MAXIMA & MINIMA

EXERCISE # 1

Question based on **Local Minima & Maxima**

Q.1 If the function $y = \dfrac{ax+b}{(x-4)(x-1)}$ has an extremum at P(2, –1), then the values of a and b are -

(A) a = 0, b = 1 (B) a = 0, b = –1

(C) a = 1, b = 0 (D) a = –1, b = 0

Sol. **[C]**

$$y = \frac{ax+b}{x^2-5x+4}$$

$$y' = \frac{(ax^2-5ax+4a)-(ax+b)(2x-5)}{(x-1)^2\ (x-4)^2}$$

$$= \frac{-ax^2+4a-2bx+5b}{(x-1)^2\ (x-4)^2}$$

$= y'_{(2,-1)} = 0$

$\Rightarrow -4a + 4a - 3b = 0 \Rightarrow b = 0$

& $2a + b = 2$

Put b = 0 in above equations

$\Rightarrow$ a = 1, b = 0

Q.2 $f(x) = \sin^p x \cos^q x$ (p, q > 0 ; $0 < x < \frac{\pi}{2}$) has point of maxima at -

(A) $x = \tan^{-1}\sqrt{\left(\frac{p}{q}\right)}$ (B) $x = \tan^{-1}\sqrt{\left(\frac{q}{p}\right)}$

(C) no such point exist (D) None of these

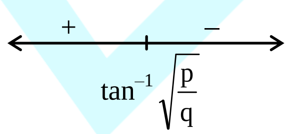
Sol. **[A]**

$f'(x) = \sin^p x \cdot q \cos^{q-1}x\,(-\sin x)$

$+ \cos^q x \cdot p \sin^{p-1}x \cos x$

$= \sin^p x \cdot \cos^q x\,(-q \tan x + p \cot x)$

$$= \frac{\sin^p x \cos^q x}{\tan x}(p - q\tan^2 x)$$



Q.3 The range of values of k for which the function $f(x) = (2k-3)(x + \tan 2) + (k-1)(\sin^4 x + \cos^4 x)$ does not possess critical points, is-

(A) $\left(-\infty, \frac{4}{3}\right) \cup (2, \infty)$ (B) $\left(\frac{4}{3}, 2\right)$

(C) $\left(\frac{4}{3}, \infty\right)$ (D) $(2, \infty)$

Sol. **[A]**

$f'(x) = (2k-3) + (k-1)$

$(4\sin^3 x \cos x - 4\cos^3 x \sin x)$

$= 2k - 3 + (k-1)\, 4 \sin x \cos x\,(\sin^2 x - \cos^2 x)$

$= 2k - 3 + (k-1)(-\sin 4x)$

$= 2k - 3 - k \sin 4x + \sin 4x$

> 0 or < 0

(1) > 0

$$k > \frac{3-\sin 4x}{2-\sin 4x}$$

$$> \frac{1}{2-\sin 4x} + 1$$

> 2

(2) < 0

$$k < \frac{1}{2-\sin 4x} + 1$$

$$k < \frac{4}{3}$$

Q.4 The function $g(x) = \dfrac{f(x)}{x}$, $x \neq 0$ has an extreme value when-

(A) $g'(x) = f(x)$ (B) $f(x) = 0$

(C) $x\, g'(x) = f(x)$ (D) $g(x) = f'(x)$

Sol. **[D]**

$$g'(x) = \frac{x f'(x) - f(x)}{x^2}$$

$$= \frac{x[f'(x) - g(x)]}{x^2}$$

Q.5 Let $f(x) = (x-a)^n g(x)$, where $g^n(a) \neq 0$; n = 1, 2, 3....then

(A) f(x) has local extremum at x = a, when n = 3

(B) f(x) has local extremum at x = a; when n = 4

(C) f(x) has neither local maximum nor local minimum at x = a, when n = 2

(D) f(x) has neither local maximum nor local minimum at x = a, when n = 4

Sol. **[B]**

$f'(x) = n(x-a)^{n-1}\, g(x) + g'(x)\,(x-a)^n$

$= (x-a)^n\left[\dfrac{ng(x)}{(x-a)} + g'(x)\right]$

$= (x-a)^{n-1}\,[ng(x) + g'(x)\,(x-a)]$

For extremum at x = a

(n – 1) must be odd

n is even

Q.6 Let $f(x) = (x^2-1)^n\,(x^2+x-1)$ then f (x) has local extremum at x = 1 when -

(A) n = 2 (B) n = 3

(C) n = 1 (D) n = 5

Sol. **[A]**

$f(x) = (x-1)^n\,(x+1)^n\,(x^2+x+1)$

(n – 1) odd

n is even

Q.7 If $y = a\log|x| + bx^2 + x$ has its extremum values at x = –1 and x = 2, then

(A) a = 2, b = –1 (B) a = 2, b = –1/2

(C) a = –2, b = 1/2 (D) None of these

Sol. **[B]**

$y = a\log x + bx^2 + x\ ;\ x \ge 0$

$a\log(-x) + bx^2 + x\ ;\ x < 0$

$y' = \dfrac{a}{x} + 2bx + 1 = 0$

$\dfrac{2bx^2+x+a}{x} = 0$

$\Rightarrow a + 2b = 1$ ….(1)

$8b + a = -2$ ….(2)

Solving (1) & (2)

$b = -\dfrac{1}{2}$

a = 2

Q.8 If h(x) = f(x) + f(–x), then h(x) has got an extreme value at a point where f′(x) is -

(A) even function (B) odd function

(C) zero (D) None of these

Sol. **[A]**

$h'(x) = f'(x) - f'(-x)$

Q.9 Equation of a straight line passing through (1, 4) if the sum of its positive intercept on the coordinate axis is the smallest is

(A) 2x + y – 6 = 0 (B) x + 2y – 9 = 0

(C) y + 2x + 6 = 0 (D) None of these

Sol. **[A]**

$y - 4 = m(x-1)$

$\Rightarrow mx - y \equiv m - 4$

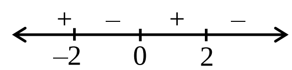
$f(m) = \dfrac{m-4}{m} + 4 - m$

$= 5 - m - \dfrac{4}{m}$

$f'(m) = -1 + \dfrac{4}{m^2}$

$= \dfrac{4-m^2}{m^2}$

$= \dfrac{(2-m)(2+m)}{m^2}$



$m = \pm 2$

(1) 2x – y + 2 = 0

(2) – 2x – y + 6 = 0

Question based on

Global Minima & Maxima

Q.10 $f(x) = 1 + [\cos x]x$, in $0 \le x \le \dfrac{\pi}{2}$, where $[.] \to$ G. I. F.

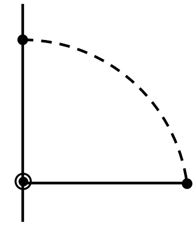
(A) has a minimum value 0

(B) has a maximum value 2

(C) is continuous in $\left[0, \dfrac{\pi}{2}\right]$

(D) is not differentiable at $x = \dfrac{\pi}{2}$

Sol. **[C]**



$f(x) = 1\ ;\ 0 \le x \le \dfrac{\pi}{2}$

continuous

Q.11 The greatest value of the function $f(x) = \tan^{-1}x - \dfrac{1}{2}\log x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$ is -

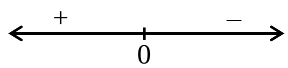
(A) $\dfrac{\pi}{6} + \dfrac{1}{4}\log 3$ (B) $\dfrac{\pi}{3} - \dfrac{1}{4}\log 3$

(C) $\frac{\pi}{6} - \frac{1}{4}$ log 3 (D) $\frac{\pi}{3} + \frac{1}{4}$ log 3

Sol. **[A]**

$f'(x) = \frac{1}{1+x^2} - \frac{1}{2x}$

$= \frac{-(x-1)^2}{2x(x^2+1)}$



f(x) ↓ in x > 0

$\therefore f(x)_{max} = f\left(\frac{1}{\sqrt{3}}\right)$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} - \frac{1}{2}\log\frac{1}{\sqrt{3}}$

Q.12 Function f(x) , g(x) are defined on [–1, 3] and $f''(x) > 0$, $g''(x) > 0$ for all $x \in [-1, 3]$, then which of the following is always true?

(A) f(x) – g(x) is concave upwards on (–1, 3)
(B) f(x) g(x) is concave upwards on (–1, 3)
(C) f (x) g(x) does not have a critical point on (–1, 3)
(D) f(x) + g(x) is concave upwards on (–1, 3)

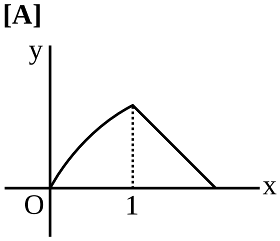
Sol. **[D]**

Clearly

Q.13 Let $f(x) = \begin{cases} \sin\frac{\pi x}{2} & , 0 \le x < 1 \\ 3 - 2x & , x \ge 1 \end{cases}$ then

(A) f(x) has local maxima at x = 1
(B) f(x) has local minima at x = 1
(C) f(x) does not have any local extrema at x =1
(D) f(x) has a global minima at x =1

Sol. **[A]**



cont. at x = 1
LHD = 0
RHD = –2

Q.14 The difference between the greatest and the least value of $f(x) = \cos^2\frac{x}{2}\sin x$, $x \in [0, \pi]$ is

(A) $\frac{3\sqrt{3}}{8}$ (B) $\frac{\sqrt{3}}{8}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2\sqrt{2}}$

Sol. **[A]**

$f'(x) = \sin x \,.\, 2\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + \cos^2\frac{x}{2}\cos x$

$= \cos x \,.\, \cos^2\frac{x}{2} - \frac{\sin^2 x}{2}$

$= \cos^2\frac{x}{2}\left[-3\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right]$

$= \cos^4\frac{x}{2}\left(1 - 3\tan^2\frac{x}{2}\right)$

f (0) = 0

f (π) = 0

$f\left(\frac{\pi}{3}\right) = \frac{3}{4} \times \frac{\sqrt{3}}{2}$

Difference = $\frac{3\sqrt{3}}{8}$

Question based on Point of Inflection

Q.15 The cubic polynomial function passing through (1, 2) with origin as the point of inflection is-

(A) $x^3 + 2x^2 + 3$ (B) $2x^3$
(C) $x^3 + 7x^2 + 2$ (D) None of these

Sol. **[B]**

$y = 2x^3$

Q.16 If the point (1, 3) serves as the point of inflection of the curve $y = ax^3 + bx^2$ then the value of 'a' and 'b' are

(A) a = 3/2 & b = – 9/2 (B) a = 3/2 & b = 9/2
(C) a = –3/2 & b = – 9/2 (D) a = –3/2 & b = 9/2

Sol. **[D]**

y"(1, 3) = 0

⇒ 6a (1) + 2b = 0

b = –3a

(A) & (D) any one can be ans.

Q.17 The set of value (s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possess a negative point of inflection-

(A) $(-\infty, -2) \cup (0, \infty)$ (B) {–4/5}
(C) {–2, 0} (D) empty set

Sol. **[A]**

f "(x) = 2ax + 2(a + 2) = 0

$$\Rightarrow x = -\frac{a+2}{a} < 0 \Rightarrow a \in (-\infty, -2) \cup (0, 0)$$

Question based on **Applications**

Q.18 In a submarine telegraph cable the speed of signaling varies as $x^2 \log (1/x)$ where x is the ratio of the radius of the cable to that of covering. Then the greatest speed is attained when this ratio is

(A) $1 : \sqrt{e}$ (B) $\sqrt{e} : 1$ (C) e : 1 (D) 1 : e

Sol. **[A]**

$$f(x) = x^2 \lambda n\left(\frac{1}{x}\right)$$

$$f'(x) = x^2 . x . \left(\frac{-1}{x^2}\right) + 2x \lambda n\left(\frac{1}{x}\right) = 0$$

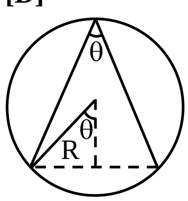
$$\frac{1}{x} = \sqrt{e}$$

$$x = \frac{1}{\sqrt{e}}$$

Q.19 A cone of maximum volume is inscribed in a given sphere. Then the ratio of the height of the cone to the diameter of the sphere is

(A) $\frac{3}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{2}{3}$

Sol. **[D]**



$$v = \frac{1}{3}\pi R^2 \cos^2\theta . R(1 + \sin\theta)$$

$$\frac{dv}{d\theta} = \frac{R^3}{3} (\cos^3\theta - 2\cos\theta \sin\theta (1 + \sin\theta))$$

$$= 0 \Rightarrow \cos^2\theta = 2\sin\theta + 2\sin^2\theta$$

$$(1 - \sin\theta)(1 + \sin\theta) = 2\sin\theta (1 + \sin\theta)$$

$$\Rightarrow \sin\theta = \frac{1}{3}$$

$$h = \frac{4}{3}R \Rightarrow \frac{h}{2R} = \frac{2}{3}$$

Q.20 Let $f(x) = \sqrt{x-1} + \sqrt{2-x}$ & $g(x) = x^2 + bx + c$ are two given functions such that f(x) and g(x) attain their maximum and minimum values respectively for same value of x then the value of b is

(A) 1 (B) 2 (C) 3 (D) –3

Sol. **[D]**

$$f'(x) = \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{2-x}} = 0$$

$$2 - x = x - 1$$

$$x = \frac{3}{2}$$

Now,

$$g'\left(\frac{3}{2}\right) = 0$$

$$\Rightarrow 2\left(\frac{3}{2}\right) + b = 0$$

$$b = -3$$

Q.21 Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are a and b respectively. Then the numerical value of a – b is-

(A) 1 (B) 2 (C) 7 (D) 8

Sol. **[D]**

$x^2 + y^2 - 4x + 3 = 0$

Let $x^2 + y^2 = \lambda$

$\lambda = 4x - 3$

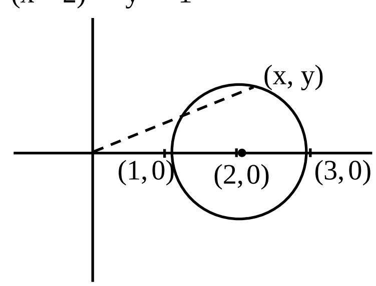
$(x - 2)^2 + y^2 = 1$

$y = \sqrt{1 - (x-2)^2}$

Domain $-1 \le x - 2 \le 1$

Alternate

$(x - 2)^2 + y^2 = 1$



$\lambda_{max} = (3)^2 = 9 = a$

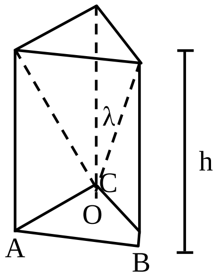
$\lambda_{min} = (1)^2 = 1 = b$

$a - b = 8$

Q.22 In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is λ. The altitude of the prism for which the volume is greatest.

(A) $\lambda/2$ (B) $\lambda/\sqrt{3}$ (C) $\lambda/3$ (D) $\lambda/4$

Sol. **[B]**



$OB = \sqrt{\lambda^2 - h^2}$

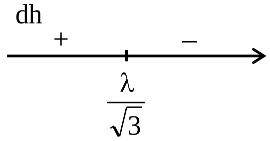
$AB = 2\ OB \cos 30°$

$= \sqrt{3}\ \sqrt{\lambda^2 - h^2}$

$V = (\Delta ABC)\ .\ h$

$= \frac{\sqrt{3}}{2} \times 3\ (\lambda^2 - h^2)\ h$

$\frac{dV}{dh} = \lambda^2 - 3h^2$



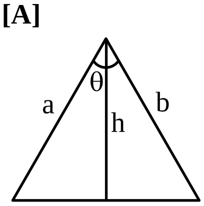
$h = \frac{\lambda}{\sqrt{3}}$

Q.23 Two sides of a triangle are to have lengths 'a' cm and 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(A) $\frac{1}{2}\sqrt{a^2 + b^2}$ (B) $\frac{2a + b}{3}$

(C) $\sqrt{\frac{a^2 + b^2}{2}}$ (D) $\frac{a + 2b}{3}$

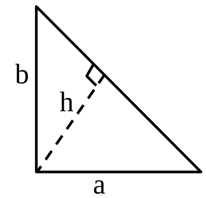
Sol. **[A]**



$\text{Area} = \frac{1}{2}\ ab \sin\theta$

$A_{max} = \frac{1}{2}\ ab$

$\theta = 90°$



$h = \frac{1}{2}\sqrt{a^2 + b^2}$

Q.24 The lateral edge of a regular rectangular pyramid is 'a' cm long. The lateral edge makes an angle α with the plane of the base. The value of α for which the volume of the pyramid is greatest, is

(A) $\frac{\pi}{4}$ (B) $\sin^{-1}\sqrt{\frac{2}{3}}$

(C) $\cot^{-1}\sqrt{2}$ (D) $\frac{\pi}{3}$

Sol. **[C]**

$V = \frac{1}{2} \times (2a\cos\alpha)^2 \times a \sin\alpha \times \frac{1}{3}$

$= \frac{2}{3}\ a^3 \cos^2\alpha \sin\alpha$

$\frac{dV}{d\alpha} = 0$

$\Rightarrow \frac{2}{3}\ a^3\ [\cos^2\alpha\ .\ \cos\alpha - \sin\alpha\ .\ 2\cos\alpha\ .\ \sin\alpha]$

$= 0$

$\cot^2\alpha = 2 \Rightarrow\ \alpha = \cot^{-1}\sqrt{2}$

Q.25 A rectangle has one side on the positive y-axis and one side on the positive x-axis. The upper right hand vertex on the curve $y = \frac{\lambda nx}{x^2}$. The maximum area of the rectangle is

(A) e^{-1} (B) $e^{-1/2}$ (C) 1 (D) $e^{1/2}$

Sol. **[A]**

$\text{Area } A = \frac{\lambda nx}{x} \Rightarrow \frac{dA}{dx} = \frac{1 - \lambda nx}{x^2}$

$\frac{dA}{dx} = 0 \Rightarrow x = e$

$A = \frac{1}{e} = 0$

➤ True or false type questions

Q.26 $f(x) = (3 - x)\ e^{2x} - 4x \cdot e^x - x$ has neither maxima nor minima at $x = 0$.

Sol. **[True]**

$f'(x) = 6x\ e^{2x} - e^{2x}$

$-2x^2 e^{2x} - 4e^x$

$-4x^2 e^x - 1$

neither maxima nor minima at x = 0

Q.27 The greatest value of the function $\log_x(1/9) - \log_3 x^2$, (x > 1) is –4.

Sol. **[True]**

$$f(x) = \frac{\lambda n\left(\frac{1}{9}\right)}{\lambda n\, x} - \frac{\lambda n\, x^2}{\lambda n\, 3}$$

$$f'(x) = \frac{-\lambda n\left(\frac{1}{9}\right)}{(\lambda n\, x)^2}\ \frac{1}{x} - \frac{2}{\lambda n 3}\ \frac{1}{x}$$

$$= \frac{-1}{x}\left[\frac{\lambda n\left(\frac{1}{9}\right)}{(\lambda n\, x)^2} + \frac{2}{\lambda n 3}\right]$$

Q.28 The shortest distance of the line y = x + 1 from $y^2 = x$ is $\frac{3\sqrt{3}}{4}$.

Sol. **[False]**

$2y\ \ y' = 1$

$$y' = \frac{1}{2y}$$

$$\frac{1}{2y} = 1$$

$$\left[\begin{array}{l} y = \frac{1}{2} \\ x = \frac{1}{4} \end{array}\right.$$

$$\text{Dist.} = \frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$$

➤ Fill in the blanks type questions

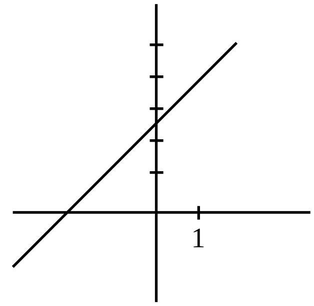
Q.29 The co-ordinates of the point on the parabola $y^2 = 8x$ which is at a minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are ......

Sol. (2, –4)

Q.30 Let $f(x) = \begin{cases} |x-1| + a\ , & x \le 1 \\ 2x + 3\ , & x > 1 \end{cases}$

If f(x) has local minimum at x = 1 and a ≥ 5 then a = ...

Sol.



a = 5

Q.31 The coordinates of the point on the curve $x^3 = y\ (x - a)^2$, (a > 0) where the ordinate is minimum is.............

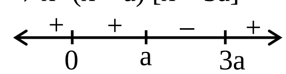
Sol. $3x^2 = 2$

$$y = \frac{x^3}{(x-a)^2}$$

$$y' = \frac{(x-a)^2\ 3x^2 - x^3\ 2(x-a)}{(x-a)^4}$$

$x^2\ (x - a)\ [3x^2\ (x - a) - 2x^3]$

$\Rightarrow x^2\ (x - a)\ [x - 3a]$



x = 3a

$$y = \frac{27a^3}{4a^2}$$

Q.32 Suppose

$$f(x) = \begin{cases} -x^3 + \lambda^2 - 3\lambda + 2 & ; \quad 0 \le x < 1 \\ 2x - 3 & ; \quad 1 \le x \le 3 \end{cases}$$

If f(x) is smallest at x = 1 then λ ∈..............

Sol. $f(1) = 2 - 3 = -1 < f(1^+)$

$f(1^-) = -(1-h)^3 + \lambda^2 - 3\lambda + 2 > -1$

$\lambda^2 - 3\lambda + 2 > -1 + (1-h)^3$

$\lambda^2 - 3\lambda + 2 \ge 0$

$\lambda \in (-\infty, 1] \cup [2, \infty)$

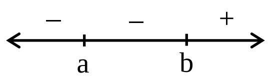
EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 If $\frac{dy}{dx} = (x-a)^{2n}(x-b)^{2p+1}$, n, p $\in$ N, then the function y = f(x) attains a min. at x = ...

