MONOTONICITY

EXERCISE # 1

Question based on **MONOTONIC FUNCTION & THEIR PROPERTIES**

Q.1 For $x > 0$, which of the following function is not monotonic -

(A) $x + |x|$ (B) e^x

(C) $\log x$ (D) $\sin x$

Sol. **[D]**

sinx

Q.2 Function $f(x) = \log \sin x$ is monotonic increasing when -

(A) $x \in (\pi/2, \pi)$ (B) $x \in (-\pi/2, 0)$

(C) $x \in (0, \pi)$ (D) $x \in (0, \pi/2)$

Sol. **[D]**

$f(x) = \log \sin x$ is monotonic

When $x \in (0, \pi/2)$

Q.3 Consider the function $f(x) = \dfrac{(x-1)}{(x^2 - 3x + 3)}$, then

(A) f(x) increase in (0, 2)

(B) f(x) decreases in $(-\infty, 0)$

(C) the interval into which the function f(x) transforms the entire real line is [3, –1]

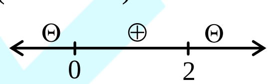
(D) $f'(x)$ is discontinuous for all $x \in R$

Sol. **[A]**

$$f'(x) = \frac{x^2 - 3x + 3 - (x-1)(2x-3)}{(x^2 - 3x + 3)^2}$$

$$= \frac{-x^2 + 2x}{(x^2 - 3x + 3)^2}$$

$$= \frac{-x(x-2)}{(x^2 - 3x + 3)^2}$$



f(x) increase in [0, 2]

f(x) decrease $x \in [-\infty, 0] \cup [2, \infty)$

Q.4 If $x \in [0, \pi]$, then $f(x) = x \sin x + \cos x + \cos^2 x$ is -

(A) increasing

(B) decreasing

(C) neither increasing nor decreasing

(D) None of these

Sol. **[C]**

$x \in [0, \pi]$

$f(x) = x \sin x + \cos x + \cos^2 x$

Differentiating w.r.t.x, we get

$f'(x) = x \cos x + \sin x - \sin x + 2 \cos x (-\sin x)$

$f'(x) = x \cos x - \sin 2x$

Let for $x > 0 \Rightarrow f(x) > f(0)$ (for increasing)

$x \sin x + \cos x + \cos^2 x > 0 + 1 + 1$

$x \sin x + \cos x + \cos^2 x > 2$

i.e. $x \sin x + \cos x + \cos^2 x > 0$

It indicates, f(x) is increasing function.

Let for $\pi > x \Rightarrow f(x) > f(\pi)$

i.e. if $x > 0 \Rightarrow f(x) > f(0)$ increasing

then it must be $\pi > x \Rightarrow f(x) > f(\pi)$ decreasing $f(x) > f(\pi)$

$\Rightarrow x \sin x + \cos x + \cos^2 x > \pi \sin\pi + \cos\pi + \cos^2 \pi$

$\Rightarrow x \sin x + \cos x + \cos^2 x > 0 + (-1) + 1$

$\Rightarrow x \sin x + \cos x + \cos^2 x > 0$

It indicates, f(x) is increasing in both sides. It is not possible.

Hence, f(x) is neither increasing nor decreasing.

$\therefore$ Option (C) is correct answer.

Q.5 If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a monotonically decreasing function of x in the largest possible interval (–2, –2/3) then -

(A) $\lambda = 4$ (B) $\lambda = 2$

(C) $\lambda = -1$ (D) λ has no real value

Sol. **[A]**

$f(x) = x^3 + 4x^2 + \lambda x + 1$

Differentiating w.r.t.x, we get

$f'(x) = 3x^2 + 8x + \lambda + 0 < 0$ … (1)

Also, $x \in (-2, -2/3)$

$\Rightarrow (x + 2)\left(x + \dfrac{2}{3}\right) < 0 \Rightarrow x^2 + 2x + \dfrac{2}{3}x + \dfrac{4}{3} < 0$

$\Rightarrow x^2 + \dfrac{8}{3}x + \dfrac{4}{3} < 0$

$\Rightarrow 3x^2 + 8x + 4 < 0 \quad \ldots (2)$

Comparing equation (1) and (2),

we get $\lambda = 4$

$\therefore$ Option (A) is correct answer.

Q.6 Let f (x) = $\tan^{-1}$ {ϕ (x)}, where ϕ (x) is monotonically increasing for $0 < x < \pi/2$. Then f (x) is -

(A) increasing in $(0, \pi/2)$

(B) decreasing in $(0, \pi/2)$

(C) increasing in $(0, \pi/4)$ and decreasing in $(\pi/4, \pi/2)$

(D) None of these

Sol. **[A]**

$f(x) = \tan^{-1}\{\phi(x)\}$

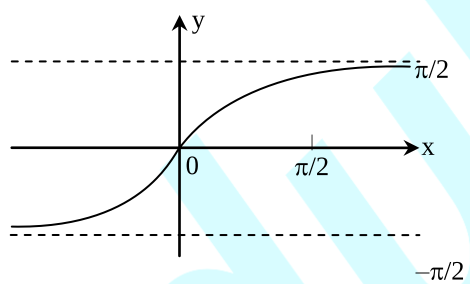
$\phi(x)$ is increasing function for $0 < x < \pi/2$

is $\phi'(x) > 0$ for $0 < x < \pi/2$

$$f'(x) = \frac{1}{1+\phi^2(x)} \times \phi'(x)$$

$\Rightarrow f'(x) > 0$ for $0 < x < \pi/2$

$\therefore$ f(x) is increasing for $x\left(0, \frac{\pi}{2}\right)$



Also, from the graph of $\tan^{-1}(x)$

$f(x) = \tan^{-1}(\phi(x))$ increasing in $x \in \left(0, \frac{\pi}{2}\right)$

Q.7 If f(x) = x + cos x – a then

(A) f(x) is an increasing function

(B) f(x) is a decreasing function

(C) f(x) = 0 has one positive roots for a < 1

(D) f(x) = 0 has no positive root for a > 1

Sol. **[A]**

$f(x) = x + \cos x - a$

Differentiating w.r.t. x, we get

$f'(x) = 1 - \sin x - 0$

$f'(x) = 1 - \sin x$

= which is increasing as $-1 \le \sin x \le 1$

$\therefore$ Option (A) is correct answer.

Q.8 Function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is monotonic increasing when -

(A) $x < 0$ (B) $x > 0$

(C) $x \in R$ (D) $x > -1$

Sol. **[D]**

$$f(x) = \log(1+x) - \frac{2x}{2+x}$$

Differentiating w.r.t. t. x, we get

$$f'(x) = \frac{1}{1+x} - \left[\frac{2(2+x)-2x.1}{(2+x)^2}\right]$$

$$f'(x) = \frac{1}{1+x} - [4 + 2x - 2x]/(2+x)^2$$

$$f'(x) = \frac{1}{1+x} - \frac{4}{(2+x)^2} > 0$$

$$\Rightarrow \frac{1}{1+x} > \frac{4}{(2+x)^2}$$

$\Rightarrow (2 + x)^2 > 4(1 + x)$

$\Rightarrow 4 + x^2 + 4x > 4 + 4x$

$\Rightarrow x^2 > 0$

It holds good for $x \in R - \{0\}$

Because 0 > 0 – non sense

Hence option (D) is correct answer.

Q.9 The function $f(x) = \frac{|x-1|}{x^2}$ is monotonically decreasing on -

(A) $(0, 1) \cup (2, \infty)$ (B) $(0, \infty)$

(C) $(-\infty, 1) \cup (2, \infty)$ (D) $(-\infty, \infty)$

Sol. **[A]**

$$f(x) = \frac{|x-1|}{x^2}$$

$$= \begin{cases} \frac{x-1}{x^2} & ; \quad x \ge 1 \\ \frac{1-x}{x^2} & ; \quad x < 1 \end{cases}$$

Differentiating above function w.r.t. x, we get

$$f'(x) = \frac{1.x^2 - (x-1) \times 2x}{x^4}$$

$$= \frac{x^2 - 2x^2 + 2x}{x^4} = \frac{2x - x^2}{x^4} \; ; \; x \ge 1$$

Since, f(x) is decreasing function, $f'(x) < 0$

$$\frac{2x - x^2}{x^4} < 0 \Rightarrow (x^2 - 2x) > 0$$

$\Rightarrow x(x-2) > 0 \quad \Rightarrow x < 0$ or $x > 2$

$\Rightarrow$ since, $x \geq 1 \quad \Rightarrow x \in (2, \infty)$

when $x < 1$, $f(x) = \dfrac{1-x}{x^2}$

Differentiating above function w.r.t. x, we get

$$f'(x) = \frac{(-1)x^2 - (1-x)\times 2x}{x^4}$$

$$= \frac{-x^2 - 2x + 2x^2}{x^4} = \frac{x^2 - 2x}{x^4} = \frac{x(x-2)}{x^4}$$

Since, f(x) is decreasing function, $\dfrac{x(x-2)}{x^4} < 0$

$\Rightarrow x(x - 2) < 0 \quad \Rightarrow 0 < x < 2$

Since, $x < 1$

Hence, option (A) is appropriate answer.

Q.10 If $f(x) = x^5 - 20x^3 + 240\,x$, then f(x) satisfies -

(A) It is monotonically decreasing everywhere

(B) It is monotonically decreasing on $(0, \infty)$

(C) It is monotonically increasing on $(-\infty, 0)$

(D) It is monotonically increasing everywhere

Sol. **[D]**

$f(x) = x^5 - 20x^3 + 240x$

Differentiating above function w.r.t. x, we get

$f'(x) = 5x^4 - 60x^2 + 240$

$= 5(x^4 - 12x^2 + 48) > 0$ for $x \in R$

which is increasing for $x \in R$.

$\therefore$ Option (D) is correct answer.

Q.11 $f(x) = \sin x - a \sin 2x - 1/3 \sin 3x + 2ax$ increases for all $x \in R$ if -

(A) $a < 0$ (B) $0 < a < 1$

(C) $a = 1$ (D) $a > 1$

Sol. **[D]**

$f(x) = \sin x - a \sin 2x - \dfrac{1}{3} \sin 3x + 2ax$

Differentiating above function w.r.t. x, we get

$f'(x) = \cos x - 2a \cos 2x - \cos 3x + 2a$

$= (\cos x - \cos 3x) + 2a(1 - \cos 2x)$

$\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$

$\Rightarrow 2 \sin^2 x = 1 - \cos 2x$

$f'(x) = 2 \sin \dfrac{x+3x}{2} . \sin \dfrac{3x-x}{2} + 2a \times 2 \sin^2 x$

$f'(x) = 2 \sin 2x. \sin x + 4a \sin^2 x$

$= 4 \sin^2 x (\cos x + a)$

$f'(x) = 4 \sin^2 x (\cos x + a) > 0;\ x \in R$

Since, $\sin^2 x \leq 1$, then it must be

$\cos x + a > 0;\ -1 \leq \cos x \leq 1$

$\left.\begin{array}{lll} -1+a>0 & \Rightarrow & a>1 \\ \text{and } 1+a>0 & \Rightarrow & a-1 \end{array}\right\} \Rightarrow a > 1$

$\therefore$ Option (D) is correct answer.

Q.12 Let f(x) be a function such that ;

$f'(x) = \log_{1/3}(\log_3(\sin x + a))$. If f(x) is decreasing for all real values of x then -

(A) $a \in (1, 4)$ (B) $a \in (4, \infty)$

(C) $a \in (2, 3)$ (D) $a \in (2, \infty)$

Sol. **[B]**

$f'(x) = \log_{1/3} (\log_3(\sin x + a)) < 0 ;\ x \in R$

$\Rightarrow \log_3(\sin x + a) > \dfrac{1}{3}^{\circ}$

$\Rightarrow \log_3 (\sin x + a) > 1$

$\Rightarrow (\sin x + a) > 3^1 \quad \Rightarrow (\sin x + a) > 3$

Since, $-1 \leq \sin x \leq 1$

$\Rightarrow -1 + a > 3$ and $1 + a > 3$

$\Rightarrow a > 4$ and $a > 2$

$\Rightarrow a \in (4, \infty)$ and $a \in (2, \infty)$

Most appropriate answer is $a \in (4, 8)$

$\therefore$ Option (B) is correct answer.

