Properties of triangle
EXERCISE # 1

Questions based on **Sine & Cosine Rule**

Q.1 In a triangle ABC, (a + b + c) (b + c – a) = λbc if -

(A) $\lambda < 0$ (B) $\lambda > 0$

(C) $0 < \lambda < 4$ (D) $\lambda > 4$

Sol.[C] Θ $(b + c)^2 - a^2 = \lambda bc$

$\Rightarrow b^2 + c^2 - a^2 + 2bc = \lambda bc$

$\Rightarrow$ 2bc cos A + 2bc = λbc (from cosine rule)

$\Rightarrow 2(\cos A + 1) = \lambda$

$\Rightarrow \cos A = \frac{\lambda}{2} - 1$

But $-1 < \cos A < 1$

$\Rightarrow -1 < \frac{\lambda}{2} - 1 < 1 \Rightarrow 0 < \frac{\lambda}{2} < 2 \Rightarrow 0 < \lambda < 4$

Q.2 Let ABC be a triangle such that $\angle A = 45^o$, $\angle B = 75^o$ then $a + c\sqrt{2}$ is equal to -

(A) 0 (B) b (C) 2b (D) –b

Sol.[C] Θ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \frac{a}{\sin 45°} = \frac{b}{\sin 75°} = \frac{c}{\sin 60°}$

$\Rightarrow a\sqrt{2} = \frac{b.2\sqrt{2}}{\sqrt{3}+1} = \frac{2c}{\sqrt{3}}$

$\Rightarrow a = \frac{2b}{\sqrt{3}+1}$, $c = \frac{\sqrt{6}b}{\sqrt{3}+1}$

$\Rightarrow a + c\sqrt{2} = \frac{2b}{\sqrt{3}+1} + \frac{2b\sqrt{3}}{\sqrt{3}+1} = 2b$

Q.3 Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then angle A is equal to -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

Sol.[C] Given that 2B = A + C and $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow \frac{b}{\sqrt{3}/2} = \frac{c}{1/\sqrt{2}} \Rightarrow \frac{b}{\sin 60°} = \frac{c}{\sin 45°}$

$\Rightarrow \angle B = 60°, \angle C = 45°$

$= 120° = A + 45° \Rightarrow A = 75° = \frac{5\pi}{12}$

Q.4 If in a triangle ABC, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is -

(A) Right angled or isosceles

(B) Right angled and isosceles

(C) Equilateral

(D) None of these

Sol.[A] $\frac{\sin(A-B)}{\sin(A+B)} = \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$

$\Rightarrow \frac{\sin(A-B)}{\sin C} = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B}$

$\Rightarrow \sin(A-B)\left[\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right] = 0$

either $\sin(A - B) = 0 \Rightarrow A = B$

or $\sin^2A + \sin^2B = \sin^2C$

$\Rightarrow a^2 + b^2 = c^2$

$\Rightarrow$ Triangle is right angled or isosceles

Q.5 In any triangle ABC,

$\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B}$

is equal to -

(A) a + b + c (B) a + b – c

(C) a – b + c (D) 0

Sol.[D] From sine rule we take 1st term

$\Rightarrow \frac{\lambda^2 \sin^2 A \sin(B-C)}{\sin B + \sin C}$

$\Rightarrow \frac{\lambda^2 \sin A \sin(B+C)\sin(B-C)}{\sin B + \sin C}$

$\Rightarrow \frac{\lambda^2 \sin A(\sin^2 B - \sin^2 C)}{\sin B + \sin C}$

$\Rightarrow \lambda^2 \sin A (\sin B - \sin C)$

similarly we solve remain term and add we get

$\Rightarrow \lambda^2[\sin A(\sin B - \sin C) + \sin B(\sin C - \sin A)$
$+ \sin C(\sin A - \sin B)] = 0$

Q.6 The expression

$$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

is equal to -

(A) $\cos^2 A$ (B) $\sin^2 A$

(C) cosA cosB cosC (D) None of these

Sol.[B] $E = \frac{[(b+c)^2 - a^2][a^2 - (b-c)^2]}{4b^2c^2}$

$= \frac{[b^2 + c^2 - a^2 + 2bc][a^2 - b^2 - c^2 + 2bc]}{4b^2c^2}$

$= \frac{-[(b^2 + c^2 - a^2)^2 - 4b^2c^2]}{4b^2c^2}$

$= -\left[\left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 - 1\right]$

$= -[\cos^2 A - 1] = \sin^2 A$

Questions based on Projection formula, Napier's Analogy, & Half-Angled Formulae

Q.7 In a triangle ABC,

$a^2 \cos 2B + b^2 \cos 2A + 2ab \cos(A - B) =$

(A) a^2 (B) c^2 (C) b^2 (D) $a^2 + b^2$

Sol.[B] $a^2(2\cos^2 B - 1) + b^2(2\cos^2 A - 1) + 2ab\cos(A - B)$

$= 2(a^2\cos^2 B + b^2\cos^2 A) - a^2 - b^2 + 2ab\cos(A - B)$

$= 2(a\cos B + b\cos A)^2 - a^2 - b^2 + 4ab\cos A\cos B + 2ab\cos(A - B)$

$= 2c^2 - a^2 - b^2 - 2ab\cos A\cos B + 2ab\sin A\sin B$

$= 2c^2 - a^2 - b^2 - 2ab\cos(A + B)$

$= 2c^2 - a^2 - b^2 + 2ab\cos C = 2c^2 - c^2 = c^2$

Q.8 If in a triangle ABC, $b = \sqrt{3}$, $c = 1$ and $B - C = 90°$ then $\angle A$ is -

(A) 30º (B) 45º (C) 75º (D) 15º

Sol.[A] $\because \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$

$\Rightarrow \tan 45° = \frac{\sqrt{3}-1}{\sqrt{3}+1}\cot\left(\frac{A}{2}\right)$

$\Rightarrow \cot\frac{A}{2} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15°$

$\Rightarrow \frac{A}{2} = 15° \Rightarrow A = 30°$

Q.9 If in a triangle ABC, $(s - a)(s - b) = s(s - c)$, then angle C is equal to -

(A) 90º (B) 45º (C) 30º (D) 60º

Sol.[A] $\frac{(s-a)(s-b)}{s(s-c)} = 1$

$\Rightarrow \tan^2\left(\frac{C}{2}\right) = 1 \Rightarrow \tan\frac{C}{2} = 1$

$\Rightarrow \frac{C}{2} = 45° \Rightarrow C = 90°$

Questions based on Area of Triangle, Solutions of Triangle, m–n theorem & Ambiguous case

Q.10 In a Δ ABC, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, and the side a = 2, then area of the triangle is -

(A) 1 (B) 2 (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$

Sol.[D] $\because \frac{\cos A}{a} = \frac{\cos B}{b}$

$\Rightarrow \frac{b^2 + c^2 - a^2}{2abc} = \frac{a^2 + c^2 - b^2}{2abc} \Rightarrow a^2 = b^2 \Rightarrow a = b$

similarly $\frac{\cos B}{b} = \frac{\cos C}{c} \Rightarrow b = c$

so a = b = c = 2 given

triangle is equilateral so

Area $= \frac{\sqrt{3}}{4}a^2 = \sqrt{3}$

Q.11 We are given b, c and sin B such that B is acute and b < c sin B. Then -

(A) No triangle is possible

(B) One triangle is possible

(C) Two triangles are possible

(D) A right angled triangle is possible

Sol.[A] We know that

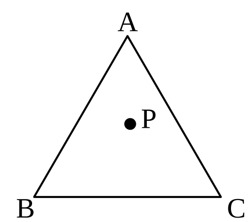
$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sin C = \frac{c\sin B}{b}$

given that b < c sin B

$\Rightarrow$ sin C > 1 which is impossible

$\Rightarrow$ No triangle is possible

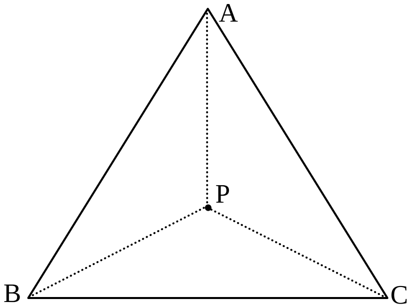
Q.12 In the adjacent figure 'P' is any interior point of the equilateral triangle ABC of side length 2 unit



If x_a, x_b and x_c represent the distance of P from the sides BC, CA and AB respectively then $x_a + x_b + x_c$ is equal to -

(A) 6 (B) $\sqrt{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $2\sqrt{3}$

Sol. **[B]**



Given that AB = BC = CA = 2

x_a, x_b and x_c are the distances of P from the sides BC, CA and AB

$\Rightarrow$ area of $\Delta BPC = \frac{1}{2} BC.\ x_a = \frac{1}{2}.2x_a = x_a$

similarly

Area of $\Delta CPA = x_b$ and Area of $\Delta APB = x_c$

then the area of ΔABC

or $(\Delta ABC) = ar(\Delta BPC)+ar(\Delta CPA) + ar(\Delta APB)$

$\because$ ABC is equilateral

ar $(\Delta ABC) = \sqrt{3}$ $\because$ AB = 2

$x_a + x_b + x_c = \sqrt{3}$

Q.13 If two sides a, b and the angle A be such that two triangles are formed, then the sum of the two values of the third side is -