(A) a (B) b
(C) 0 (D) a + b

Sol. **[B]**



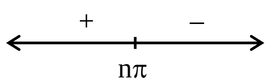
f_{min} at x = b

Q.2 The greatest value of the function
$f(x) = 8 - \tan^4 x - 4\sec^2 x$ is-

(A) 4 (B) 2
(C) 6 (D) None of these

Sol. **[A]**

$f'(x) = -4\tan^3 x \sec^2 x - 8\sec^2 x \tan x$
$= -4\tan x \sec^2 x\,(2 + \tan^2 x)$



$f(x)_{max}$ at $x = n\pi$
$f_{max} = 4$

Q.3 Let f (x) =

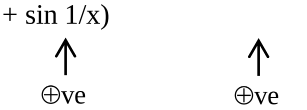
$$\begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then value of point where f (x) attains its minimum is -

(A) 0 (B) 1 (C) 3 (D) infinite

Sol. **[A]**

f (x) > 0
for x = 0 + and x = 0 –
Θ (3 + sin 1/x)



∴ f (x) is min = 0
at x = 0

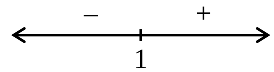
Q.4 The minimum value of the function

$$f(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1 \text{ is-}$$

(A) 1 (B) 0
(C) 2 (D) None of these

Sol. **[A]**

$f'(x) = x^{p-1} - x^{-q-1}$
$= x^{-q-1}(x^{p+q} - 1)$



$f_{min} = f(1) = \frac{1}{p} + \frac{1}{q} = 1$

Q.5 If a < b < c < d & x $\in$ R then the least value of the function f(x) = |x – a| + |x – b| + |x – c| + |x – d| is

(A) c – d + b – a (B) c + d – b – a
(C) c + d – b + a (D) c – d + b + a

Sol. **[B]**

$f(x) = 4x - (a + b + c + d)$; $x \geq d$
$= 3x - (a + b + c + d)$; $c \leq x < d$
$= -(a + b - c - d)$; $b \leq x \leq c$
$= -2x - (a - b - c - d)$; $a < x < b$
$= (a + b + c + d) - 4x$; $x < a$
$f_{min} = -a - b + c + d$

Q.6 The set of values of p for which the points of extremum of the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval (–2, 4), is

(A) (– 3, 5) (B) (– 3, 3)
(C) (– 1, 3) (D) (– 1, 5)

Sol. **[C]**

$f'(x) = 3x^2 - 6px + 3(p^2 - 1)$
(1) D > 0
$36p^2 - 36(p^2 - 1) > 0$
1 > 0
$p \in R$
(2) $-2 < p < 4$
(3) $f(-2) > 0$
$\Rightarrow -8 - 12p + 6p^2 - 6 + 1 > 0$
$6p^2 - 12p + 13 > 0$
$p \in R$
(4) $f(4) > 0$
$64 - 48p + 12p^2 - 12 + 1 > 0$
$\Rightarrow 12p^2 - 48p + 57 > 0$
$p \in R$
$\therefore p \in (-2, 4)$

or

$f'(x) = 0$

$\Rightarrow x = p - 1, p + 1$

$p - 1 \in (-2, 4)$

$p \in (-1, 5)$

$p + 1 \in (-2, 4)$

$p \in (-3, 3)$

$p \in (-1, 3)$

Q.7 Number of solution (s) satisfying the equation,
$3x^2 - 2x^3 = \log_2 (x^2 + 1) - \log_2 x$ is

(A) 1 (B) 2

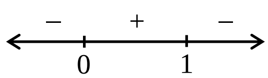
(C) 3 (D) none

Sol. **[A]**

$f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$= 6x (1 - x)$

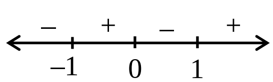


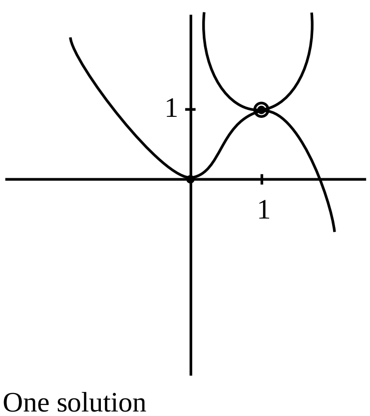
Domain $x > 0$

$g(x) = \log_2(x^2 + 1) - \log_2 x$

$$g'(x) = \frac{1}{\lambda n2} \times \frac{2x}{(x^2+1)} - \frac{1}{x\,\lambda n2}$$

$$= \frac{1}{\lambda n2} \times \frac{x^2 - 1}{x(x^2+1)}$$





One solution

Q.8 The equation of the line through (3, 4) which cuts the first quadrant a triangle of minimum area is

(A) $4x + 3y - 24 = 0$ (B) $3x + 4y - 12 = 0$

(C) $2x + 4y - 12 = 0$ (D) $3x + 2y - 24 = 0$

Sol. **[A]**

$y - 4 = m (x - 3)$

x-intercept $= 3 - \frac{4}{m}$

y-intercept $= 4 - 3m$

$A = \frac{1}{2} (4 - 3m) (3 - \frac{4}{m})$

$A = \frac{1}{2} (24 - 9m - \frac{16}{m})$

for A_{min} $9m = \frac{16}{m}$

$m = -\frac{4}{3}$

$\Rightarrow 4x + 3y - 24 = 0$

Q.9 The minimum value of $a \tan^2 x + b \cot^2 x$ equals the maximum value of $a \sin^2\theta + b \cos^2\theta$ where $a > b > 0$, when

(A) a = b (B) a = 2b

(C) a = 3b (D) a = 4b

Sol. **[D]**

$f(x) = a \tan^2\theta + b \cot^2\theta$

$AM \geq GM$

$f(x) \geq 2\sqrt{ab}$

$g(x) = a \sin^2\theta + b \cos^2\theta$

$= (a - b) \sin^2\theta + b$

$g(x)_{max} = a$

Now,

$a = 2\sqrt{ab}$

$\Rightarrow$ $a^2 = 4ab$

$\Rightarrow$ $a(a - 4b) = 0$

$a = 4b$

Q.10 Let h be a twice continuously differentiable positive function on an open interval J. Let $g(x) = \ln(h(x))$ for each $x \in J$.
Suppose $(h'(x))^2 > h''(x)\, h(x)$ for each $x \in J$. Then

(A) g is increasing on J

(B) g is decreasing on J

(C) g is concave up on J

(D) g is concave down on J

Sol. **[D]**

$g'(x) = \dfrac{h'(x)}{h(x)}$

$g''(x) = \dfrac{h(x)h''(x) - (h'(x))^2}{(h(x))^2}$ = negative

g(x) concave down in J

Q.11 If a differentiable function f(x) has a local minimum at x = 0, then the function y = f(x) + ax + b has a local minimum at x = 0 for

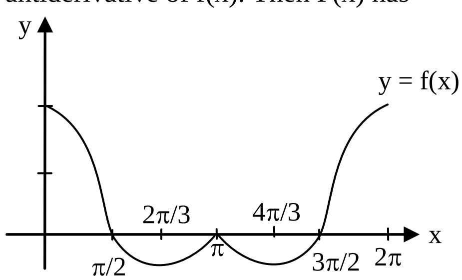
(A) all a > 0 (B) all b > 0

(C) all a and b (D) all b if a = 0

Sol. **[D]**

$y' = f'(x) + a$

Q.12 The graph of y = f(x) is shown. Let F(x) be an antiderivative of f(x). Then F(x) has



(A) points of inflexion at $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and 2π, a local maximum at $x = \frac{\pi}{2}$, and a local minimum at $x = \frac{3\pi}{2}$

(B) points of inflexion at $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and 2π, a local minimum at $x = \frac{\pi}{2}$ and a local maximum at $x = \frac{3\pi}{2}$

(C) point of inflexion at $x = \pi$, a local maximum at $x = \frac{\pi}{2}$, and a local minimum at $x = \frac{3\pi}{2}$

(D) point of inflexion at $x = \pi$, a local minimum at $x = \frac{\pi}{2}$, and a local maximum at $x = \frac{3\pi}{2}$

Sol. **[C]**

Clearly from the graph

Q.13 The value of the real number 'a' having the property f(a) = a, is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$, is

(A) 1 (B) 2 (C) 3 (D) – 1

Sol. **[A]**

$f'(a) = 4a^3 - 3a^2 - 2a + a = 0$

$\Rightarrow a = 0, 1, -\frac{1}{4}$

$f''(a) = 12a^2 - 6a - 1$

$f''(0) = -1, f''(1) = +ve, f''\left(-\frac{1}{4}\right) = +ve$

a = 0 not possible for minima

$f(a) = a \Rightarrow a = 1$

Q.14 For $a \in [\pi, 2\pi]$ and $n \in Z$, the critical points of

$f(x) = \frac{1}{3}\sin a \tan^3 x + (\sin a - 1)\tan x + \sqrt{\frac{a-2}{8-a}}$

are

(A) $x = n\pi$ (B) $x = 2n\pi$

(C) $x = (2n + 1)\pi$ (D) none of these

Sol. **[D]**

$f'(x) = \sin a \tan^2 x \sec^2 x + (\sin a - 1)\sec^2 x$

$= \sin a \sec^4 x - \sec^2 x$

$= \sec^2 x(\sin a - \sec^2 x)$

Q.15 For $0 < a \le 1$ and $b \in R$, then in (–a, a) the function, $f(x) = ax^3 - 3ax + b$

(A) has exactly 2 roots

(B) can not have a root

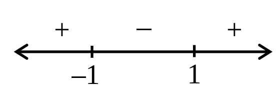
(C) has atmost one root

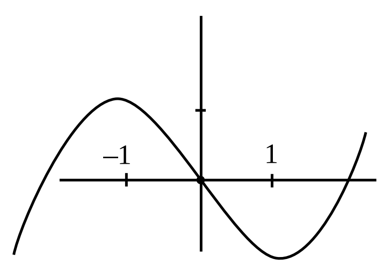
(D) more than two roots

Sol. **[C]**

$f'(x) = 3ax^2 - 3a$

$= 3a(x + 1)(x - 1)$





Part-B	**One or more than one correct answer type questions**

Q.16 The function $f(x) = \dfrac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if

(A) $b - a = n\pi, n \in I$

(B) $b - a = (2n+1)\pi, n \in I$

(C) $b - a = 2n\pi, n \in I$

(D) none of these

Sol. **[A, B, C]**

$$f'(x) = \frac{\sin(x+b)\cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)} = 0 \Rightarrow b - a = n\pi$$

Q.17 The coordinates of the point P on the graph of the function $y = e^{-|x|}$ where the portion of the tangent intercepted between the coordinate axis has the greatest area, is

(A) $\left(1, \frac{1}{e}\right)$ (B) $\left(-1, \frac{1}{e}\right)$

(C) (e, e^{-e}) (D) none of these

Sol. **[A, B]**

$y = e^{-x}$

$y' = e^{-x_1}$

tangent $y - e^{-x_1} = -e^{-x_1}(x - x_1)$

Area A = $\frac{1}{2}e^{-x_1}(x_1+1)^2 \Rightarrow A_{max} \Rightarrow x_1 = 1$

P $\equiv \left(1, \frac{1}{e}\right)$ curve is symmetric so other point $\left(-1, \frac{1}{e}\right)$ will also

Q.18 The parabola $y = x^2 + px + q$ cuts the straight line $y = 2x - 3$ at a point with abscissa 1. If the distance between the vertex of the parabola and the x-axis is least then

(A) $p = 0$ & $q = -2$

(B) $p = -2$ & $q = 0$

(C) least distance between the parabola and x-axis is 2

(D) least distance between the parabola and x-axis is 1

Sol. **[B]**

(1, –1)

$\Rightarrow -1 = 1 + p + q$

$p + q = -2$ …..(1)

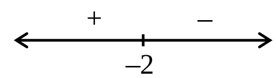
Now.

Dist. = $\dfrac{p^2 - 4q}{-4}$

$= q - \dfrac{p^2}{4}$

$= -2 - p - \dfrac{p^2}{4}$

$\dfrac{d(\text{dist.})}{dp} = -1 - \dfrac{2}{4} = -\dfrac{1}{2}(2+p)$



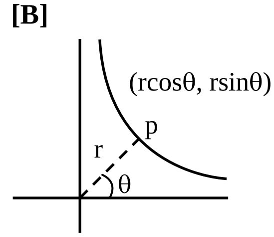
dist. max at p = – 2

Q.19 The coordinates of the points on the curve, $5x^2 - 6xy + 5y^2 = 4$ which are the nearest to the origin are

(A) $\left(0, \frac{2}{\sqrt{5}}\right), \left(0, \frac{-2}{\sqrt{5}}\right)$ (B) $\left(\frac{1}{2}, \frac{-1}{2}\right), \left(\frac{-1}{2}, \frac{1}{2}\right)$

(C) $\left(\frac{2}{\sqrt{5}}, 0\right), \left(-\frac{2}{\sqrt{5}}, 0\right)$ (D) None of these

Sol. **[B]**



$5r^2 - 6r^2 \sin\theta \cos\theta = 4$

$r^2 = \dfrac{4}{5 - 3\sin 2\theta}$

$r^2_{min} = \dfrac{4}{8} = \dfrac{1}{2}$

$r_{min} = \dfrac{1}{\sqrt{2}}$

when $2\theta = \dfrac{-\pi}{2}, \dfrac{3\pi}{2}$

$\theta = \dfrac{-\pi}{4}, \dfrac{3\pi}{4}$

Q.20 A particle is moving in a straight line such that its distance at any time t is given by

$x = \dfrac{t^4}{4} - 2t^3 + 4t^2 + 7$. Then

(A) velocity is max. at $t = (6 - 2\sqrt{3})/3$
(B) acceleration is min. at t = 2
(C) min. distance is at t = 0, 4
(D) None of these

Sol. **[A, B, C]**

$v = t^3 - 6t^2 + 8t$

$a = 3t^2 - 12t + 8$

v_{max} at $t = \dfrac{12 - \sqrt{48}}{6} = \dfrac{6 - 2\sqrt{3}}{3}$

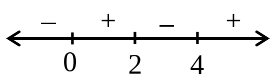
$a' = 6t - 12$

a_{min} at t = 2

v = 0

$\Rightarrow t(t^2 - 6t + 8) = 0$

$\Rightarrow t(t - 2)(t - 4) = 0$



d_{min} at x = 0, 4

Q.21 Let $f(x) = \log(2x - x^2) + \sin \pi x/2$. Then -
(A) graph of f is symmetrical about the line x =1
(B) graph of f is symmetrical about the line x = 2
(C) maximum value of f is 1
(D) minimum value of f does not exist

Sol. **[A, C, D]**

$2x - x^2$ in (0, 2)

is symmetric about x = 1 and also $\sin\dfrac{\pi x}{2}$

so (A) is true also for both at x = 1 point of maxima so C is true.

Q.22 Let $f(x) = \begin{cases} \tan^{-1} x, & |x| < 1 \\ 0, & |x| = 1 \\ 1 - |x|, & |x| > 1 \end{cases}$ then -

(A) f(x) has no point of local minimum

(B) f(x) has one point of local maximum

(C) f(x) has two points of local maximum

(D) f(x) has one point of local minimum

Sol. **[A, B]**

$$f'(x) = \begin{cases} \dfrac{1}{1+x^2} & ; \quad -1 < x < 1 \\ = 0 & ; \quad x = \pm 1 \\ = -1 & ; \quad x > 1 \\ = 1 & ; \quad x < -1 \end{cases}$$

Q.23 If $f(x) = \sin^3 x + \lambda \sin^2 x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ then-

(A) f(x) has a point of inflexion if $\lambda = 0$

(B) f(x) has exactly one point of maximum & exactly one point of minimum if $|\lambda| < 3/2$

(C) f(x) has exactly one point of maximum and exactly one point of minimum if $\lambda \in (-3/2, 0) \cup (0, 3/2)$

(D) all above

Sol. **[A, C]**

$f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x$

$= \sin x \cos x (3 \sin x + 2\lambda)$

Q.24 Let $f(x) = \begin{cases} x^3(1-x), & x \le 0 \\ x \log x + 3x, & x > 0 \end{cases}$ then -

(A) there is no critical point

(B) f is continuous at x = 0

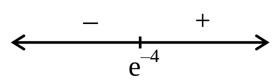
(C) $x = e^{-4}$ is a point of minimum

(D) $f'(x)$ is continuous at x = 0

Sol. **[B, C]**

$f'(x) = -x^3 + 3x^2(1 - x) - 4x^3 + 3x^2$; $x \le 0$

$4 + \ln x$; $x > 0$

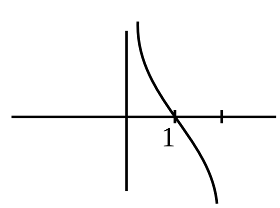


Q.25 For the function $f(x) = \ln(1 - \ln x)$ which of the following do not hold good ?
(A) increasing in (0, 1) and decreasing in (1, e)
(B) decreasing in (0, 1) and increasing in (1, e)
(C) x = 1 is the critical number for f(x)
(D) f has two asymptotes

Sol. **[A, B, C]**

$f'(x) = \dfrac{1}{1 - \ln x}\left(-\dfrac{1}{x}\right)$

$= \dfrac{1}{x(\ln x - 1)}$



Q.26 Let the function f(x) be defined as follows :

$$f(x) = \begin{cases} x^3 + x^2 - 10x & ; \quad -1 \le x < 0 \\ \cos x & ; \quad 0 \le x < \pi/2 \\ 1 + \sin x & ; \quad \pi/2 \le x \le \pi \end{cases}$$

Then which of the following statement(s) is/are correct

(A) Local maximum at x = 0

(B) Local maximum at x = π/2

(C) Absolute maxima at x = – 1

(D) Absolute minima at x = π

Sol. **[A, B, C]**

$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & ; -1 \le x < 0 \\ -\sin x & ; \quad 0 \le x < \frac{\pi}{2} \\ \cos x & ; \quad \frac{\pi}{2} \le x \le \pi \end{cases}$$

$f(0) = 1$, $f(0^-) = 0$

$f(0^+) < 1 \Rightarrow A$

$f(-1) = 10 \Rightarrow$ absolute maxima

Part-C Assertion-Reason type questions

The following questions 27 to 29 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.27 **Assertion** : $e^\pi > \pi^e$

Reason : The function $x^{1/x}$ (x > 0) has a local maximum at x = e.

Sol. **[A]**

To prove

(A) $e^{1/e} > \pi^{1/\pi}$

$f(x) = x^{1/x}$

has maxima at x = e

Q.28 **Assertion** : If a is positive rational number and b is irrational number then the maximum value of sin ax + sin bx cannot be 2.

Reason : The number $\frac{b}{a}$ is irrational.

Sol. **[A]**

sin bx + sin ax = 2

$\Rightarrow$ bx & ax both must be $2n\pi + \frac{\pi}{2}$ form which is possible

if both b & a are rational.

Q.29 **Assertion** : The local maximum of the function x cos x will occur between π/4 and π/3.

Reason : The function g(x) = x tan x is increasing in $\left(0, \frac{\pi}{2}\right)$

$1 - g\left(\frac{\pi}{4}\right) > 0$ and $1 - g\left(\frac{\pi}{3}\right) < 0$.

Sol. **[A]**

f(x) = x cos x

f '(x) = – x sin x + cos x

= cos x (– x tan x + 1)

$f'\left(\frac{\pi}{4}\right)$ +ve

$f'\left(\frac{\pi}{3}\right)$ –ve

Part-D Column Matching type questions

Q.30 Four points A, B, C and D lie in the order on the parabola $y = ax^2 + bx + c$ and the coordinates of A, B and D are known A(–2, 3); B(–1, 1); D(2, 7).

Column I	**Column II**
(A) The value of a + b + c =	(P) –1
(B) If roots of the equation $ax^2 + bx + c = 0$ are α & β then $\alpha^{19} + \beta^7 =$	(Q) 8
(C) Least value of $(a + 2)x^2 + \frac{(b+2)}{x^2} + c =$	(R) 3
(D) If area of quadrilateral ABCD is greatest and co-ordinates of C are (p, q) then 2p + 4q =	(S) 7

Sol. **[A → R, B → P, C → S, D → Q]**

(A) $4a - 2b + c = 3$

$a - b + c = 1$

$4a + 2b + c = 7$

$b = 1$

$a = 1, \quad c = 1$

$a + b + c = 3$

(B) $x^2 + x + 1 = 0 \begin{cases} \omega \\ \omega^2 \end{cases}$

$\omega^{19} + \omega^{14}$

$\Rightarrow \omega + \omega^2 = -1$

(C) $f(x) = 3x^2 + \dfrac{3}{x^2} + 1$

$= 3\left(x^2 + \dfrac{1}{x^2}\right) + 1$

$\geq 7 \qquad\qquad (AM \geq GM)$

(D) $A(-2, 3), D(2, 7)$

$B(-1, 1), C(p, q)$

$$\begin{vmatrix} -2 & 3 \\ -1 & 1 \\ p & q \\ 2 & 7 \\ -2 & 3 \end{vmatrix}$$

$\text{Area} = \dfrac{1}{2}\left| -2 - q + 7p + 6 + 3 - p - 2q + 14 \right|$

$A = \dfrac{1}{2}\left| 6p - 3q + 21 \right|$

$q = p^2 + p + 1$

$\dfrac{dA}{dp} = 0 \Rightarrow p = \dfrac{1}{2}, q = \dfrac{7}{4}$

$2p + 4q = 8$

Q.31 For the function $f(x) = x^4 (12 \lambda nx - 7)$, match the following

Column I	**Column II**
(A) If (a, b) is the point of inflection then $a - b$ is equal to	(P) 3
(B) If e^t is point of minima then 12t is equal to	(Q) 1
(C) If graph is concave downward in (d, e) then d + 3e is equal to	(R) 4
(D) If graph is concave upward in (p, ∞), then p is equal to	(S) 8

Sol. **[A → S, B → R, C → P, D → Q]**

(A) $f'(x) = x^4\left(\dfrac{12}{x}\right) + (12\lambda nx - 7)(4x^3)$

$f'(x) = -16x^3 + 48x^3 \lambda nx$

$f''(x) = -48x^2 + 48x^2 + 48 \lambda n x \cdot 3x^2$

$x = 1, y = -7$

$a - b = 8$

(B) $f'(x) = 16x^3 (3\lambda nx - 1)$

$\qquad + \qquad - \qquad +$

$\longleftarrow\!\!\!-\!\!-\!\!\!|_{0}\!\!-\!\!-\!\!-\!\!-\!\!|_{e^{1/3}}\!\!-\!\!\!\longrightarrow$

$12 \times \dfrac{1}{3} = 4$

(C) $f''(x) = 144x^2 \lambda nx$

concave ↓ in (0, 1)

$d + 3e = 3$

(D) concave ↑ in $(1, \infty)$

$p = 1$

Q.32 Let $f(x) = (x - 1)^m (2 - x)^n$; $m, n \in N$ and $m, n > 2$

Column I	**Column II**
(A) Both x = 1 and x = 2 are the points of minima if	(P) m is even
(B) x = 1 is a point of minima and x = 2 is a point of inflection if	(Q) m is odd
(C) x = 2 is a point of minima and x = 1 is a point of inflection if	(R) n is even
(D) Both x = 1 and x = 2 are the points of inflection if	(S) n is odd
	(T) m and n both are even

Sol. **[A → P,R,T, B → P,S, C → R,Q, D → Q,S]**

at point of extrema f ' (x)

has odd power factor

i.e. f(x) has even power factor.