Q.13 The intervals of decrease of the function $f(x) = 3 \cos^4 x + 10 \cos^3 x + 6 \cos^2 x - 3$, $0 \leq x \leq \pi$ is -

(A) $\left(0, \dfrac{2\pi}{3}\right)$ (B) $\left(\dfrac{2\pi}{3}, \pi\right)$

(C) $\left(\dfrac{\pi}{2}, \pi\right)$ (D) $\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$

Sol. **[B]**

$f(x) = 3 \cos^4 x + 10 \cos^3 x + 6 \cos^2 x - 3$

Differentiating above function w.r.t. x, we get

$f'(x) = 12\cos^3 x (-\sin x) + 30 \cos^2 x(-\sin x) + 12 \cos x (-\sin x) - 0$

$= -6 \sin x [2 \cos^3 x + 5 \cos^2 x + 2 \cos x]$

$= -6 \sin x \times \cos x [2 \cos^2 x + 5 \cos x + 2]$

$f'(x) = -6 \sin x. \cos x [2 \cos^2 x + 5 \cos x + 2]$

$= -3 \sin 2x [2 \cos^2 x + 4 \cos x + \cos x + 2]$

$= -3 \sin 2x [2 \cos x (\cos x + 2) + 1(\cos x + 2)]$

$= -3 \sin 2x [(\cos x + 2) (2 \cos x + 1)]$

f(x) is decreasing function in $x \in [0, \pi]$

$f'(x) = -3 \sin 2x [(\cos x + 2) (2 \cos x + 1)] < 0$

$= 3 \sin 2x [(\cos x + 2) (2 \cos x + 1)] > 0$

Since, $\sin 2x$, $(\cos x + 2)$, $(2 \cos x + 1)$

All increasing functions in $[0, \pi]$

Hence, $(2 \cos x + 1) > 0$

$\Rightarrow \cos x > -\frac{1}{2} \quad \Rightarrow x > \frac{2\pi}{3}$

Also, $\sin 2x > 0$

$\Rightarrow x > \frac{\pi}{2}$

We choose, $x > \pi/2$, $x > 2\pi/3$ among best option.

Hence, $x \in \left(\frac{2\pi}{3}, \pi\right)$ is the most appropriate answer.

$\therefore$ Option (B) is the correct answer.

Q.14 Let $f(x) = \begin{cases} \frac{4-x}{2-\sqrt{x}} & \text{for } 0 < x < 4 \\ 4 & \text{for } x = 4 \\ 16-3x & \text{for } 4 < x < 6 \end{cases}$ which of the following properties does f have on the interval (0, 6) ?

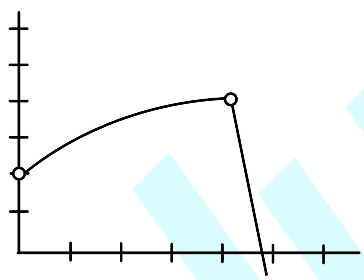
(i) λn f(x) exists (ii) f is continuous

(iii) f is monotonic

(A) i only (B) ii only

(C) iii only (D) none

Sol. **[B]**



f (x) is continuous

Q.15 The function $f(x) = \cos x - 2px$ is monotonically decreasing for

(A) $p < \frac{1}{2}$ (B) $p > \frac{1}{2}$

(C) $p < 2$ (D) $p > 2$

Sol. **[B]**

$f'(x) = -\sin x - 2p < 0$

$p > \frac{-\sin x}{2}$

$\left| p > \frac{1}{2} \right|$

Q.16 The length of largest continuous interval in which function $f(x) = 4x - \tan 2x$ is monotonic, is

(A) $\pi/2$ (B) $\pi/4$

(C) $\pi/8$ (D) $\pi/16$

Sol. **[B]**

$f'(x) = 4 - 2\sec^2 2x = 2(2 - \sec^2 2x)$

for monotonic f ' (x) should not change sign

$\Rightarrow 2x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$x \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right) \Rightarrow \text{length} = \frac{\pi}{4}$

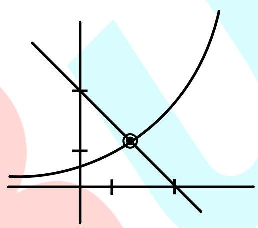
Q.17 The equation $e^{x-1} + x - 2 = 0$ has :

(A) one real root (B) two real roots

(C) three real roots (D) infinite real roots

Sol. **[A]**

$e^{x-1} = -x + 2$



one real root

Q.18 The function $f(x) = \cos(\pi/x)$ is increasing in the interval -

(A) $(2n + 1, 2n)$, $n \in N$

(B) $\left(\frac{1}{2n+1}, \frac{1}{2n}\right)$, $n \in N$

(C) $\left(\frac{1}{2n+3}, \frac{1}{2n+1}\right)$, $n \in N$

(D) None of these

Sol. **[D]**

$f(x) = \cos(\pi/x)$

Differentiating w.r.t. x, we get

$f'(x) = -\sin\frac{\pi}{x} \times \left(\frac{-\pi}{x^2}\right)$

$f'(x) = \pi \sin(\pi/x) \times \frac{1}{x^2} > 0$

$f'(x) = \frac{\pi \sin(\pi/x)}{x^2} > 0$

$\Rightarrow \sin(\pi/x)$ will be increasing for $0 < \frac{\pi}{x} < \frac{\pi}{2}$

$\Rightarrow \sin 0 < \sin(\pi/x) < \sin \pi/2$

$\Rightarrow n\pi < \frac{\pi}{x} < n\pi + \frac{\pi}{2}$; $n \in N$

$\Rightarrow n < \frac{1}{x} < (2n+1)\frac{1}{2}$; $n \in N$

$\Rightarrow \frac{2}{(2n+1)} < x < \frac{1}{n}; n \in N$

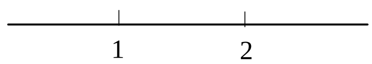
$\Rightarrow x \in \left(\frac{2}{2n+1}, \frac{1}{n}\right); n \in N$

True or False type Questions

Q.19 In the interval (1, 2), function $f(x) = 2|x - 1| + 3|x - 2|$ is monotonically increasing.

Sol. **False**

$f(x) = 2|x - 1| + 3|x - 2|$



$f(x) = 2(x - 1) + 3(2 - x)$

$f(x) = 2x - 2 + 6 - 3x$

$f(x) = 4 - x$

Differentiating w.r.t.x, we get

$f'(x) = -1$

$f'(x) < 0$. A decreasing function.

$\therefore$ Option is false.

Q.20 The function $y = \frac{x}{x^2 - 6x - 16}$ is a decreasing function in $R - \{-2, 8\}$.

Sol. **True**

$y = \frac{x}{x^2 - 6x - 16}$

Differentiating w.r.t.x, we get

$$\frac{dy}{dx} = \frac{1.(x^2 - 6x - 16) - x.(2x - 6)}{(x^2 - 6x - 16)^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 6x - 16) - 2x^2 + 6x}{(x^2 - 6x - 16)^2}$$

$$\frac{dy}{dx} = \frac{-x^2 - 16}{(x^2 - 6x - 16)^2} = \frac{-(x^2 + 16)}{(x^2 - 6x - 16)^2}$$

$x^2 - 6x - 16 = 0$

$\Rightarrow x^2 - 8x + 2x - 16 = 0$

$\Rightarrow x(x - 8) + 2(x - 8) = 0$

$\Rightarrow (x + 2)(x - 8) = 0$

$\Rightarrow x = -2, 8$

Hence, f(x) is decreasing function for $x \in R$ except, $x = -2, 8$

Hence, $x \in R - \{-2, 8\}$

Therefore, option is true.

EXERCISE # 2

Part-A Only single correct answer type questions

Q.1 y = log x satisfies for x > 1, the inequality-

(A) $x - 1 > y$ (B) $x^2 + 1 > y$

(C) $y > x - 1$ (D) $(x + 1)/x < y$

Sol. **[A]**

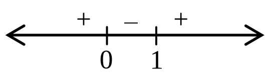
Check options (x > 1)

$f(x) = x - 1 - \ln x$

$f'(x) = 1 - \frac{1}{x}$

$= \frac{x-1}{x}$

$f(x) \uparrow$ in $[1, \infty)$



$f(x) > 1$ for $[1, \infty)$

Q.2 Function $f(x) = \left(\frac{\sqrt{3a-5}}{1-a} - 1\right)x^5 - 3x + \log 3$, $\forall x \in R$ is decreasing function then value of a is in the interval of-

(A) $\left[\frac{3}{5}, \infty\right)$ (B) $(-\infty, 1)$

(C) $\left[\frac{5}{3}, \infty\right)$ (D) $(1, \infty)$

Sol. **[C]**

$f(x) = \left(\frac{\sqrt{3a-5}}{1-a} - 1\right)x^5 - 3x + \log 3,$

Differentiating w.r.t.x, we get

$f'(x) = \left(\frac{\sqrt{3a-5}}{1-a} - 1\right)5x^4 - 3 + 0$

$= \left(\frac{\sqrt{3a-5}}{1-a} - 1\right)5x^4 - 3 < 0$

$= \left(\frac{\sqrt{3a-5}}{1-a} - 1\right)x^4 < 3/5$; $\forall x \in R$

It must be $3a - 5 \ge 0$; $a \ne 1$

$a \ge 5/3$; $a \ne 1$

Hence $a \in \left[\frac{5}{3}, \infty\right)$

$\therefore$ Option (C) is correct answer.

Q.3 Function $f(x) = x^{100} + \sin x - 1$ is increasing in the interval -

(A) (0, 1) (B) $(-\pi/2, \pi/2)$

(C) (–1, 1) (D) None of these

Sol. **[A]**

$f(x) = x^{100} + \sin x - 1$

Differentiating w.r.t. x, we get

$f'(x) = 100x^{99} + \cos x - 0$

$f'(x) = 100x^{99} + \cos x > 0$

We have to check for every option

For A : $x > 0 \Rightarrow \phi(x) > \phi(0)$

Let $\phi(x) = 100x^{99} + \cos x$

$\phi(0) = 0 + 1 = 1$

$\phi(x) > 0 \Rightarrow \phi(x) > \phi(0)$

$\Rightarrow 100x^{99} + \cos x > 1$

$\Rightarrow 100x^{99} + \cos x > 0$

Q.4 The number of solutions of the equation $a^{f(x)} + g(x) = 0$, where $a > 0$, $g(x) \ne 0$ and g (x) has minimum value 1/4, is

(A) one (B) two

(C) infinitely many (D) zero

Sol. **[D]**

$a^{f(x)} + g(x) = 0$, $a > 0$, $g(x) \ne 0$

$g(x)|_{min} = \frac{1}{4}$

$a^{f(x)} + \frac{1}{4} = 0 \Rightarrow a^{f(x)} = -\frac{1}{4} \Rightarrow f(x) = \log_a\left(-\frac{1}{4}\right)$

= Does not exist.

$\therefore$ Option (D) is correct answer.

Q.5 Function $f(x) = \frac{\lambda \sin x + 3\cos x}{2\sin x + 6\cos x}$ is monotonic increasing when -

(A) $\lambda < 1$ (B) $\lambda > 1$

(C) $\lambda < 2$ (D) $\lambda > 2$

Sol. **[B]**

$$f(x) = \frac{\lambda \sin x + 3\cos x}{2\sin x + 6\cos x}$$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{(\lambda \cos x - 3\sin x)(2\sin x + 6\cos x) - (\lambda \sin x + 3\cos x)(2\cos x - 6\sin x)}{(2\sin x + 6\cos x)^2}$$

Since, f(x) is monotonically increasing function i.e. $f'(x) > 0$

$\Rightarrow [2\lambda \sin x \cos x - 6\sin^2 x + 6\lambda\cos^2 x - 18 \sin x \cos x]$

$- [2\lambda \sin x \cos x + 6 \cos^2 x - 6\lambda \sin^2 x - 18 \sin x \cos x] > 0$

$\Rightarrow - 6 + 6\lambda (1) > 0$

$\lambda > 1$

Q.6 Given that f is a real valued differentiable function such that $f(x) \cdot f'(x) < 0$, for all real x it follows that -

(A) $f^2(x)$ is increasing function

(B) $f^2(x)$ is decreasing function

(C) f (x) is increasing function

(D) f (x) is decreasing function

Sol. **[B]**

Let $\phi(x) = f^2(x)$

Differentiate, w.r.t. x, we get

$\phi'(x) = 2f(x).\ f'(x) < 0;\ x \in R$

It means, $\phi(x)$ is decreasing function for all $x \in R$.

$\therefore$ Option (B) is correct answer.