(A) $b^2 - a^2$ (B) 2b cosA

(C) 2b sinA (D) $\frac{b-c}{b+c}$

Sol. **[B]**

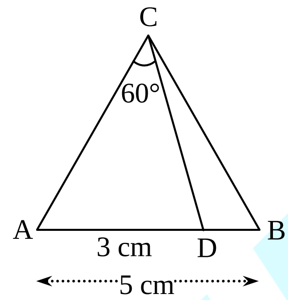
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$

Let c_1 and c_2 be two values of c then

$c_1 + c_2 = 2b \cos A$

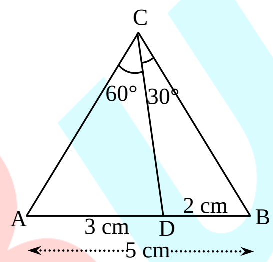
Q.14 In the figure, ABC is triangle in which C = 90º and AB = 5 cm. D is a point on AB such that AD = 3 cm and $\angle ACD$ = 60º. Then the length of AC is



(A) $5\sqrt{\frac{3}{7}}$ cm (B) $\sqrt{\frac{3}{7}}$ cm

(C) $\frac{3}{\sqrt{7}}$ cm (D) none of these

Sol. **[A]**



Using $(m + n) \cot \theta = n \cot \beta - m \cot \alpha$ we get

$(3 + 2) \cot (\angle CDA) = 2 \cot 30 - 3 \cot 60$

$\Rightarrow \cot (\angle CDA) = \frac{\sqrt{3}}{5} \Rightarrow \sin (\angle CDA) = \frac{5}{2\sqrt{7}}$

Apply sine rule in ΔACD, we get

$$\frac{AC}{\sin\left(\sin^{-1}\frac{5}{2\sqrt{7}}\right)} = \frac{AD}{\sin 60}$$

$$AC = 3.\ \frac{2}{\sqrt{3}} . \frac{5}{2\sqrt{7}} = 5\sqrt{\frac{3}{7}}$$

Fill in The Blanks type Questions

Q.15 If one angle of a triangle be 60º, the area $10\sqrt{3}$ sq. cm and the perimeter 20 cm, then the lengths of the sides are ....

Sol. Let A = 60° given $\Delta = 10\sqrt{3}$ sq. cm. and a + b + c = 20 cm

We know that $\Delta = \frac{1}{2}$ bc sin A

$\Rightarrow \frac{1}{2}$ bc sin 60° = $10\sqrt{3}$ $\Rightarrow$ bc = 40

$$\because \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - 2bc - a^2}{2bc}$$

$\because$ b + c = 20 – a, A = 60°, bc = 40°

$$\therefore\ \frac{1}{2} = \frac{(20-a)^2 - 80 - a^2}{80}$$

$\Rightarrow 40 = 400 - 80 - 40a \ \Rightarrow a = 7$

Θ $a + b + c = 20 \Rightarrow b + c = 13$

$\Rightarrow (b + c)^2 = 169 \Rightarrow (b - c)^2 = 169 - 4bc \Rightarrow b - c = 3$

on solving we get b = 8, c = 5

side are 7, 8, 5

Q.16 If the sines of the angles of a triangle are in the ratio 3 : 5 : 7, their cotangents are in the ratio ....

Sol. By sine rule we have

a : b : c = 3 : 5 : 7 or a = 3k, b = 5k, c = 7k

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$Θ\ \Delta = \frac{1}{2} bc \sin A \quad \Rightarrow \sin A = \frac{2\Delta}{bc}$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{4\Delta} = \frac{65k^2}{4\Delta}$$

$$\text{similarly } \cot B = \frac{c^2 + a^2 - b^2}{4\Delta} = \frac{33k^2}{4\Delta}$$

$$\text{and } \cot C = \frac{a^2 + b^2 - c^2}{4\Delta} = \frac{-15k^2}{4\Delta}$$

$\therefore$ cot A : cot B : cot C = 65 : 33 : – 15

➤ True or False type Questions

Q.17 If one side of a triangle is double of another side and the angles opposite to these differ by 60°, then the triangle is right angled.

Sol. Let a = 2b and A – B = 60° Then by sine rule

$$\frac{2b}{\sin(60 + B)} = \frac{b}{\sin B}$$

$$\Rightarrow \frac{\sin(60 + B)}{\sin B} = 2$$

$$\Rightarrow \frac{\frac{\sqrt{3}}{2}\cos B + \frac{1}{2}\sin B}{\sin B} = 2$$

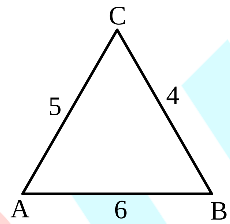
$$\Rightarrow \frac{\sqrt{3}}{2}\cot B = \frac{3}{2}$$

$\Rightarrow \cot B = \sqrt{3} \Rightarrow B = 30°$

and A = 90° (True)

Q.18 If the sides of a triangle are 4m, 5m, 6m then the smallest angle is half of the greatest angle.

Sol.



$$\cos A = \frac{36 + 25 - 16}{60} = \frac{45}{60} = \frac{3}{4}$$

$$\cos C = \frac{25 + 16 - 36}{40} = \frac{1}{8}$$

$$\cos\frac{C}{2} = \sqrt{\frac{1 + \cos C}{2}} = \frac{3}{4}$$

$\Rightarrow A = \dfrac{C}{2}$ True

EXERCISE # 2

Part-A Only Single Correct Answer type Questions

Q.1 If in a triangle ABC, sinA = $\sin^2$B and $2\cos^2$A = $3\cos^2$B, then the Δ ABC is -

(A) right angled (B) obtuse angled

(C) isosceles (D) equilateral

Sol. **[B]**

$\sin A = \sin^2 B$ .. (i)

$2\cos^2 A = 3\cos^2 B$ …(ii)

from (i) and (ii)

$2(1 - \sin^2 A) = 3(1 - \sin A)$

$\Rightarrow 2\sin^2 A - 3\sin A + 1 = 0$

$\Rightarrow (2\sin A - 1)(\sin A - 1) = 0$

$\Rightarrow \sin A = \frac{1}{2}$ or $\sin A = 1$

But $\sin A \neq 1$

$\Rightarrow \angle A = 30°$

from (i) $\sin^2 B = \frac{1}{2}$

$\Rightarrow \angle B = 45°$

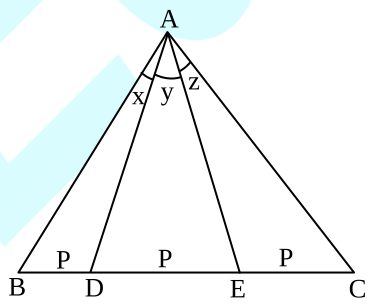
and $\angle C = 180° - 75° = 105°$

ΔABC is obtuse angled.

Q.2 Points D, E are taken on the side BC of a triangle ABC such that BD = DE = EC. If ∠BAD = x, ∠DAE = y, ∠EAC = z, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} =$

(A) 1 (B) 2 (C) 4 (D) none

Sol. **[C]**



Θ $\frac{AD}{BD} = \frac{\sin B}{\sin x}$ and $\frac{AD}{DC} = \frac{\sin C}{\sin(y+z)}$ …(i)

and $\frac{AE}{BE} = \frac{\sin B}{\sin(x+y)}$ and $\frac{AE}{EC} = \frac{\sin C}{\sin z}$ … (ii)

From (i) $\frac{DC}{BD} = \frac{\sin B}{\sin C} \cdot \frac{\sin(y+z)}{\sin x}$ …(iii)

From (ii) $\frac{BE}{EC} = \frac{\sin C}{\sin B} \cdot \frac{\sin(x+y)}{\sin z}$ …(iv)

Θ $\frac{DC}{BD} = 2$ and $\frac{BE}{EC} = 2$

Then from (iii) and (iv), we get

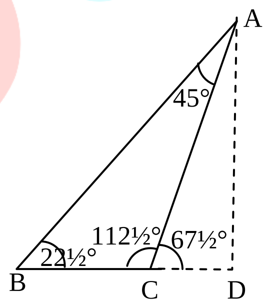
$\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} = 4$

Q.3 If the base angles of a triangle are 22½° and 112½°, the height of the triangle is equal to -

(A) half the base (B) the base

(C) twice the base (D) four times the base

Sol. **[A]**



From sine rule, we have

$\frac{BC}{\sin 45°} = \frac{AC}{\sin 22½°}$ …(i)

Also $\frac{AD}{AC} = \sin 67½°$

$\Rightarrow AD = AC \cos 22½°$

$\Rightarrow AD = \frac{BC}{\sin 45°} \sin 22½° \cos 22½°$ from (i)

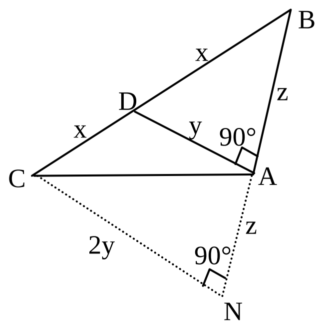
$\Rightarrow AD = \frac{BC}{2} \cdot \frac{\sin 45°}{\sin 45°}$

$\Rightarrow AD = \frac{1}{2} BC = \frac{1}{2}$ of base

Q.4 In a ΔABC if median AD is perpendicular to AB, then tan A + 2 tan B is equal to -