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Let f(x) = $\begin{cases} x^2 + 2x + \dfrac{a^2-1}{a^4-5}; x < 0 \\ x + 2\ ;\ x \geq 0 \end{cases}$

Find all possible real values of a such that f(x) possesses the smallest value at x = 0.

Sol. f(x) = $\begin{cases} x^2 + 2x + \dfrac{a^2-1}{a^4-5}; x < 0 \\ x + 2\ ;\ x \geq 0 \end{cases}$

$f'(x) = 2x + 2 + 0 \Rightarrow$ put $f'(x) = 0$

$f''(x) = 2 \Rightarrow x = -1$

For minima, $f(a) \leq f(a + h) \Rightarrow f(0) \leq f(0 + h)$

$f(a) \leq f(a - h) \Rightarrow f(0) \leq f(0 - h)$

$\Rightarrow f(0 + h) \geq f(0 - h)$

$(-1)^2 + 2(-1) + \dfrac{a^2-1}{a^4-5} \leq 0 + 2;\ a^4 - 5 \neq 0$

$1 - 2 + \dfrac{a^2-1}{a^4-5} \leq 2 \quad ;\ a^4 - 5 \neq 0$

$\dfrac{a^2-1}{a^4-5} \leq 3 \quad ;\ a^4 - 5 \neq 0$

$\Rightarrow 3a^4 - 15 \geq a^2 - 1;\ a^4 - 5 \neq 0$

$\Rightarrow 3a^4 - a^2 - 14 \geq 0;\ a^4 - 5 \neq 0$

Now, we have to find roots of $3a^4 - a^2 - 14 = 0$

$3a^4 - a^2 - 14 = 0 \Rightarrow a^2 = \dfrac{1 \pm \sqrt{1+168}}{2\times 3}$

$a^2 = \dfrac{1 \pm \sqrt{169}}{2\times 3}$

$a^2 = \dfrac{1 \pm 13}{2\times 3} = \dfrac{7}{3}$

$a = \pm\sqrt{\dfrac{7}{3}}$

$\Rightarrow \left(a + \sqrt{\dfrac{7}{3}}\right)\left(a - \sqrt{\dfrac{7}{3}}\right) \leq 0$

$a \in \left[-\sqrt{\dfrac{7}{3}} - 5\dfrac{1}{4}\right) \cup \left(5\dfrac{1}{4}, \sqrt{\dfrac{7}{3}}\right]$

Q.2 Find all the values of k for which the point of minimum of the function $f(x) = 1 + k^2x - x^3$ satisfy the inequality $\dfrac{x^2+x+2}{x^2+5x+6} < 0$.

Sol. $f(x) = 1 + k^2x - x^3$

Differentiating w.r.t. x, we get

$f'(x) = 0 + k^2 = 3x^2$

Put $f'(x) = 0 \Rightarrow k^2 - 3x^2$

$\Rightarrow |x| = \dfrac{k}{\sqrt{3}}$

$\Rightarrow x = \pm\dfrac{k}{\sqrt{3}}$

$\dfrac{(x^2+x+2)}{(x^2+5x+6)} < 0$

$\Rightarrow \dfrac{x^2 + x + \frac{1}{4} - \frac{1}{4} + 2}{(x^2+5x+6)} < 0$

$\Rightarrow \dfrac{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}}{(x^2+5x+6)} < 0$

$\left(x + \dfrac{1}{2}\right)^2 + \dfrac{7}{4}$ always positive.

Hence, $(x^2 + 5x + 6) < 0$

$\Rightarrow (x + 2)(x + 3) < 0$

$\Rightarrow -3 < x < -2$

Put $x = +\dfrac{k}{\sqrt{3}}$ in above inequality, we get

$\Rightarrow -3 < \dfrac{k}{\sqrt{3}} < -2$

$\Rightarrow -3\sqrt{3} < k < -\dfrac{2}{\sqrt{3}}$

$\Rightarrow k \in \left(-\dfrac{3}{\sqrt{3}}, -\dfrac{2}{\sqrt{3}}\right)$

Put $x = -\dfrac{k}{\sqrt{3}}$ involves inequality, we get

$\Rightarrow -3 < -\dfrac{k}{\sqrt{3}} < -2$

$\Rightarrow 2 < \dfrac{k}{\sqrt{3}} < 3$

$\Rightarrow 2\sqrt{3} < k < 3\sqrt{3}$

$\Rightarrow k \in (2\sqrt{3}, 3\sqrt{3})$

Hence, $k \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$ which is required.

Q.3 If $f(x) = x^3 + ax^2 + bx + c$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. Determine the constants a, b, c.

Sol. $f(x) = x^3 + ax^2 + bx + c$

Differentiating above function w.r.t = x, we get

$f'(x) = 3x^2 + 2ax + b + 0$

$f'(x) = 3x^2 + 2ax + b$

$f'(x)|_{x=-1} = 3 - 2a + b = 0$

$\Rightarrow 2a - b = 3$

$f'(x)|_{x=3} = 27 + 6a + b = 0$

$\Rightarrow 6a + b = -27$

Solving above two equations for a and b.

$2a - b = 3$

$6a + b = -27 \quad \Rightarrow a = -3$

$8a + 0 = -24 \quad b = 2a - 3 = -6 - 3$

$b = -a$

Hence, $a = -3$

$b = -9$ and $c \in R$

Q.4 Show that $\left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$ has just one maximum and one minimum value. Show also that the difference between them is $\frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$. What is the least value of this difference.

Sol. Let $f(x) = \left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$

Differentiating w.r.t. x, we get

$f'(x) = (-1)(4 - 3x^2) + \left(\alpha - \frac{1}{\alpha} - x\right)(0 - 6x)$

$= (3x^2 - 4) + 6x\left[x - \left(\alpha - \frac{1}{\alpha}\right)\right]$

$= 3x^2 - 4 + 6x^2 - 6x\left(\alpha - \frac{1}{\alpha}\right) - 4 \quad \ldots \text{(i)}$

Again differentiating w.r.t. x, we get

$f''(x) = 18x - 6\left(\alpha - \frac{1}{\alpha}\right) - 0$

$f''(x) = 18x - 6\left(\alpha - \frac{1}{\alpha}\right) \quad \ldots\text{(ii)}$

from equation (1), put $f'(x) = 0$

$9x^2 - 6x\left(\alpha - \frac{1}{\alpha}\right) - 4 = 0$

$$\Rightarrow x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) \pm \sqrt{36\left(\alpha - \frac{1}{\alpha}\right)^2 + 16 \times 9}}{2 \times 9}$$

$$x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) \pm 6\sqrt{\left(\alpha - \frac{1}{\alpha}\right)^2 + 4}}{2 \times 9}$$

$$x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) \pm 6\left(\alpha - \frac{1}{\alpha}\right)^2}{18}$$

$$\Rightarrow x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) \pm 6\left(\alpha + \frac{1}{\alpha}\right)^2}{18}$$

Taking + ve sign

$$x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) + 6\left(\alpha + \frac{1}{\alpha}\right)}{18}$$

$$x = \frac{6\alpha - \frac{6}{\alpha} + 6\alpha + \frac{6}{\alpha}}{18}$$

$x = \frac{12}{18}\alpha = \frac{2}{3} - \alpha$

Taking – ve sign

$$x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) - 6\left(\alpha + \frac{1}{\alpha}\right)}{18}$$

$$x = \frac{6\alpha - \frac{6}{\alpha} - 6\alpha + \frac{6}{\alpha}}{18}$$

$x = \frac{-12}{18} \times \frac{1}{\alpha} = \frac{-2}{3} \times \frac{1}{\alpha}$

Hence, $x = \frac{-2}{3}\frac{1}{\alpha}, \frac{2}{3}\alpha$

From equation (2),

$f''(x) = 18x - 6\left(\alpha - \frac{1}{\alpha}\right)$

$f''(x)|_{x=\frac{-2}{3}=\frac{1}{\alpha}} = 18 \times \left(\frac{-2}{3}\alpha\right) - 6\alpha + \frac{6}{\alpha}$

$= -\frac{12}{\alpha} - 6\alpha + \frac{6}{\alpha}$

$= -6\alpha - \frac{6}{\alpha} = -\text{ve}$

Hence, at $x = \frac{-2}{3}\frac{1}{\alpha}$, f(x) would be maximum.

$$f''(x)|_{x=\frac{2}{3}\alpha} = 18 \times \frac{2}{3}\alpha - 6\left(\alpha - \frac{1}{\alpha}\right)$$

$$= 12\alpha - 6\alpha + \frac{6}{\alpha}$$

$$= 6\alpha + \frac{6}{\alpha}$$

$$= 6\left(\alpha - \frac{1}{\alpha}\right)$$

= +ve

Hence, at $x = \frac{2}{3}\alpha$, f(x) will be minimum

$$\therefore\ f(x)|_{x=\frac{2}{3}\alpha} = \left(\alpha - \frac{1}{\alpha} - x\right)(4-3x^2)\Big|_{x=\frac{2}{3}\alpha}$$

$$= \left(\alpha - \frac{1}{\alpha} - \frac{2}{3} = \alpha\right)(4 - 3x\frac{4}{9}\alpha^2)$$

$$f_{min} = \left(\frac{\alpha}{3} - \frac{1}{\alpha}\right)\left(4 - \frac{4}{3}\alpha^2\right)$$

$$f(x)|_{x=-\frac{2}{3}\times\frac{1}{\alpha}} = f_{max} = \left(\alpha - \frac{1}{\alpha} + \frac{2}{3\alpha}\right)$$

$$\left(4 - 3\times\frac{4}{9} - \frac{1}{\alpha^2}\right)$$

$$= \left(\alpha - \frac{1}{3\alpha}\right)\left(4 - \frac{12}{9} - \frac{1}{\alpha^2}\right)$$

$$= 4\left(\alpha - \frac{1}{3\alpha}\right)\left(1 - \frac{1}{3} - \frac{1}{\alpha^2}\right)$$

Hence, $f_{max} - f_{min}$ =

$$4\left(\alpha - \frac{1}{3\alpha}\right)\left(1 - \frac{1}{3\alpha^2}\right) - 4\left(\frac{\alpha}{3} - \frac{1}{\alpha}\right)\left(1 - \frac{\alpha^2}{3}\right)$$

$$= 4\left[\left(\alpha - \frac{1}{3\alpha} - \frac{1}{3\alpha} + \frac{1}{9\alpha^3}\right) - \left(\frac{2}{3} - \frac{1}{\alpha} - \frac{\alpha^3}{9} + \frac{\alpha}{3}\right)\right]$$

$$= 4\left[\alpha - \frac{2}{3\alpha} + \frac{1}{9\alpha^3} - \frac{2\alpha}{3} + \frac{1}{2} + \frac{\alpha^3}{9}\right]$$

$$= 4\left[\frac{\alpha^3}{9} + \frac{1}{9\alpha^3} + \frac{\alpha}{3} + \frac{1}{3\alpha}\right]$$

$$= \frac{4}{9}\left[\alpha^3 + \frac{1}{\alpha^3} + 3\alpha + \frac{3}{\alpha}\right] = \frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$$

$$\Rightarrow f_{max} - f_{min} = \frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$$. proved.

Let $f(\alpha) = \frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$

Differentiating $f(\alpha)$ w.r.t.= α, we get

$$f'(\alpha) = \frac{4}{9} \times 3\left(\alpha + \frac{1}{\alpha}\right)^2 \cdot \left(\alpha - \frac{1}{\alpha^2}\right)$$

Again differentiating w.r.t α_1, we get

$$f''(\alpha) = \frac{4}{3} \times 2 \times \left(\alpha + \frac{1}{\alpha}\right)\left(1 - \frac{1}{\alpha^2}\right)\left(1 - \frac{1}{\alpha^2}\right)$$

$$+ \frac{4}{3} \times \left(\alpha + \frac{1}{\alpha^2}\right)^2 \times \left(\frac{2}{\alpha^3}\right)$$

$$f''(\alpha) = \frac{8}{3}\times\left(\alpha + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\alpha^2}\right)^2 + \frac{8}{3}\times$$

$$\left(\alpha + \frac{1}{\alpha^2}\right)^2 \times \frac{1}{\alpha^3}$$

Put $f'(\alpha) = 0$

$$\Rightarrow \frac{4}{9} \times 3 \times \left(\alpha + \frac{1}{\alpha}\right)^2\left(1 - \frac{1}{\alpha^2}\right)^2 \times \frac{1}{\alpha^3}$$

Put $f'(\alpha) = 0$

$$\Rightarrow \frac{4}{9} \times 3 \times \left(\alpha + \frac{1}{\alpha}\right)^2\left(1 - \frac{1}{\alpha^2}\right)^2 = 0$$

$\Rightarrow \alpha = \pm 1$

At $\alpha = +1$, $f''(\alpha) = \frac{8}{3} \times 2 \times (0) + \frac{8}{3} \times 4 \times \frac{1}{1}$

$= \frac{32}{3} > 0$

At $\alpha = -1$, $f''(\alpha) = \frac{8}{3} \times 2 \times (0) + \frac{8}{3} \times 4 \times \frac{1}{1}$

$= \frac{32}{3} > 0$

At $\alpha = -1$, $f''(\alpha) = 0 + 0$

Hence, $f(\alpha)$ would be least at $\alpha = +1$

Least value of $f(\alpha) = \frac{4}{9}\left(1 + \frac{1}{1}\right)^3 = \frac{32}{9}$

Q.5 If b > a find the minimum value of
$f(x) = |(x-a)^3| + |(x-b)^3|$, $x \in R$.

Sol. b > a

$f(x) = |(x-a)^3| + |(x-b)^3|$, $x \in R$.

$$\underset{a}{|}\qquad\qquad \underset{b}{|}$$

when $x < a$,

$f(x) = -(x-a)^3 - (x-b)^3$

Differentiating w.r.t.x, we get

$f'(x) = -3(x-a)^2 - 3(x-b)^2$

Again differentiating w.r.t. x, we get

$f''(x) = -6(x-a) - 6(x-b)$.

Now, put $f'(x) = 0$

$\Rightarrow -3(x-a)^2 - 3(x-b)^2 = 0$

$\Rightarrow (x-a)^2 = -(x-b)^2$

which is not possible due to negative sign.

when $a < x < b$, then

$f(x) = (x-a)^3 - (x-b)^3$

Differentiating w.r.t. x, we get

$f'(x) = 3(x-a)^2 - 3(x-b)^2$

Again differentiating w.r.t.x, we get

$f''(x) = 6(x-a) - 6(x-b)$

Now put, $f'(x) = 0$

$\Rightarrow 3(x-a)^2 - 3(x-b)^2 = 0$

$\Rightarrow (x-a)^2 = (x-b)^2$

$\Rightarrow (x-a) = \pm(x-b)$

$\Rightarrow x - a = -(x-b)$

$\Rightarrow x - a = -x + b$

$\Rightarrow 2x = a + b$

$\Rightarrow x = \dfrac{a+b}{2}$

$f''(x) = 6(x-a) - 6(x-b)$

$f''(x) = 6x - 6a - 6x + 6b$

$f''(x) = 6(b-a)$

$= +ve$

Hence f(x) would be minimum

$$f(x)\Big|_{x=\left(\frac{a+b}{2}\right)} = [(x-a)^3 - (x-b)^3]_{x=\left(\frac{a+b}{2}\right)}$$

$$= \left(\frac{a+b}{2} - a\right)^3 - \left(\frac{a+b}{2} - b\right)^3$$

$$= \left(\frac{a+b-2a}{2}\right)^3 - \left(\frac{a+b-2b}{2}\right)^3$$

$$= \left(\frac{b-a}{2}\right)^3 - \left(\frac{a+b-2b}{2}\right)^3$$

$$= \left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3 = \left(\frac{b-a}{2}\right)^3 + \left(\frac{b-a}{2}\right)^3$$

$= (b-a)^3/4$

$f(x)|_{min} = (b-a)^3/4$

When, $x > b$

$f(x) = (x-a)^3 + (x-b)^3$

Differentiating w.r.t.x, we get

$f'(x) = 3(x-a)^2 + 3(x-b)^2$

Again differentiating w.r.t.x, we get

$f''(x) = 6(x-a) + 6(x-b)$

Now, put $f'(x) = 0$

$\Rightarrow 3(x-a)^2 + 3(x-b)^2 = 0$

$\Rightarrow (x-a)^2 = -(x-b)^2$

which is not possible.

Q.6 Find the range of the function,

$f(x) = \lambda n((\cos x)^{\cos x} + 1)$, $x \in \left(0, \dfrac{\pi}{2}\right)$

Sol. $g(x) = (\cos x)^{\cos x} + 1$

$g'(x) = (\cos x)^{\cos x}$.

$$\left[\frac{\cos x}{\cos x}(-\sin x) + \lambda n \cos x.(-\sin x)\right]$$

$= -\sin x \,.\, (\cos x)^{\cos x}(1 + \lambda n \cos x)$

$$\overset{+}{\longleftarrow}\!\!\underset{\cos^{-1}\frac{1}{e}}{|}\!\!\overset{-}{\longrightarrow}$$

$g(0) = 2$

$$g\left(\cos^{-1}\frac{1}{e}\right) = 1 + \left(\frac{1}{e}\right)^{1/e}$$

Range

$$f(x)\left[\lambda n\left(1 + \left(\frac{1}{e}\right)^{1/e}\right), \lambda n\, 2\right]$$

Q.7 A function y = f(x) is given by the parametric equations $x = t^5 - 5t^3 - 20t + 7$ and $y = 4t^3 - 3t^2 - 18t + 3$, where $-2 < t < 2$. Investigate the maxima and minima.

Sol. $\dfrac{dy}{dx} = \dfrac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20}$

$= \dfrac{6(2t^2 - t - 3)}{5(t^4 - 3t^2 - 4)}$

$= \dfrac{6(2t^2 - 3t + 2t - 3)}{5(t^4 - 4t^2 + t^2 - 4)}$

$= \dfrac{6(t+1)\,(2t-3)}{5(t^2+1)(t+2)(t-2)}$

+   –   +   –   +

–2   –1   3/2   2

Min. at $t = -1$

Max. at $t = \dfrac{3}{2}$

Q.8 Find the point of local maxima/minima of following functions

(a) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(b) $f(x) = -(x-1)^3 (x+1)^2$

(c) $f(x) = x\lambda nx$

Sol. (a) $f'(x) = 6x^3 - 42x + 36$

$= 6(x^2 - 7x + 6)$

$= 6(x-1)(x-6)$

+   –   +

1   6

Min. x = 6

Max. x = 1

(b) $f'(x) = -(x-1)^3 = (x+1)$

$-(x+1)^2\, 3(x-1)^2$

$= -(x+1)(x-1)^2(2x - 2 + 3x + 3)$

$= -(x+1)(x-1)^2(5x+1)$

–   +   –   –

–1   –1/5   1

Min. x = –1

Max. $x = -\dfrac{1}{5}$

(c) $f'(x) = 1 + \lambda n\, x$

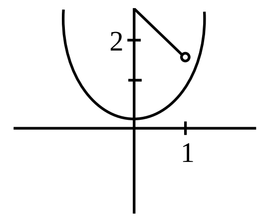
–   +

$\dfrac{1}{e}$

Min. $x = \dfrac{1}{e}$

Q.9 Let $f(x) = \begin{cases} 3 - x & 0 \le x < 1 \\ x^2 + \lambda nb & x \ge 1 \end{cases}$. Find the set of value of b such that f(x) has a local minima at x = 1

Sol.