Q.7 A function y = f(x) is given by $x = \frac{1}{1+t^2}$ & $y = \frac{1}{t(1+t^2)}$ for all t > 0 then f is

(A) increasing in (0, 3 / 2) & decreasing in (3 / 2, ∞)

(B) increasing in (0, 1)

(C) increasing in (0, ∞)

(D) decreasing in (0, 1)

Sol. **[B]**

$$y = \frac{x\sqrt{x}}{\sqrt{1-x}}$$

(1) $x \in (0, 1)$

$$y' = \frac{\dfrac{-1-2t}{(1+t^2)^2}}{\dfrac{-2t}{(1+t^2)^2}} = \frac{2t+1}{2t}$$

(t > 0) given

y' = (+)ve

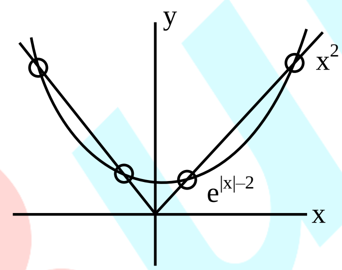
$\therefore$ f (x) ↑ in (0, 1)

Q.8 Number of roots of the equation $x^2.e^{2-|x|} = 1$ is

(A) 2 (B) 4 (C) 6 (D) infinite

Sol. **[B]**

$x^2 = e^{|x|-2}$ (symmetric about y-axis)



Q.9 Number of solution of the equation $3 \tan x + x^3 = 2$ in $\left(0, \frac{\pi}{4}\right)$ is

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **[C]**

$f(x) = 3 \tan x + x^3 - 2$

$f'(x) = 3 \sec^2 x + 3x^2$ +ve in $\left(0, \frac{\pi}{4}\right)$

f(0) = –2, f (1) +ve

⇒ only 1 root

Q.10 Consider the function $f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

then the number of points in (0, 1) where the derivative $f'(x)$ vanishes, is

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. **[D]**

$$f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$$

$$= \cos\frac{\pi}{x}\left(\tan\frac{\pi}{x} - \frac{\pi}{x}\right)$$

= 0 at infinite points

Q.11 Number of roots of the function $f(x) = \frac{1}{(x+1)^3} - 3x + \sin x$ is

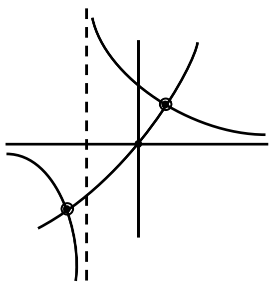
(A) 0 (B) 1

(C) 2 (D) more than 2

Sol. **[C]**

$f(x) = \frac{1}{(x+1)^3} - 3x + \sin x = 0$

$\frac{1}{(x+1)^3} = 3x - \sin x$



2 roots

Q.12 Let f : [–1, 2] → R be differentiable such that $0 \le f'(t) \le 1$ for $t \in [-1, 0]$ and $-1 \le f'(t) \le 0$ for $t \in [0, 2]$. Then

(A) $-2 \le f(2) - f(-1) \le 1$

(B) $1 \le f(2) - f(-1) \le 2$

(C) $-3 \le f(2) - f(-1) \le 0$

(D) $-2 \le f(2) - f(-1) \le 0$

Sol. **[D]**

$t \in [-1, 0]$

$f'(t) = \frac{f(0) - f(-1)}{1}$ …..(i)

$t \in [0, 2]$

$f'(t) = \frac{f(2) - f(0)}{2}$ ….(ii)

(i) $\Rightarrow 0 \le f(0) - f(-1) \le 1$

(ii) $\Rightarrow -2 \le f(2) - f(0) \le 0$

add

$\Rightarrow -2 \le f(2) - f(-1) \le 1$

Q.13 Let $f'(\sin x) < 0$ & $f''(\sin x) > 0$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then g(x) is decreasing in-

(A) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{4}\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

Sol. **[B]**

$x \in \left(0, \frac{\pi}{2}\right)$

$g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

$\therefore f'(\sin x) < 0$

$\therefore f'(\cos x) < 0$

$f''(\sin x) > 0$

$\therefore f''(\cos x) > 0$

$g'' = f''(\sin x)\cos^2 x - f'(\sin x)\sin x$

$- f'(\cos x)\cos x + f''(\cos x)\sin^2 x$

$g''(x) = +ve$

For critical point of g(x)

$g'(x) = 0$

$\Rightarrow x = \frac{\pi}{4}$

So, $x = \frac{\pi}{4}$ is point of minima

$g(x) \downarrow \left(0, \frac{\pi}{4}\right)$

$\uparrow \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Q.14 Which of the following six statements are true about the cubic polynomial

$P(x) = 2x^3 + x^2 + 3x - 2$?

(i) It has exactly one positive real root

(ii) It has either one or three negative roots

(iii) It has a root between 0 and 1

(iv) It must have exactly two real roots

(v) It has a negative root between – 2 and –1

(vi) It has no complex roots

(A) only (i), (iii) and (vi)

(B) only (ii), (iii) and (iv)

(C) only (i) and (iii)

(D) only (iii), (iv) and (v)

Sol. **[C]**

$P'(x) = 6x^2 + 2x + 3 > 0$

$P(x) \uparrow$

Check options

Q.15 If $\log_e x > \frac{x-2}{x}$ then $x \in$

(A) $(1, \infty)$ (B) $(1, 2)$

(C) $(-\infty, \infty)$ (D) $(2, \infty)$

Sol. **[A]**

Clearly $x > 1$

Part-B **One or more than one correct answer type questions**

Q.16 If $f(x) = \tan^{-1}(\sin x + \cos x)$, then f(x) is increasing in -

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ (B) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(C) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (D) $\left(-2\pi, -\frac{7\pi}{4}\right)$

Sol. **[A, B, C]**

$f(x) = \tan^{-1}(\sin x + \cos x)$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x - \sin x)$$

$$f'(x) = \frac{\cos x - \sin x}{1+(\sin x+\cos x)^2} > 0$$

$$= \frac{(\cos x - \sin x)}{2+\sin 2x} > 0$$

$-1 \le \sin 2x \le 1 \Rightarrow -1 + 2 \le 2 + \sin 2x \le 1 + 2$

$\Rightarrow 1 \le 2 + \sin 2x \le 3$

Hence, $(\cos x - \sin x) > 0$

$\cos x > \sin x$

$\Rightarrow 1 > \tan x$

$\Rightarrow \tan x < 1$

$\Rightarrow x < \pi/4$

$\Rightarrow x \in \left(\frac{-\pi}{2}, \frac{\pi}{4}\right)$

Since, $-\infty < \tan x < \infty \Rightarrow x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ Options A and B are correct answers.

We have to check options (C) and (D) as well.

For C :

$\frac{5\pi}{4}, \frac{3\pi}{2}$

$5\pi/4$ — $4\pi/3$ — $3\pi/2$

$$\frac{\cos x - \sin x}{2+\sin 2x} = \frac{\cos(\pi+\pi/3)-\sin(\pi+\pi/3)}{2+\sin(2\pi+2\pi/3)}$$

$$= \frac{-\frac{1}{2}+\sqrt{3}/2}{2+\sqrt{3}/2} > 0$$

Option (C) is also correct answer.

For D : $-7\pi/2$ — -3π — -2π

$$\frac{\cos x - \sin x}{2+\sin 2x} = \frac{\cos(-3\pi)-\sin(-3\pi)}{2+\sin 2(-3\pi)}$$

$$= \frac{\cos(2\pi+\pi)+\sin(2\pi+\pi)}{2-\sin 6\pi}$$

$$= \frac{-1+0}{2-0} = -\frac{1}{2}$$

Option (D) is not correct answer.

$\therefore$ Options A, B, C only correct answer.

Q.17 If $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)\ \forall\ x \in (-3, 4)$, where $f''(x) > 0\ \forall\ x \in (-3, 4)$, then h (x) is -

(A) increasing in $\left(\frac{3}{2}, 4\right)$

(B) increasing in $\left(-\frac{3}{2}, 0\right)$

(C) decreasing in $\left(-3, -\frac{3}{2}\right)$

(D) decreasing in $\left(0, \frac{3}{2}\right)$

Sol. **[A,B,C, D]**

$h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)\ \forall\ x \in (-3, 4)$,

$f''(x) > 0\ \forall\ x \in (-3, 4)$

Differentiating above function w.r.t. x, we get

$$h'(x) = 3f'\left(\frac{x^2}{3}\right) \times \frac{2x}{3} + f'(3 - x^2) \times (-2x)$$

$$= 2x\left[f'\left(\frac{x^2}{3}\right) - f'(3-x^2)\right]$$

Since, $f'(x)$ is increasing in $(-3, 4)$

For $x > 0$, $h'(x)$ would be increasing.

For $x < 0$, $h'(x)$ would be decreasing.

For increasing function

$x > 0 \Rightarrow h'(x) > h'(0)$

$$\Rightarrow f'\left(\frac{x^2}{3}\right) - f'(3 - x^2) > 0$$

$$\Rightarrow f'\left(\frac{x^2}{3}\right) > f'(3 - x^2) \Rightarrow \frac{x^2}{3} > 3 - x^2$$

$$\Rightarrow \frac{4}{3}x^2 > 3 \Rightarrow x^2 > 9/4 \Rightarrow x < -3/2 \text{ or } x > 3/2$$

-3 — $-3/2$ — 0 — $3/2$ — 4

Hence, h(x) would be increasing in (3/2, 4)

h(x) would be decreasing in (0, 3/2)

h(x) would be increasing in (–3/2, 0)

h(x) would be decreasing in (–3, –3/2)

Hence, options (A), (B), (C) and (D) are correct answers.

Q.18 If $f(x) = (x - 1)\,|(x - 2)(x - 3)|$ then 'f ' decreases in-

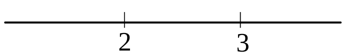
(A) $\left(2-\frac{1}{\sqrt{3}},\ 2\right)$ (B) $\left(2,\ 2+\frac{1}{\sqrt{3}}\right)$

(C) $\left(2+\frac{1}{\sqrt{3}},\ 3\right)$ (D) $(3, \infty)$

Sol. **[A, C]**

When x < 2

$f(x) = (x - 1)(2 - x)(3 - x)$



$= (x - 1)(x - 2)(x - 3)$

$= (x^2 - x - 2x + 2)(x - 3)$

$= (x^2 - 3x + 2)(x - 3)$

$= (x^3 - 3x^2 + 2x - 3x^2 + 9x - 6)$

$= (x^3 - 6x^2 + 11x - 6)$

$f(x) = (x^3 - 6x^2 + 11x - 6)$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 12x + 11 - 0$

$f'(x) = (3x^2 - 12x + 11)$

$f'(x) < 0$

$\Rightarrow (3x^2 - 12x + 11) < 0$

We can find roots of above quadratic equation as follows

$$x = \frac{12 \pm \sqrt{144 - 132}}{2 \times 3}$$

$$x = \frac{12 \pm \sqrt{12}}{6}$$

$$x = 2 \pm \frac{1}{\sqrt{3}}$$

Hence, $x \in \left(2-\frac{1}{\sqrt{3}}, 2\right)$

∴ Option (A) is correct answer.

For 2 < x < 3

$f(x) = (x - 1)(x - 2)(3 - x)$

$= -[x^3 - 6x^2 + 11x - 6]$

Differentiating w.r.t. x, we get

$f'(x) = -(3x^2 - 12x + 11) < 0$

$\Rightarrow (3x^2 - 12x + 11) > 0$

$\Rightarrow x < 2 - \frac{1}{\sqrt{3}}$ or $x > 2 + \frac{1}{\sqrt{3}}$

Hence, $x \in \left(2+\frac{1}{\sqrt{3}}, 3\right)$

∴ Option (C) is correct answer.

For x > 3,

$f(x) = (x - 1)(x - 2)(x - 3)$

$f(x) = (x^3 - 6x^2 + 11x - 6)$

$f'(x) = (3x^2 - 12x + 11 - 0)$

$f'(x) < 0 \Rightarrow 3x^2 - 12x + 11 < 0$

$2 - \frac{1}{\sqrt{3}} < x < 2 + \frac{1}{\sqrt{3}}$

Since, x > 3

Hence, options (A) and (C) are correct answers.

Q.19 Let f be the function $f(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$ then-

(A) f(x) is an increasing function in $\left(0, \frac{\pi}{2}\right)$

(B) f(x) is a decreasing function in $(-\infty, \infty)$

(C) f(x) is an increasing function in the interval $-\infty < x \le 0$ and decreasing in the interval $0 \le x < \infty$

(D) f(x) is a decreasing function in the interval $-\infty < x \le 0$ and increasing in the interval $0 \le x < \infty$

Sol. **[A, D]**

$f(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$

Differentiating w.r.t. x, we get

$f'(x) = -\sin x - \left(0 - \frac{2x}{2}\right)$

$f'(x) = -\sin x + x$

$f'(x) = x - \sin x$

We have to check for every option as follows:

For A : x > 0,

For increasing function, f(x) > f(0)

$\cos x - \left(1-\frac{x^2}{2}\right) > 1-(1-0)$

$\cos x - \left(1-\frac{x^2}{2}\right) > 0 \Rightarrow \cos x > \left(1-\frac{x^2}{2}\right)$

$\therefore$ Option (A) is correct answer.

For option C:

$-\infty < x \le 0 \rightarrow$ increasing

$f(x) \le f(0) \Rightarrow \cos x - \left(1-\frac{x^2}{2}\right) \le 0$

which is decreasing. Hence wrong.