(A) 1 (B) 3 (C) 0 (D)1/2

Sol. **[C]**



Given that BD = DC ; $\angle DAB = 90°$

Draw a perpendicular CN at extend BA

In ΔBCN ; DA = $\frac{1}{2}$ CN = y and AB = AN = z

$\therefore$ tan A = tan ($\pi - \angle CAN$) = – tan ($\angle CAN$)

$= -\frac{CN}{AN} = -\frac{2y}{z}$

$\Rightarrow \tan A = -\frac{2AD}{AB}$

$\Rightarrow$ tan A = – 2 tan B

$\Rightarrow$ tan A + 2 tan B = 0

Q.5 In a triangle, $a^2 + b^2 + c^2 = ca + ab\sqrt{3}$, then the triangle is -

(A) equilateral

(B) right-angled and isosceles

(C) right-angled with A = 90º, B = 60º, C = 30º

(D) None of these

Sol. **[C]**

Write $a^2 = \frac{a^2}{4} + \frac{3a^2}{4}$

then

$$\left(\frac{a^2}{4} + c^2 - ca\right) + \left(\frac{3a^2}{4} - ab\sqrt{3} + b^2\right) = 0$$

$$\Rightarrow \left(\frac{a}{2} - c\right)^2 + \left(\frac{\sqrt{3}a}{2} - b\right)^2 = 0$$

$$\Rightarrow c = \frac{a}{2} \text{ and } b = \frac{\sqrt{3}a}{2}$$

then $b^2 + c^2 = \frac{3a^2}{4} + \frac{a^2}{4} = a^2$

triangle is right triangled, A = 90°

cos B = $\frac{C}{1} = \frac{1}{2}$; B = 60° and C = 30°

Q.6 If in a ΔABC, $a^2 \cos^2 A = b^2 + c^2$, then-

(A) $A < \frac{\pi}{2}$ (B) $\frac{\pi}{4} < A < \frac{\pi}{2}$

(C) $A > \frac{\pi}{2}$ (D) $A = \frac{\pi}{2}$

Sol. **[C]**

Given $a^2 \cos^2 A = b^2 + c^2$

we know that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\Rightarrow \cos A = \frac{a^2 \cos^2 A - a^2}{2bc}$$

or $\cos A = -\frac{a^2 \sin^2 A}{2bc} = -$ ve

$A > \frac{\pi}{2}$

Q.7 In a ΔABC, if $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$, then $\angle C$ is equal to-

(A) 60º (B) 135º (C) 90º (D) 75º

Sol. **[B]**

$a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2 = 0$

$\Rightarrow a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2 + 2a^2b^2 = 2a^2b^2$

$\Rightarrow (a^2 + b^2 - c^2)^2 = 2a^2b^2$

$$\Rightarrow \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2 = \frac{1}{2}$$

$\Rightarrow \cos^2 C = \frac{1}{2} \Rightarrow \cos C = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ C = 45°, 135°

Q.8 In ΔABC, a = 5, b = 4 and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$. The side c is -

(A) 6 (B) 3 (C) 2 (D) None

Sol. **[A]**

$\because \cos C = \dfrac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}$ $\quad \because \tan\frac{C}{2} = \sqrt{\frac{7}{9}}$

$\Rightarrow \cos C = \dfrac{1-\frac{7}{9}}{1+\frac{7}{9}} \Rightarrow \cos C = \dfrac{1}{8}$

$\because \cos C = \dfrac{a^2+b^2-c^2}{2ab}$ $\quad \because a = 5, b = 4$

$\Rightarrow \dfrac{1}{8} = \dfrac{25+16-c^2}{40}$

$\Rightarrow c^2 = 36 \qquad \Rightarrow c = 6$

Q.9 In a ΔABC, $A = \dfrac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and ar(ΔABC) = $\dfrac{9\sqrt{3}}{2}$ cm^2. Then a is -

(A) $6\sqrt{3}$ cm (B) 9 cm

(C) 18 cm (D) None of these

Sol. **[B]**

Given $A = \dfrac{2\pi}{3}$, $b - c = 3\sqrt{3}$ & ar (ΔABC) = $\dfrac{2\sqrt{3}}{2}$

$\because \Delta = \dfrac{1}{2} bc \sin A \Rightarrow \dfrac{1}{2} bc \sin\dfrac{2\pi}{3} = \dfrac{9\sqrt{3}}{2}$

$\Rightarrow bc = 18$ and $b - c = 3\sqrt{3}$

$\Rightarrow b^2 + c^2 = 27 + 2bc = 27 + 36 \Rightarrow b^2 + c^2 = 63$

$\because \cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\Rightarrow \cos\dfrac{2\pi}{3} = \dfrac{63-a^2}{2\times18} \Rightarrow -\dfrac{1}{2} = \dfrac{63-a^2}{36}$

$\Rightarrow a^2 = 63 + 18 = 81 \Rightarrow a = 9$

Q.10 In a ΔABC, $\cos B.\cos C + \sin B.\sin C \sin^2 A = 1$. Then the triangle is-

(A) right-angle isosceles

(B) isosceles whose equal angles are greater than $\pi/4$

(C) equilateral

(D) none of these

Sol. **[A]**

$\sin^2 A = \dfrac{1-\cos B\cos C}{\sin B\sin C} \le 1 \qquad \ldots$ (i)

$\Rightarrow 1 \le \cos B \cos C + \sin B \sin C$

$\Rightarrow 1 \le \cos (B - C)$

$\Rightarrow 1 = \cos (B - C) \Rightarrow B - C = 0$

$\Rightarrow B = c \qquad \therefore \cos\theta \not> 1$

then from (i)

$\sin^2 A = \dfrac{1-\cos^2 B}{\sin^2 B} = 1$

$\Rightarrow \sin A = 1 \qquad \Rightarrow A = 90°$

$B = C = 45°$

triangle is right angled isosceles.

Q.11 If A is the area and 2s the sum of the sides of a triangle, then-

(A) $A \le \dfrac{s^2}{3\sqrt{3}}$ (B) $A \ge \dfrac{s^2}{3\sqrt{3}}$

(C) $A > \dfrac{s^2}{\sqrt{3}}$ (D) none of these

Sol. **[A]**

$\therefore A^2 = s(s-a)(s-b)(s-c)$ and $2s = a + b + c$

Now A.M. $\ge$ G.M.

$\Rightarrow \dfrac{(s-a)+(s-b)+(s-c)}{3} \ge [(s-a)(s-b)(s-c)]^{1/3}$

$\Rightarrow \left(\dfrac{s}{3}\right)^3 \ge \dfrac{A^2}{s} \Rightarrow A \le \dfrac{s^2}{3\sqrt{3}}$

Part-B One or More Than one correct Answer Type Questions

Q.12 In a ΔABC, tan A and tan B satisfy the in-equation $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$. Then-

(A) $a^2 + b^2 + ab > c^2$

(B) $a^2 + b^2 - ab < c^2$

(C) $a^2 + b^2 > c^2$

(D) none of these

Sol. **[A, B]**

$\sqrt{3}x^2 - 4x + \sqrt{3} < 0$

$\Rightarrow (x - \sqrt{3})(\sqrt{3}x - 1) < 0$

$\Rightarrow \dfrac{1}{\sqrt{3}} < x < \sqrt{3}$

$\therefore \dfrac{1}{\sqrt{3}} < \tan A < \sqrt{3}$, $\dfrac{1}{\sqrt{3}} < \tan B < \sqrt{3}$

$\Rightarrow 30° < \tan A < 60°$, $30° < B < 60°$

$\therefore 60° < C < 120°$

$\Rightarrow -\dfrac{1}{2} < \cos C < \dfrac{1}{2}$

$\Rightarrow -\dfrac{1}{2} < \dfrac{a^2+b^2-c^2}{2ab} < \dfrac{1}{2}$

$\Rightarrow -ab < a^2 + b^2 - c^2 < ab$

$\Rightarrow a^2 + b^2 - ab < c^2$ and $a^2 + b^2 + ab > c^2$

Q.13 In a ΔABC, $A = \frac{\pi}{3}$ and $b : c = 2 : 3$.

If $\tan\alpha = \frac{\sqrt{3}}{5}$, $0 < \alpha < \frac{\pi}{2}$, then-

(A) $B = 60^o + \alpha$ (B) $C = 60^o + \alpha$
(C) $B = 60^o - \alpha$ (D) $C = 60^o - \alpha$

Sol. **[B, C]**

$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$

$\Rightarrow \tan\frac{C-B}{2} = \frac{1}{5}\cot 30° = \frac{\sqrt{3}}{5} = \tan\alpha$ given.