$f(1) \le 2$

$\Rightarrow 1 + \lambda n\, b \le 2$

$0 < b \le e$

Q.10 Find the absolute maxima/minima value of following functions

(a) $f(x) = 4x - \dfrac{x^2}{2}$; $x \in \left[-2, \dfrac{9}{2}\right]$

(b) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

(c) $f(x) = \sin x + \dfrac{1}{2}\cos 2x$; $x \in \left[0, \dfrac{\pi}{2}\right]$

(d) $f(x) = x + \sqrt{x}$; $x \in (0, 4)$

(e) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

Sol. (a) $f'(x) = 4 - x$

+   –

4

Max. at x = 4

$f(4) = 16 - \dfrac{16}{2} = \dfrac{16}{2}$

$f(-2) = -8 - 2 = -10$

$f\left(\dfrac{9}{2}\right) = 18 - \dfrac{81}{8} = \dfrac{63}{8}$

f min. = – 10

f max. = 8

(b) $f'(x) = 12x^3 - 24x^2 + 24x - 48$

$= 12(x^3 - 2x^2 + 2x - 4)$

$= 12(x-2)(x^2+2)$

–   +

2

min. x = 2

$f(0) = 25$

$f(3) = 16$

$f(2) = -39$

(c) $f'(x) = \cos x - \sin 2x$

$= \cos x\,(1 - 2\sin x)$

$= -2\cos x\left(\sin x - \frac{1}{2}\right)$

$$\overset{+}{\longleftarrow}\underset{\pi/6}{|}\overset{-}{\longrightarrow}$$

max. at $x = \frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$f(0) = \frac{1}{2}$

$f\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

(d) $f'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$

$f(x)\uparrow$

$$\left[\begin{array}{l} f(x)_{min} = f(0) \\ \qquad\quad = 0 \\ f(x)_{max} = f(4) \\ \qquad\quad = 4 + 2 = 6 \end{array}\right.$$

(e) $f'(x) = 16x^{1/3} - \frac{2x}{x^{2/3}}$

$= \frac{16}{x^{2/3}}\left(x - \frac{1}{8}\right)$

$$\overset{-}{\longleftarrow}\underset{1/8}{|}\overset{+}{\longrightarrow}$$

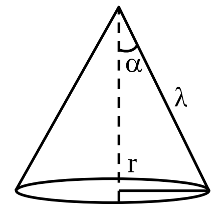
min. $x = \frac{1}{8}$

$f(1) = 6$

$f(-1) = 18$

Q.11 For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

Sol.



$s = \pi r\lambda$

$v = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2\sqrt{\lambda^2 - r^2}$

$= \frac{1}{3}\pi r^2\sqrt{\frac{s^2}{\pi r^2} - r^2}$

$= \frac{1}{3} r\sqrt{s^2 - \pi^2 r^4}$

$\frac{dv}{dr} = \frac{1}{3}\left(\sqrt{s^2 - \pi^2 r^4} + \frac{r(-4\pi^2 r^3)}{2\sqrt{s^2 - \pi^2 r^4}}\right)$

$= \frac{1(s^2 - \pi^2 r^4 - 2\pi^2 r^4)}{3\sqrt{s^2 - \pi^2 r^4}}$

$$\overset{+}{\longleftarrow}\underset{\sqrt{\frac{s}{\pi}\frac{1}{3}} = r}{|}\overset{-}{\longrightarrow}$$

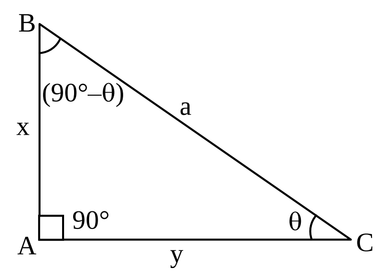
$\alpha = \sin^{-1}\left(\frac{r}{\lambda}\right)$

$s = 3^{1/2} r^2\pi = \pi r\lambda$

$\frac{r}{\lambda} = \frac{1}{\sqrt{3}}$

Q.12 Prove that the area of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Sol. $x^2 + y^2 = a^2$



a is given as hypotenuse

Area of triangle, $A = \frac{1}{2}x.\,y$

$A = \frac{1}{2} \times x \times \sqrt{a^2 - x^2}$

$A = \frac{1}{2} x \times \sqrt{a^2 - x^2}$

Differentiating above expression w.r.t. x, we get

$\frac{dA}{dx} = \frac{1}{2}(1) \times \sqrt{a^2 - x^2} + \frac{x}{2} \times \frac{1}{2\sqrt{a^2 - x^2}} \times (-2x)$

$\frac{dA}{dx} = \frac{\sqrt{a^2 - x^2}}{2} - \frac{x^2}{2\sqrt{a^2 - x^2}}$

$= \frac{1}{2}[a^2 - x^2 - x^2]/\sqrt{a^2 - x^2}$

$\frac{dA}{dx} = \frac{1}{2}\frac{(a^2 - 2x^2)}{\sqrt{a^2 - x^2}}$

Again differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{1}{2}\frac{(0-4x)\sqrt{a^2 - x^2} - (a^2 - 2x^2)\frac{1(-2x)}{2\sqrt{a^2 - x^2}}}{(a^2 - x^2)}$$

$$\frac{d^2A}{dx^2} = \frac{1}{2}\frac{(-4x)\sqrt{a^2 - x^2} + \frac{(a^2 - 2x^2)x}{\sqrt{a^2 - x^2}}}{(a^2 - x^2)}$$

$$= \frac{1}{2}\frac{(-4x)(a^2 - x^2) + (a^2 - 2x^2)x}{(a^2 - x^2)\sqrt{a^2 - x^2}}$$

$$\frac{d^2A}{dx^2} = \frac{1}{2}\frac{[-4xa^2 + 4x^3 + xa^2 - 2x^3]}{(a^2 - x^2)^{3/2}}$$

$\frac{d^2A}{dx^2} = \frac{1}{2}[2x^3 - 3xa^2]/(a^2 - x^2)^{3/2}$

Put $\frac{dA}{dx} = 0 \Rightarrow \frac{1}{2}\frac{(a^2 - 2x^2)}{\sqrt{a^2 - x^2}} = 0$; but $x \neq \pm a$

$\Rightarrow x^2 = \frac{a^2}{2}$

$\Rightarrow |x| = \frac{a}{\sqrt{2}}$

$$\left.\frac{d^2A}{dx^2}\right|_{|x| = \frac{a}{\sqrt{2}}} = \frac{1}{2}\frac{\left[2 \times \frac{a^3}{2\sqrt{2}} - 3 \times \frac{a}{\sqrt{2}} \times a^2\right]}{\left(a^2/2\right)^{3/2}}$$

$$= \frac{1}{2}\frac{\left[\frac{a^3}{\sqrt{2}} - \frac{3a^3}{\sqrt{2}}\right]}{\left(a^2/\sqrt{2}\right)^3}$$

$$= \frac{1}{2}\frac{\left[-\frac{2a^3}{\sqrt{2}}\right]}{a^3/2\sqrt{2}}$$

$$= -\frac{2a^3}{2\sqrt{2}} \times \frac{2\sqrt{2}}{a^3}$$

$= -2$

Hence, A would be maximum

from $y^2 = a^2 - x^2 = a^2 - a^2/2$

$= a^2/2$

$\Rightarrow |y| = a/\sqrt{2} \Rightarrow |x| = |y|$

$\Rightarrow \theta = 45°$

$\Rightarrow$ triangle will be isosceles.

$A_{max} = \frac{1}{2} \times x \times \sqrt{a^2 - x^2}$

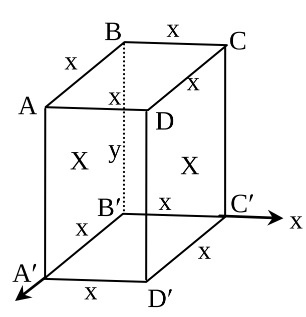
$= \frac{1}{2} \times \frac{a}{\sqrt{2}} \times \frac{a}{\sqrt{2}}$

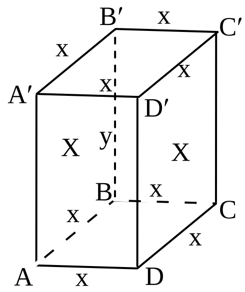
$= \frac{a^2}{4}$

$A_{max} = a^2/4$

Q.13 A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 Rs. the top is 25 Rs. and for the sides 20 Rs. The labour charges for making the box are Rs.3/-. Find the dimensions of the box when the cost is minimum.

Sol.





Let squares ABCD & A′B′C′D′ have dimensions of x & AA′ = BB′ = CC′ = DD′ = y foot

Material cost for bottom square = $15x^2$ paise

Material cost for top square = $25x^2$ paise

Material cost for side rectangle = (20xy) × 4

= 80xy paise

labour charges = Rs.3/- = 300 paise

Total cost, c = $(15x^2 + 25x^2 + 80xy + 300)$ paise

Also, $x^2 \times y = 1000\ (\text{feet})^3$

$$\Rightarrow y = \frac{1000}{x^2}$$

Put value of y in the expression of total cost,

$$c = 40x^2 + 80xy + 300$$

$$c = 40x^2 + 80x \times \frac{1000}{x^2} + 300$$

$$c = 40x^2 + \frac{80000}{x} + 300$$

Differentiating w.r.t. x, we get

$$\frac{dc}{dx} = 80x - \frac{80000}{xz} + 0$$

Again differentiating w.r.t x, we get

$$\frac{d^2c}{dx^2} = 80 + \frac{160000}{x^3}$$

Put $\frac{dc}{dx} = 0 \Rightarrow 800x = \frac{80000}{x^2}$

$$\Rightarrow x^3 = 1000$$

$$\Rightarrow x = 10 \text{ feet}$$

Also, $y = \frac{1000}{x^2} = \frac{1000}{100} = 10$

From, $\frac{d^2c}{dx^2} = 80 + \frac{160000}{1000}$

= 80 + 160 = 240 = +ve

Hence cost will be minimum

C_{min} = minimum cost

$= (40x^2 + 80xy + 300)\big|_{x=10}$ paise

= 40 × 100 + 80 × 100 + 300

= 4000 + 8000 + 300

= 12000 + 300

= 12300 paise

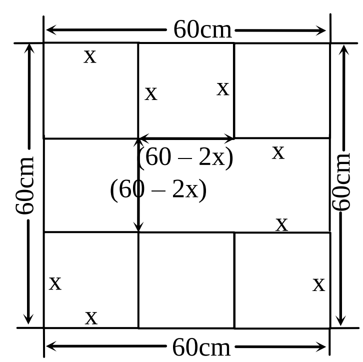
Dimension will be 10, 10, 10 feet respectively.

Q.14 A box is constructed from a square metal sheet of side 60 cm by cutting out identical squares from the four corners and turning up the sides. Find the length of the side of the square to be cut so that the box is of maximum volume.

Sol. Hence, volume of box = $x \times (60 - 2x)^2$

$$V = x \times (60 - 2x)^2$$

Differentiating w.r.t. x, we get



$$\frac{dv}{dx} = x \times 2x\,(60 - 2x)\,(-2) + 1.\,(60 - 2x)^2$$

$$\frac{dv}{dx} = (60 - 2x)\,[60 - 2x - 4x]$$

$$\frac{dv}{dx} = (60 - 2x)\,(60 - 6x)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2v}{dx^2} = (-2)\,(60 - 6x) + (60 - 2x)\,(-6)$$

$$\frac{d^2v}{dx^2} = -120 + 12x - 360 + 12x$$

$$\frac{d^2v}{dx^2} = 24x - 480$$

Put, $\frac{dv}{dx} = 0$

$\Rightarrow (60 - 2x)(60 - 6x) = 0$

$\Rightarrow x = 10, 30$

$$\left.\frac{d^2v}{dx^2}\right|_{x=10cm} = 240 - 480 = -240$$

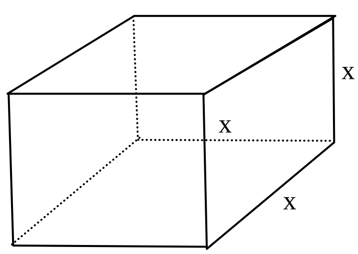
$$\left.\frac{d^2v}{dx^2}\right|_{x=30cm} = 24 \times 30 - 480 = 720 - 480 = +ve$$

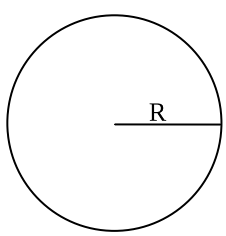
Hence, volume would be maximum when $x = 10$ cm

Q.15 Given the sum of surfaces of a cube and a sphere. Show that the edge of the cube is equal to the diameter of the sphere, if the sum of their volumes is minimum.

Sol. Surface of cube = $6x^2$

Volume of cube = x^3





Surface of sphere = $4\pi R^2$

Volume of sphere = $\frac{4}{3}\pi R^3$

Given, sum of surface of cube and sphere,

$6x^2 + 4\pi R^2 = 5$ …(i)

sum of volume = $x^3 + \frac{4}{3}\pi R^3$ ….(ii)

from (i), $x^2 = \left(\frac{S - 4\pi R^2}{6}\right) \Rightarrow x = \sqrt{\frac{S - 4\pi R^2}{6}}$

Put value of x in equation (ii), we get

$$V = \left(\frac{S - 4\pi R^2}{6}\right)^{3/2} + \frac{4}{3}\pi R^3$$

Differentiating w.r.t. R, we get

$$\frac{dV}{dR} = 4\pi R^2 + \frac{1}{6^{3/2}}\left(\frac{3}{2}\right)(S - 4\pi R^2)^{1/2}(-8\pi R)$$

$$= 4\pi R^2 + \frac{1}{6^{3/2}}(-12\pi R)(S - 4\pi R^2)^{1/2}$$

Again differentiating w.r.t. R, we get

$$\frac{d^2V}{dR^2} = 8\pi R + \frac{(-12\pi)}{6^{3/2}}$$

$$\left[1.(S - 4\pi R^2)^{1/2} + R\frac{1(-8\pi R)}{\sqrt{S - 4\pi R^2}}\right]$$

$$= 8\pi R - \frac{12\pi}{6^{3/2}}\left[(S - 4\pi R^2)^{1/2} - \frac{4\pi R^2}{\sqrt{S - 4\pi R^2}}\right]$$

Put, $\frac{dV}{dR} = 0$

$$\Rightarrow 4\pi R^2 + \frac{(-12\pi R)}{6^{3/2}}(S - 4\pi R^2)^{1/2} = 0$$

$$\Rightarrow 4\pi R^2 = \frac{12\pi R}{6^{3/2}}(S - 4\pi R^2)^{1/2}$$

$$\Rightarrow R = \frac{3}{6^{3/2}}(S - 4\pi R^2)^{1/2}$$

$\Rightarrow 6^{3/2}R = 3(S - 4\pi R^2)^{1/2}$

Squaring both sides, we get

$\Rightarrow 6^3R^2 = 9(S - 4\pi R^2)$

$\Rightarrow 36 \times 6 \times R^2 = 9(S - 4\pi R^2)$

$\Rightarrow 24 \times R^2 = (S - 4\pi R^2)$

$\Rightarrow 4R^2(6 + \pi) = S$

$\Rightarrow R^2 = S/4(\pi + 6)$

$\Rightarrow R = \sqrt{S/4(\pi+6)}$

$\Rightarrow 2R = D = \sqrt{S/(\pi+6)}$

Put value of R in $\frac{d^2v}{dR^2}$

$$\frac{d^2v}{dR^2} = 8\pi\sqrt{\frac{S}{4(\pi+6)}} - \frac{12\pi}{6^{3/2}}\left[\sqrt{S-\frac{4\pi\times s}{4(\pi+6)}} - \frac{4\pi\frac{S}{4(\pi+6)}}{\sqrt{S-4\pi\times\frac{S}{4(\pi+6)}}}\right]$$

$$= \frac{8\pi}{2}\sqrt{\frac{S}{\pi+6}} - \frac{12\pi}{6^{3/2}}\left[\sqrt{\frac{S}{4(\pi+6)}} - \frac{\frac{\pi S}{\pi+6}}{\sqrt{\frac{6S}{\pi+6}}}\right]$$

$$= \frac{8\pi}{2}\sqrt{\frac{S}{(\pi+6)}} - \frac{12\pi}{6^{3/2}}\left[\sqrt{\frac{6S}{4(\pi+6)}} - \frac{\pi}{\sqrt{6}}\times\sqrt{\frac{S}{\pi+6}}\right]$$

$$= 4\pi\sqrt{\frac{S}{(\pi+6)}} - \frac{12\pi}{6^{3/2}}\left[\frac{\sqrt{6}}{2}\sqrt{\frac{S}{\pi+6}} - \frac{\pi}{\sqrt{6}}\sqrt{\frac{S}{\pi+6}}\right]$$

$$= 4\pi\sqrt{\frac{S}{\pi+6}} - \frac{12\pi}{6^{3/2}}\sqrt{\frac{S}{(\pi+6)}}\left[\frac{6-2\pi}{2\sqrt{6}}\right]$$

$$= 4\pi\sqrt{\frac{S}{\pi+6}} - \frac{12\pi}{36\times 2}\sqrt{\frac{S}{(\pi+6)}}\ [6-2\pi]$$

$$\frac{d^2v}{dR^2} = \sqrt{\frac{S}{(\pi+6)}}\times 4\pi\left[1-\frac{6-2\pi}{24}\right]$$

$$= 4\pi\times\sqrt{\frac{S}{\pi+6}}\left[\frac{24-6+2\pi}{24}\right]$$

= +ve.

Hence, sum of their volumes will be minimum from

$6x^2 + 4\pi R^2 = S$

$$6x^2 + 4\pi\times\frac{S}{4(\pi+6)} = S$$

$$6x^2 = S - \frac{\pi S}{(\pi+6)}$$

$$6x^2 = \frac{S\pi+6S-\pi S}{(\pi+6)}$$

$$x^2 = \frac{S}{(\pi+6)} \Rightarrow x = \sqrt{\frac{S}{(\pi+6)}}$$

$$\therefore\ 2R = \text{Diameter} = x = \sqrt{\frac{S}{\pi+6}}$$

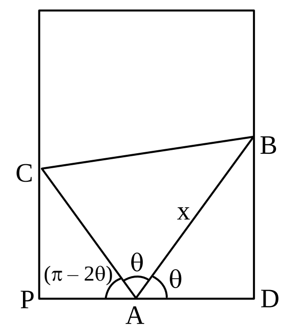
Hence proved.

Q.16 One corner of long rectangular sheet of paper of width 1m, is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

Sol. Let length of crease to be x m.

i.e. AB = x metre

when point disfolded over opposite edge at point C.



Angle $\angle BAD = \angle CAB = \theta$

$\angle CAP = (p - 2\theta)$

Θ AD = AC

From ΔABD, $\cos\theta = \frac{AD}{x}$

$\Rightarrow AD = x\cos\theta = AC$

Also, from ΔPAC, $\cos(\pi - 2\theta) = \frac{PA}{AC}$

$\Rightarrow PA = -AC\cos 2\theta$

$\Rightarrow PA = -x\cos\theta\cos 2\theta$

Hence, PD = PA + AD

$1 = x\cos\theta - x\cos\theta.\cos 2\theta$

$$\Rightarrow x = \frac{1}{\cos\theta(1-\cos 2\theta)}$$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{0.\cos\theta(1-\cos 2\theta) - 1.(\cos\theta.2\sin 2\theta - -\sin\theta(1-\cos 2\theta)}{(\cos\theta(1-\cos 2\theta))^2}$$

$$\frac{dx}{d\theta} = \frac{-\cos\theta.2\sin 2\theta + \sin\theta(1-\cos 2\theta)}{\cos^2\theta(1-\cos 2\theta)^2}$$

Since $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = (1 - \cos 2\theta)$

$$\frac{dx}{d\theta} = \frac{-\cos\theta\times 4\sin\cos\theta + \sin\theta\times 2\sin^2\theta}{\cos^2\theta\times 4\sin^4\theta}$$

$$= \frac{-4\cos^2\theta + 2\sin^2\theta}{4\cos^2\theta \times \sin^3\theta}$$

Again differentiating w.r.t. θ, we get

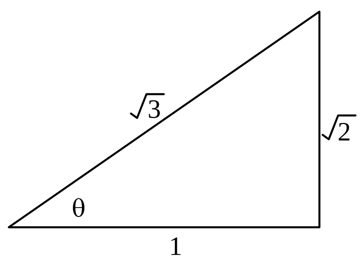
$$\frac{d^2x}{d\theta^2} = \frac{(8\sin\theta\cos\theta + 4\sin\theta\cos\theta)(4\cos^2\theta \times \sin^3\theta) - (2\sin^2\theta - 4\cos^2\theta)\times[-8\cos\theta\sin^4\theta + 12\cos^3\theta\sin^2\theta]}{(4\cos^2\theta \times \sin^3\theta)^2}$$

Now, put $\frac{dx}{d\theta} = 0$

$$\Rightarrow \frac{2\sin^2\theta - 4\cos^2\theta}{4\cos^2\theta \times \sin^3\theta} = 0$$

$\Rightarrow 2\sin^2\theta = 4\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$



$\Rightarrow \tan\theta = \pm\sqrt{2}$

$\sin\theta = \sqrt{2}/\sqrt{3}$

$\cos\theta = 1/\sqrt{3}$

$$\left.\frac{d^2x}{d\theta^2}\right|_{\theta=\tan^{-1}\sqrt{2}} =$$

$$\frac{\left(8\times\frac{\sqrt{2}}{\sqrt{3}}\times\frac{1}{\sqrt{3}} + 4\times\frac{\sqrt{2}}{\sqrt{3}}\times\frac{1}{\sqrt{3}}\right)\left(4\times\frac{1}{3}\times\frac{2}{3}\right) - \left(2\times\frac{2}{3} - 4\times\frac{1}{3}\right)\times\left[-8\times\frac{1}{\sqrt{3}}\times\frac{4}{9} + 12\times\frac{1}{3\sqrt{3}}\times\frac{2}{3}\right]}{\left(4.\frac{1}{3}\times\frac{2\sqrt{2}}{3\sqrt{3}}\right)^2}$$

$$= \frac{12\times\frac{\sqrt{2}}{3}\times\frac{8}{9} - 0}{(8\sqrt{2}/9\sqrt{3})^2} = +ve$$

Hence, x to be minimum.