$0 \le x < \infty \rightarrow$ decreasing

$x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \cos x - \left(1-\frac{x^2}{2}\right) \ge 0 \;\Rightarrow \cos x - \left(1-\frac{x^2}{2}\right) \ge 0$

Which is increasing. Hence, wrong.

$\therefore$ Option (C) is not correct answer.

For option D : $-\infty < x \le 0 \rightarrow$ decreasing

$x \le 0 \Rightarrow f(x) \le f(0)$

$\Rightarrow \cos x - \left(1-\frac{x^2}{2}\right) \le 0$

which is correct. Hence right.

$0 \le x < \infty \rightarrow$ increasing

$x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \cos x - \left(1-\frac{x^2}{2}\right) \ge 0$

Which is increasing. Hence right.

$\therefore$ Option (D) is correct answer.

$\therefore$ Options (A) and (D) are correct answers.

Q.20 If $p'(x) > p(x)$ for all $x \ge 1$ and $p(1) = 0$ then-

(A) $e^{-x}p(x)$ is an increasing function

(B) $e^{-x}\,p(x)$ is a decreasing function

(C) $p(x) > 0$ for all x in $(1, \infty)$

(D) $p(x) < 0$ for all x in $(1, \infty)$

Sol. **[A, C]**

$p'(x) > p(x)$; for all $x \ge 1$

$p(1) = 0$

Multiplying both sides by e^{-x}, we get

$e^{-x}\,p'(x) > e^{-x}\,p(x)$

$\Rightarrow e^{-x}\,p'(x) - e^{-x}\,p(x) > 0$

$\Rightarrow \frac{d}{dx}(e^{-x}\,p(x)) > 0$

Integrating, we get

$e^{-x}\,p(x) >$ constant

Put $x = 1$, $e^{-1}\,p(1) >$ constant

$\Rightarrow$ constant < 0

Hence, $e^{-x}\,p(x) > 0$, an increasing function.

Also, $p(x) > \frac{0}{e^{-x}}$

$p(x) > 0$ for all $x \in (1, \infty)$

$\therefore$ Options (A) and (C) are correct answers.

Q.21 Let $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$ for every real number 'x', then

(A) 'h' is increasing whenever 'f' is increasing

(B) 'h' is increasing whenever 'f' is decreasing

(C) 'h' is decreasing whenever 'f' is decreasing

(D) nothing can be said in general

Sol. **[A, C]**

$h'(x) = f'(x) - 2f(x)f'(x) + 3f^2(x)f'(x)$

$= f'(x)\{1 - 2f(x) + 3f^2(x)\}$

$f(x)\uparrow$ $\qquad f'(x) = 0$

$h'(x) > 0$ $\qquad h(x)\uparrow$

Q.22 The function $y = \frac{2x-1}{x-2} (x \ne 2)$

(A) is its own inverse

(B) decrease for all values of x

(C) has a graph entirely above x-axis

(D) is bounded for all x

Sol. **[A, B]**

$y' = \frac{(x-2)2-(2x-1)}{(x-2)^2}$

$= \frac{-3}{(x-2)^2} < 0$

$y\downarrow \quad x \in R$

$y = \frac{2x-1}{x-2}$

$xy - 2y = 2x - 1$

$\Rightarrow x = \frac{2y-1}{y-2}$

$f^{-1}(x) = \frac{2x-1}{x-2}$

Q.23 Let $g(x) = 2f(x/2) + f(1-x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g (x)

(A) decreasing in [0, 2/3]

(B) decreasing in [2/3, 1]

(C) increasing in [0, 2/3]

(D) increasing in (2/3, 1)

Sol. **[B, C]**

$g'(x) = f'\left(\frac{x}{2}\right) - f'(1-x)$

$f'(x) \downarrow$

$\frac{x}{2} = 1 - x$

$x = 2 - 2x$

$x = \frac{2}{3}$

$\xleftarrow{\quad + \quad} \frac{2}{3} \xrightarrow{\quad - \quad}$

Q.24 If $f(x) = a^{\{a^{|x|}\operatorname{sgn} x\}}$; $g(x) = a^{[a^{|x|}\operatorname{sgn} x]}$ for a > 0, a ≠ 1 and x ∈ R – {0}, where {} & [] denote the fractional part and integral part functions respectively, then which of the following statements holds good for the function h(x), where (λna) h(x) = (λnf(x) + λng(x))

(A) 'h' is even and increasing ∀ a > 1

(B) 'h' is odd and decreasing ∀ 0 < a < 1

(C) 'h' is even and decreasing ∀ 0 < a < 1

(D) 'h' is odd and increasing ∀ a > 1

Sol. **[B, D]**

λn a . h(x) = λn f(x) + λn g(x)

λn a h(x) = λn a $[a^{|x|} \operatorname{sgn} x]$

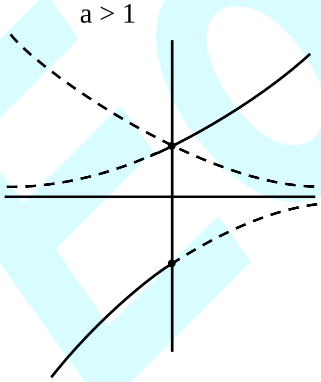
$h(x) = a^x$; x > 0

$= -a^{-x}$; x < 0

$h'(x) = a^x \lambda na$; x > 0

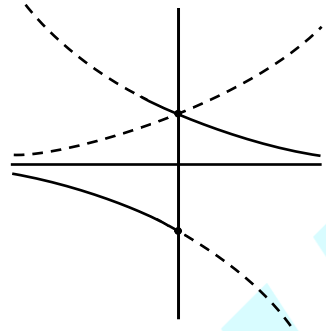
$= a^{-x} \lambda na$; x < 0

a > 1



increasing

0 < a < 1



decreasing

Q.25 Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x$ ∀ x ∈ R' then

(A) ϕ is increasing whenever f is increasing

(B) ϕ is increasing whenever f is decreasing

(C) ϕ is decreasing whenever f is decreasing

(D) ϕ is decreasing if $f'(x) = -11$

Sol. **[A, D]**

$\phi'(x) = f'(x)\,[3f^2(x) - 6f(x) + 4] + [5 + (3\cos x - 4\sin x)]$

≥ 0

(C) cannot be concluded

Part-C Assertion - Reason type Questions

The following questions 26 to 28 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.26 **Assertion** : Let f : R → R be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then f is one one.

Reason : f(x) neither increasing nor decreasing function

Sol. **[C]**

Assertion : f : R → R

$f(x) = x^3 + x^2 + 3x + \sin x$

For each value of x, f(x) has unique value.

Hence, f(x) would be one-one function.

$\therefore$ Assertion is correct option.

Reason :

$f(x) = x^3 + x^2 + 3x + \sin x$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 + 2x + 3 + \cos x$

$= (2x^2 + 2) + (x^2 + 2x + 1) + \cos x$

$= 2(x^2 + 1) + (x + 1)^2 + \cos x$

(since $-1 \le \cos x \le 1$)

$\therefore f'(x) > 0$

Hence, it would be increasing function. Reason is false.

$\therefore$ option (C) is correct answer.

Q.27 **Assertion** : The greatest of the numbers $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}, 7^{1/7}$, is $3^{1/3}$

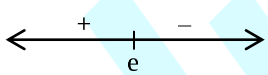
Reason : $x^{1/x}$ is increasing for $0 < x < e$ and decreasing for $x > e$

Sol. **[A]**

$f(x) = x^{1/x}$

$$f'(x) = x^{1/x}\left(\frac{1}{x^2} - \frac{\lambda nx}{x^2}\right)$$

$$= \frac{x^{1/x}}{x^2}(1 - \lambda nx)$$



$f(x)_{max} = f(e) = e^{1/e}$

Q.28 **Assertion** : If f(0) = 0, $f'(x) = \log(x + \sqrt{1+x^2})$, then f(x) is positive for all x.

Reason : f(x) is increasing for x > 0 and decreasing for x < 0.

Sol. **[A]**

Assertion : f(0) = 0

$f'(x) = \log(x + \sqrt{1+x^2})$

Integrating w.r.t. x, we get

$$\int f'(x)\,dx = \int \log(x + \sqrt{1+x^2})\,dx$$

$f(x) = x.\log(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$

f(x) would be positive for all x.

Hence, assertion is correct option.

Reason :

$f'(x) = \log(x + \sqrt{1+x^2})$

for $x > 0$, $f'(x) = \log(x + \sqrt{1+x^2}) > 0$

Hence, increasing

For $x < 0$, $f'(x) = \log(x + \sqrt{1+x^2}) < 0$

Hence, decreasing

$\therefore$ reason is true.

$\therefore$ option (A) is correct choice.

Part-D Column Matching Questions

Match the entry in Column 1 with the entry in Column 2.

Q.29

Column 1	Column 2
(A) $x - \frac{x^3}{3} \le \tan^{-1}x$	(P) $x \in R$
(B) $2x\tan^{-1}x \ge \log_e(1 + x^2)$	(Q) $x > 4$
(C) $\tan x > x + x^3/3$	(R) $x \ge 0$
(D) $x^3 - 3x^2 - 9x + 20 > 0$	(S) $x \in \left(0, \frac{\pi}{2}\right)$ for

Sol. **A → R, B → P, C → S, D → Q**

(A) $x - \frac{x^3}{3} \le \tan^{-1}x$

Let $f(x) = \tan^{-1}x - x + x^3/3 \ge 0$

$$f'(x) = \frac{1}{1+x^2} - 1 + x^2 \ge 0$$

$$f'(x) = \frac{1 - x^2 - 1 + x^2 + x^4}{1+x^2} \ge 0$$

$$f'(x) = \frac{x^4}{1+x^2} \ge 0$$

(B) $2x\tan^{-1}x \ge \log_e(1 + x^2)$

Let $f(x) = 2x\tan^{-1}x - \log_e(1 + x^2) \ge 0$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{2x}{1+x^2} + 2\tan^{-1}x - \frac{1}{1+x^2}(2x) \ge 0$$

$f'(x) = 2\tan^{-1}x \ge 0$

which is true for $x \in R$.

Let $f(x) = \tan x - x - x^3/3 > 0$

(C) Differentiating w.r.t. x, we get

$f'(x) = \sec^2 x - 1 - x^2 > 0$

which is true only for $x \in (0, \pi/2)$

Part-E Fill in The Blanks type Questions

Q.30 The value of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x is given by................

Sol. $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

Differentiating w.r.t.x, we get

$f'(x) = \sqrt{3}\cos x + \sin x - 2a \le 0$

$= 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) - 2a \le 0$

$= 2(\sin 60° \cos x + \cos 60° \sin x) - 2a \le 0$

$= 2[\sin(x + \pi/3)] \le 2a \Rightarrow \sin(x + \pi/3) \le a \Rightarrow a = 1$

Q.31 If $f(x) = \left(\frac{\alpha^2 - 1}{\alpha^2 + 1}\right) x^3 - 3x + \ln 2$ is a decreasing function of x in R then the set of possible values of α (independent of x) is..............