$\Rightarrow C - B = 2\alpha$ and $C + B = 180° - 60° = 120°$

Solving we get $C = 60° + \alpha$, $B = 60° - \alpha$

Q.14 If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$ then-

(A) $C = \frac{\pi}{4}$ (B) $A = \sin^{-1}\frac{2}{\sqrt{5}}$
(C) ar $(\Delta ABC) = 9$ (D) none of these

Sol. **[B, C]**

$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{1}{3}\cot\frac{C}{2}$

$\therefore \cos(A-B) = \dfrac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}$

$\Rightarrow \frac{4}{5} = \dfrac{1-\frac{1}{9}\cot^2\frac{C}{2}}{1+\frac{1}{9}\cot^2\frac{C}{2}} \Rightarrow \tan^2\frac{C}{2} = 1 \Rightarrow C = \frac{\pi}{2}$

ar $(\Delta ABC) = \frac{1}{2}ab = \frac{1}{2}.6.3 = 9$

Also $\sin A = \dfrac{6}{\sqrt{3^2+6^2}} = \dfrac{6}{3\sqrt{5}}$

$A = \sin^{-1}\frac{2}{\sqrt{5}}$

Part-C Assertion Reason type Questions

The following questions 15 to 16 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.15 **Assertion (A) :** If $y^2 + y + 1$, $2y + 1$ and $y^2 - 1$ represents lengths of the sides of Δ, then $y > 1$.

Reason (R) : If a, b, c be the length of sides of Δ, then $a + b > c$, $b + c < a$, $c + a > b$.

Sol. **[C]**

We know that in any triangle

$a + b > c$, $b + c > a$, $c + a > b$

(R) is false.

and $y^2 + y + 1 + 2y + 1 > y^2 - 1$

$\Rightarrow 3y + 3 > 0 \Rightarrow y > -1$ … (i)

and $y^2 + y + 1 + y^2 - 1 > 2y + 1$

$\Rightarrow 2y^2 - y - 1 > 0$

$y \in \left(-\infty, -\frac{1}{2}\right) \cup (1, \infty)$ … (ii)

and

$2y + 1 + y^2 - 1 > y^2 + y + 1 \Rightarrow y > 1$ ….(iii)

from (i), (ii) and (iii) we get $y > 1$

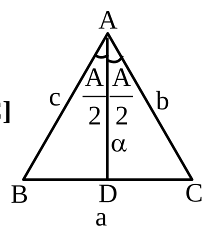
(A) is true.

Q.16 **Assertion (A) :** If α, β, γ are the lengths of internal bisector of the angles A, B and C respectively of a triangle, then

$\frac{1}{\alpha}\cos\left(\frac{A}{2}\right) + \frac{1}{\beta}\cos\left(\frac{B}{2}\right) + \frac{1}{\gamma}\cos\left(\frac{C}{2}\right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Reason (R) : Since, the length of angle bisectors are $\frac{bc}{b+c}\cos\left(\frac{A}{2}\right)$, $\frac{ca}{c+a}\cos\left(\frac{B}{2}\right)$, $\frac{ab}{a+b}\cos\left(\frac{C}{2}\right)$

Sol. **[C]**



Θ $\frac{1}{2}bc\sin A = \frac{1}{2}c\alpha\sin\frac{A}{2} + \frac{1}{2}b\alpha\sin\frac{A}{2}$

$\Rightarrow 2bc\sin\frac{A}{2}\cos\frac{A}{2} = c\alpha\sin\frac{A}{2} + b\alpha\sin\frac{A}{2}$

$\Rightarrow 2bc\cos\frac{A}{2} = \alpha(b + c)$

$\Rightarrow \dfrac{\cos\frac{A}{2}}{\alpha} = \frac{1}{2b} + \frac{1}{2c}$

similarly $\dfrac{\cos\frac{B}{2}}{\beta} = \frac{1}{2a} + \frac{1}{2c}$

and $\dfrac{\cos\frac{C}{2}}{\gamma} = \dfrac{1}{2a} + \dfrac{1}{2b}$

So $\dfrac{1}{\alpha}\cos\dfrac{A}{2} + \dfrac{1}{\beta}\cos\dfrac{B}{2} + \dfrac{1}{\gamma}\cos\dfrac{C}{2} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

$\because$ length of bisector is

$\alpha = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

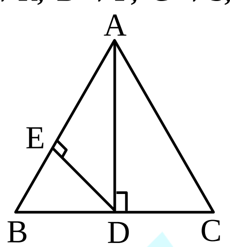
$\Rightarrow$ A is true but R is false

Part-D Column Matching type Questions

Q.17 In a triangle ABC, AD is perpendicular to BC and DE is perpendicular to AB.

Column-I	**Column-II**
(A) Area of Δ ADB	(P) $(b^2/4)\sin 2C$
(B) Area of Δ ADC	(Q) $(c^2/4)\cos^2 B \sin 2B$
(C) Area of Δ ADE	(R) $(c^2/4)\sin 2B$
(D) Area of Δ BDE	(S) $(c^2/4)\sin^2 B \sin 2B$

Sol. **A $\to$ R, B $\to$ P, C $\to$ S, D $\to$ Q**



(A) Area of $\Delta ADB = \dfrac{1}{2}$ (AD) (BD)

$\because$ c cos B = BD, c sin B = AD

Area $= \dfrac{1}{2}c^2 \sin B \cos B = \dfrac{c^2}{4}\sin 2B$

(B) Area of $\Delta ADC = \dfrac{1}{2}$ (AD) (DC)

$\because$ AD = b sin C, DC = b cos C

Area $= \dfrac{1}{2}b^2 \sin C \cos C = \dfrac{b^2}{4}\sin 2C$

(C) Area of $\Delta ADE = \dfrac{1}{2}$ (AE) (ED)

$\because$ ED = BD sin B = C sin B cos B

$\because (AE)^2 = (AD)^2 - (ED)^2 = c^2 \sin^4 B$

$\Rightarrow AE = c\sin^2 B$

Area $= \dfrac{1}{4}c^2 \sin^2 B \sin^2 B$

(D) Area of $\Delta BDE = \dfrac{1}{2}$ (ED) (BE)

ED = c sin B cos B

$(BE)^2 = (BD)^2 - (ED)^2$

$(BE)^2 = c^2\cos^2 B - c^2 \sin^2 B \cos^2 B$

$= c^2 \cos^4 B$

$BE = c\cos^2 B$

Area $= \dfrac{1}{4}c^2\cos^2 B \sin 2B$

Q.18 Match the following :

Column-I	**Column-II**
(A) In a triangle ABC, if a is the arithmetic mean and b, c ($b \neq c$) are the two geometric means between any two positive real numbers then $\dfrac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C}$ is equal to	(P) 9
(B) In a triangle ABC; $\dfrac{(a^2-b^2-c^2)\tan A + (a^2+c^2-b^2)\tan B}{2}$ is equal to	(Q) 2
(C) In a triangle ABC, if B = 30º and C = $\sqrt{3}$ b, then $\dfrac{A}{45^\circ}$ can be equal to	(R) 1
(D) In a ΔABC, if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then $\Sigma\, 4\sin A \sin B =$	(S) 0

Sol. **A $\to$ Q, B $\to$ S, C $\to$ Q, D $\to$ P**

(A) Let number are P, Q

then $\dfrac{P+Q}{2} = a$, $b = P\left(\dfrac{Q}{P}\right)^{1/3}$, $c = P\left(\dfrac{Q}{P}\right)^{2/3}$

$\because \dfrac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C} = \dfrac{b^3 + c^3}{abc}$

put the values of a,b,c then we get

$\dfrac{2(P+Q)PQ}{(P+Q)PQ} = 2$

(B) $\dfrac{(a^2-b^2-c^2)\tan A + (a^2+c^2-b^2)\tan B}{2}$

$= \dfrac{-2bc\cos A\tan A + 2ac\cos B\tan B}{2}$

$= -$ bc sin A + ac sin B

$= -\lambda$ abc + λ abc = 0

(C) b sin C = c sin B

$\because c = \sqrt{3}\,b$, B = 30°

$\sin C = \dfrac{\sqrt{3}}{2} \Rightarrow C = 60°$

$\Rightarrow$ A = 180° – (90°) = 90°

$\Rightarrow \frac{A}{45°} = 2$

(D) solving we get

$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

but $a + b + c \neq 0$ (side of Δ)

$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

$\Rightarrow a = b = c$

Δ is equilateral

$A = B = C = 60°$

$\sum \sin A \sin B = 3 \sin^2 A = 3 . \frac{3}{4} = \frac{9}{4}$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 In any ΔABC, prove that

$$\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$$

Sol. $\because \frac{a\sin(B-C)}{b^2-c^2} = \frac{\sin A\sin(B-C)}{k(\sin^2 B-\sin^2 C)}$

$$= \frac{\sin(B+C)\sin(B-C)}{k(\sin^2 B-\sin^2 C)} = \frac{\sin^2 B-\sin^2 C}{k(\sin^2 B-\sin^2 C)}$$

$$= \frac{1}{k}$$

similarly $\frac{b\sin(C-A)}{c^2-a^2} = \frac{1}{k}$ and $\frac{c\sin(A-B)}{a^2-b^2} = \frac{1}{k}$

Hence proved.

Q.2 In a triangle ABC, the side c has two values. Prove that both the values satisfy the equation

$$\frac{(a+b)^2}{1+\cos C} + \frac{(b-a)^2}{1-\cos C} = \frac{2a^2}{\sin^2 A}$$

Sol. Taking L.H.S. we have

$$\frac{(1-\cos C)(a+b)^2 + (1+\cos C)(b-a)^2}{1-\cos^2 C}$$

$$= \frac{2(a^2+b^2)-4ab\cos C}{\sin^2 C} = \frac{2c^2}{\sin^2 C}$$

Then we get

$$2\left(\frac{c}{\sin C}\right)^2 = 2\left(\frac{a}{\sin A}\right)^2$$

⇒ Hence both values of c satisfy the equation.