When, $\tan\theta = -\sqrt{2}$,

$\Rightarrow \sin\theta = -\sqrt{2}/\sqrt{3}$

$\cos\theta = +1/\sqrt{3}$

$$\frac{d^2x}{d\theta^2} =$$

$$\frac{\left(12\times\left(\frac{-\sqrt{2}}{\sqrt{3}}\right)\times\frac{1}{\sqrt{3}}\right)\left(-4\times\frac{1}{3}\times\frac{2\sqrt{2}}{3\sqrt{3}}\right) - 0}{\left(-4\times\frac{1}{3}\times\frac{2\sqrt{2}}{3\sqrt{3}}\right)^2}$$

= +ve but $\sin\theta \neq 0$ and $\cos\theta \neq 0$

Hence, only possible value, $\tan\theta = +\sqrt{2}$

$$(x)_{min} = \left.\frac{1}{\cos\theta\times(1-\cos 2\theta)}\right|_{\tan\theta=\sqrt{2}}$$

$$= \left.\frac{1}{\cos\theta\times 2\sin^2\theta}\right|_{\theta=\tan^{-1}\sqrt{2}}$$

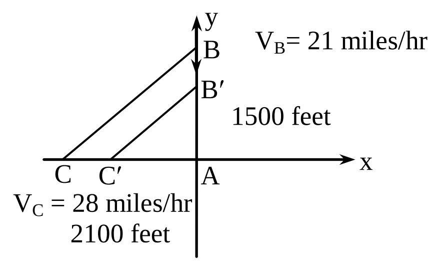
$$= \frac{1}{\frac{1}{\sqrt{3}}\times 2\times\frac{2}{3}}$$

$$(x)_{min} = \frac{3\sqrt{3}}{4}$$

Q.17 Two cars are travelling along two roads which cross each other at right angles at A. One car is travelling towards A at 21 meter/h, while the other is travelling towards it at 28 meter/h. If initially their distances from A are 1500 meter and 2100 meter respectively, prove that the least distance between them is 60 meter.

Sol. AB = 1500 feet

AC = 2100 feet



Let least distance is x feet i.e.

x = B′C′

Let in time t, car B travels to B′ and car C travels to C′

Then BB′ = 21 × t ⇒ BB′ = 21 × $\frac{CC'}{28}$

CC′ = 28 × t ⇒ CC′ = BB′ × $\frac{4}{3}$

Hence, $x^2 = (AB')^2 + (AC')^2$

$x^2 = (1500 - BB')^2 + (2100 - 4/3\ BB')^2$

Differentiating x^2 w.r.t. (BB′), we get

$$\frac{d(x^2)}{d(BB')} = 2(1500 - BB')(-1)$$
$$+ 2\,(2100 - 4/3\ BB') \times (-4/3)$$

Again differentiating w.r.t. (BB′), we get

$$\frac{d^2(x^2)}{d(BB')^2} = 2 + 2 \times \frac{16}{9} = \frac{18+32}{9} = \frac{50}{9} = +ve$$

Hence, distance x to be least.

Now, put $\frac{d(x^2)}{d(BB')} = 0$

$\Rightarrow 2(1500 - BB')(-1) +$
$\quad 2\,(2100 - 4/3\ BB')(-4/3) = 0$

$\Rightarrow 2(1500 - BB') = 2(4/3\ BB' - 2100) \times 4/3$

$\Rightarrow 1500 - BB' = 16/9\ BB' - 2800$

$\Rightarrow 4300 = 25/9\ BB'$

$\Rightarrow BB' = 43 \times 36$

$BB' = 1548$ feet

Hence, $x^2 = (1500 - 1548)^2 +$
$\quad (2100 - 4/3 \times 1548)^2$

$= (-48)^2 + (2100 - 2064)^2$

$= (48)^2 + (36)^2$

$x^2 = 2304 + 1296$

$x^2 = 3600$

x = 60 feet proved.

Q.18 Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$ then the value of p(2) is

Sol. $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$$\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$$

$$\Rightarrow \lim_{x\to 0}\left[1 + \left(\frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2}\right)\right] = 2$$

d = 0, e = 0

c = 1

Now,

$p'(x) = 4ax^3 + 3bx^2 + 2x$

$\Rightarrow 4a + 3b = -2 \quad \ldots(1)$

$\Rightarrow 32a + 12b = -4 \quad \ldots.(2)$

$\Rightarrow 16a = 4$

$a = \frac{1}{4}$

$b = -1$

$p(x) = \frac{1}{4}x^4 - x^3 + x^2$

$p(2) = 0$

Q.19 Find the intervals of monotonicity of the function f(x) = 2sinx + cos2x ; $(0 \le x \le 2\pi)$. Also find the point of local maxima & minima.

Sol. $f(x) = 2\sin x + \cos 2x\ ;\ (0 \le x \le 2\pi)$

Differentiating w.r.t.x, we get

$f'(x) = 2\cos x - 2\sin 2x$

$= 2\cos x - 2 \times 2\sin x \times \cos x$

$= 2\cos x\,(1 - 2\sin x)$

Put $\cos x = 0 = \cos \pi/2$

$\Rightarrow x = 2n\pi \pm \pi/2;\ n = 0, \pm 1, \pm 2$

$n = 0,\ x = \pi/2$ (only)

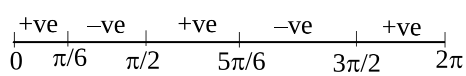
$n = 1,\ x = 3\pi/2$ (only)

Put $(1 - 2\sin x) = 0 \Rightarrow \sin x = 1/2 = \sin \pi/6$

$\Rightarrow x = n\pi + (-1)^n\,\pi/6;\ n \in$ Integer

Only acceptable values

$x = \pi/6,\ 5\pi/6$



Hence, monotonically increasing function in

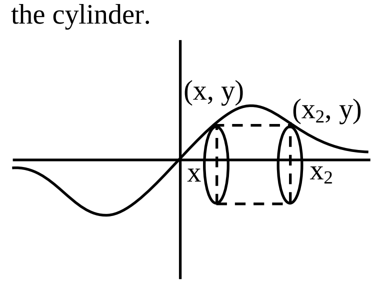
$x \in (0, \pi/6) \cup (\pi/2, 5\pi/6) \cup (3\pi/2, 2\pi)$

monotonically decreases function in

$$x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$$

Q.20 A cylinder is obtained by revolving a rectangle about the x-axis, the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = \frac{x}{x^2+1}$ & the x-axis. Find the maximum possible volume of the cylinder.

Sol.



$xx_2 = 1$

$$v = \pi\left(\frac{x}{x^2+1}\right)^2\left(\frac{1}{x} - x\right)$$

$$v = \pi\,\frac{(x - x^3)}{(x^2+1)^2}$$

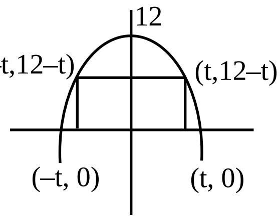
$$\frac{dv}{dx} = \pi \cdot \frac{(1-3x^2)(x^2+1)^2 - 2(x^2+1)2x}{(x^2+1)^4}$$

$$\frac{dv}{dx} = 0 \Rightarrow (1 - 3x^2)(x^2+1) = 4x^2 - 4x^4$$

$$\Rightarrow x = \sqrt{2} - 1 \Rightarrow \text{volume} = \frac{\pi}{4}$$

Q.21 Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

Sol.



$$\text{Area} = 2t \times 12 - t$$
$$= 24t - 2t^2$$

$$\frac{dA}{dt} = 24 - 4t$$

$+ \quad\quad -$

$\longleftarrow\!\!\!\!\!\overset{}{|}\!\!\!\!\!\longrightarrow$ 6

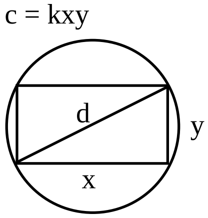
max. at $t = 6$

$A_{max} = (24 \times 6) - 72$

$= 144 - 72 = 72$

Q.22 A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression, (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

Sol. (a) $c = kxy$



$$c = kx\sqrt{d^2 - x^2}$$

$$c' = \frac{kx(-2x)}{2\sqrt{d^2 - x^2}} + k\sqrt{d^2 - x^2}$$

$$= \frac{k}{\sqrt{d^2 - x^2}}\,(-2x^2 + d^2)$$

$$= \frac{-2k}{\sqrt{d^2 - x^2}}\left(x^2 - \frac{d^2}{2}\right)$$

$- \quad + \quad -$

$-d \quad\quad \frac{d}{\sqrt{2}}$

$$x = \frac{d}{\sqrt{2}} = y$$

(b) $B = kxy^2$

$= kx(d^2 - x^2)$

$B' = k(d^2 - 3x^2)$

$- \quad + \quad -$

$-\frac{d}{\sqrt{3}} \quad\quad \frac{d}{\sqrt{3}}$

$$x = \frac{d}{\sqrt{3}} \qquad y = \sqrt{\frac{2}{3}}\,d$$

Q.23 The value of 'a' for which $f(x) = x^3 + 3(a-7)x^2 + 3(a^2 - 9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

Sol. $f'(x) = 3x^2 + 6(a-7)x + 3(a^2 - 9)$

$x > 0$

roots are > 0

$\frac{-b}{2a} > 0 \qquad \frac{c}{a} > 0$

$-2(a - 7) > 0$

$a < 7$

$(a + 3)(a - 3) > 0$

$a \in (-\infty, -3) \cup (3, \infty)$

$a \in (-\infty, -3) \cup (3, 7)$

$\quad a_1 \quad a_2 \quad a_3 \quad a_4$

$\Rightarrow a_2 + 11a_3 + 70a_4$

$\Rightarrow -3 + 33 + 490$

$= 520$

Q.24 The mass of a cell culture at time t is given by,

$$M(t) = \frac{3}{1 + 4e^{-t}}$$

(a) Find $\lim\limits_{t\to-\infty} M(t)$ and $\lim\limits_{t\to\infty} M(t)$

(b) Show that $\dfrac{dM}{dt} = \dfrac{1}{3}M(3-M)$

Sol. (a) $\lim\limits_{t\to-\infty} \dfrac{3}{1+4e^{-t}}$

$\lim\limits_{t\to-\infty} \dfrac{3e^t}{e^t+4} = 0$

$\lim\limits_{t\to\infty} \dfrac{3}{1+4e-t} = 3$

(b) $\dfrac{dM}{dt} = \dfrac{-3(-4e-t)}{(1+4e^{-t})^2}$

$= \dfrac{12e^{-t}}{(1+4e^{-t})^2}$

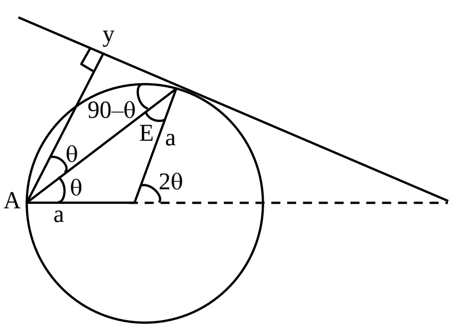
$= M^2\left(\dfrac{4}{3}e^{-t}\right)$

$\dfrac{dM}{dt} = \dfrac{M^2}{3}\left(\dfrac{3-M}{M}\right)$

$= \dfrac{M}{3}(3-M)$

Q.25 From a fixed point A on the circumference of a circle of radius 'a', let the perpendicular AY fall on the tangent at a point P on the circle, prove that the greatest area which the ΔAPY can have is $3\sqrt{3}\dfrac{a^2}{8}$ sq. units.

Sol.



$AP = \sqrt{2a^2 + 2a^2(-\cos 2\theta)}$

$= \sqrt{2a^2(1-\cos 2\theta)}$

$= 2a \sin\theta$

$AY = 2a \sin\theta \cos\theta$

$PY = 2a \sin^2\theta$

Area $= 2a^2 (\sin^3\theta \cos\theta)$

$\dfrac{dA}{d\theta} = 2a^2 [-\sin^4\theta + \cos^2\theta \,.\, 3\sin^2\theta]$

$= 2a^2 \sin^2\theta [-\sin^2\theta + 3\cos^2\theta]$

$= -2a^2 \sin^2\theta \cos^2\theta [\tan^2\theta - 3]$

for A_{max} $\theta = \dfrac{\pi}{3}$

$A_{max} = 2a^2 \times \dfrac{3\sqrt{3}}{8} \times \dfrac{1}{2}$

$= \dfrac{3\sqrt{3}}{8} a^2$

Q.26 A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum, the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi + 2)$.

Sol. $v = \dfrac{\pi r^2}{2} \times h$

$A = \pi rh + \pi r^2 + 2rh$

$= \pi \times \dfrac{2v}{\pi r} + \pi r^2 + 2 \times \dfrac{2v}{\pi r}$

$= \dfrac{2v}{r} + \pi r^2 + \dfrac{4v}{\pi r}$

$\dfrac{dA}{dr} = \dfrac{-2v}{r^2} + 2r\pi + \dfrac{4v}{\pi r^2}$

$-\dfrac{2v}{r^2} + 2r\pi - \dfrac{4v}{\pi r^2} = 0$

$2r\pi = \dfrac{2v}{r^2} + \dfrac{4v}{\pi}$

$r^3 = \dfrac{v}{\pi} + \dfrac{2v}{\pi^2}$

$\Rightarrow r^3 = \dfrac{r^2h}{2} + \dfrac{r^2h}{\pi}$

$\Rightarrow \dfrac{2r}{h} = \dfrac{2+\pi}{\pi}$

$\dfrac{h}{2r} = \dfrac{\pi}{\pi+2}$

Q.27 Consider the function $y = f(x) = \lambda n\,(1 + \sin x)$ with $-2\pi \le x \le 2\pi$. Find

(a) the zeroes of f(x)

(b) inflection points if any on the graph

(c) local maxima and minima of f(x)

(d) asymptotes of the graph

(e) sketch the graph of f(x

Sol. (a) $\lambda n(1 + \sin x) = 0$

$\sin x = 0$

$x = -2\pi, -\pi, 0, \pi, 2\pi$

(b) $f'(x) = \dfrac{\cos x}{1+\sin x}$ $[\sin x \neq 1]$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$f''(x) = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

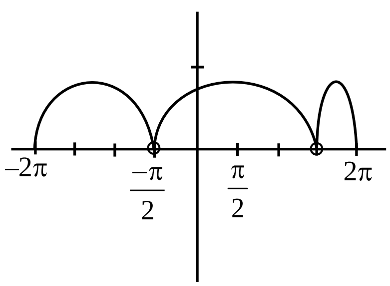
No. inflection pt.

(c) $+ \quad - \quad + \quad - \quad +$ at $\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

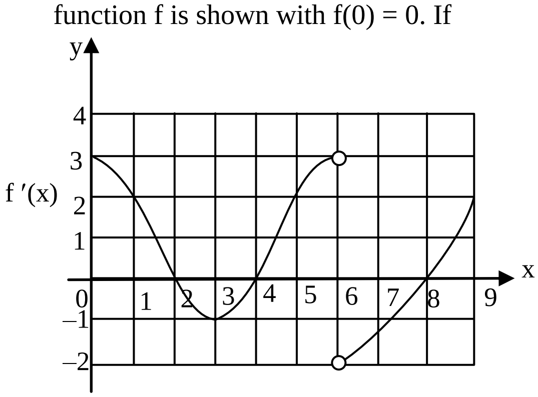
Max at $x = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}$

$f(x)_{max} = \lambda n\, 2$

(d) $x = -\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

(e) 

Q.28 The graph of the derivative f ′ of a continuous function f is shown with f(0) = 0. If



(i) f is monotonic increasing in the interval $[a, b) \cup (c, d) \cup (e, f]$ and decreasing in $(p, q) \cup (r, s)$.

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

(iii) f is concave up in $(\lambda, m) \cup (n, t]$

(iv) f has inflection point at x = k

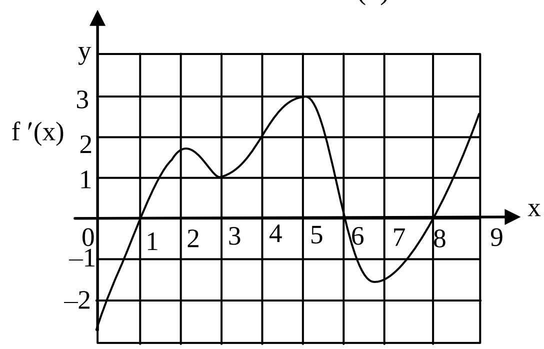
(v) number of critical points of y = f(x) is 'w'

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (\lambda + m + n) + (x_1 + x_2) + (k + w)$.

Sol.

a = 0	c = 4
b = 2	d = 6
e = 8	p = 2
f = q	q = 4
r = 6	$x_1 = 4$
s = 8	$x_2 = 8$
$\lambda = 3$	n = 6
m = 6	t = 9
k = 3	w = 4

Q.29 The graph of the derivative f ′ of a continuous function f is shown with f (0) = 0



(i) On what intervals is f increasing or decreasing?

(ii) At what values of x does f have a local maximum or minimum ?

(iii) On what intervals is f concave upward or downward ?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that f (0) = 0, sketch a graph of f.

Sol. (i) $f \uparrow [1, 6] \cup [8, 9]$

$\downarrow [0, 1] \cup [6, 8]$

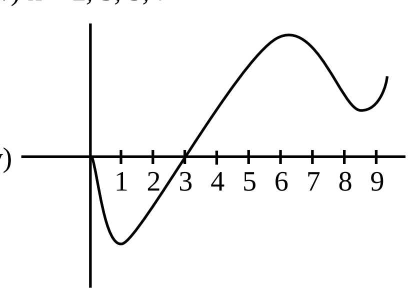
(ii) max $\Rightarrow$ 6

min $\Rightarrow$ 1, 8

(iii) cu $\Rightarrow [0, 2] \cup [3, 5] \cup [7, 9]$

(i) $\Rightarrow [2, 3] \cup [5, 7]$

(iv) x = 2, 3, 5, 7

(v) 

Q.30 The function f(x) defined for all real numbers x has the following properties

$f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is increasing and decreasing and any local maximum or minimum values.

(b) the intervals on which the graph f is concave down and concave up.

(c) the function f(x) and plot its graph.

Sol. (a) $f'(x) = k(2x - x^2)e^{-x}$

$= ke^{-x}x(2 - x)$

$-\quad + \quad -$

$0 \quad 2$

$f \uparrow [0, 2]$

$\downarrow [-\infty, 0] \cup (2, \infty)$

(b) $f''(x) = -k(2x - x^2)e^{-x} + ke^{-x}(2 - 2x)$

$= ke^{-x}[-2x + x^2 + 2 - 2x]$

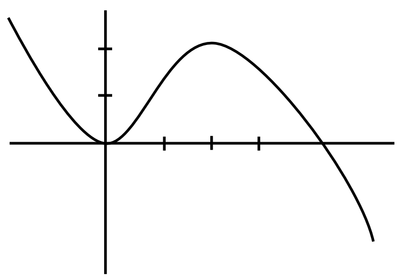
$= ke^{-x}[x^2 - 4x + 2]$

$+ \quad - \quad +$

$\frac{4-\sqrt{8}}{2} \quad 2+\sqrt{2}$

at $\Rightarrow (-\infty,\ 2-\sqrt{2}) \cup [2+\sqrt{2}, \infty]$

CD $\Rightarrow [2-\sqrt{2},\ 2+\sqrt{2}]$

(c) 

Q.31 Use calculus to prove the inequality,

$\sin x \ge 2x/\pi$ in $0 \le x \le \pi/2$

Use this inequality to prove that,

$\cos x \le 1 - x^2/\pi$ in $0 \le x \le \pi/2$

Sol. $f(x) = \sin x - \frac{2x}{\pi}$

$f'(x) = \cos x - \frac{2}{\pi}$

$+ \quad -$

$\cos^{-1}\left(\frac{2}{\pi}\right)$

$\therefore f_{max}$ at $\cos^{-1}\left(\frac{2}{\pi}\right)$

$f(0) = 0$

$f\left(\frac{\pi}{2}\right) = 0$

$\therefore f(x) \ge 0$

Now,

$g(x) = 1 - \frac{x^2}{\pi} - \cos x$

$g'(x) = -\frac{2x}{\pi} + \sin x$

(using above result)

$\therefore g(x) \uparrow$

g_{min}

$= g(0) = 0$

$\therefore g(x) \ge 0$

Part-B Passage based objective questions

Passage I (Q.32 to 34)

Suppose f(x) is a real valued function of degree 6 satisfying the following condition

(A) ‘f ’ has minimum value at x = 0 & 2

(B) ‘f ’ has maximum value at x = 1

(C) for all x, $\lim\limits_{x\to 0} \frac{1}{x} \lambda n \begin{vmatrix} f(x) & 1 & 0 \\ 0 & 1/x^2 & 1 \\ 1 & 0 & 1/x \end{vmatrix} = 2$

On the basis of above information, answer the following questions.