Sol. $f(x) = \left(\frac{\alpha^2 - 1}{\alpha^2 + 1}\right) x^3 - 3x + \ln 2$

Differentiating w.r.t.x, we get

$f'(x) = \frac{(\alpha^2 - 1)}{(\alpha^2 + 1)} \times 3x^2 - 3 + 0 < 0$

$\Rightarrow \frac{(\alpha^2 - 1)}{(\alpha^2 + 1)} x^2 < 1; x \in R$

It must be $(\alpha^2 - 1) < 0$

$\Rightarrow \alpha^2 < 1 \Rightarrow -1 < \alpha < 1$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Find the intervals in which the following function are (i) Increasing (ii) Decreasing

(a) $f(x) = x^3 + 3x^2 - 105x + 25$

(b) $f(x) = 2x^3 - 24x + 7$

(c) $f(x) = \ln(1 - x^2)$

(d) $f(x) = x^3 - 6x^2 + 9x + 1$; $x \in R$

(e) $f(x) = \dfrac{1-x+x^2}{1+x+x^2}$

(f) $f(x) = 5x^{3/2} - 3.x^{5/2}$; $x > 0$

Sol. (a) $f(x) = x^3 + 3x^2 - 105x + 25$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 + 6x - 105 + 0$

$= 3x^2 + 6x - 105$

Increasing function : $f'(x) = 3x^2 + 6x - 105 > 0$

$\Rightarrow 3(x^2 + 2x - 35) > 0$

$\Rightarrow 3(x^2 + 7x - 5x - 35) > 0$

$\Rightarrow 3[x(x + 7) - 5(x + 7)] > 0$

$\Rightarrow 3[(x - 5)(x + 7)] > 0$

$\Rightarrow$ either $x < -7$ or $x > 5$

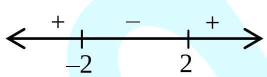
$\Rightarrow x \in (-\infty, -7) \cup (5, \infty)$

Decreasing function

$f'(x) = 3(x + 7)(x - 5) < 0$

$\Rightarrow -7 < x < 5 \quad \Rightarrow x \in (-7, 5)$

(b) $f'(x) = 6x^2 - 24 = 6(x - 2)(x + 2)$

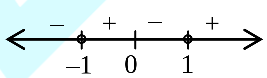


$f(x) \uparrow (-\infty, -2] \cup [2, \infty)$

$\downarrow [-2, 2]$

(c) $f'(x) = \dfrac{-2x}{1-x^2} = \dfrac{-2x}{(1+x)(1-x)}$

$= \dfrac{2x}{(x-1)(x+1)}$

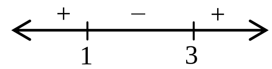


Domain $x \in (-1, 1)$

$f(x) \uparrow (-1, 0]$

$\downarrow [0, 1)$

(d) $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

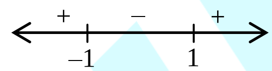


$f(x) \uparrow (-\infty, 1] \cup [3, \infty)$

$\downarrow [1, 3]$

(e) $f'(x) = (1 + x + x^2)(2x - 1)$

$$f'(x) = \frac{2x + 2x^2 + 2x^3 - 1 - x - x^2 - 2x - 1 + x + 2x^2 - x^2 - 2x^2}{(1+x+x^2)^2}$$

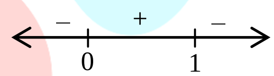


$f(x) \uparrow (-\infty, -1] \cup [1, \infty)$

$\downarrow [-1, 1]$

(f) $f'(x) = \dfrac{15}{2}x^{1/2} - \dfrac{15}{2}x^{3/2}$

$= \dfrac{15}{2}x^{1/2}(1 - x)$



$f(x) \uparrow [0, 1]$

$\downarrow [1, \infty]$

Q.2 (a) If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for every $x \in R$ then find the range of values of a.

(b) Find the set of all values of the parameter 'a' for which the function $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$

Sol. (a) $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$

Differentiating w.r.t.x, we get

$f'(x) = 2e^x + ae^{-x} + (2a + 1) - 0$

for, $f'(x) > 0$

$\Rightarrow 2e^x + ae^{-x} + (2a + 1) > 0$

$\Rightarrow 2e^{2x} + (2a + 1)e^x + a > 0$

$$\Rightarrow e^x = \frac{-(2a+1) \pm \sqrt{(2a+1)^2 - 4 \times 2 \times a}}{2 \times 2}$$

$$\Rightarrow e^x = \frac{-(2a+1) \pm \sqrt{4a^2 + 1 + 4a - 8a}}{4}$$

$\Rightarrow e^x = \dfrac{-(2a+1) \pm \sqrt{4a^2 - 4a + 1}}{4}$

$\Rightarrow e^x = \dfrac{-(2a+1) \pm (2a-1)}{4}$

$\Rightarrow e^x = \dfrac{-2a-1+2a-1}{4}, \dfrac{-2a-1-2a+1}{4}$

$\Rightarrow e^x = \dfrac{-1}{2}, -a$

$\Rightarrow \left(e^x + \dfrac{1}{2}\right)(e^x + a) > 0$

$\Rightarrow a \geq 0$, because e^x is positive for every $x \in R$.

(b) $f'(x) = 2\cos 2x - 3(a+1)\cos x$
$+ 4a^2 + 8a - 14$

$\Rightarrow 2\cos^2 x - 1 - 4(a+1)\cos x + 2a^2 + 4a - 7 > 0$

$\Rightarrow \cos^2 x - 2(a+1)\cos x + (a+1)^2 - 5 > 0$

$\Rightarrow (\cos x - (a+1))^2 > 5$

$\cos x - (a+1) < -\sqrt{5}$

or

$\cos x - (a+1) > -\sqrt{5}$

(1) $a > \cos x - 1 + \sqrt{5}$

$a > \sqrt{5}$

(2) $a < \cos x - 1 - \sqrt{5}$

$a < -2 - \sqrt{5}$

Q.3 (a) Find the set of all real values of μ so that the function $f(x) = (\mu + 1)x^3 + 2x^2 + 3\mu x - 7$ is
(i) always increasing and
(ii) always decreasing.

(b) Find b if $f(x) = \sin x - bx + c$ is always an increasing or a decreasing function.

Sol. (a) $f(x) = (\mu + 1)x^3 + 2x^2 + 3\mu x - 7$

Differentiating w.r.t. x, we get

$f'(x) = 3(\mu + 1)x^2 + 4x + 3\mu - 0$

$f'(x) = 3(\mu + 1)x^2 + 4x + 3\mu$

(i) for increasing function i.e. $f'(x) > 0$

$\Rightarrow 3(\mu + 1)x^2 + 4x + 3\mu > 0$

$\Rightarrow 3x^2 + \dfrac{4}{(1+\mu)}x + \dfrac{3\mu}{(1+\mu)} > 0$

recall from quadratic theory

$ax^2 + bx + c > 0, a > 0 \Rightarrow D < 0$

Here $a = 3 > 0, D < 0$

$\Rightarrow B^2 - 4AC < 0$

$\Rightarrow \left(\dfrac{4}{1+\mu}\right)^2 - 4 \times 3 \times \dfrac{3\mu}{(1+\mu)} < 0$

$\Rightarrow \dfrac{16}{(1+\mu)^2} - \dfrac{4 \times 9\mu}{(1+\mu)} < 0$

$\Rightarrow 16 - 4 \times 9\mu(1+\mu) < 0$

$\Rightarrow 9 \times 4\mu(1+\mu) - 16 > 0$

$\Rightarrow 9\mu(1+\mu) - 4 > 0$

$\Rightarrow 9\mu^2 + 9\mu - 4 > 0$

$\Rightarrow \mu = \dfrac{-9 \pm \sqrt{81 + 16 \times 9}}{2 \times 9}$

$\mu = \dfrac{-9 \pm 3\sqrt{9+16}}{2 \times 9}$

$\mu = \dfrac{-9 \pm 15}{18}$

$= \dfrac{-24}{18}, \dfrac{6}{18}$

$\mu = \dfrac{-4}{3}, \dfrac{1}{3}$

$\Rightarrow \left(\mu + \dfrac{4}{3}\right)\left(\mu - \dfrac{1}{3}\right) > 0$

$\Rightarrow$ Either $\mu < -\dfrac{4}{3}$ or $\mu > \dfrac{1}{3}$

(ii) For decreasing function, $f'(x) < 0$

$\Rightarrow 3(\mu + 1)x^2 + 4x + 3\mu < 0$

$\Rightarrow 3x^2 + \dfrac{4}{(1+\mu)}x + \dfrac{3\mu}{(1+\mu)} < 0$

$\Rightarrow -3x^2 - \dfrac{4}{(1+\mu)}x - \dfrac{3\mu}{(1+\mu)} > 0$

Here, $a = -3 < 0, D > 0$

$\Rightarrow B^2 - 4AC > 0$

$\left(\dfrac{-4}{1+\mu}\right)^2 - 4 \times (-3) \times \left(\dfrac{-3\mu}{1+\mu}\right) > 0$

$\Rightarrow \dfrac{16}{(1+\mu)^2} - \dfrac{9 \times 4\mu}{(1+\mu)} > 0$

$\Rightarrow 16 - 9 \times 4\mu \times (1+\mu) > 0$

$\Rightarrow 9 \times 4\mu(1+\mu) - 16 < 0$

$\Rightarrow 9\mu(1+\mu) - 4 < 0$

$\Rightarrow 9\mu^2 + 9\mu - 4 < 0$

$(\mu + 4/3)(\mu - 1/3) < 0$

$\Rightarrow -4/3 < \mu < 1/3$

(b) $f(x) = \sin x - bx + c$

Differentiating w.r.t.x, we get

$f'(x) = \cos x - b$

for increasing function, $f'(x) > 0$

$\Rightarrow \cos x - b > 0$

$\Rightarrow \cos x > b$, but $-1 \le \cos x \le 1$

$\Rightarrow -1 > b \;\Rightarrow b < -1$

For decreasing function,

$f'(x) < 0 \Rightarrow \cos x - b < 0$

$\Rightarrow \cos x < b$; but $-1 \le \cos x \le 1 \Rightarrow 1 < b \Rightarrow b > 1$

Q.4 Show that the function $f(x) = x/\sqrt{(x+1)} - \lambda n(1+x)$ is an increasing function for $x > -1$.

Sol. $f(x) = x/\sqrt{(x+1)} - \lambda n\,(1+x); x \ne -1$

Differentiating w.r.t.x, we get

$$f'(x) = \frac{1.\sqrt{x+1} - x\dfrac{1}{2\sqrt{x+1}}}{(x+1)} - \frac{1}{1+x}\;;\; x \ne -1$$

$$f'(x) = \frac{\sqrt{x+1} - \dfrac{x}{2\sqrt{x+1}}}{(x+1)} - \frac{1}{1+x}\;;\; x \ne -1$$

$$f'(x) = \frac{2(x+1) - x}{2\sqrt{x+1}\;(x+1)} - \frac{1}{1+x}\;;\; x \ne -1$$

$$f'(x) = \frac{(x+2) - 2(\sqrt{x+1})}{2\sqrt{x+1}\;(x+1)}\;;\; x \ne -1$$

for increasing function, $f'(x) > 0$

i.e. $\dfrac{(x+2) - 2(\sqrt{x+1})}{2\sqrt{x+1}\;(x+1)} > 0;\; x \ne -1$

$\Rightarrow (x+2) > 2\sqrt{x+1}$

Squaring both sides, we get

$\Rightarrow x^2 + 4 + 4x > 4(x+1)$; but $x \ne -1$

$\Rightarrow x^2 > 0\;;\; x \ne -1$

$\Rightarrow x > 0\;;\; x \ne -1$, Hence, $x > -1$

Q.5 If $ax + (b/x) \ge c$ for all positive values of x, where a, b and c are positive constants, show that $ab \ge c^2/4$

Sol. $f(x) = ax + (b/x) \ge c\;;\; x > 0$

Differentiate w.r.t.x, we get

$f'(x) = a - b/x^2 - 0 \ge 0$

$\Rightarrow a - b/x^2 \ge 0$

$\Rightarrow a \ge b/x^2$

$\Rightarrow x^2 \ge b/a \Rightarrow x \ge \sqrt{b/a}$

Hence, put value of $x \ge \sqrt{b/a}$ in $ax + b/x \ge c$

$\Rightarrow a \times \sqrt{b/a} + b \times \sqrt{a/b} \ge c$

$\Rightarrow \sqrt{ab} + \sqrt{ab} \ge c$

$\Rightarrow 2\sqrt{ab} \ge c \Rightarrow \sqrt{ab} \ge c/2 \Rightarrow ab \ge c^2/4$

Q.6 Find the intervals in which the function $f(x) = \sin(\log_e x) + \cos(\log_e x)$ is decreasing.

