)}\right)} = \left(\frac{abc}{a+b+c}\right)^{1/2}$$

**Q.5** If $2\phi_1$, $2\phi_2$, $2\phi_3$ are the angles subtended by the circle escribed to the side a of a triangle at the centres of the inscribed circle and the other two escribed circles. Prove that

$$\sin\phi_1 \sin\phi_2 \sin\phi_3 = \frac{r_1^2}{16R^2}$$

**Sol.** If I, $I_1$, $I_2$, $I_3$ are the centres of the incircle and three escribed circles then

$$I I_1 = \frac{a}{\cos\left(\frac{A}{2}\right)} \text{ and } I_1 I_2 = \frac{c}{\sin\left(\frac{C}{2}\right)},$$

$$I_3 I_1 = \frac{b}{\sin\left(\frac{B}{2}\right)}$$

LHS $\therefore$ $\sin\phi_1 \sin\phi_2 \sin\phi_3$

$$= \frac{r_1}{II_1} \times \frac{r_2}{I_1 I_2} \times \frac{r_3}{I_3 I_1}$$

$$= \frac{r_1 \cos\left(\frac{A}{2}\right)}{a} \times \frac{r_1 \sin\left(\frac{C}{2}\right)}{c} \times \frac{r_1 \sin\left(\frac{B}{2}\right)}{b}$$

$$= \frac{r_1^2}{abc} 4R \sin\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right) \cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)$$

$$= \frac{r_1^2}{abc} \frac{R}{2} \sin A \sin B \sin C$$

$$= \frac{r_1^2}{abc} \cdot \frac{R}{2} \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{r_1^2}{16R^2} = \text{RHS.}$$

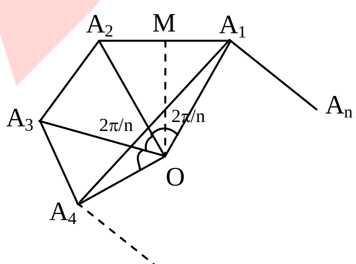
**Q.6** Let $A_1, A_2, A_3, \ldots\ldots A_n$ be the vertices of an n-sides regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$ Find the value of n.

**Sol.** If O is the centre of the n-sided regular polygon, then

$$\angle A_1OA_2 = \angle A_2OA_3 = \ldots\ldots \frac{2\pi}{n}$$

Let $\quad OA_1 = OA_2 = \ldots\ldots = OA_n = r$




Clearly we get $A_1 A_2 = 2A_1 M = 2r \sin\frac{\pi}{n}$

Similarly $A_1 A_3 = 2r \sin\frac{2\pi}{n}$

and $\quad A_1 A_4 = 2r \sin\frac{3\pi}{n}$

$\therefore$ $\quad$ we get (from the question)

$$\frac{1}{2r\sin\frac{\pi}{n}} = \frac{1}{2r\sin\frac{2\pi}{n}} + \frac{1}{2r\sin\frac{3\pi}{n}}$$

or, $$\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$$

or $\quad \dfrac{1}{\sin\frac{\pi}{n}} - \dfrac{1}{\sin\frac{3\pi}{n}} = \dfrac{1}{\sin\frac{2\pi}{n}}$

or, $\quad \sin\dfrac{2\pi}{n}\left(\sin\dfrac{3\pi}{n} - \sin\dfrac{\pi}{n}\right) = \sin\dfrac{\pi}{n}.\sin\dfrac{3\pi}{n}$

or, $\quad \sin\dfrac{2\pi}{n}.\,2\cos\dfrac{2\pi}{n}.\sin\dfrac{\pi}{n} = \sin\dfrac{\pi}{n}.\sin\dfrac{3\pi}{n}$

or, $\quad \sin\dfrac{4\pi}{n} = \sin\dfrac{3\pi}{n} \qquad \left\{\because \sin\dfrac{\pi}{n} \neq 0\right\}$

$\therefore \quad \dfrac{4\pi}{n} = \pi - \dfrac{3\pi}{n} \qquad \{\because 4 \neq 3\}$

or, $\quad \dfrac{7\pi}{n} = \pi; \qquad \therefore \quad n = 7$

**Ans.**

**Q.7** In a cyclic quadrilateral ABCD, prove that

$\tan^2\dfrac{B}{2} = \dfrac{(s-a)(s-b)}{(s-c)(s-d)}$ where a, b, c and d being the lengths of the sides AB, BC, CD and DA, respectively.