Hence proved.

Q.3 In a triangle, the angles A, B, C are in AP. Show that $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{[a^2-ac+c^2]}}$

Sol. A, B, C are in A.P.

so 2B = A + C

But $A+B+C=\pi \Rightarrow B = 60°$ and A + C = 120°

Now $b^2 = c^2 + a^2 - 2ca\cos B$

$b^2 = c^2 + a^2 - ac$ $\qquad \because B = 60°$

From R.H.S. we have

$$\frac{a+c}{\sqrt{a^2-ac+c^2}} = \frac{a+c}{b} = \frac{\sin A+\sin C}{\sin B}$$

using sine rule

$$= \frac{2\sin\frac{A+C}{2} + \cos\frac{A-C}{2}}{\sin B}$$

$$= \frac{2\sin 60°\cos\frac{A-C}{2}}{\sin 60°} = 2\cos\frac{A-C}{2}$$

L.H.S. Hence proved.

Q.4 Find the greatest angle of the triangle whose sides are $x^2 + x + 1$, $2x + 1$, $x^2 - 1$.

Sol. $\because$ sides are positive.

$\Rightarrow x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0$

$x > 1$ or $x < -1$

But $x \not< -1$ $\qquad \because 2x + 1$ becomes – ve

So $x > 1$

$\because a = x^2 + x + 1 > x + x + 1$

$\Rightarrow x^2 + x + 1 > 2x + 1$

$\Rightarrow a > b$

and $a - c = x^2 + x + 1 - x^2 + 1 > 0$

⇒ greatest side is a

so $\cos A = \frac{b^2+c^2-a^2}{2bc}$

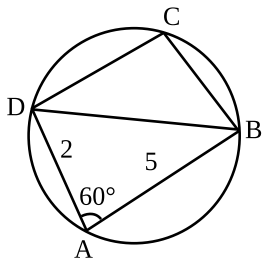
$$= \frac{(2x+1)^2+(x^2-1)^2-(x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

On solving we get

$\cos A = -\frac{1}{2} \Rightarrow A = 120°$

Q.5 The two adjacent sides of cyclic quadrilateral are 2 and 5 and the angle between them is 60º. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Sol.



Let BC = a, DC = b, BD = c

then from ΔABD weight

$\cos 60 = \frac{25+4-c^2}{20} \Rightarrow c^2 = 19$ ….(1)

given that area of ABCD = $4\sqrt{3}$

$\Rightarrow \frac{1}{2} . 2.5 \sin 60 + \frac{1}{2} . a.b \sin 120 = 4\sqrt{3}$

$\Rightarrow ab = 6$ …….(2)

from ΔBCD we have

$\cos 120 = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow a^2 + b^2 = 13$ …….(3)

from (2) & (3) we get

a = 2, b = 3 or a = 3, b = 2

Q.6 In any Δ ABC, prove that

$$\frac{b\sec B + c\sec C}{\tan B + \tan C} = \frac{c\sec C + a\sec A}{\tan C + \tan A}$$

$$= \frac{a\sec A + b\sec B}{\tan A + \tan B}.$$

Sol. Taking $\frac{b\sec B + c\sec C}{\tan B + \tan C}$

$$= \frac{b\cos C + c\cos B}{\sin B\cos C + \cos C\sin B}$$

$$= \frac{a}{\sin(B+C)} = \frac{a}{\sin A}$$

(By projection formula)

Similarly $\frac{c\sec C + a\sec A}{\tan C + \tan A} = \frac{b}{\sin B}$

and $\frac{a\sec A + b\sec B}{\tan A + \tan B} = \frac{c}{\sin C}$

from sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q.7 In any Δ ABC, prove that, if θ be any angle, then b cosθ = c cos(A – θ) + a cos(C + θ).

Sol. Taking R.H.S. we have

k sin C cos (A – θ) + k sin A cos (C + θ)

= k [sin C cos A cos θ + sin C sin A sin θ

+ sin A cos C cos θ – sin A sin C sin θ]

= k cos θ [sin C cos A + cos C sin A]

= k cos θ sin (A + C)

= k cos θ sin B = b cos θ

L.H.S. Hence proved.

Q.8 In any ΔABC, prove that a(cos B + cos C – 1) + b(cos C + cos A – 1) + c(cos A + cos B – 1) = 0.

Sol. Taking L.H.S. we have

a cos B + a cos C – a + b cos C + b cos A – b

+ c cos A + c cos B – C

= (a cos B + b cos A) + (a cos C + c cos A)

+ (b cos C + c cos B) – a – b – c

= c + b + a – a – b – c

by projection formula = 0

Q.9 In any ΔABC, prove that

$$\frac{1}{(a-b)(a-c)}\tan\left(\frac{A}{2}\right) + \frac{1}{(b-c)(b-a)}\tan\left(\frac{B}{2}\right) + \frac{1}{(c-a)(c-b)}\tan\left(\frac{C}{2}\right) = \frac{1}{\Delta}.$$

Sol. Taking L.H.S.

$\because \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ etc.

$$\Rightarrow \frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{(a-b)(a-c)} + \frac{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}{(b-c)(b-a)} + \frac{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}{(c-a)(c-b)}$$

$$\Rightarrow \frac{(s-b)(s-c)(c-b) + (a-c)(s-a)(s-c) + (b-a)(s-a)(s-b)}{(a-b)(b-c)(c-a)\sqrt{s(s-a)(s-b)(s-c)}}$$

On solving we get

$$= \frac{1}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{1}{\Delta}$$

R.H.S. Hence proved.

Q.10 In any ΔABC, if $3\tan\left(\frac{A}{2}\right).\tan\left(\frac{C}{2}\right) = 1$, prove that a, b, c are in A. P.

Sol. $3\tan\frac{A}{2}.\tan\frac{C}{2} = 1$

$\Rightarrow 3\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = 1$

$\Rightarrow 3(s-b) = s$

$\Rightarrow 2s = 3b$

$\Rightarrow 2b = a + c$

a, b, c are in A.P.

Hence proved.

Part-B Passage based objective questions

Passage-I (Q. 11 to 13)

In a Δ ABC two sides a, b and $\angle$A are given. Then the numbers of triangles that are formed in the following cases are

On the basis of above passage, answer the following questions :

Q.11 If $a < b\sin A$

(A) 1 (B) 2

(C) 0 (D) Infinite

Sol. **[C]**

From sine rule, we have

$\sin B = \frac{b}{a}\sin A$ …(i)

If $a < b\sin A$

$\Rightarrow \sin B > 1$ which is impossible.

No triangles are formed.

Q.12 If $a = b\sin A$ and $\angle$A is acute

(A) 1 (B) 2

(C) 0 (D) Infinite

Sol. **[A]**

If $a = b\sin A$ and $\angle$A acute

then from (i), we get

$\sin B = 1$

one triangle we can formed which is right angled at B.

Q.13 If $a = b\sin A$ and $\angle$A is obtuse

(A) 1 (B) 2

(C) 0 (D) Infinite

Sol. **[C]**

If $a = b\sin A$ and $\angle$A obtuse.

No any triangle are formed.

Passage-II (Q. 14 to 16)

The internal bisectors of the angle of ΔABC meet the sides BC, CA and AB at D, E and F respectively. Then

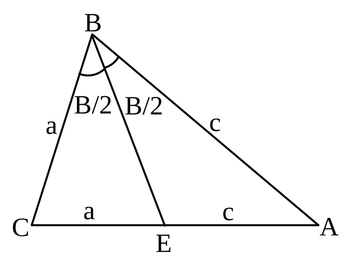
On the basis of above passage, answer the following questions :

Q.14 The length BE is given by

(A) $\frac{2bc}{b+c}\cos\left(\frac{A}{2}\right)$ (B) $\frac{bc}{b+c}\cos\left(\frac{A}{2}\right)$

(C) $\frac{2ac}{a+c}\cos\left(\frac{B}{2}\right)$ (D) $\frac{ca}{c+a}\cos\left(\frac{B}{2}\right)$

Sol. **[C]**



Let BE = x

$\because \Delta = \Delta_1 + \Delta_2$

$\Rightarrow \frac{1}{2}ac\sin B = \frac{1}{2}ax\sin\frac{B}{2} + \frac{1}{2}cx\sin\frac{B}{2}$

$\Rightarrow x = \frac{ac\sin B}{(a+c)\sin\frac{B}{2}} = \frac{2ac\cos\frac{B}{2}}{a+c}$

Q.15 $\frac{1}{AD}\cos\frac{A}{2} + \frac{1}{BE}\cos\frac{B}{2} + \frac{1}{CF}\cos\frac{C}{2} =$

(A) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ (B) abc

(C) a + b + c (D) ab + bc + ca

Sol. **[A]**

we know that

$BE = \frac{2ac}{a+c}\cos\frac{B}{2}$, $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$,

$CF = \frac{2ab}{a+b}\cos\frac{C}{2}$

$\because \frac{1}{AD}\cos\frac{A}{2}+\frac{1}{BE}\cos\frac{B}{2}+\frac{1}{CF}\cos\frac{C}{2}$

$= \frac{b+c}{2bc}+\frac{a+c}{2ac}+\frac{a+b}{2ab}$

$= \frac{2(ab+bc+ca)}{2abc}$

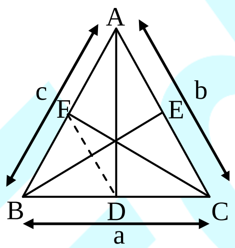
$= \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Q.16 Area of ΔBDF is given by