Sol. $f(x) = \sin(\log_e x) + \cos(\log_e x)$

Differentiating w.r.t.x, we get

$$f'(x) = \frac{\cos(\log_e x)}{x} - \frac{\sin(\log_e x)}{x} < 0;\; x > 0$$

$$\Rightarrow \frac{\cos(\log_e x)}{x} < \frac{\sin(\log_e x)}{x}\;;\; x > 0$$

$\Rightarrow \tan(\log_e x) > 1;\; x > 0$

$\Rightarrow 2n\pi + \pi/4 < \log_e x < 2n\pi + 5\pi/4;\; n \in I$

$\Rightarrow e^{2n\pi + \frac{\pi}{4}} < x < e^{2n\pi + \frac{5\pi}{4}}\;;\; n \in I$

Q.7 Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1-5x)$

Sol. $f'(x) = -1 - 3x^2$

$f(x) \downarrow$

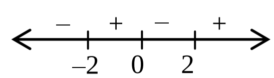
Now,

$f(f(x)) > f(1-5x)$

$\Rightarrow 1 - x - x^3 < 1 - 5x$

$\Rightarrow x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$x \in (-2, 0) \cup (2, \infty)$

Q.8 Establish the following inequalities

(a) $x^2 - 1 > 2x\,\lambda n\,x > 4(x-1) - 2\,\lambda n\,x$ for $x > 1$

(b) $\tan^2 x + 6\,\lambda n \sec x + 2\cos x + 4 > 6\sec x$

for $x \in \left(\dfrac{3\pi}{2}, 2\pi\right)$

(c) $x - (x^3/3) < \tan^{-1}x < x - x^3/6$ for $0 < x \le 1$

(d) $x^2 > (1+x)\,[\lambda n\,(1+x)]^2$ for $x > 0$

Sol. (a)

(1) $x^2 - 1 > 2x\,\lambda n\,x$

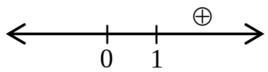
$f(x) = x^2 - 2x\,\lambda n\,x - 1$

$f'(x) = 2x - 2 - 2\lambda n\,x - 1$

$= 2x - 2\lambda n\,x - 3$

$f''(x) = 2 - \frac{2}{x}$

$= \frac{2(x-1)}{x}$



$f'(x) \uparrow \; ; \forall\, x > 1$

$f(x)_{min} = f(1) = 0$

$\therefore f(x) > 0$

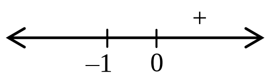
(2) $2x \lambda n\, x > 4(x-1) - 2\lambda n\, x$

$f(x) = 2x\lambda n\, x - 4x + 4 - 2\lambda n\, x$

$f'(x) = 2 + 2\lambda n\, x - 4 - \frac{2}{x}$

$f''(x) = \frac{2}{x} + \frac{2}{x^2}$

$= \frac{2}{x^2}(x+1)$



$f'(x) \uparrow$

$f(x)_{min} = f(1) = 0$

$\therefore f(x) > 0$

(b)

$f(x) = \tan^2 x + 6\lambda n \sec x + 2\cos x + 4 - 6\sec x$

$f'(x) = 2\tan x \,.\, \sec^2 x + 6\tan x - 2\sin x - 6\sec x \tan x$

$= \frac{2\sin x}{\cos^3 x} + \frac{6\sin x}{\cos x} - 2\sin x - \frac{6\sin x}{\cos^2 x}$

$\frac{2\sin x}{\cos^3 x}[1 + 3\cos^2 x - \cos^3 x - 3\cos x]$

$= \frac{2\sin x}{\cos^3 x}(1-\cos x)^3$

(c) (1) $f(x) = \tan^{-1} x - x + \frac{x^3}{3}$

$f'(x) = \frac{1}{1+x^2} - 1 + x^2$

$= \frac{1 - 1 - x^2 + x^2 + x^4}{1+x^2}$ ⊕ ve

$f(x) > 0$

(2) $f(x) = \tan^{-1} x - x + \frac{x^3}{6}$

$f'(x) = \frac{1}{1+x^2} - 1 + \frac{x^2}{2}$

$\frac{2 - 2 - 2x^2 + x^3}{1+x^2}$

$= \frac{-x^2}{1+x^2}$ –ve

$f(x) < 0$

(d) $f(x) = x^2 - (1+x)[\lambda n(1+x)]^2$

$f'(x) = 2x - 2\lambda n(1+x) - [\lambda n(1+x)]^2$

$f''(x) = 2 - \frac{2}{1+x}$

$= \frac{2\lambda n(1+x)}{1+x} = \frac{2[1 + x - 1 - \lambda n(1+x)]}{1+x}$

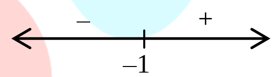
$= \frac{2[x - \lambda n(1+x)]}{(1+x)}$

Q.9 Show that $xe^x = 2$ has one and only one root between 0 and 1.

Sol. $f(x) = xe^x - 2$

$f'(x) = xe^x + e^x$

$= e^x(x+1)$



$f(x) \uparrow x > -1$

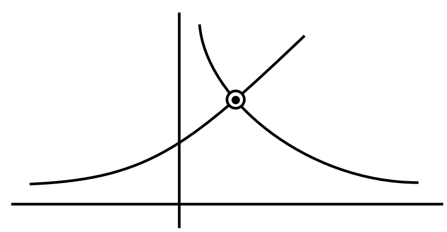
$f(0) = -2$

$f(1) = e - 2 = ⊕$

Alternate

Graph

$e^x = \frac{2}{x}$



Part-B Passage based objective questions

Passage I (Question 10 to 12)

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where $a, b \in R$

Q.10 For a = 1 if y = f(x) is strictly increasing $\forall\, x \in R$ then the maximum range of values of b is.

(A) $\left(-\infty, \frac{1}{3}\right]$ (B) $\left(\frac{1}{3}, \infty\right)$

(C) $\left[\frac{1}{3}, \infty\right)$ (D) $(-\infty, \infty)$

Sol. **[B]**

$f(x) = 8x^3 + 4x^2 + 2bx + 1$

$f'(x) = 24x^2 + 8x + 2b$

$> 0 \ \forall x \in R$

$D < 0$

$64 - 24 \times 8b < 0$

$3b > 1$

$b > \frac{1}{3}$

Q.11 For b = 1, if y = f(x) is non monotonic then the sum of all the integral values of $a \in [1, 100]$ is

(A) 4950 (B) 5049

(C) 5050 (D) 5047

Sol. **[B]**

$f(x) = 8x^3 + 4ax^2 + 2x + a$

$f'(x) = 24x^2 + 8ax + 2$

Reason when $f'(x) \geq 0$ or ≤ 0

$\forall x \in R$

(1) $f'(x) \geq 0$

$64a^2 - 8 \times 24 < 0$

$-\sqrt{3} < a < \sqrt{3}$

(2) $f'(x) \leq 0$

$64a^2 - 8 \times 24 = 0$

$a = \pm\sqrt{3}$

$\therefore$ for non monotonic

$a \in R - [-\sqrt{3}, \sqrt{3}]$

$a \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

sum

$\Rightarrow 2 + 3 + \text{----------} + 100$

$\Rightarrow \frac{99}{2}[4 + 93]$

$= 5049$

Q.12 If the sum of the base 2 logarithms of the roots of the cubic f(x) = 0 is 5 then the value of 'a' is

(A) – 64 (B) – 8

(C) – 128 (D) – 256

Sol. **[D]**

$f(x) = 8x^3 + 4ax^2 + 2bx + a$

$\alpha\beta\gamma = \frac{-a}{8}$

$\Rightarrow \log_2 \alpha + \log_2 \beta + \log_2 \gamma = \log_2\left(\frac{-a}{8}\right)$

$\log_2\left(\frac{-a}{8}\right) = 5$

$a = -8 \times 2^5$

$= -256$

Passage II (Question 13 to 15)

Let $f(x) = \lambda n\, mx \ (m > 0)$

$g(x) = px$

Q.13 The equation $|f(x)| = g(x)$ has only one solution for

(A) $0 < p < \frac{m}{e}$ (B) $p < \frac{e}{m}$

(C) $0 < p < \frac{e}{m}$ (D) $p > \frac{m}{e}$

Sol. **[D]**

$f(x) = \lambda n\, mx \ (m > 0)$ & $(x > 0)$

$g(x) = px$

$|\lambda n\, mx| = px$

$\Rightarrow \frac{|\lambda n\, mx|}{x} = p$

As, $x > 0$

$\left|\frac{\lambda n\, mx}{x}\right| \Rightarrow p$

Let $y = \frac{\lambda n\, mx}{x}$

$y' = \frac{1 - \lambda n\, mx}{x^2}$

+ | –

e/m

$y\left(\frac{e}{m}\right) = \left(\frac{m}{e}\right)$

$\therefore \left(\frac{1}{m}, 0\right)$ & $\left(\frac{e}{m}, \frac{m}{e}\right)$

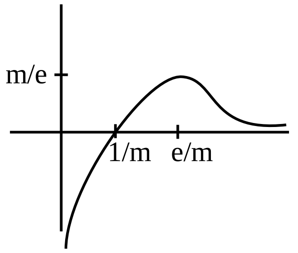
lie on the graph

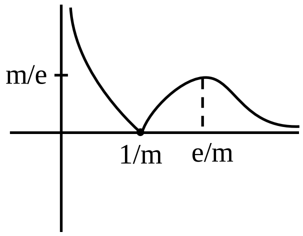
$\lim\limits_{x \to 0} \frac{\lambda n\, mx}{x} = -\infty$

$\lim\limits_{x \to \infty} \frac{\lambda n\, mx}{x}$

D.L.H. Hospital rule

$\lim\limits_{x \to \infty} \frac{1/x}{1} = 0$





$p \geq \dfrac{m}{e}$

Q.14 The equation |f(x)| = g (x) has exactly two solutions (not necessarily distinct) for

(A) $p = \dfrac{m}{e}$ (B) $p = \dfrac{e}{m}$

(C) $0 < p \leq \dfrac{e}{m}$ (D) $0 < p \leq \dfrac{m}{e}$

Sol. **[A]**

$p = \dfrac{m}{e}$

Q.15 The equation |f(x)| = g(x) has exactly three solutions for

(A) $p = \dfrac{m}{e}$ (B) $0 < p < \dfrac{m}{e}$

(C) $0 < p < \dfrac{e}{m}$ (D) $p < \dfrac{e}{m}$

Sol. **[B]**

$0 < p < \dfrac{m}{e}$

Passage III (Question 16 to 18)

Let here define two functions

$f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\cos x); \ 0 \leq x \leq 2\pi$

$g(x) = e^x \qquad ; 0 \leq x \leq 2\pi$

Q.16 Find the value of x such that f(x) is increases-

(A) $[0, \pi]$ (B) $\left[0, \dfrac{\pi}{4}\right]$

(C) $\left[0, \dfrac{\pi}{2}\right]$ (D) None of these

Sol. **[C]**

$f'(x) = \dfrac{\cos x}{|\cos x|} + \dfrac{\sin x}{|\sin x|}$

$x \neq 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

$f'(x) = 2 \ ; x \in \left(0, \dfrac{\pi}{2}\right)$

$= 0 \ ; x \in \left(\dfrac{\pi}{2}, \pi\right)$

$= -2 \ ; \left(\pi, \dfrac{3\pi}{2}\right)$

$= 0$

$f(x) \uparrow \left[0, \dfrac{\pi}{2}\right]$

Q.17 Find the value of x such that f(g(x)) is decreases-

(A) $(0, \pi)$ (B) $\left(0, \dfrac{\pi}{2}\right)$

(C) $\left[\lambda n\,\pi, \lambda n \dfrac{3\pi}{2}\right]$ (D) None of these

Sol. **[C]**

$f'(g(x)) \quad g'(x)$

$= e^x \left[\dfrac{\cos e^x}{|\cos e^x|} + \dfrac{\sin e^x}{|\sin e^x|}\right]$

$e^x \in \left(\pi, \dfrac{3\pi}{2}\right)$

$x \in \left[\lambda n \pi, \lambda n \dfrac{3\pi}{2}\right]$

Q.18 Find the total length of interval of x such that f(x) is neither increases nor decreases-

(A) $\pi/2$ (B) $\pi/4$

(C) π (D) None of these

Sol. **[C]**

$f(x) \downarrow$

$\Rightarrow x \in \left[\dfrac{\pi}{2}, \pi\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$

$\therefore$ Total path $= \pi$

EXERCISE # 4

Old IIT-JEE Questions

Q.1 Let f (x) = x $e^{x(1-x)}$, then f (x) is- **[IIT Scr. 2001]**

(A) Increasing on [–1/2, 1]

(B) Decreasing on R

(C) Increasing on R

(D) Decreasing on [–1/2, 1]

Sol. **[A]**

$f(x) = x.\, e^{x(1-x)}$

$= x.\, e^{(x-x^2)}$

Differentiating w. r. t. x, we get

$f'(x) = 1.e^{(x-x^2)} + x.e^{(x-x^2)}(1-2x)$

$= e^{(x-x^2)}[1 + x - 2x^2]$

$f'(x) = -e^{(x-x^2)}[2x^2 - x - 1]$

Put $2x^2 - x - 1 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1+4\times 2}}{2\times 2}$

$x = \dfrac{1\pm 3}{4}$

x = 1, –1/2

$\therefore\ f'(x) = -e^{(x-x^2)}(x-1)(x+1/2)$

–ve | –1/2 | +ve | 1 | –ve

Hence, $f'(x) > 0$ for $x \in \left[-\dfrac{1}{2}, 1\right]$

∴ Option (A) is correct answer.

Q.2 Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval [1/2, 1] and identify it. **[IIT 2001]**

Sol. Given that $-1 \le p \le 1$.

Consider $f(x) = 4x^3 - 3x - p = 0$

Now, $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2} - \dfrac{3}{2} - p = -1 - p \le 0$ as $(-1 \le p)$

Also $f(1) = 4 - 3 - p = 1 - p \ge 0$ as $(p \le 1)$

∴ f(x) has at least one real root between [1/2,1].

Also $f'(x) = 12x^2 - 3 > 0$ on [1/2, 1]

⇒ f is increasing on [1/2, 1]

⇒ f has only one real root between [1/2, 1]

To find the root, we observe f(x) contains $4x^3 - 3x$ which is multiple angle formula of cos 3θ of we put x = cos θ.