Q.32 Number of solutions of the equation $8f(x) - 1 = 0$ is

(A) one (B) two

(C) three (D) four

Sol.[D] $\lim\limits_{x\to 0} \frac{1}{x} \lambda n\left(\frac{f(x)}{x^3} + 1\right) = 2$

$f(x) = ax^6 + bx^5 + gx^3 + cx^4 + dx^2 + ex + f$

here

$d = e = f = 0 = g$

$\lim\limits_{x\to 0} \frac{\lambda n(cx + bx^2 + ax^3 + 1)}{x} = 2$

$\lim\limits_{x\to 0} \frac{\lambda n(cx + bx^2 + ax^3 + 1)}{x(c + bx + ax^2)}$

$(c + bx + ax^2) = 2$

$c = 2$

$f(x) = ax^6 + bx^5 + 2x^4$

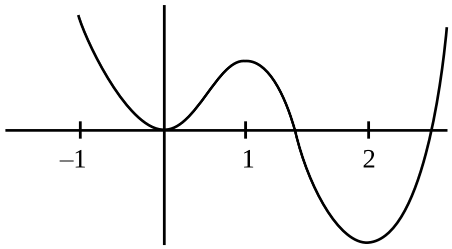
$f'(x) = 6ax^5 + 5bx^4 + 8x^3$

$6a + 5b + 8 = 0$

$24a + 10b + 8 = 0$

$a = 2/3,\ b = \frac{-12}{5}$

$$f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$$



Q.33 Range of f(x) is -

(A) $\left[-\frac{32}{15},\infty\right)$ (B) $\left[-\frac{4}{15},\infty\right)$

(C) $\left(-\infty,\frac{2}{15}\right]$ (D) None of these

Sol. **[A]**

$f(2) = \frac{8}{3} - \frac{3}{5}$. 128 + 32

$= 128\left(\frac{1}{3} - \frac{3}{5}\right) + 32 = -\frac{32}{15}$

Range $\left[-\frac{32}{15},\infty\right)$

Q.34 If the area bounded by y = f(x), x-axis, x = ± 1; is $\frac{a}{b}$, where a & b are relatively prime then the value of $\tan^{-1}(a - b)$ is -

(A) π/4 (B) –π/4

(C) π/3 (D) π/6

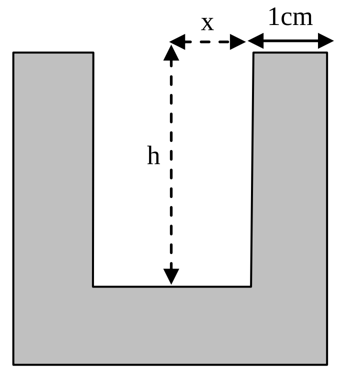
Sol. **[B]**

$$\int_{-1}^{1} f(x)dx = \frac{4}{21} + \frac{4}{5} = \frac{104}{105} = \frac{a}{b}$$

$\tan^{-1}(-1) = -\frac{\pi}{4}$

Passage II (Q. 35 to 37)

Let there is a cylindrical capacitor whose cross sectional area is as shown in the figure. The thickness of the wall of capacitor is 1 cm. and volume is 27π cm^3. Let x and h are the radius and height of the cylinder respectively and $v(x) = \pi\left\{(x+a)^2 + \frac{b}{x} + \frac{c}{x^2}\right\}$, where v(x) shows the volume of the material required to form the cylinder, which is expressed as a function of radius of the cylinder.



On the basis of above passage, answer the following questions :

Q.35 The value of $\frac{ab}{c}$ is

(A) prime but not even (B) even but not prime

(C) even prime (D) irrational

Sol. [C] $27\pi = \pi x^2 h$

$v(x) = \pi\left\{(x+a)^2 + \frac{b}{x} + \frac{c}{x^2}\right\}$

$v = \pi[(x+1)^2(h+1) - x^2 h]$

$= \pi\left[(x+1)^2\frac{27}{x^2} + (x+1)^2 - 27\right]$

$= \pi\left[(x+1)^2\frac{54}{x} + \frac{27}{x^2}\right]$

a = 1, b = 54,

c = 27

Q.36 If the cost of the material to form the cylinder is minimum, then which of the following relation in x and h is true

(A) x = h (B) 2x = h

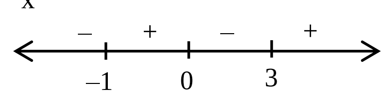
(C) x = 2h (D) none of these

Sol. **[A]**

$\frac{dV}{dx} = \pi\left[2(x+1) - \frac{54}{x^2} - \frac{54}{x^3}\right]$

$\Rightarrow \frac{2\pi}{x^3}\left[x^4 + x^3 - 27x - 27\right]$

$\Rightarrow \frac{2\pi}{x^3}(x^3 - 27)(x+1)$



V_{min} at x = 3

Now,

$27\pi = \pi x^2 h$

h = 3

$\therefore$ x = h

Q.37 If function v(x) is redefined such that $v : R_0 \to R$ then the number of solutions of v(x) = 0 will be

(A) 1 (B) 2

(C) 3 (D) 4

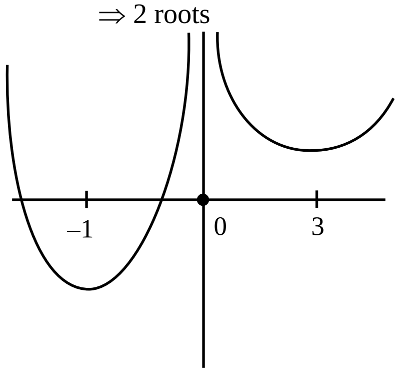
Sol. **[B]**

$$v = \pi\left((x+1)^2 + \frac{54}{x} + \frac{27}{x^2}\right)$$

$v_{min} = \pi(-27)$

$\Rightarrow$ v(x) = 0

$\Rightarrow$ 2 roots



Passage III (Q. 38 to 40)

Two functions are defined as follows

$f(x) = ax^2 + bx + c$, a, b, c $\in$ R, a $\neq$ 0

$g(x) = dx^2 + ex + f$, d, e, f $\in$ R, d $\neq$ 0

Real part of complex roots of both the equations f(x) = 0 and g(x) = 0 are same. Minimum value of f(x) is same as negative of maximum value of g(x).

On the basis of above passage, answer the following questions :

Q.38 Which statement is correct -

(A) $bd = e^2$ (B) bd = ea

(C) bc = ef (D) None of these

Sol. **[B]**

$b^2 - 4ac < 0$

$e^2 - 4fd < 0$

$$f\frac{b}{2a} = -\frac{3}{2d}$$

$\Rightarrow$ bd = ae

Q.39 If y = f(|x|) has only one critical point, then-

(A) minimum value of y is same as minimum value of f(x)

(B) minimum value of y is greater than minimum value of f(x)

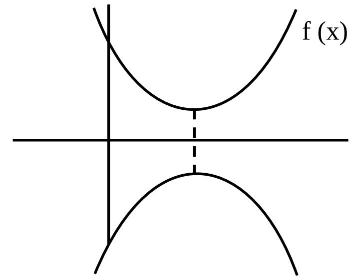
(C) minimum value of y is smaller than minimum value of f(x)

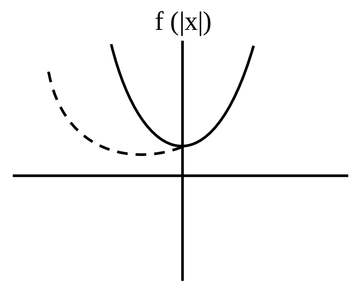
(D) (A) or (B) is correct

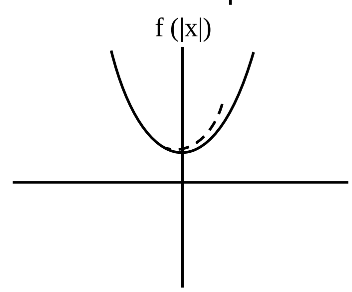
Sol. **[D]**

a > 0

d < 0







Q.40 If y = |g(|x|)| has three critical points, then -

(A) maximum value of y can be evaluated

(B) minimum value of y is same as minimum value of f(x).

(C) minimum value of y can be same as minimum value of f(|x|)

(D) (B) and (C) are correct

Sol. **[D]**

Small cases as above

EXERCISE # 4

Old IIT-JEE Questions

Q.1 Let f (x) = $(1 + b^2)x^2 + 2bx + 1$ and m (b) is minimum value of f (x). As b varies, the range of m (b) is- **[IIT Scr. 2001]**

(A) [0, 1] (B) (0, 1/2] (C) [1/2, 1] (D) (0, 1]

Sol. **[D]**

$f(x) = (1 + b^2)x^2 + 2bx + 1$

Differentiating w.r.t.x, we get

$f'(x) = (1 + b^2)\times 2x + 2b + 0$

Again differentiating w.r.t. x, we get

$f''(x) = 2(1 + b^2)$

$= +ve$ for all $b \in R$

Now, put $f'(x) = 0 \Rightarrow (1 + b^2) \times 2x + 2b = 0$

$\Rightarrow (1 + b^2) \times 2x = -2b \Rightarrow x = -b\big|_{(1+b^2)}$

Hence, $f(x)\big|_{x=\frac{-b}{(1+b^2)}} = f_{min}$

$= (1+b^2).\ \dfrac{b^2}{(1+b^2)^2} + 2b\left(\dfrac{-b}{1+b^2}\right)+1$

$= \dfrac{b^2}{(1+b^2)} - \dfrac{2b^2}{1+b^2} + 1$

$= \dfrac{b^2-2b^2+1+b^2}{(1+b^2)} = \dfrac{1}{(1+b^2)}$

$\Rightarrow m(b) = \dfrac{1}{1+b^2} \Rightarrow 1 + b^2 = \dfrac{1}{m(b)}$

$\Rightarrow b = \sqrt{\dfrac{1}{m(b)}-1} \Rightarrow b = \sqrt{\dfrac{1-m(b)}{m(b)}}$

$b = \sqrt{\dfrac{1-m(b)}{m(b)}}$; $m(b) \neq 0$

For b to be exist, $1 - m(b) \geq 0$

$\Rightarrow m(b) \leq 1$

$\Rightarrow m(b) \in (0, 1]$

$\therefore$ option [D] is correct answer.

Q.2 The max. value of $(\cos \alpha_1) \cdot (\cos \alpha_2)...(\cos \alpha_n)$, under the restrictions $0 \leq \alpha_1, \alpha_2,.....\alpha_n \leq \pi/2$ and $(\cot \alpha_1)\cdot(\cot \alpha_2).....(\cot \alpha_n) = 1$ is : **[IIT 2001]**

(A) $\dfrac{1}{2^{n/2}}$ (B) $\dfrac{1}{2^n}$ (C) $\dfrac{1}{2n}$ (D) 1

Sol. **[A]**

$f(\alpha) = \cos \alpha_1. \cos \alpha_2 \ldots\ldots \cos \alpha_n$

under the restrictions

$0 \leq \alpha_1. \alpha_2.\alpha_3 \ldots\ldots\ldots \alpha_n \leq \pi/2$

and $(\cot \alpha_1) . (\cot \alpha_2) \ldots..(\cot \alpha_n) = 1$

$\Rightarrow \cos \alpha_1. \cos \alpha_2 \ldots\ldots \cos \alpha_n = \sin \alpha_1. \sin \alpha_2 \ldots.$
$\sin \alpha_n$

Differentiating above function $f(\alpha)$ w.r.t.α_1, $\alpha_2 \ldots\ldots, \alpha_n$

$f'(\alpha) = ((-\sin \alpha_1). \cos \alpha_2.\cos \alpha_3\ldots..\cos \alpha_n) +$
$(\cos \alpha_1. (-\sin \alpha_2). \cos \alpha_3\ldots\ldots\cos \alpha_n) +$
$(\cos \alpha_1. \cos \alpha_2(-\sin \alpha_3) \ldots.\cos \alpha_n) + \ldots\ldots\ldots +$
$(\cos \alpha_1. \cos \alpha_2. \cos \alpha_3\ldots\ldots\ldots(-\sin \alpha_n))$

$= \cos \alpha_1. \cos \alpha_2.\cos \alpha_3\ldots\ldots\cos \alpha_n$
$[-\tan \alpha_1 - \tan \alpha_2 - \ldots\ldots\ldots - \tan \alpha_n]$

$= f(\alpha) [-\tan \alpha_1 -\tan \alpha_2 - \ldots\ldots. -\tan \alpha_n]$

$f'(\alpha) = f(\alpha) [-\tan \alpha_1 - \tan \alpha_2 - \ldots..-\tan \alpha_n]$

Again differentiating w.r.t. $\alpha_1, \alpha_2,\ldots.., \alpha_n$ we get

$f''(\alpha) = f'(\alpha) [-\tan \alpha_1 - \tan \alpha_2 - \ldots..-\tan \alpha_n]$
$+ f(\alpha) [-\sec^2 \alpha_1 - \sec^2 \alpha_2 - \ldots\ldots - \sec^2\alpha_n]$

$= f(\alpha) [-\tan \alpha_1 - \tan \alpha_2 - \tan \alpha_3 - \ldots\ldots -\tan \alpha_n]^2$
$+ f(\alpha) [-\sec^2 \alpha_1 - \sec^2 \alpha_2 - \ldots\ldots - \sec^2\alpha_n]$

Now, put $f'(\alpha) = 0$

$\Rightarrow f(\alpha)[-\tan \alpha_1 - \tan \alpha_2 - \ldots\ldots.. - \tan \alpha_n] = 0$

$\Rightarrow \tan \alpha_1 + \tan \alpha_2 + \ldots + \tan \alpha_n = 0$ but $f(\alpha) \neq 0$

This is only possible when

$\tan \alpha_1 = 0 \Rightarrow \alpha_1 = 0$

$\tan \alpha_2 = 0 \Rightarrow \alpha_2 = 0$

$\tan \alpha_3 = 0 \Rightarrow \alpha_3 = 0$

$\vdots$

$\vdots$

$\tan \alpha_n = 0 \Rightarrow \alpha_n = 0$

But, we use restrictions

$\cot \alpha_1, \cot \alpha_2, \cot \alpha_3,\ldots\ldots., \cot \alpha_n = 1$

$\Rightarrow \cot (0). \cot (0)\ldots\ldots\cot (0) = 1$ doesn't exist

Hence, $\cos \alpha_1. \cos \alpha_2. \cos \alpha_3 \ldots\ldots.\cos \alpha_n$
$= \sin \alpha_1. \sin \alpha_2. \sin \alpha_3\ldots\ldots..\sin \alpha_n$
holds only for

$\alpha_1 = \alpha_2 = \alpha_3 = \ldots\ldots\ldots = \alpha_n = \pi/4$

Hence,

$f''(x)\big|_{(\pi/4)} = -ve$

$f(\alpha)$ would be maximum.

$f(\alpha)\big|_{(\pi/4)} = f(\alpha)_{max}$

$= \cos\dfrac{\pi}{4} . \cos\dfrac{\pi}{4} . \cos\dfrac{\pi}{4} \ldots..\cos\dfrac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} \ldots\ldots\ldots \dfrac{1}{\sqrt{2}}$

$= \dfrac{1}{2^{n/2}}$

$\therefore$ option [A] is correct answer.

Q.3 A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. **[IIT-2002]**

Sol. L : (y – 2) = m(x – 8), m < 0

The points P and Q are $\left(8 - \dfrac{2}{m}, 0\right)$ and (0, 2 – 8m) respectively.

Then OP + OQ = $\left(8 - \dfrac{2}{m}\right)$ + (2 – 8m)

$= 10 + \left(-\dfrac{2}{m} + (-8m)\right)$

A.M. $\geq$ G.M.

$\Rightarrow \left(\dfrac{2}{-m}\right) + (-8m) \geq 2\sqrt{16}$

(as $-\dfrac{2}{m}$ and – 8m are +ve)

$\Rightarrow -\left(\dfrac{2}{m} + 8m\right) \geq 8$

$\Rightarrow 10 - \left(\dfrac{2}{m} + 8m\right) \geq 10 + 8$

$\Rightarrow$ OP + OQ $\geq$ 18

$\therefore$ Minimum value of OP + OQ is 18, which occurs only when $-8m = \dfrac{-2}{m}$ i.e.

$m = \dfrac{-1}{2}$ (as m < 0)

Q.4 The value of 'θ'; $\theta \in [0, \pi]$ for which the sum of intercepts on coordinate axes cut by tangent at point $(3\sqrt{3}\cos\theta, \sin\theta)$ to ellipse $\dfrac{x^2}{27} + y^2 = 1$ is minimum is : **[IIT Scr. 2003]**

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{8}$

Sol. **[A]**

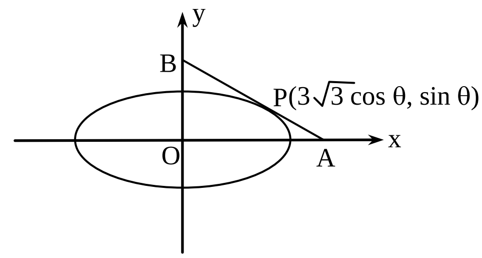
$\dfrac{x^2}{27} + y^2 = 1$

Equation of tangent at $P(3\sqrt{3}\cos\theta, \sin\theta)$

$\dfrac{x(3\sqrt{3}\cos\theta)}{27} + y\sin\theta = 1$

$\dfrac{\sqrt{3}\cos\theta}{9}x + y\sin\theta = 1$

$\Rightarrow \dfrac{x}{\left(\dfrac{9}{\sqrt{3}\cos\theta}\right)} + \dfrac{y}{\left(\dfrac{1}{\sin\theta}\right)} = 1$



sum of intercept, $S = \dfrac{9}{\sqrt{3}\cos\theta} + \dfrac{1}{\sin\theta}$

Differentiating w.r.t. θ, we get

$\dfrac{ds}{d\theta} = \dfrac{9}{\sqrt{3}}\sec\theta.\tan\theta - \text{cosec}\,\theta\cot\theta$

Again differentiating w.r.t. θ, we get

$\dfrac{d^2s}{d\theta^2} = \dfrac{9}{\sqrt{3}}[\sec\theta.\tan^2\theta + \sec^3\theta]$

$-\ [-\text{cosec}\,\theta\cot^2\theta - \text{cosec}^3\theta]$

$= \dfrac{9}{\sqrt{3}}[\sec\theta.\tan^2\theta + \sec^3\theta]$

$+ [\text{cosec}\,\theta\cot^2\theta + \text{cosec}^3\theta]$

Now, put $\dfrac{ds}{d\theta} = 0$

$\Rightarrow \dfrac{9}{\sqrt{3}}\sec\theta.\tan\theta - \text{cosec}\,\theta\cot\theta = 0$

$\dfrac{9}{\sqrt{3}}\dfrac{1}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sin\theta} \times \dfrac{\cos\theta}{\sin\theta}$

$3\sqrt{3}\ \sin^3\theta = \cos^3\theta$

$\tan^3\theta = \dfrac{1}{3\sqrt{3}} = \dfrac{1}{(\sqrt{3})^3}$

$\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \pi/b$

$\dfrac{d^2s}{d\theta^2} = \dfrac{9}{\sqrt{3}}\left[\dfrac{2}{\sqrt{3}} \times \dfrac{1}{3} + \dfrac{8}{3\sqrt{3}}\right] + [2 \times 3 + 8] = +ve$

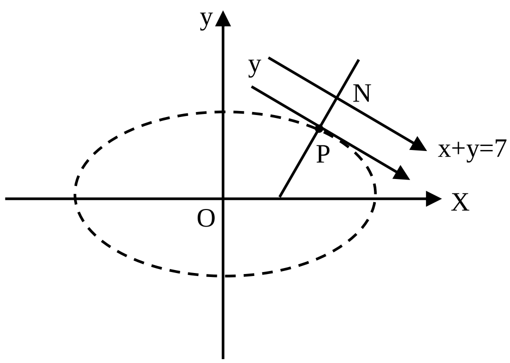
Hence, s to be minimum.

$\therefore$ [A] is correct answer.

Q.5 Find the point on $x^2 + 2y^2 = 6$ nearest to the line x + y = 7. **[IIT 2003]**

Sol. Let us taken a point P($\sqrt{6}$ cosθ, $\sqrt{3}$ sin θ) on $\frac{x^2}{6}+\frac{y^2}{3} = 1$.

Now to minimize the distance from P to given straight line x + y = 7, shortest distance exists along the common normal.