(A) $\frac{ac\sin B}{2(a+b)}$ (B) $\frac{\Delta ac}{(a+b)(b+c)}$

(C) $\frac{ac\sin B}{(a+b)(b+c)}$ (D) $\frac{ab\sin C}{(a+b)}$

Sol.[B] Area of $\Delta BDF = \frac{1}{2}\times BD\times BF\times \sin B$



$\frac{BD}{CD} = \frac{c}{b} = x$ (let)

$\Rightarrow \frac{BD}{c} = \frac{CD}{b} = x$

$\Rightarrow BD = cx$ .....(1)

and CD = bx ......(2)

$\Rightarrow BD + CD = cx + bx = a$ (from (1) & (2))

$\Rightarrow (x)(b + c) = a$

$\Rightarrow x = \frac{a}{b+c}$

$BD = cx = \frac{ac}{b+c}$ (from (1))

$BD = \frac{ac}{b+c}$ ....(3)

Similarly,

$\frac{AF}{b} = \frac{BF}{a} = y$ (let)

AF = by ......(4)

and BF = ay ......(5)

AF + BF = y(a + b) = c {From (4) & (5))

$\Rightarrow y = \frac{c}{a+b}$

$BF = ay = \frac{ac}{a+b}$ (from (5)

$BF = \frac{ac}{a+b}$ .....(6)

$ar(\Delta BDF) = \frac{1}{2}\times BD\times BF\sin B$

$= \frac{1}{2}\times\frac{ac}{b+c}\times\frac{ac}{a+b}\sin B$

$= \frac{\Delta ac}{(a+b)(b+c)}$, (Δ = area of ΔABC)

EXERCISE # 4

Old IIT-JEE Questions

Q.1 In a triangle ABC, $\angle A = 45^\circ$, b = 10 cm and $c = 10\sqrt{2}$ cm, then - **[REE 2000]**

(A) a = 10 cm (B) $a = 10\sqrt{2}$ cm

(C) $\angle C = 90^\circ$ (D) $\angle B = 45^\circ$

Sol. **[A, C, D]**

$b = 10,\ c = 10\sqrt{2}\ ,\ \angle A = 45^\circ$

$\Rightarrow \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{100 + 200 - a^2}{2.10.10\sqrt{2}}$

$\Rightarrow a^2 = 100 \Rightarrow a = 10$

Again from sine rule

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow \dfrac{10\sqrt{2}}{1} = \dfrac{10}{\sin B} = \dfrac{10\sqrt{2}}{\sin C}$

$\Rightarrow \sin B = \dfrac{1}{\sqrt{2}} \Rightarrow B = 45^\circ$

and $\sin C = 1 \Rightarrow C = 90^\circ$

Option A, C, D are correct.

Q.2 If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side to the perimeter is **[IIT 2003]**

(A) $\sqrt{3} : (2 + \sqrt{3})$ (B) 1 : 6

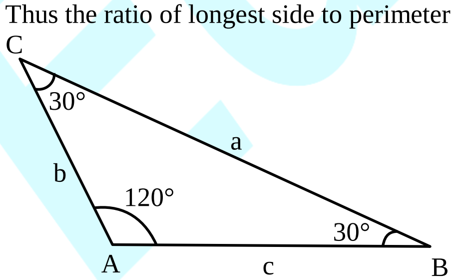
(C) $1 : 2 + \sqrt{3}$ (D) 2 : 3

Sol.[A] Here, ratio of angles are 4 : 1 : 1

$\Rightarrow$ 4x + x + x = 180° $\Rightarrow$ x = 30°

$\therefore\ \angle \geq A = 120^\circ,\ \angle B = \angle C = 30^\circ$

Thus the ratio of longest side to perimeter



$= \dfrac{a}{a + b + c}$

Let b = c = x

$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$

$\Rightarrow a^2 = 2x^2 - 2x^2 \cos A = 2x^2(1 - \cos A)$

$\Rightarrow a^2 = 4x^2 \sin^2 A/2 \Rightarrow a = 2x \sin A/2$

$\Rightarrow a = 2x \sin 60^\circ = \sqrt{3}x$

Thus required ratio is

$\dfrac{a}{a + b + c} = \dfrac{\sqrt{3}x}{x + x + \sqrt{3}x} = \dfrac{\sqrt{3}}{2 + \sqrt{3}}$

Q.3 In any Δ ABC having sides a, b, c opposite to angles A, B, C respectively **[IIT Scr. 2005]**

(A) $a \sin\left(\dfrac{B - C}{2}\right) = (b - c) \cos\dfrac{A}{2}$

(B) $a \cos\dfrac{A}{2} = (b - c) \sin\dfrac{B - C}{2}$

(C) $a \cos\dfrac{A}{2} = (b + c) \sin\dfrac{B + C}{2}$

(D) $a \sin\dfrac{B + C}{2} = (b + c) \cos\dfrac{A}{2}$

Sol. **[A]**

$\because \dfrac{b - c}{a} \cos\dfrac{A}{2} = \dfrac{\sin B - \sin C}{\sin A} \cos\dfrac{A}{2}$

$= \dfrac{2\cos\frac{B + C}{2} \sin\frac{B - C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} \cos\dfrac{A}{2} = \sin\dfrac{B - C}{2}$

L.H.S. option A is correct.

(B) Clearly B is wrong.

(C) $\because \dfrac{a}{b + c} \cos\dfrac{A}{2}$

$= \dfrac{\sin A}{\sin B + \sin C} \cos\dfrac{A}{2}$

$= \dfrac{2\cos\frac{A}{2}\sin\frac{A}{2}}{2\sin\frac{B + C}{2}\cos\frac{B - C}{2}} \cos\dfrac{A}{2} = \dfrac{\sin A}{2\cos\frac{B - C}{2}}$

Clearly (C) and (D) are wrong.

Q.4 Internal angle bisector of $\angle A$ of triangle ABC, meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at P and the side AB at Q. If a, b, c represent the sides of ΔABC then **[IIT-2006]**

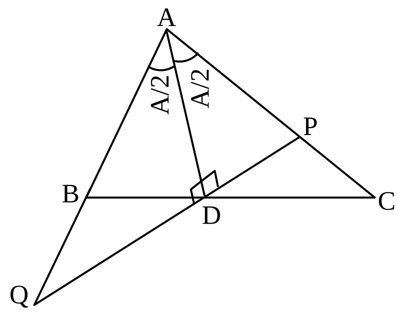
(A) $AD = \dfrac{2bc}{b + c} \cos\dfrac{A}{2}$

(B) $PQ = \dfrac{4bc}{b + c} \sin\dfrac{A}{2}$

(C) The triangle APQ is isosceles

(D) AP is HM of b and c

Sol. **[A, B, C, D]**



By geometry in ΔAPQ, we have

$AP = AQ$, ΔAPQ is isosceles

Now ar (ΔABC) = ar (ΔABD) + ar(ΔADC)

$\Rightarrow \frac{1}{2} bc \ \sin \ A = \frac{1}{2} c.(AD) \ \sin \frac{A}{2} + \frac{1}{2} b. \ (AD) \ \sin \frac{A}{2}$

$\Rightarrow AD = \dfrac{2bc \cos \frac{A}{2}}{b+c}$

Also $AD = AP \cos \frac{A}{2}$

$\Rightarrow AP = \frac{2bc}{b+c}$ = H.M. of b and c.

Again PQ = 2DP= 2.AD tan A/2

$\Rightarrow PQ = 2.\frac{2bc}{b+c}\cos\frac{A}{2} . \dfrac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$

$\Rightarrow PQ = \frac{4bc}{b+c} \sin \frac{A}{2}$

Q.5 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then **[IIT-2008]**

(A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$

(B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$

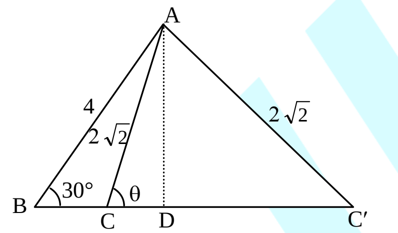
(D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Sol. **[B, D]**

By using A.M $\geq$ G.M. inequality, we get the answers

Q. 6 Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles, is........... **[IIT- 2009]**

Sol. **[4]**



AD = 4 sin 30° = 2

difference of areas = $\Delta ACC'$

$2 = 2\sqrt{2} \sin \theta \Rightarrow \theta = 45°$

side CC′ = 2 ($2\sqrt{2}$ cos 45°) = 4

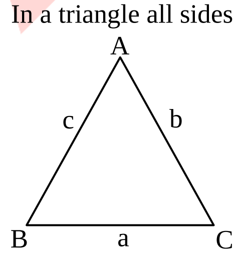
$\Delta ACC' = \frac{1}{2}$. 4. 2 = 4

Q.7 Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) **[IIT-2010]**

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$

(C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

Sol. In a triangle all sides are positive.