**Sol.** From $\Delta ABC$, we get

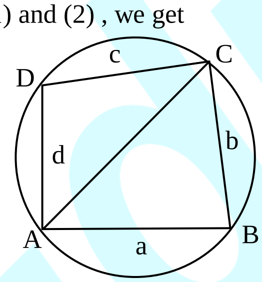
$AC^2 = a^2 + b^2 - 2ab\cos B$ .......(1)

and from $\Delta$ ADC, we have

$AC^2 = d^2 + c^2 - 2cd\cos D = d^2 + c^2 - 2cd\cos(180^o - B)$

$= d^2 + c^2 + 2cd\cos B$ .......(2)

From (1) and (2), we get




$\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

Now, since $\tan^2\dfrac{B}{2} = \dfrac{1-\cos B}{1+\cos B}$

we get, $\quad \tan^2\dfrac{B}{2}$

$= \dfrac{2(ab+cd)-(a^2+b^2-c^2-d^2)}{2(ab+cd)+(a^2+b^2-c^2-d^2)}$

$= \dfrac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}$

$= \dfrac{(c+d+a-b)(c+d-a+b)}{(a+b+c-d)(a+b-c+d)}$

$= \dfrac{(2s-2b)(2s-2a)}{(2s-2d)(2s-2c)} = \dfrac{(s-a)(s-b)}{(s-c)(s-d)}$

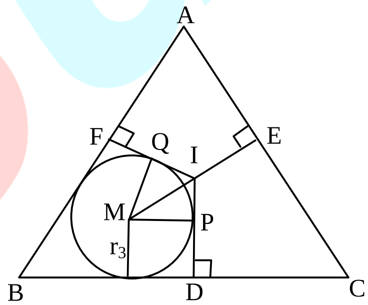
$[\because a+b+c+d = 2s]$

**Q.8** Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendicular from I to the sides BC, CA and AB respectively. If $r_1$, $r_2$ and $r_3$ are the radii of circles inscribed in the quadrilaterals AFIE, BDIF and CEID respectively. Prove that

$\dfrac{r_1}{r-r_1} + \dfrac{r_2}{r-r_2} + \dfrac{r_3}{r-r_3} = \dfrac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$

**[IIT 2000]**

**Sol.**




$\because$ ID = r, DP = $r_3$ = MP

$\Rightarrow$ IP = $r - r_3$

Clearly IP and IQ are tangents to circle with center M

$\Rightarrow$ IM must be angle bisector of $\angle$PI $\theta$

$\Rightarrow \angle PIM = \angle QIM = \theta_3$

so from $\Delta$IPM, we have

$\tan\theta_3 = \dfrac{r_3}{r-r_3} = \dfrac{PM}{IP}$

similarly, we get $\tan\theta_2 = \dfrac{r_2}{r-r_2}$ and

$\tan\theta_1 = \dfrac{r_1}{r-r_1}$

Also $2\theta_1 + 2\theta_2 + 2\theta_3 = 2\pi$

$\Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi$

$\Rightarrow \tan\theta_1 + \tan\theta_2 + \tan\theta_3 = \tan\theta_1\tan\theta_2\tan\theta_3$

putting values, we get

$\dfrac{r_1}{r-r_1} + \dfrac{r_2}{r-r_2} + \dfrac{r_3}{r-r_3} = \dfrac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$

Hence proved.

# ANSWER KEY

## EXERCISE # 1

| Q.No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ans. | C | A | A | B | A | C | B | C | A | A | D | C |

**13.** 0 **14.** 3s **15.** True **16.** True

## EXERCISE # 2

### PART-A

| Q.No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Ans. | B | C | A | C | D | A | B | A | B | B |

### PART-B

| Q.No. | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|
| Ans. | A,B,C,D | A,B,C,D | B | B,C | A,B,C | A,D |

### PART-C

| Q.No. | 17 | 18 |
|---|---|---|
| Ans. | B | D |

### PART-D

**19.** $A \to Q$, $B \to P$, $C \to P$, $D \to P$ **20.** $A \to Q$, $B \to R$, $C \to P$, $D \to S$

**21.** $A \to R$, $B \to P$, $C \to S$, $D \to Q$

## EXERCISE # 3

**2.** $8(\sqrt{2} - 1)$ **4.** a = 3 cm, b = 4 cm, c = 5 cm; Largest angle = 90º

**8.** (A) **9.** (B) **10.** (B)

## EXERCISE # 4

| Q.No. | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Ans. | A | D | C | D | A |

**6.** $\sqrt{5}$ **7.** (D) **8.** 3

## EXERCISE # 5

**6.** n = 7