Slope of normal at P = $\frac{\sqrt{6}\sec\theta}{\sqrt{6}\,\text{cosec}\,\theta} = \sqrt{2}$ tanθ = 1

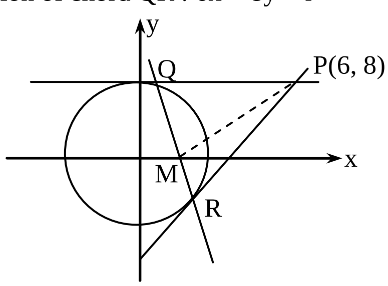
So, $\cos\theta = \sqrt{\frac{2}{3}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

Hence, P(2, 1)

Q.6 For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum. **[IIT 2003]**

Sol. $x^2 + y^2 = r^2$

Equation of chord QR : $6x + 8y = r^2$



Let PM be perpendicular on QR

$$|PM| = \left|\frac{36+64-r^2}{\sqrt{36+64}}\right| = \left|\frac{100-r^2}{10}\right|$$

Now, solving $x^2 + y^2= r^2$ and $6x + 8y = r^2$

$$\Rightarrow x = \frac{r^2-8y}{6}$$

$$\left(\frac{r^2-8y}{6}\right)^2 + y^2 = r^2$$

$(r^2 - 8y)^2 + 36y^2 = 36r^2$

$r^4 + 64y^2 - 16yr^2 + 36y^2 = 36r^2$

$100\,y^2 - 16r^2y + (r^4 - 36r^2) = 0$

sum of roots,

$$y_1 + y_2 = \frac{16r^2}{100}$$

product of roots, $= \frac{r^4-36r^2}{100} = y_1y_2$

$(y_1- y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$

$(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$

$$= \left(\frac{16r^2}{100}\right)^2 - 4 \times \frac{(r^4-36r^2)}{100}$$

$$= \frac{256r^4}{100\times100} - \frac{4(r^4-36r^2)}{100}$$

$$= \frac{256r^4-400(r^4-36r^2)}{100\times100}$$

$$= \frac{400\times36r^2-144r^4}{100\times100}$$

$$= \frac{144r^2(100-r^2)}{100\times100}$$

$$(y_1 - y_2) = \frac{12r}{100} \times \sqrt{100-r^2}$$

Now, from $6x + 8y = r^2$

$6x_1 + 8y_1 = r^2$

$6x_2 + 8y_2 = r^2$

$\Rightarrow 6x_1 + 8y_1 = 6x_2 + 8y_2$

$\Rightarrow 6(x_1 - x_2) = 8(y_2 - y_1)$

$$\Rightarrow (x_1 - x_2) = \frac{-8}{6}(y_1- y_2) = \frac{-4}{3}(y_1- y_2)$$

$$\Rightarrow (x_1 - x_2) = \frac{-4}{3}(y_1 - y_2)$$

$$\Rightarrow (x_1 - x_2)^2 = \frac{16}{9} \times (y_1 - y_2)^2$$

$$= \frac{16}{9} \times \frac{144r^2}{100\times100}(100 - r^2)$$

Hence, $QR^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$= \frac{16}{9} \times \frac{144r^2}{100\times100} \times (100 - r^2) + \frac{144r^2}{100\times100}(100 - r^2)$$

$$= \frac{144r^2}{100\times100}(100 - r^2) \times \left[\frac{16}{9}+1\right]$$

$$= \frac{144r^2}{100\times100}(100 - r^2) \times \frac{25}{9}$$

$$|QR| = \frac{12r}{100} \times \frac{5}{3} \times \sqrt{100-r^2}$$

Hence, Area of ΔPQR, $A = \frac{1}{2} \times QR \times PM$

$= \frac{1}{2} \times \left(\frac{100-r^2}{10}\right) \times \frac{12r}{100} \times \frac{5}{3}(100-r)^{1/2}$

$= \frac{12\times5}{100\times6\times10} r\,(100-r^2)^{3/2}$

Differentiating w.r.t. r, we get

$$\frac{dA}{dr} = \frac{1}{100} \times \left[1.(100-r^2)^{3/2} + r.\frac{3}{2}(100-r^2)^{1/2}(-2r)\right]$$

$$\frac{dA}{dr} = \frac{1}{100} \times \left[(100-r^2)^{3/2} + r.\frac{3}{2}.(100-r^2)^{1/2}(-2r)\right]$$

$= \frac{1}{100} \times [(100-r^2)^{3/2} - 3r^2\,(100-r^2)^{1/2}$

Again differentiating w.r.t.r, we get

$$\frac{d^2A}{dr^2} = \frac{1}{100} \times \left[\frac{3}{2}(100-r^2)^{1/2}(-2r) - 3\left\{2r(100-r^2)^{1/2} + r^2\frac{1}{2\sqrt{100-r^2}}(-2r)\right\}\right]$$

$$= \frac{1}{100} \times \left[-3r\,(100-r^2)^{1/2} - 3\left\{2r(100-r^2)^{1/2} - \frac{r^3}{\sqrt{100-r^2}}\right\}\right]$$

$$= \frac{1}{100} \times \left[-3r\,(100-r^2)^{1/2} - 3\left\{\frac{2r(100-r^2)-r^3}{\sqrt{100-r^2}}\right\}\right]$$

$$= \frac{1}{100} \times \left[-3r\,(100-r^2)^{1/2} - 3\left\{\frac{200r-3r^3}{\sqrt{100-r^2}}\right\}\right]$$

$$= \frac{1}{100} \times \left[-3r(100-r^2)^{1/2} + \frac{9r^3-600r}{\sqrt{100-r^2}}\right]$$

$$= \frac{1}{100} \times \left[\frac{-3r(100-r^2)+9r^3-600r}{\sqrt{100-r^2}}\right]$$

$$= \frac{1}{100} \times \left[\frac{-300r+3r^3+9r^3-600r}{\sqrt{100-r^2}}\right]$$

$$= \frac{1}{100} \times \left[\frac{12r^3-900r}{\sqrt{100-r^2}}\right]$$

Now, put $\frac{dA}{dr} = 0$

$\Rightarrow (100-r^2)^{3/2} = 3r^2\,(100-r^2)^{1/2}$

$\Rightarrow 100-r^2 = 3r^2 \quad \Rightarrow 4r^2 = 100$

$\Rightarrow r^2 = 25 \Rightarrow |r| = 5$

$$\left.\frac{d^2A}{dr^2}\right|_{|r|=5} = \frac{1}{100} \times \left[\frac{12r^3-900r}{\sqrt{100-r^2}}\right]_{|r|=5}$$

$= \frac{1}{100} \times \left[\frac{1500-4500}{\sqrt{75}}\right] = -\text{ve}$

Hence, area would be maximum.

Q.7 If p(x) be the cubic polynomial satisfying p (–1) = 10, p(1) = – 6 and p(x) has maximum at x = –1 and p′(x) has minima at x = 1. Find the points of local maxima and minima, also find the distance between these two points.

[IIT 2005]

Sol. Let $P(x) = ax^3 + bx^2 + cx + d$

$P(-1) = -a + b - c + d = 10$

$\Rightarrow b + d = 10 + a + c$ …(i)

$P(1) = a + b + c + d = -6$

$\Rightarrow a + b + c + d = -6$ …(ii)

$P'(x) = 3ax^2 + 2bx + c = 0$

$P'(x)\big|_{(x=-1)} = 3a - 2b + c = 0$

$\Rightarrow c = 2b - 3a$ …(iii)

$P''(x) = 6ax + 2b + 0$

$\Rightarrow P''(x)\big|_{(x=1)} = 6a + 2b = 0$

$\Rightarrow b = 3a = 0$ ….(iv)

Now solving equations (i), (ii), (iii) and (iv), we get

$b = -3a$

$c = 2b - 3a = -6a - 3a$

$c = -99$

$b + d = 10 + a + c$

$a + b + c + d = -6$

$(b + d) + (a + c) = -6$

$10 + (a + c) + (a + c) = -6$

$2(a + c) = -16 \Rightarrow (a + c) = -8$

$\Rightarrow (a - 9a) = -8$

$\Rightarrow a = 1$

$b = -3$

$c = -9$

$d = 10 + 1 - 9 + 3$

$d = 5$

Hence, $P(x) = x^3 - 3x^2 - 9x + 5$

$P'(x) = 3x^2 - 6x - 9 + 0$

Now, put $P'(x) = 0$

$\Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - 3x + x - 3 = 0$
$\Rightarrow x(x-3) + 1(x-3) = 0$
$\Rightarrow (x-3)(x+1) = 0$
$\Rightarrow x = -1, 3$
$P''(x) = 6x - 6 + 0$
$P''(x) = 6(x-1)$
$P''(x)\big|_{(x=-1)} = 6(-2) = -12$
x = –1 give maxima
from, $P(x) = x^3 - 3x^2 - 9x + 5$
$P(x)\big|_{(x=-1)} = -1 - 3 + 9 + 5 = 10$
Hence, point of maxima (–1, 10)
$P''(x)\big|_{(x=3)} = 6(3-1) = 12$
x = 3 gives minima
$P(x)\big|_{(x=3)} = 27 - 27 - 27 + 5 = -22$
Hence, point of minima (3, –22)
(–1, 10) = (3, – 22)
Distance = $\sqrt{(3+1)^2 + (-22-10)^2}$
$= \sqrt{16+1024} = \sqrt{1040}$
$= 4\sqrt{65}$ unit.

Q.8 f(x) is a cubic polynomial such that f(3) = 18, f(–1) = 2 and f(x) has local maximum at x = –1. If $f'(x)$ has local maximum at x = 0, then **[IIT 2006]**

(A) f(x) is increasing for $x \in [1, 2\sqrt{5}]$
(B) the distance between (–1, 2) and (a, f(a)) where x = a is the point of local minimum is $2\sqrt{5}$
(C) f(x) has local minima at x = 1
(D) the value of f(0) = 5

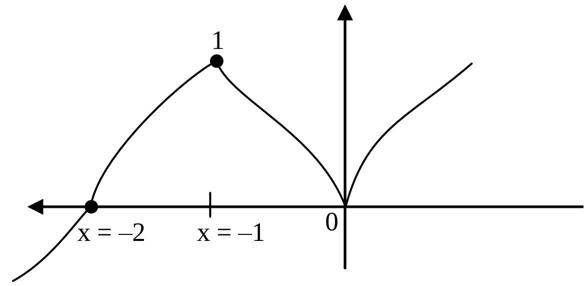
Sol. **[A, B,C]**

Q.9 The total number of local maxima and local minima of the function **[IIT 2008]**

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases} \text{ is -}$$

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **[C]**



Q.10 The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x | x^2 + 20 \le 9x\}$ is ..... **[IIT 2009]**

Sol. **[7]**

$f(x) = 2x^3 - 15x^2 + 36x - 48$
$f'(x) = 6(x-2)(x-3)$
$\Theta\ A = \{x | x^2 + 20 - 9x \le 0\}$
$4 \le x \le 5$
so f(x) is increasing for $x \in [3, \infty)$
so $(f(x))_{max}$ at $x \in [4, 5]$ is f(5)
so $(f(x))_{max} = f(5) = 7$

Q.11 Let f , g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[IIT 2010]**

(A) a = b and c $\ne$ b (B) a = c and a $\ne$ b
(C) a $\ne$ b and c $\ne$ b (D) a = b = c

Sol. **[D]**

$f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right)$
$g'(x) = e^{x^2}\left(2x^2 - 2x + 1\right)$
$h'(x) = 2x^3e^{x^2}$
Θ all f'(x) , g'(x), h'(x) are positive so all attains absolute maxima at x = 1
So Θ f (1) = g(1) = h(1) = e + e^{-1} = a = b = c

Q.12 Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \lambda n\{g(x)\}$, for all $x \in R$, then the number of points in R at which g has a local maximum is **[IIT 2010]**

Sol. $g(x) = e^{f(x)}$

$g'(x) = e^{f(x)} f'(x)$

$g'(x) = 0 \Rightarrow f'(x) = 0 \Rightarrow x = 2009, 2010, 2011, 2012$

Points of local maxima = 2009, $\Rightarrow$ only one point

Q.13 Let $f : IR \to IR$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT 2012]**

Sol.[5]

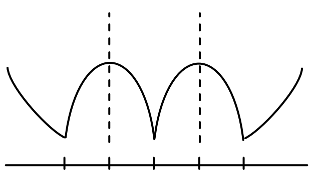
$$f(x) = |x| + |x - 1||x + 1|$$

$x \geq 1$ $\quad f(x) = x^2 + x - 1$ $\quad f'(x) = 2x + 1$ $\quad$ +ve

$0 \leq x < 1$ $\quad f(x) = 1 - x^2 + x$ $\quad f'(x) = 1 - 2x$ $\quad x > \frac{1}{2}$ –ve

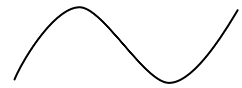
$-1 < x < 0$ $\quad f(x) = 1 - x^2 - x$ $\quad f'(x) = -2x - 1$ $\quad x > -\frac{1}{2}$ –ve ; $x < -\frac{1}{2}$ +ve

$x \leq -1$ $\quad f(x) = x^2 - x - 1$ $\quad f'(x) = 2x - 1$ $\quad$ –ve



Q.14 Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then $p'(0)$ is **[IIT 2012]**

Sol. [9] $P'(1) = 0, P'(3) = 0$



$P'(x) = K(x - 1)(x - 3)$

$= K(x^2 - 4x + 3)$ $\qquad P'(0) = 3K$

$$P(x) = \frac{K}{3}x^3 - 2Kx^2 + 3Kx + \lambda$$

$\frac{K}{3} - 2K + 3K + \lambda = 6, \quad K - 18K + 9K + \lambda = 2$

$\frac{4}{3}K + \lambda = 6, \qquad \frac{4}{3}K = 4$

$K = 3$

$P'(0) = 9$

EXERCISE # 5

Q.1 The function $f(x) = 3 + 2(a + 1)x + (a^2 + 1)x^2 - x^3$ has a local minimum at $x = x_1$ and local maximum at $x = x_2$ such that $x_1 < 2 < x_2$, then a belongs to interval(s)

(A) $\left(-\infty, -\frac{3}{2}\right)$ (B) $\left(-\frac{3}{2}, 1\right)$

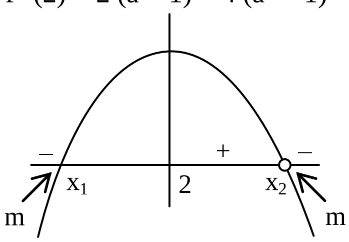
(C) $(0, \infty)$ (D) $(1, \infty)$

Sol. **[A, D]**

$f'(x) = 2(a + 1) + 2(a^2 + 1)x - 3x^2$

Clearly it is opening downward graph (coeff of x^2 – ve)

$f'(2) = 2(a + 1) + 4(a^2 + 1) - 12 > 0$



$4a^2 + 2a - 6 > 0$

$2a^2 + a - 3$ s> 0

$2a^2 + 3a - 2a - 3 > 0$

$(a + 1)(2a + 3) > 0 \Rightarrow a \in (-\infty, -3/2) \cup (1, \infty)$

Q.2 Let $g'(x) > 0$ and $f'(x) < 0\ \forall\ x \in R$, then

(A) $f(f(x + 1)) > f(f(x - 1))$

(B) $f(g(x - 1)) > f(g(x + 1))$

(C) $g(f(x + 1)) < g(f(x - 1))$

(D) $g(g(x + 1)) > g(g(x - 1))$

Sol. **[A, B, C, D]**

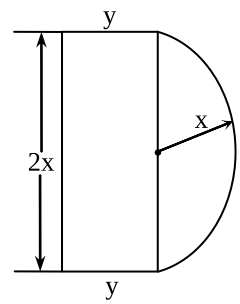
g (x) ↑ and f (x) ↓

then fog and gof will be ↓

while fof and gog will be ↑ Hence A, B, C, D all the true

Q.3 A housing corporation wants to build some flats. The ground plane of a flat consists of a semicircle with a rectangle constructed on its diameter. Given that the perimeter of the flat is 50 meter, find its dimensions in order that the area covered is maximum.

Sol. Let radius of semi circle be x meter and dimensions of rectangle be y meter and 2x metres.



Given perimeter of plate,

$\pi x + 2y + 2x = 50$

$y + x + \frac{\pi}{2}x = 25$

$y = 25 - x\left(1 + \frac{\pi}{2}\right)$

Area of flat, $A = \frac{\pi x^2}{2} + 2xy$

$A = \frac{\pi x^2}{2} + 2x\,[25 - x(1 + \pi/2)]$

$A = \frac{\pi x^2}{2} + 50x - 2x^2(\pi/2 + 1)$

$A = \frac{\pi x^2}{2} + 50x - \pi x^2 - 2x^2$

$A = \frac{\pi x^2}{2} + 50x - \pi x^2 - 2x^2$

$A = 50x - 2x^2 - \pi x^2/2$

Differentiating above function w.r.t. x, we get

$\frac{dA}{dx} = 50 - 4x - \pi x$

$= 50 - (4 + \pi)x$

Again differentiating w.r.t. x, we get

$\frac{d^2A}{dx^2} = 0 - (4 + \pi)$

$= -(4 + \pi)$

$= -$ve

Put $\frac{dA}{dx} = 0 \Rightarrow 50 - (4 + \pi)x = 0$

$\Rightarrow x = 50/(4 + \pi)$

$\Rightarrow x = \dfrac{50}{4+\frac{22}{7}} = \dfrac{50}{50/7}$

$\Rightarrow x = 7$ meters

$y = 25 - \dfrac{x}{2}(\pi + 2)$

$= 25 - \dfrac{7}{2}\left(\dfrac{22}{7} + 2\right)$

$= 25 - \dfrac{7}{2}\left(\dfrac{22+14}{7}\right) = 25 - \dfrac{7}{2} \times \dfrac{36}{7} = 25 - 18$

= 7 meters.

x = 7 meters and y = 7 meters.

Q.4 Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Find the intervals in which λ should lies in order that f(x) has exactly one minimum and exactly one maximum. **[IIT 1985]**

Sol. The given function is,

$f(x) = \sin^3 x + \lambda \sin^2 x$ for $-\pi/2 < x < \pi/2$

$f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x$

$= \dfrac{1}{2} \sin 2x\,(3\sin x + 2\lambda)$

So, from $f'(x) = 0$, we get $x = 0$ or $3\sin x + 2\lambda = 0$

Also, $f''(x) = \cos 2x\,(3\sin x + 2\lambda) + \dfrac{3}{2}\sin 2x \cos x$

Therefore, for $\lambda = \dfrac{-3}{2}\sin x$, we have

$f''(x) = 3\sin x \cos^2 x = -2\lambda \cos^2 x$

Now, if $0 < x < \pi/2$, then $-3/2 < \lambda < 0$ and therefore $f''(x) > 0$,

$\Rightarrow$ f(x) has one minimum for this value of λ.

Also for x = 0, we have $f''(0) = 2\lambda < 0$, That is, f(x) has a maximum at x = 0

Again, if $-\pi/2 < x < 0$, then $0 < \lambda < 3/2$ and therefore $f''(x) = -2\lambda \cos^2 x < 0$.

So that f(x) has a maximum.

Also for x = 0, $f''(a) = 2\lambda > 0$ so that f(x) has a minimum.

Thus, for exactly one maximum and minimum value of f(x), λ must lie in the interval

$-3/2 < \lambda < 0$ or $0 < \lambda < 3/2$

i.e., $\lambda \in (-3/2, 0) \cup (0, 3/2)$.

Q.5 Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \ldots < a_n$. The function P(x) has- **[IIT-1986]**

(A) neither a maximum nor a minimum

(B) only one maximum

(C) only one minimum

(D) only one maximum and only one minimum

Sol. **[C]**

The given polynomial is

$p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$, $x \in R$

and $0 < a_0 < a_1 < a_2 < \ldots < a_n$.

Method 1 : Here we observe that all coefficients of different powers of x, i.e., a_0, a_1, a_2, …… a_n are positive.

Also only even powers of x are involved.

$\therefore$ P(x) can not have any max. value.

More over P(x) is minimum, when x = 0, i.e., a_0.

$\therefore$ P(x) has only one minimum.

$\therefore$ 'C' is the correct answer.

Method 2 : We have $P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_nx^{2n-1}$ clearly P(x) increases for all x > 0 and decreases for all x < 0.

$\therefore P'(x)$ has no max. value and min. value at x = 0

$\therefore$ 'C' is the correct answer.

Method 3 : We have $P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_nx^{2n-1}$

$P'(x) = 0 \Rightarrow x = 0$

$P''(x) = 2a_1 + 12a_2x^2 + \ldots + 2n(2n-1)a_nx^{2n-2}$

$P''(x)|_{x=0} = +$ ve as $a_1 > 0$

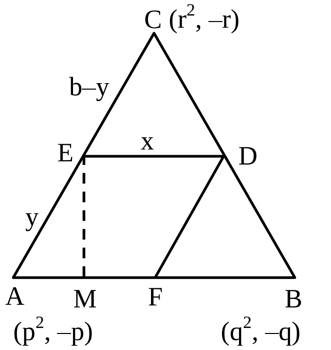
$\therefore$ P(x) has only one minimum at x = 0.

$\therefore$ 'C' is the correct answer.

Q.6 Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is $\dfrac{1}{4}(p+q)(q+r)(p-r)$.

[IIT 1986]

Sol.



$\because$ AEDF is a parallelogram.

Let AF = x = ED

and AE = y = FD

Then from similar Δ 's CED and CAB, we have

$$\frac{CE}{ED} = \frac{CA}{AB} \quad \text{or} \quad \frac{b-y}{x} = \frac{b}{c} \quad \ldots..(1)$$

Let P denote the area of $| |^{gm}$ AFDE.

Then $\quad P = 2\Delta\, AEF = 2 \cdot \frac{1}{2}\, xy \sin A$

$= y \cdot \frac{c}{b}\,(b - y) \sin A$ [Using eq. (1)]

$\Rightarrow P = \frac{c}{b} \sin A\,(by - y^2)\ \therefore \frac{dP}{dy} = \frac{c}{b} \sin A\,(b - 2y) = 0$

Above gives $y = \frac{b}{2}$ and $\frac{d^2P}{dy^2} = \frac{c}{b} \sin A(-2) = -ve$

Hence area is maximum when y = b/2 and its value is

$$P = \frac{b}{2}\cdot\frac{c}{b}\left(b - \frac{b}{2}\right) \sin A = \frac{1}{4}\, bc \sin A = \frac{1}{2}\ .\Delta ABC$$

Now, $\quad \Delta = \frac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{vmatrix}$

Operating $R_2 - R_1$ and $R_3 - R_1$, we get

$$= \frac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 - p^2 & q + p & 0 \\ r^2 - p^2 & p - r & 0 \end{vmatrix}$$

$$= \frac{1}{2}\,(q + p)\,(p - r)\begin{vmatrix} q - p & 1 \\ -(r + p) & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,(q + p)\,(p - r)\,(q + r)$$

Thus max. area of $| |^{gm}$ AFDE is

$$= \frac{1}{2}\Delta = \frac{1}{4}\,(p + q)\,(q + r)\,(p - r)$$

Q.7 The smallest positive root of the equation, $\tan x - x = 0$ lies in- **[IIT-1987]**

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{2}, \pi\right)$

(C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(\frac{3\pi}{2}, 2\pi\right)$

Sol. **[C]**

Let $f(x) = \tan x - x$

(A) If $x \in (0, \pi/2)$ then

$f'(x) = \sec^2 x - 1 > 0,\ 0 < x < \pi/2$

$\Rightarrow f(x)$ is monotonically increasing in $(0, \pi/2)$

Also $f(0) = 0$ and $f(x) > 0$ for $0 < x < \pi/2$

[Keeping in mind that $\tan x > x$ for $x \in (0, \pi/2)$]

$\therefore$ $f(x) = 0$ has no root in $(0, \pi/2)$

(B) If $x \in (\pi/2, \pi)$ i.e. $\pi/2 < x < \pi$ then consider

$x = \pi/2 + \alpha$

$\Rightarrow \pi/2 < \pi/2 + \alpha < \pi \Rightarrow 0 < \alpha < \pi/2$

$\therefore f(x) = \tan(\pi/2 + \alpha) - (\pi/2 + \alpha)$

$= -[\cot \alpha + \pi/2 + \alpha] < 0$

As the above is true for any $\alpha \in (\pi/2, \pi)$

$\therefore$ $f(x) = 0$ has no root in $(\pi/2, \pi)$.