$\therefore$ c > 0

$\Rightarrow x^2 - 1 > 0$

$\Rightarrow x < -1$ or $x > 1$ .....(1)

b > 0

$\Rightarrow 2x + 1 > 0$

$\Rightarrow x > -\frac{1}{2}$ .....(2)

from (1) & (2) x > 1 .....(3)

Now, a – b

$\Rightarrow x^2 + x + 1 - 2x - 1$

$\Rightarrow x^2 - x \Rightarrow x(x-1)$

as, x > 1 (from (3))

x(x – 1) > 0

$\Rightarrow$ a > b

Also a – b = x + 2 > 0

$\therefore$ a is the greatest side.

$\Rightarrow$ A is the largest angle.

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(x^2-1)^2+(2x+1)^2-(x^2+x+1)^2}{2(x^2-1)(2x+1)}$$

$$= \frac{(x^2-1)^2+(2x+1+x^2+x+1)(2x+1-x^2-x-1)}{2(x^2-1)(2x+1)}$$

$$= \frac{(x^2-1)^2+(3x+2+x^2)(x-x^2)}{2(x^2-1)(2x+1)}$$

$$= \frac{(x^2-1)^2+(-x)(x^2+3x+2)(x-1)}{2(x^2-1)(2x+1)}$$

$$= \frac{(x^2-1)^2+(-x)(x+1)(x+2)(x-1)}{2(2x+1)}$$

$$= \frac{(x^2-1)-(x)(x+2)}{2(2x+1)}$$

$$= \frac{x^2-1-x^2-2x}{2(2x+1)} = \frac{-(2x+1)}{2(2x+1)} = -\frac{1}{2}$$

$\cos A = -\frac{1}{2}$

A = 120º $\Rightarrow$ B = 30º

$\angle B = \angle C = 30º$

$\therefore$ b = c

$\Rightarrow$ $x^2 - 1 = 2x + 1$ $\Rightarrow$ $x^2 - 2x - 2 = 0$

$\Rightarrow$ $x = 1 + \sqrt{3}$

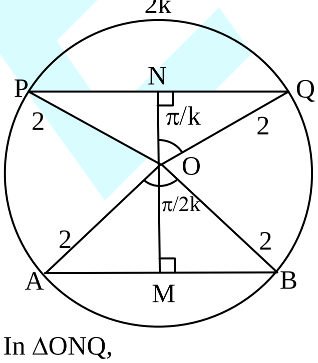
& $x = 1 - \sqrt{3}$ Neglect using (3)

Q.8 Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where k > 0, then the value of [k] is

[Note : [k] denotes the largest integer less than or equal to k] **[IIT-2010]**

Sol.[3] In ΔOMB

$OM = 2\cos\frac{\pi}{2k}$ ....(1)



In ΔONQ,

$ON = 2\cos\frac{\pi}{k}$ .....(2)

(1) + (2)

$$OM + ON = 2\left(\cos\frac{\pi}{k}+\cos\frac{\pi}{2k}\right)$$

$$\frac{\sqrt{3}+1}{2} = 2\cos\frac{3\pi}{4k}\cos\frac{\pi}{4k}$$

$$\frac{\sqrt{3}+1}{4} = \cos\frac{3\pi}{4k}\cos\frac{\pi}{4k}$$

$$\frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4k}\cos\frac{\pi}{4k}$$

comparing,

$$\cos\frac{\pi}{4k} = \cos\frac{\pi}{12}$$

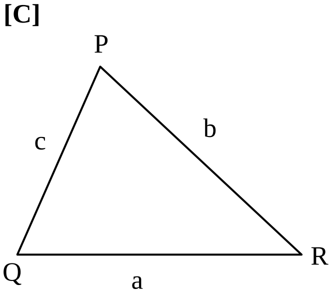
$$\frac{\pi}{4k} = \frac{\pi}{12} \Rightarrow k = 3$$

[k] = 3

Q.9 Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and c = $\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **[IIT-2012]**

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Sol. **[C]**



$$\left.\begin{array}{l} a = 2 \\ b = 7/2 \\ c = 5/2 \end{array}\right\} \Rightarrow s = 4$$

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{1-\cos P}{1+\cos P}$$

$$\Delta = \sqrt{4.2.\frac{1}{2}.\frac{3}{2}} = \sqrt{6} = \tan^2\frac{P}{2}$$

$$= \left(\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\right)^2 = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{4(2)} = \frac{3}{32}$$

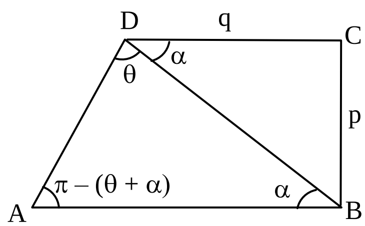
$$= \left(\frac{9}{16.6}\right) = \left(\frac{3}{4\Delta}\right)^2$$

EXERCISE # 5

Q.1 ABCD is a trapezium such that AB, DC are parallel and BC is perpendicular to them. If angle ADB = θ, BC = p & CD = q.

Show that $AB = \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

Sol.



Let ∠ BDC = α = ∠ DBA

so ∠ DAB = π – (θ + α) and $\tan\alpha = \dfrac{p}{q}$

Now from ΔABD

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\pi-(\theta+\alpha))} = \frac{AD}{\sin\alpha}$$

$$\Rightarrow \frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta+\alpha)}$$

$$\Rightarrow AB = \frac{BD\sin\theta}{\sin(\theta+\alpha)}$$

$$\Rightarrow AB = \frac{BD^2\sin\theta}{BD\sin\theta\cos\alpha + BD\cos\theta\sin\alpha} \quad \ldots(i)$$

∵ from right triangle DCB, we have

$BD^2 = p^2 + q^2$, p = BD sin α, q = BD cos α

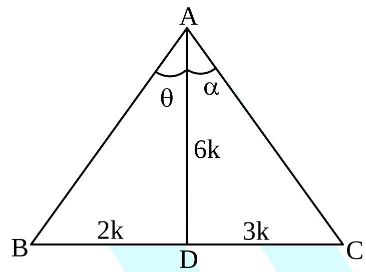
put these values in (i), we get

$$AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Hence proved.

Q.2 The perpendicular AD to the base of a triangle ABC divides it into segments such that BD, CD and AD are in the ratio of 2, 3, and 6; prove that the vertical angle of the triangle is 45º.

Sol.



Let ∠BAD = θ and ∠DAC = α ⇒ A = θ + α

Given BD : CD : AD = 2 : 3 : 6

⇒ BD = 2k, CD = 3k, AD = 6k

from right triangle ADB, we have

$$\tan\theta = \frac{2k}{6k} = \frac{1}{3}$$

and from ΔADC, we have

$$\tan\alpha = \frac{3k}{6k} = \frac{1}{2}$$

$$\because \tan(\theta+\alpha) = \tan A = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

⇒ tan A = 1

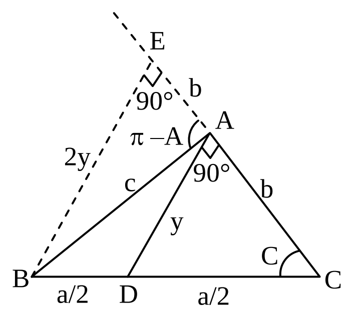
A = 45°

Hence proved.

Q.3 ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that

$$\cos A\,.\cos C = \frac{2(c^2-a^2)}{3ac}.$$

Sol.



∵ Median AD perpendicular to AC

Draw a perpendicular on extended AC

such that AE = AC = b

From property of triangle

If AD = y then BE = 2y

$\because \cos C = \frac{b}{a/2} = \frac{2b}{a}$

$\therefore \cos(\pi - A) = \frac{b}{c} \Rightarrow \cos A = -\frac{b}{c}$

$\Rightarrow \cos A \,.\, \cos C = -\frac{2b^2}{ac}$ … (i)

in ΔBEA and ΔDAC, we have

$4y^2 + b^2 = c^2$ and $y^2 + b^2 = \frac{a^2}{4}$

On subtracting, we get

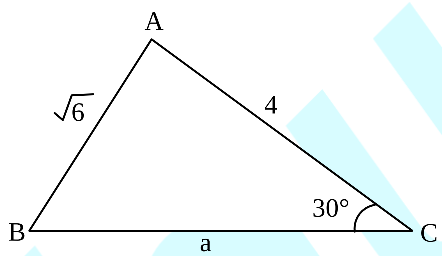
$3b^2 = a^2 - c^2 \Rightarrow b^2 = \frac{a^2 - c^2}{3}$

so from (i) $\cos A.\, \cos C = \frac{2(c^2 - a^2)}{3ac}$

Hence proved.

Q.4 Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller sides is 30°. How many such triangles are possible? Find the lengths of their third side and area. **[REE 98]**

Sol.



$\because \frac{4}{\sin B} = \frac{\sqrt{6}}{\sin 30°}$

$\Rightarrow \sin B = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

$\Rightarrow \angle B$ have two values which is supplementary so that two triangle are possible.