∴ Let the req. root be cos θ then,

$$4\cos^3\theta - 3\cos\theta - p = 0$$

$\Rightarrow \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1} p \Rightarrow \theta = \dfrac{1}{3}\cos^{-1}(p)$

∴ Root is $\cos\left(\dfrac{1}{3}\cos^{-1}(p)\right)$

Q.3 The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is - **[IIT Scr. 2002]**

(A) π/3 (B) π/2

(C) 3π/2 (D) π

Sol. **[A]**

Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Differentiating w. r. t. x, we get

$f'(x) = \cos 3x \times 3 > 0$

$\Rightarrow \cos 3x > 0$

$\Rightarrow 2n\pi + \pi/2 < 3x < 2n\pi + \dfrac{3\pi}{2}$

where, n ∈ integer

$\Rightarrow \dfrac{2n\pi}{3} + \dfrac{\pi}{6} < x < \dfrac{2n\pi}{3} + \dfrac{\pi}{2}$

Hence, length of longest interval

$= \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{3\pi - \pi}{6} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$

∴ Option (A) is correct answer.

Q.4 Using the result $2(1 - \cos x) < x^2$, $x \ne 0$. Prove that $\sin(\tan x) > x$, for $\forall\ x \in (0, \pi/4)$. **[IIT 2003]**

Sol. Let $f(x) = \sin(\tan x) - x$

Differentiating w.r.t.x, we get

$f'(x) = \cos(\tan x) \times \sec^2 x - 1$

$= \cos(\tan x) \times (1 + \tan^2 x) - 1$

$= \tan^2 x \cos(\tan x) + \cos(\tan x) - 1$

$f'(x) = \tan^2 x \cos(\tan x) - (1 - \cos(\tan x))$

using the relation,

$2(1 - \cos x) < x^2$

$f'(x) > \tan^2 x \cos(\tan x) - \dfrac{1}{2}\tan^2 x$

$f'(x) > \tan^2 x\,[\cos(\tan x) - 1/2]$

Again using, $2(1 - \cos x) < x^2$

$\Rightarrow 1 - \dfrac{x^2}{2} < \cos x$

$\Rightarrow 1 - \frac{\tan^2 x}{2} < \cos(\tan x)$

$f'(x) > \tan^2 x \left[1 - \frac{\tan^2 x}{2} - \frac{1}{2}\right]$

$f'(x) > \tan^2 x \left[\frac{1}{2} - \frac{\tan^2 x}{2}\right]$

$\Rightarrow f'(x) > 0$ for $x \in (0, \pi/4)$

Let $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \sin(\tan x) - x > \sin(\tan 0) - 0$

$\Rightarrow \sin(\tan x) - x > 0$

$\Rightarrow \sin(\tan x) > x$. for $x \in (0, \pi/4)$

Hence, proved.

Q.5 If f(x) is differentiable and strictly increasing function, then the value of $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is-

[IIT 2004]

(A) 1 (B) 0 (C) –1 (D) 2

Sol. **[C]**

Since, f(x) is differentiable function i.e.

$\lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$

$= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$

Then $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$

$\left(\frac{0}{0}\text{form, apply L – H Rule, we get}\right)$

$= \lim_{x\to 0} \frac{f'(x^2)\times 2x - f'(x)}{f'(x) - 0}$

$= \frac{2(0)\times f'(0) - f'(0)}{f'(0) - 0} = \frac{0 - f'(0)}{f'(0)} = -1$

$\therefore$ Option (C) is correct answer.

Q.6 Prove that :

$\sin x + 2x \geq \frac{3x(x+1)}{\pi}, \forall x \in \left[0, \frac{\pi}{2}\right]$

(Justify the inequality, if any used)

[IIT 2004]

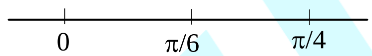
Sol. $\sin x + 2x \geq \frac{3x(x+1)}{\pi}, \forall x \in \left[0, \frac{\pi}{4}\right]$

Let $f(x) = \sin x + 2x - \frac{(3x^2 + 3x)}{\pi}$

Differentiating w. r. t. x, we get

$f'(x) = \cos x + 2 - \frac{(6x+3)}{\pi}$

$f'(x) = (\cos x - 6x/\pi) + \left(2 - \frac{3}{\pi}\right)$



Put x = 0, $f'(x)|_{x=0} = (1 - 0) + 2 - \frac{3}{\pi} = 3 - \frac{3}{\pi}$

$f'(x)|_{x=0} = 3\left(1 - \frac{1}{\pi}\right) > 0$

Put $x = \frac{\pi}{6}$, $f'(x)|_{x=\pi/6} = \left(\cos\frac{\pi}{6} - \frac{6}{\pi}\times\frac{\pi}{6}\right) + 2 - \frac{3}{\pi}$

$= \left(\frac{\sqrt{3}}{2} - 1\right) + 2 - \frac{3}{\pi} = +\text{ ve}$

$\Rightarrow f'(x)|_{x=\pi/6} > 0$

Put $x = \pi/4$, $f'(x)|_{x=\pi/4}$

$= \left(\cos\frac{\pi}{4} - \frac{6}{\pi}\times\frac{\pi}{4}\right) + 2 - \frac{3}{\pi}$

$= \left(\frac{1}{\sqrt{2}} - \frac{3}{2}\right) + 2 - \frac{3}{\pi}$

$= \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{3}{\pi}$

$= \frac{2\pi + \sqrt{2}\pi - 6\sqrt{2}}{2\sqrt{2}\pi}$

$= \frac{\pi(2+\sqrt{2}) - 6\sqrt{2}}{2\sqrt{2}\pi}$

$= \frac{3.14\times 3.41 - 6\times 1.41}{2\sqrt{2}\pi} > 0$

$\Rightarrow f'(x)|_{x=\pi/4} > 0$

$\therefore f'(x)$ is increasing function in $x \in \left[0, \frac{\pi}{4}\right]$

Let $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \sin x + 2x - \frac{(3x^2 + 3x)}{\pi} > 0 + 0 - 0$

$\sin x + 2x - \frac{(3x^2 + 3x)}{\pi} > 0$

$\Rightarrow \sin x + 2x > \frac{3x(x+1)}{\pi}$; $\forall\ x \in \left[0, \frac{\pi}{4}\right]$

Hence, proved.

Q.7 f is a set of polynomial of degree ≤ 2 ; f(0) = 0; f(1) = 1; $f'(x) > 0$; $x \in [0, 1]$ then set f =

[IIT 2005]

(A) ϕ

(B) $ax + (1 - a)x^2$; $a \in R$

(C) $ax + (1 - a)x^2$; $0 < a < \infty$

(D) $ax + (1 - a)x^2$; $0 < a < 2$

Sol. **[D]**

Let $f(x) = ax^2 + bx + c$

$f(0) = 0 = 0 + 0 + c \Rightarrow c = 0$

$f(1) = 1 = a + b + c \Rightarrow a + b = 1$

$f'(x) = 2ax + b + 0 > 0$

$= 2ax + (1-a) > 0$ for $x \in [0, 1] \Rightarrow x > \frac{a-1}{2a}$

When $x = 0$, $0 > \frac{a-1}{2a}$

$\Rightarrow a < 1$

when $x = 1$, $1 > \frac{a-1}{2a}$

$\Rightarrow 2a > a - 1 \Rightarrow a > -1$

Then, $f(x) = ax^2 + (1 - a)x$ for $-1 < a < 1$

or $f(x) = (1 - a)x^2 + ax$ for $0 < 1 - a < 2$

$\therefore$ Option (D) is correct answer.

Q.8 Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then g is-

[IIT 2008]

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

Sol. **[C]**

$g'(u) = \frac{2e^u}{1+e^{2u}} > 0$

So g(u) increases

Now $g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2}$

$= 2\tan^{-1}\left(\frac{1}{e^u}\right) - \frac{\pi}{2}$

$= 2\left(\frac{\pi}{2} - \tan^{-1}(e^u)\right) - \left(\frac{\pi}{2}\right)$

$= \frac{\pi}{2} - 2\tan^{-1}e^u = -g(u)$

$\Rightarrow$ g(u) is odd increasing.

Q.9 The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT 2011]**

Sol.[2] Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

Let $\alpha, \beta, \gamma, \delta$ are the root of equation.

$\therefore$ $\alpha\beta\gamma\delta = -1$ so the equation has at least two real roots. …..(i)

$f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24 = 12((x+1)^2 + 1)$

so $f''(x) > 0$ so $f'(x) = 0$ has only one real roots so $f(x) = 0$ has at most two real roots. ….(ii)

from (i) & (ii)

$f(x) = 0$ has exactly two real roots.

Q.10 Match the statements given in Column-I with the intervals/union of intervals given in Column-II.

[IIT 2011]

Column-I

(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) ; z \text{ is a complex number}, |z| = 1, z \neq \pm 1\right\}$ is

(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is

(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is

(D) If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then $f(x)$ is increasing in

Column-II

(P) $(-\infty, -1) \cup (1, \infty)$

(Q) $(-\infty, 0) \cup (0, \infty)$

(R) $[2, \infty)$

(S) $(-\infty, -1] \cup [1, \infty)$

(T) $(-\infty, 0] \cup [2, \infty)$

Sol. **[A → s ; B → t ; C → r ; D → r]**

(A) Let $z = \cos\theta + i\sin\theta$

so $\dfrac{2iz}{1-z^2} = \dfrac{2i(\cos + i\sin\theta)}{1-\cos 2\theta - i\sin 2\theta} = -\operatorname{cosec}\theta$

$\forall\ \theta \neq (2n+1)\ \dfrac{\pi}{2}$

so

$$\text{Re}\left(\frac{2iz}{1-z^2}\right) = -\operatorname{cosec}\theta \in (-\infty, -1] \cup [1, \infty)$$

(B) $\dfrac{8\times 3^{x-2}}{1-3^{2x-2}} = \dfrac{8\times 3^x}{9-3^{2x}}$

Let $3^x = t$

So $f(x) = \sin^{-1}\left(\dfrac{8\times 3^x}{9-3^{2x}}\right) = \sin^{-1}\left(\dfrac{8t}{9-t^2}\right)$

$-1 \le \dfrac{8t}{9-t^2} \le 1$ on solving

$x \in (-\infty, 0] \cup [2, \infty) \cup \{1\}$

(C) $f(\theta) = 2\sec^2\theta$

so $f(\theta) \in [2, \infty)$

(D) $f(x) = 3x^{5/2} - 10x^{3/2}$

$f'(x) = \dfrac{15\sqrt{x}}{2}(x-2)$

So $f(x)$ is increasing for $f'(x) \ge 0$

$x \in [2, \infty)$

EXERCISE # 5

Q.1 Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0)= 0$ then $h(x) - h(1)$ is- **[IIT-1987]**

(A) always zero (B) always negative

(C) always positive (D) strictly increasing

Sol. **[A]**

Since g is decreasing in $[0, \infty)$

$\therefore$ For $x \ge y \ge 0,\ g(x) \le g(y)$ ….(1)

Also $g(x), g(y) \in [0, \infty)$ and f is increasing from $[0, \infty)$ to $[0, \infty)$

$\therefore$ For $g(x), g(y) \in [0, \infty)$

s.t. $g(x) \le g(y)$

$\Rightarrow$ $f(g(x)) \le f(g(y))$ where $x \ge y$

$\Rightarrow$ $h(x) \le h(y)$

$\Rightarrow$ h is decreasing function from $[0, \infty)$

$\therefore$ $h(x) \le h(0),\ \forall\ x \ge 0$

But $h(0) = 0$ (given)

$h(x) \le 0\ \ \forall\ x \ge 0$ ….(2)

$h(x) \ge 0,\ \forall\ x \ge 0$ ….(3)

Also

[as $h(x) \in [0, \infty)$]

From (2) and (3) we get

$h(x) = 0,\ \forall\ x \ge 0$

Hence, $h(x) - h(1) = 0 - 0 = 0\ \ \forall\ x \ge 0$

Q.2 The function f defined by $f(x) = (x + 2)e^{-x}$ is **[IIT 94]**

(A) Decreasing for all x

(B) Decreasing in $(-\infty, -1)$ and increasing $(-1, \infty)$

(C) Increasing for all x

(D) Decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

Sol. **[D]**

$f(x) = (x + 2)e^{-x}$

Differentiating w. r. t. x, we get

$f'(x) = 1.\, e^{-x} + (x + 2)\, e^{-x}(-1)$

$= e^{-x} - (x + 2)\, e^{-x}$

$= e^{-x}\,[1 - x - 2]$

$f'(x) = e^{-x}\,[-x - 1]$

$f'(x) = -(1 + x)\, e^{-x}$

If $(1 + x) > 0$ i.e. $x > -1$

$f'(x) < 0$

i.e. It decreases for $(-1, \infty)$

If $(1 + x) < 0$ i.e. $x < -1$

$\Rightarrow f'(x) > 0$

i.e. It increases for $(-\infty, -1)$

Hence, f(x) decreases for $(-1, \infty)$ and increases for $(-\infty, -1)$.