Now

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{a^2 + 16 - 6}{2.4.a}$

$\Rightarrow a^2 - 4\sqrt{3}\, a + 10 = 0$

it gives $a = \frac{4\sqrt{3} \pm \sqrt{48 - 40}}{2}$

$\Rightarrow a = 2\sqrt{3} \pm \sqrt{2}$

area $= \frac{1}{2}$ ab sin 30

$= 2\sqrt{3} \pm \sqrt{2}$

Q.5 If in a triangle ABC, 3 sin A = 6 sin B $= 2\sqrt{3}$ sin C, then the angle A is ….. **[REE 96]**

Sol. **[D]**

Given 3 sin A = 6 sin B = $2\sqrt{3}$ sin C

$\Rightarrow \frac{\sin A}{\sin B} = \frac{2}{1}$ and $\frac{\sin B}{\sin C} = \frac{1}{\sqrt{3}}$

$\Rightarrow$ sin A : sin B : sin C = a : b : c

$= 2 : 1 : \sqrt{3}$

$\Rightarrow a = 2k,\ b = k,\ c = \sqrt{3}\,k$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{k^2 + 3k^2 - 4k^2}{2\sqrt{3}k^2}$

cos A = 0

A = 90°

Q.6 If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cms, then the area of the triangle is .... **[IIT-1988]**

Sol.

Q.7 If in a triangle ABC, **[IIT-1993]**

$\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$

then the value of the angle A is ....

Sol. $\frac{2(b^2 + c^2 - a^2)}{2abc} + \frac{a^2 + c^2 - b^2}{2abc}$

$+ \frac{2(a^2 + b^2 - c^2)}{2abc} = \frac{a^2 + b^2}{abc}$

$\Rightarrow 2(b^2 + c^2 - a^2 + a^2 + b^2 - c^2) + a^2 + c^2 - b^2$

$= 2a^2 + 2b^2$

$\Rightarrow 4b^2 - b^2 + a^2 + c^2 = 2a^2 + 2b^2$

$\Rightarrow b^2 + c^2 = a^2$; A = 90°

Q.8 In a triangle ABC, AD is the altitude from A. Given b > c, ∠C = 23° and AD = $\frac{abc}{b^2 - c^2}$, then ∠B = ... **[IIT-1994]**

Sol. We have AD = b sin C

given AD = $\frac{abc}{b^2 - c^2}$

$\therefore \frac{abc}{b^2 - c^2} = b \sin C \Rightarrow \frac{\sin C}{c} = \frac{a}{b^2 - c^2}$

$\Rightarrow \frac{\sin A}{a} = \frac{a}{b^2 - c^2}$ $\quad \because \frac{\sin C}{c} = \frac{\sin A}{a}$

$\Rightarrow \sin A = \frac{a^2}{b^2 - c^2} \Rightarrow \sin A = \frac{\sin^2 A}{\sin^2 B - \sin^2 C}$

$\Rightarrow \sin A = \frac{\sin^2 A}{\sin(B+C)\sin(B-C)}$

$\Rightarrow \sin(B - C) = 1 \Rightarrow B - C = 90°$

$\Rightarrow B = 90° + C = 90° + 23° = 113°$

Q.9 If Δ denotes the area of any triangle and s its semi perimeter, prove that $\Delta < \frac{s^2}{4}$.

Sol. Using A.M. > G.M.

$\frac{s + s - a + s - b + s - c}{4} > \sqrt[4]{(s)(s-a)(s-b)(s-c)}$

$\frac{4s - (a + b + c)}{4} > \sqrt{\Delta}$

$\frac{4s - 2s}{4} > \sqrt{\Delta}$

$\frac{s}{2} > \sqrt{\Delta}$

$\frac{s^2}{4} > \Delta \Rightarrow \Delta < \frac{s^2}{4}$

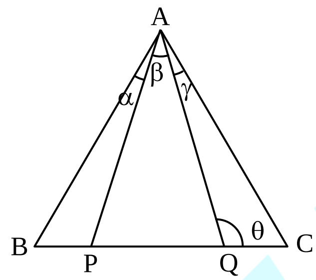
Q.10 The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that

$$\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right)$$

Sol. Let BP = PQ = QC = x

Also,

$t_1 = \tan \alpha$, $t_2 = \tan \beta$, $t_3 = \tan \gamma$



Apply m : n theorem in ΔABC we get

$(2x + x) \cot \theta = 2x \cot(\alpha + \beta) - x \cot \gamma$ .....(1)

Also

Apply m : n Theorem in ΔAPC

$(x + x)\cot \theta = x \cot \beta - x \cot \gamma$ .....(2)

Divide (1) by (2)

$\frac{3}{2} = \frac{2\cot(\alpha + \beta) - \cot\gamma}{\cot\beta - \cot\gamma}$

$3 \cot \beta - 3 \cot \gamma = \frac{4(\cot\alpha \cot\beta - 1)}{\cot\alpha + \cot\beta} - 2 \cot \gamma$

$\Rightarrow$ $3 \cot^2\beta - \cot \beta \cot \gamma + 3 \cot \alpha \cot \beta - \cot \alpha \cot \gamma = 4 \cot \alpha \cot \beta - 4$

$\Rightarrow$ $4 + 4 \cot^2\beta = \cot^2\beta + \cot \alpha \cot \beta + \cot \alpha \cot \gamma + \cot \beta \cot \gamma$

$\Rightarrow$ $4(1 + \cot^2\beta) = (\cot \beta + \cot \alpha)(\cot \beta + \cot \gamma)$

$$4\left(1 + \frac{1}{t_2^2}\right) = \left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right)$$

Q.11 If in a triangle ABC, θ is the angle determined by cosθ = (a – b)/c, prove that

$\cos\frac{1}{2}(A - B) = \frac{(a + b)\sin\theta}{2\sqrt{ab}}$ and

$\cos\frac{1}{2}(A + B) = \frac{c\sin\theta}{2\sqrt{ab}}$

Sol. $\because \cos\theta = \frac{a - b}{c}$ (given)

On squaring

$\cos^2\theta = \frac{a^2 + b^2 - 2ab}{c^2}$

$1 - \cos^2\theta = 1 - \left(\frac{a^2 + b^2 - 2ab}{c^2}\right)$

$\sin^2\theta = \frac{2ab - (a^2 + b^2 - c^2)}{c^2}$

$\sin^2\theta = \frac{2ab(1 - \cos C)}{c^2}$

$$\Rightarrow \frac{\sin\theta}{\sqrt{ab}} = \frac{\sqrt{2}}{c}\sqrt{1-\cos C}$$

$$= \frac{\sqrt{2}\left(\sqrt{2}\sin\frac{C}{2}\right)}{c}$$

$$\Rightarrow \frac{\sin\theta}{2\sqrt{ab}} = \frac{\sin C/2}{c} \qquad .....(1)$$

$$\Rightarrow \frac{(a+b)\sin\theta}{2\sqrt{ab}} = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$$

$$= \left(\frac{\sin A+\sin B}{\sin C}\right)\sin\frac{C}{2} \quad \text{[use sine rule]}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\sin\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \cos\left(\frac{A-B}{2}\right) \qquad \text{Proved.}$$

Also from (1)

$$\frac{c\ \sin\theta}{2\sqrt{ab}} = \sin\frac{C}{2} = \cos\left(\frac{A+B}{2}\right) \qquad \text{Proved.}$$

Q.12 Given B = 30º, c = 150 and b = $50\sqrt{3}$, prove that of the two triangles which satisfy the data, one will be isosceles and the other right angled. Find the greater value of the third side. Would the solution have been ambiguous had the values been B = 30º, c = 150 and b = 75 ?

Sol. Given : ∠ B = 30º, b = $50\sqrt{3}$, c = 150

using sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{50\sqrt{3}}{1/2} = \frac{150}{\sin C}$$

$$\sin C = \frac{\sqrt{3}}{2}$$

∠C = 60º or 120º

If ∠C = 60º

then ∠A = 90º

⇒ triangle is right angled

and if, ∠C = 120º

then ∠A = 30º = ∠B ⇒ a = b

⇒ triangle is isosceles

when, ∠A = 90º, ∠C = 60º

using sine rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{150}{\sqrt{3}/2} = a$$

$$\Rightarrow a = 100\sqrt{3}$$

If B = 30º, C = 150 and b = 75

then, using sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{75}{1/2} = \frac{150}{\sin C}$$

sin C = 1

⇒ C = 90º

∴ only one triangle is possible.

⇒ This would not be an ambiguous solution.

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ans.	C	C	C	A	D	B	B	A	A	D	A	B	B	A

15. 5, 7, 8 **16.** 65 : 33 : –15 **17.** True **18.** True

EXERCISE # 2

PART-A

Q.No.	1	2	3	4	5	6	7	8	9	10	11
Ans.	B	C	A	C	C	C	B	A	B	A	A

PART-B

Q.No.	12	13	14
Ans.	A,B	B,C	B,C

PART-C

Q.No.	15	16
Ans.	C	C

PART-D

17. $A \rightarrow R$, $B \rightarrow P$, $C \rightarrow S$, $D \rightarrow Q$ **18.** $A \rightarrow Q$, $B \rightarrow S$, $C \rightarrow Q$, $D \rightarrow P$

EXERCISE # 3

4. 120^o **5.** 2 & 3 **11.** (C) **12.** (A) **13.** (C) **14.** (C) **15.** (A) **16.** (B)

EXERCISE # 4

Q.No.	1	2	3	4	5
Ans.	A,C,D	A	A	A,B,C,D	B,D

6. 4 **7.** (B) **8.** 3 **9.** (C)

EXERCISE # 5

4. Two triangles are possible, $2\sqrt{3} \pm \sqrt{2}$ **5.** 90° **6.** $\frac{\sqrt{3}+1}{2}$ Sq. units **7.** 90^o **8.** 113^o

12. $100\sqrt{3}$, No