$\therefore$ Option (D) is correct answer.

Q.3 Function $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ is decreasing in the interval - **[IIT 95]**

(A) $(-\infty, \infty)$ (B) $(0, \infty)$

(C) $(-\infty, 0)$ (D) No where

Sol. **[B]**

$$f(x) = \frac{\log_e(\pi + x)}{\log_e(e + x)}$$

Differentiating w. r. t. x, we get

$$f'(x) = \frac{\dfrac{1}{\pi + x}.\log_e(e + x) - \dfrac{1}{(e + x)}\log_e(\pi + x)}{(\log_e(e + x))^2}$$

$$\frac{(e + x)\log_e(e + x) - (\pi + x)\log_e(\pi + x)}{(\pi + x)(e + x)(\log_e(e + x))^2}$$

Since, $f'(x) < 0$

$$\Rightarrow \frac{(e + x)\log_e(e + x) - (\pi + x)\log_e(\pi + x)}{(\pi + x)(e + x)(\log_e(e + x))^2} < 0$$

$$\Rightarrow \frac{\log_e(e + x)^{(e+x)} - \log_e(\pi + x)^{(\pi+x)}}{(\pi + x)(e + x)(\log_e(e + x))^2} < 0$$

$\Rightarrow \log_e(e + x)^{(e+x)} < \log_e(\pi + x)^{(\pi+x)}$

$\Rightarrow$ taking antilog both sides, we get

$\Rightarrow (\pi + x)^{(\pi + x)} > (e + x)^{(e + x)}$

Since $\pi > e$

$\Rightarrow \pi + x > e + x$

Hence, above inequality only holds good for +ve values of x i.e. $x \in (0, \infty)$

$\therefore$ Option (B) is correct statement.

Q.4 Let $f(x) = \begin{cases} xe^{ax} & , x \le 0 \\ x + ax^2 - x^3 & , x > 0 \end{cases}$; where ' a ' is positive constant. Find the interval in which f ' (x) is increasing. **[IIT 96]**

Sol. $f(x) = \begin{cases} xe^{ax} & , \ x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$;

Differentiating above function w. r. t. x, we get

$$f'(x) = \begin{cases} 1.e^{ax} + x.e^{ax}.a; & x \le 0 \\ 1 + 2ax - 3x^2; & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} e^{ax}(xa + 1); & x \le 0 \\ (1 + 2ax - 3x^2); & x > 0 \end{cases}$$

Again differentiating w.r.t. x, we get

$$f''(x) = \begin{cases} e^{ax}.a.(xa + 1) + e^{ax}.a; & x \le 0 \\ 2a - 6x; & x > 0 \end{cases}$$

$$f''(x) = \begin{cases} e^{ax}(a^2x + 2a); & x \le 0 \\ 2a - 6x; & x > 0 \end{cases}$$

Since, f ′(x) is increasing function i.e. $f''(x) > 0$

$e^{ax}(a^2x + 2a) > 0$; $x \le 0$

e^{ax} will be positive for $x \le 0$

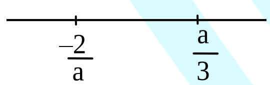
$\therefore$ $(a^2x + 2a) > 0$ for $x \le 0$

$\Rightarrow x \ge -\frac{2}{a}$ for $x \le 0$

Also, $2a - 6x > 0$ for $x > 0$

$\Rightarrow 6x < 2a$ for $x > 0$

$\Rightarrow x < a/3$ for $x > 0$



Hence, required interval is $x \in \left(-\frac{2}{a}, \frac{a}{3}\right)$

Q.5 If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \le 1$, then in this interval - **[IIT 97]**

(A) Both f(x) and g(x) are increasing functions

(B) Both f(x) and g(x) are decreasing function

(C) f(x) is an increasing function

(D) g(x) is an increasing function

Sol. **[C]**

$f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$; $0 < x \le 1$

Differentiating above functions w. r. t. x, we get

$$f'(x) = \frac{1.\sin x - x.\cos x}{(\sin x)^2} = \frac{\sin x - x\cos x}{(\sin x)^2}$$

Since, $(\sin x)^2$ is positive for $0 < x \le 1$

Let $h(x) = \sin x - x \cos x$

Differentiating w.r.t.x, we get

$h'(x) = \cos x - (\cos x + x(-\sin x))$

$= \cos x - \cos x + x \sin x$

$h'(x) = x \sin x > 0$ for $0 < x \le 1$

$\Rightarrow f'(x) > 0$ for $0 < x \le 1$

$\Rightarrow$ f(x) is increasing function in $0 < x \le 1$;

$g(x) = \frac{x}{\tan x}$

Differentiating w. r. t. x, we get

$$g'(x) = \frac{1.\tan x - x.\sec^2 x}{(\tan x)^2}$$

$$g'(x) = \frac{\tan x - x.\sec^2 x}{(\tan x)^2}$$

since, $(\tan x)^2$ + ve for $0 < x \le 1$

Let $\phi(x) = \tan x - x. \sec^2 x$

Differentiating w. r. t. x, we get

$\phi'(x) = \sec^2 x - [1.\sec^2 x + x. 2 \sec x. \tan x \sec x]$

$= \sec^2 x - \sec^2 x - 2x \sec^2 x \tan x$

$\phi'(x) = -2x \sec^2 x. \tan x$

$\phi'(x) < 0$ for $0 < x \le 1 \Rightarrow g'(x) < 0$ for $0 < x \le 1$

$\Rightarrow$ g(x) is decreasing function for $0 < x \le 1$

$\therefore$ Option (C) is correct answer.

Q.6 The function $f(x) = \sin^4 x + \cos^4 x$ increases if- **[IIT Sc. 99]**

(A) $0 < x < \frac{\pi}{8}$ (B) $\frac{\pi}{4} < x < \frac{3\pi}{8}$

(C) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (D) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

Sol. **[B]**

$f(x) = \sin^4 x + \cos^4 x$

Differentiating w. r. t. x, we get

$f'(x) = 4 \sin^3 x \cos x + 4 \cos^3 x (-\sin x)$

$= 4 \sin x \cos x [\sin^2 x - \cos^2 x]$

$= 2 \sin 2x (-\cos 2x)$

$f'(x) = -\sin 4x$

Since, $f'(x) \ge 0 \Rightarrow -\sin 4x \ge 0$

$\Rightarrow \sin 4x \le 0 \Rightarrow \pi \le 4x \le 2\pi$

$\Rightarrow \pi/4 \le x \le \pi/2$

$\therefore$ option (B) is correct answer.

Q.7 Consider the following statement **S** and **R** - **[IIT Sc. 2000]**

S : Both sin x and cos x are decreasing function in the interval $\left(\frac{\pi}{2}, \pi\right)$

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b)

Which of the following is true ?

(A) Both S and R are wrong

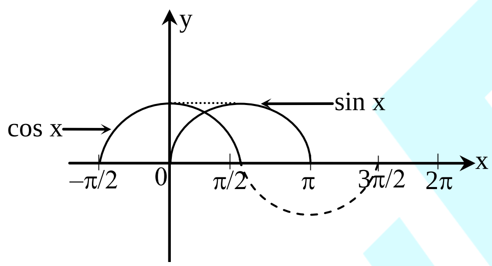
(B) Both S and R are correct, but R is not the correct explanation for S

(C) S is correct and R is the correct explanation for S

(D) S is correct and R is wrong

Sol. **[D]**

Hence, from graph, we observe that, sin x and cos x are decreasing functions in $\left(\frac{\pi}{2}, \pi\right)$. Hence, S is correct answer.



If any differentiable function increases or decreases in an interval (a, b), then its derivative will gets reversed i.e. Derivative must be decreases or increases in an interval (a, b).

Hence, R is not correct answer.

∴ Option (D) is correct answer.

Q.8 If $0 < x < 1$ prove that $y = x\lambda n\, x - (x^2/2) + (1/2)$ is a function such that $d^2y/dx^2 > 0$.

Deduce that $x\lambda nx > (x^2/2) - (1/2)$.

Sol. $y' = 1 + \lambda n\, x - x$

$y'' = \frac{1}{x} - 1 > 0$ as $x \in (0, 1)$ Hence Proved

also it means

$y'(x) \uparrow$

$y'(x) < y'(1) = 0$

$\Rightarrow y'(x) \downarrow$

$\Rightarrow y(x) > y(1) = 0$

$\Rightarrow y(x) > 0 \ \forall\, x \in (0, 1)$

$\Rightarrow x\, \lambda n\, x > \frac{x^2}{2} - \frac{1}{2} \quad \forall\, x \in (0, 1)$

Hence Proved

Passage (Question 9 to 10)

Answer the questions on the basis of the function given below :

$f : (0, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be defined as,

$f(x) = \text{arc}\tan(\lambda n\, x)$

Q.9 The above function can be classified as -

(A) injective but not surjective

(B) Surjective but not injective

(C) neither injective nor surjective

(D) both injective as well as surjective

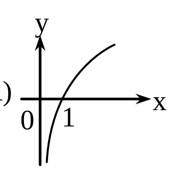
Sol.[D] $f(x) = \tan^{-1}(\lambda n\, x)$

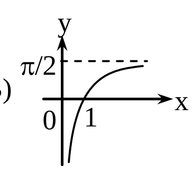
$f'(x) = \frac{1}{x} \cdot \frac{a}{1 + (\lambda n)^2} > 0 \quad \forall\, x \in (0, \infty)$ injective

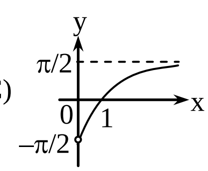
also $R_f \in (-\pi/2, \pi/2)$

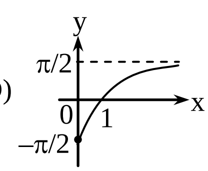
So (D) is true

Q.10 The graph of y = f(x) is best represented as -

(A)


(B)


(C)


(D)


Sol.[C] $f(x) = \tan^{-1}(\lambda n\, x)$

$f'(x) = \frac{1}{x} \cdot \frac{a}{1 + (\lambda n)^2} > 0 \quad \forall\, x \in (0, \infty)$ injective

On the basis of Range and domain of f (x) the correct graph is (C).

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ans.	D	D	A,B	C	A	A	A	D	A	D	D	B	B	B	B	B	A	B

19. False **20.** True

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	A	C	A	D	B	B	B	B	B	D	C	A	B	C	A

(PART-B)

Q.No.	16	17	18	19	20	21	22	23	24	25
Ans.	A,B,C,D	A,B,C,D	A,C	A,D	A,C	A,C	A,B	B,C	B,D	A,D

(PART-C)

Q.No.	26	27	28
Ans.	C	A	A

(PART-D)

29. (A) $\rightarrow$ R (B) $\rightarrow$ P (C) $\rightarrow$ S (D) $\rightarrow$ Q

(PART-E)

30. $a \in [1, \infty)$ **31.** $\alpha \in [-1, 1]$

EXERCISE # 3

1. (a) Increasing in $(-\infty, -7] \cup [5, \infty)$ and decreasing in $[-7, 5]$

(b) Increasing in $(-\infty, -2] \cup [2, \infty)$ and decreasing in$[-2, 2]$

(c) Increasing in $[-1, 0]$ and decreasing in $[0, 1]$

(d) Increasing in $(-\infty, 1]$ or $[3, \infty)$ and decreasing in $[1, 3]$

(e) Increasing in $(-\infty, -1] \cup [1, \infty)$, decreasing in $[-1, 1]$

(f) Increasing in $[0, 1]$ and decreasing in $[1, \infty)$

2. (a) $a \geq 0$ (b) $a < -(2 + \sqrt{5})$ or $a > \sqrt{5}$

3. (a) (i) $\mu \geq 1/3$ (ii) $\mu \leq -4/3$

(b)Increasing if $b \leq -1$ and decreasing if $b \geq 1$

6. $[e^{2n\pi + \pi/4}, e^{2n\pi + 5\pi/4}]$, $n \in Z$.

7. $(-2, 0) \cup (2, \infty)$

Q.No.	10	11	12	13	14	15	16	17	18
Ans.	C	B	D	D	A	B	C	C	C

EXERCISE # 4

1. (A) **3.** (A) **5.** (C) **7.** (D) **8.** (C) **9.** 2

10. **A → S ; B → T ; C → R ; D → R**

EXERCISE # 5

1. (A) **2.** (D) **3.** (B) **4.** [– 2/a, a/3] **5.** (C) **6.** (B) **7.** (D)
9. (D) **10.** (C)