(C) If $x \in (\pi, 3\pi/2)$ then $f(x + 0)$

$= \lim_{h\to 0} f(\pi + h) = \lim_{h\to 0} [\tan(\pi + h) - (\pi + h)]$

$= \lim_{h\to 0} [\tan h - \pi - h] = -\pi < 0$

And $\quad f(3\pi/2 - 0) = \lim_{h\to 0} f(3\pi/2 - h)$

$= \lim_{h\to 0} [\tan(3\pi/2 - h) - (3\pi/2 - h)]$

$= \lim_{h\to 0} [\cot h - 3\pi/2 + h]$

$= +\infty$

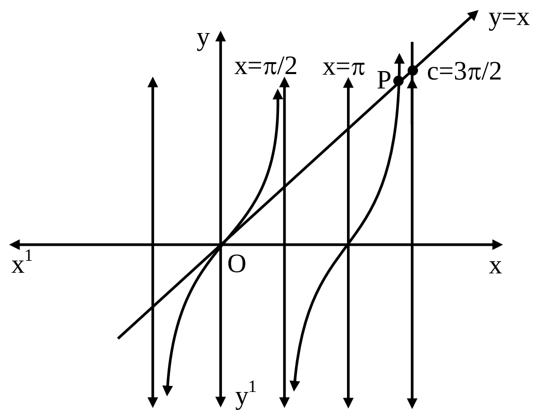
i.e. sign of f(x) changes from –ve to +ve in the interval $(\pi, 3\pi/2)$

$\therefore \exists\, x \in (\pi, 3\pi/2)$ such that $f(x) = 0$

Hence the root of $f(x) = 0$ lies in $(\pi, 3\pi/2)$

Thus the smallest +ve root of the given equation lies in $(\pi, 3\pi/2)$

$\therefore$ (C) is the correct answer.



Alternative

It is clear from the graph that the curves

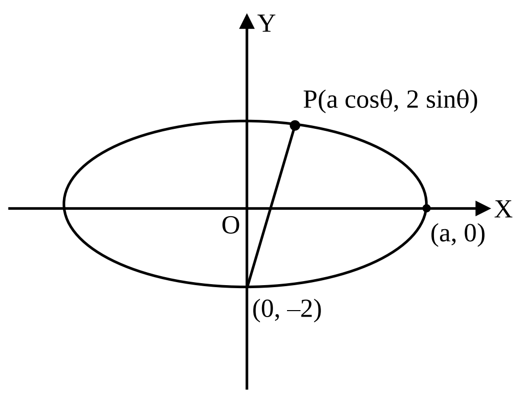
$y = \tan x$ and $y = x$ intersect at P in $(\pi, 3\pi/2)$

Thus the smallest +ve interval in which tan x has solution is $(\pi, 3\pi/2)$

$\therefore$ (C) is the correct answer.

Q.8 Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. **[IIT 1987]**

Sol.



The equation of given curve can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1$$

where $4 < a^2 < 8$

Clearly it is the equation of an ellipse

Let us consider a pt P $(a \cos\theta, 2\sin\theta)$ on the ellipse.

Let the distance of P $(a \cos\theta, 2 \sin\theta)$ from $(0, -2)$ is L

Then, $L^2 = (a \cos\theta - 0)^2 + (2 \sin\theta + 1).\cos\theta$

$\Rightarrow$ Differentiating with respect to θ, we have

$$\frac{d(L^2)}{d\theta} = 0 \Rightarrow \cos\theta\,[-2a^2 \sin\theta + 8\sin\theta + 8\,]$$

For max or min value of L we should have

$$\frac{d(L^2)}{d\theta} = 0 \Rightarrow \cos\theta\,[-2a^2 \sin\theta + 8 \sin\theta + 8] = 0$$

$\Rightarrow$ either $\cos\theta = 0$ or $(8 - 2a^2)\sin\theta + 8 = 0$

$$\Rightarrow \quad \theta = \frac{\pi}{2} \text{ or } \sin\theta = \frac{4}{a^2 - 4}$$

Since $a^2 < 8 \Rightarrow a^2 - 4 < 4$

$$\Rightarrow \quad \frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1$$

where is not possible

Also $\dfrac{d^2(L^2)}{d\theta^2} = \cos\theta\,[-2a^2\cos\theta + 8\cos\theta] +$

$(-\sin\theta)\,[-2a^2\sin\theta + 8\sin\theta + 8]$

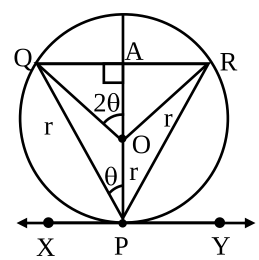
At $\theta = \dfrac{\pi}{2}$, $\dfrac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8)$

< 0 as $a^2 < 8$

$\therefore$ L is max. at $\theta = \pi/2$ and the farthest pt is (0, 2)

Q.9 A point P is given on the circumference of a circle of radius r. Chords QR are parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. **[IIT-1990]**

Sol.



As QR || XY diameter through P is $\perp$ QR.

Now area of ΔPQR is given by

$$A = \frac{1}{2} QR \,.\, AP$$

But $QR = 2\,.\,QA$

$= 2r \sin 2\theta$

and $PA = OA + OP$

$= r\cos 2\theta + r$

$$\therefore A = \frac{1}{2}\,.\,2r\sin 2\theta.\,(r + r\cos 2\theta)$$

$= r^2\,.\,2\sin\theta\cos\theta\,.\,2\cos^2\theta = 4r^2\sin\theta\cos^3\theta$

For max. value of area $\dfrac{dA}{d\theta} = 0$

$\Rightarrow 4r^2\,[\cos^4\theta - 3\sin^2\theta\cos^2\theta] = 0$

$\Rightarrow$ $\cos^2\theta(\cos^2\theta - 3\sin^2\theta) = 0$

$\Rightarrow$ $\tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow$ $\theta = 30°$

Also $\frac{d^2A}{d\theta^2} = 4r^2[-4\cos^3\theta\sin\theta - 6\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$= 4r^2[-10\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$$\frac{d^2A}{d\theta^2}\Big|_{\theta=30°} = 4r^2\left[-10.\frac{1}{2}.\frac{3\sqrt{3}}{8} + 6.\frac{1}{8}.\frac{\sqrt{3}}{2}\right]$$

$$= 4r^2\left[\frac{-15\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right]$$

$$= 4r^2\left(\frac{-12\sqrt{3}}{8}\right) = -ve$$

$\therefore$ A is max. at $\theta = 30°$

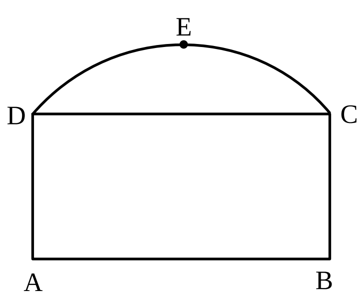
And $A_{max} = 4r^2\sin 30°\cos^3 30°$

$$= 4r^2 \times \frac{1}{2} \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}r^2$$

Q.10 A window of perimeter P (including the base of the arch) is in the form of a rectangle surmounted by a semi circle. The semi circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light?

[IIT-1991]

Sol.



Let ABCEDA be the window as shown in the figure and let

AB = x m

BC = y m

Then its perimeter including the base DC of arch

$$= \left(2x + 2y + \frac{\pi x}{2}\right) \text{ m}$$

$\therefore$ $P = \left(2 + \frac{\pi}{2}\right)x + 2y$ ....(1)

Now area of rectangle ABCD = xy

and area of arch

$$DCED = \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

Let λ be the light transmitted by coloured glass per sq. m, Then 3λ will be the light transmitted by clear glass per sq. m

Hence the area of light transmitted

$$= 3\lambda(xy) + \lambda\left[\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right]$$

$\Rightarrow$ $A = \lambda\left[3xy + \frac{\pi x^2}{8}\right]$ ...(2)

Substituting the value of y from (1) in (2), we get

$$A = \lambda\left[3x\frac{1}{2}\left[P - \left(\frac{4+\pi}{2}\right)x\right] + \frac{\pi x^2}{8}\right]$$

$$= \lambda\left[\frac{3Px}{2} - \frac{3(4+\pi)}{4}x^2 + \frac{\pi x^2}{8}\right]$$

$$\frac{dA}{dx} = \lambda\left[\frac{3P}{2} - \frac{3(4+\pi)}{2}x + \frac{\pi x}{4}\right]$$

For A to be maximum $\frac{dA}{dx} = 0$

$$\Rightarrow x = \frac{\left(\frac{3P}{2}\right)}{\left[\frac{-\pi}{4} + \left(\frac{12+3\pi}{2}\right)\right]}$$

$\Rightarrow$ $x = \frac{3P}{2} \times \frac{4}{5\pi + 24}$ $\Rightarrow$ $x = \frac{6P}{5\pi + 24}$

Also $\frac{d^2A}{dx^2} = \lambda\left[\frac{-3(4+\pi)}{2} + \frac{\pi}{4}\right] < 0$

$\therefore$ A is max when $x = \frac{6P}{5\pi + 24}$

$\Rightarrow$ $5\pi x + 24x = 6\left[\left(\frac{4+\pi}{2}\right)x + 2y\right]$

[Using value of P from (1)]

$\Rightarrow$ $(5\pi + 24 - 12 - 3\pi)x = 12y$

$\Rightarrow$ $(2\pi + 12)x = 12y \Rightarrow \frac{y}{x} = \frac{\pi + 6}{6}$

$\therefore$ The required ratio of breadth to length of the rectangle $= \left(\frac{6+\pi}{6}\right)$

Q.11 What normal to the curve $y = x^2$ forms the shortest chord? **[IIT-1992]**

Sol. The given curve is $y = x^2$ ….(1)

Consider any pt. $A(t, t^2)$ on (1) at which normal chord drawn is shortest.

Then eq. of normal to (1) at $A(t, t^2)$ is

$$y - t^2 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(t,t^2)}}(x - t)$$

[where $\frac{dy}{dx} = 2x$ from (1)]

$$y - t^2 = -\frac{1}{2t}(x-t)$$

$\Rightarrow x + 2ty = t + 2t^3$ ….(2)

This normal meet the curve again at pt B which can be obtained by solving (1) and (2) as follows :

Putting $y = x^2$ in (2), we get

$2t\,x^2 + x - (t + 2t^3)\,d = 0$

$D = 1 + 8t(t + 2t^3) = 1 + 8t^2 + 16t^4 = (1 + 4t^2)^2$

$\therefore \quad x = \frac{-1+1+4t^2}{4t}, \frac{-1-1-4t^2}{4t}$

$= t, -\frac{1}{2t} - t$

$\therefore \quad y = t^2, t^2 + \frac{1}{4t^2} + 1$

Thus, $B\left(-t-\frac{1}{2t}, t^2+\frac{1}{4t^2}+1\right)$

$\therefore$ Length of normal chord

$$AB = \sqrt{\left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2}$$

Consider $Z = AB^2 = \left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2$

$Z = \frac{1}{16t^4}+\frac{3}{4t^2} + 3 + 4t^2$, $\frac{dz}{dt} = 0$

$\Rightarrow -\frac{1}{4t^5}-\frac{3}{2t^3} + 8t = 0 \Rightarrow -1 - 6t^2 + 32t^6 = 0$

$\Rightarrow 32(t^2)^3 - 6t^2 - 1 = 0 \Rightarrow (2t^2 - 1)(16t^4 + 8t^2 + 1) = 0$

$\Rightarrow \quad t^2 = \frac{1}{2}$ (leaving –ve values of t^2)

$\Rightarrow \quad t = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \Rightarrow \frac{d^2Z}{dt^2} = \frac{5}{4t^6}+\frac{9}{2t^4}+8$

$\left.\frac{d^2Z}{dt^2}\right|_{t=\frac{1}{\sqrt{2}}}$ = + ve also $\left|\frac{d^2Z}{dt^2}\right|_{t=-\frac{1}{\sqrt{2}}}$ = +ve

$\therefore$ Z is min at $t = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

For $t = \frac{1}{\sqrt{2}}$ normal chord is (from (2))

$x + \sqrt{2}y = \sqrt{2}$

For $t = -\frac{1}{\sqrt{2}}$ normal chord is

$$x - \sqrt{2}y = -\sqrt{2}$$

Q.12 Let $f(x) = \begin{cases} -x^3 + \frac{(b^3-b^2+b-1)}{(b^2+3b+2)}, & 0 \le x \le 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at x = 1. **[IIT-1993]**

Sol. We have $f(x) = \begin{cases} -x^3 + \frac{b^3-b^2+b-1}{b^3+3b+2}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$

We can see from definition of the function, that

$f(1) = 2(1) - 3 = -1$

Also f(x) is increasing on [1, 3], $f'(x)$ being 2>0

$\therefore$ f(1) = – 1 is the smallest value of f(x)

Again $f'(x) = -3x^2$ for $x \in [0, 1)$ s.t. $f'(x) < 0$

$\Rightarrow$ f(x) is decreasing on [0, 1)

$\therefore$ For fixed value of b, its smallest occur at $x \to 1$

i.e., $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} -(1-h)^3 + \frac{b^3-b^2+b-1}{b^2+3b+2} = -1 + \frac{b^3-b^2+b-1}{b^2+3b+2}$

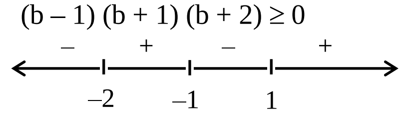
As given that the smallest value of f(x) occur at x = 1

$\therefore$ Any other smallest value $\ge$ f(1)

$\Rightarrow \quad -1+\frac{b^3-b^2+b-1}{b^2+3b+2} \ge -1 \Rightarrow \frac{b^3-b^2+b-1}{b^2+3b+2} \ge 0$

$\Rightarrow \quad \frac{(b^2+1)(b-1)}{(b+2)(b+1)} \ge 0$

$\Rightarrow \quad (b-1)(b+1)(b+2) \ge 0$

$\Rightarrow$ 

$\Rightarrow b \in (-2, -1) \cup (1, \infty)$

Q.13 If f(x) = $\begin{cases} 3x^2+12x-1, & -1\le x\le 2 \\ 37-x, & 2<x\le 3 \end{cases}$ then;

[IIT-1993]

(A) f(x) is increasing on [–1, 2]
(B) f(x) is continuous on [–1, 3]
(C) $f'(2)$ does not exist
(D) f(x) has the maximum value at x = 2

Sol. **[A, B, C, D]**

We are given that

f(x) = $\begin{cases} 3x^2+12x-1, & -1\le x\le 2 \\ 37-x, & 2<x\le 3 \end{cases}$

Then on [–1, 2], $f'(x) = 6x + 12$

for $\quad -1 \le x \le 2, -6 \le 6x \le 12$

$\Rightarrow \quad 6 \le 6x + 12 \le 24$

$\Rightarrow \quad f'(x) > 0, \forall x \in [-1, 2]$

$\therefore$ f is increasing on [–1, 2]

Also f(x) being polynomial for $x\varepsilon[-1,2) \cup (2, 3]$

f(x) is cont. on [–1, 3] except possibly at

At x = 2, LHL = $\lim\limits_{h\to 0} f(2-h) = \lim\limits_{h\to 0} 3(2-h)^2 + 12(2-h) - 1 = 35$

RHL = $\lim\limits_{h\to 0} f(2+h) = \lim\limits_{h\to 0} 37-(2+h) = 35$

and $f(2) = 3.2^2 + 12.2 - 1 = 35$

LHL = RHL = f(2) $\Rightarrow$ f(x) is continuous at x = 2

Hence f(x) is constinuous on [–1, 3]

Again at x = 2

$$RD = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{37-(2+h)-35}{h} = 1$$

$$LD = \lim_{h\to 0} \frac{f(2)-f(2-h)}{h}$$

$$= \lim_{h\to 0} \frac{35-3(2-h)^2-12(2-h)+1}{h}$$

$$= \lim_{h\to 0} \frac{35-3(4-4h+h^2)-24+12h+1}{h}$$

$$= \lim_{h\to 0} \frac{35-12+12h-3h^2-23+12h}{h}$$

$$= \lim_{h\to 0} \frac{-3h^2+24h}{h} = 24$$

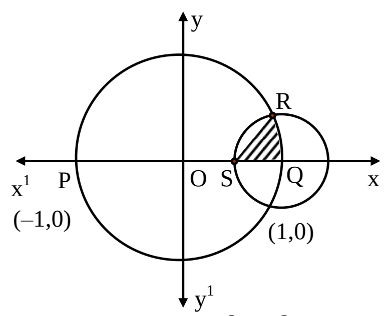
As $\quad LD \ne RD$

$\therefore \quad f'(2)$ does not exist. Hence f(x) can not have max. value at x = 2

Thus (A), (B), (C) are the correct options.

Q.14 The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR. **[IIT-1994]**

Sol.



The given circle is $\quad x^2 + y^2 = 1 \quad$ ….(1)

which intersect x-axis at P(–1, 0) and Q(1, 0).

Let radius of circle with centre at Q(1, 0) be r, where r is variable.

Then equation of this circle is, $(x-1)^2 + y^2 = r^2$ ….(2)

Subtracting (1) from (2) we get,

$$(x-1)^2 - x^2 = (r^2 - 1)$$

$\Rightarrow -2x + 1 = r^2 - 1 \Rightarrow x = 1 - \frac{r^2}{2}$

Substituting this value of x in (2), we get

$$\frac{r^2}{4} + y^2 + r^2 \Rightarrow y = \pm r\sqrt{1-\frac{r^2}{4}}$$

$\therefore \quad R\left(1-\frac{r^2}{2}, r\sqrt{1-\frac{r^2}{4}}\right)$ pt. above x-axis.

$\therefore$ Area of $\Delta QRS = \frac{1}{2}$ SQ × ordinate of pt. R

$\Rightarrow \quad A = \frac{1}{2} \times r \times r\sqrt{1-\frac{r^2}{4}}$

A will be max. if A^2 is max.

$$A^2 = \frac{r^4}{4}\left(1-\frac{r^2}{4}\right) = \frac{r^4}{4} - \frac{r^6}{16}$$

Differentiating A^2 w.r.t. to r, we get

$$\frac{dA^2}{dr} = r^2 - \frac{3}{8}r^5$$

For A^2 to be max. $\dfrac{dA^2}{dr} = 0 \Rightarrow r^3\left(1-\dfrac{3}{8}r^2\right) = 0$

$\Rightarrow r = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$\dfrac{d^2(A^2)}{dr^2} = 3r^2 - \dfrac{15}{8}\, r^4$

$$\Rightarrow \left.\frac{d^2(A^2)}{dr^2}\right|_{r^2=\frac{8}{3}} = 3\times\frac{8}{3} - \frac{15}{8}\times\frac{64}{9} = -\text{ve}$$

$\therefore$ A^2 and hence A is max. when, $r = \dfrac{2\sqrt{2}}{\sqrt{3}}$

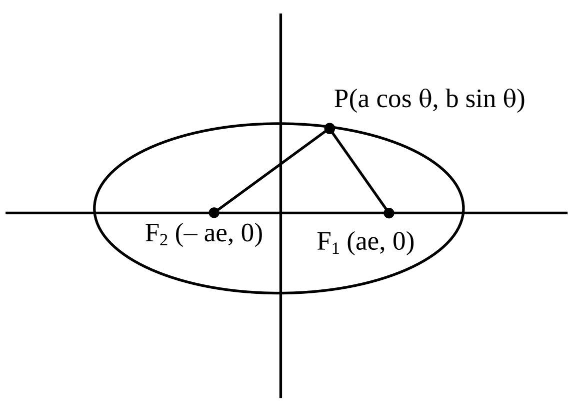
$$\therefore\ \text{Max. area} = \sqrt{\frac{1}{4}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^4 - \frac{1}{16}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^6}$$

$$= \sqrt{\frac{1}{4}\times\frac{64}{9} - \frac{1}{16}\times\frac{512}{27}} = \sqrt{\frac{16}{9} - \frac{32}{27}}$$

$$= \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9} \text{ sq. units.}$$

Q.15 Let P be a variable point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2 then the maximum value of A is.............. **[IIT-1994]**

Sol.



Let P (a cosθ, b sinθ) be any point on the ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci F_1 (ae, 0) and F_2 (– ae, 0)

Then area of $\Delta P\, F_1\, F_2$ is given by

$$A = \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ |-b\sin\theta\,(ae + ae)| = abe\,|\sin\theta|$$

Θ $|\sin\theta| \le 1$ $\therefore$ $A_{max} = abe$

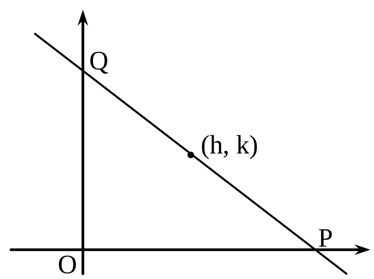
Q.16 Let (h,k) be a fixed point, where h > 0, k > 0, A straight line passing through this point cuts the positive direction of the coordinates axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.

[IIT 1995]

Sol. Let equation of line be y = mx + c

k = mh + c ⇒ c = (k –mh)

y = mx + (k – mh) ⇒ – mx + (k –mh)



$$\Rightarrow \frac{x}{\frac{(mh-k)}{m}} + \frac{y}{(k-mh)} = 1$$

Hence, $OP = \left(\dfrac{mh-k}{m}\right)$

$OQ = (k - mh)$

Area of triangle OPQ, $A = \dfrac{1}{2} \times OP \times OQ$

$A = \dfrac{1}{2} \times \left(\dfrac{mh-k}{m}\right) \times (k - mh)$

$A = \dfrac{1}{2} \times \dfrac{1}{m} \times (-1) \times (mh - k)^2$

$A = -\dfrac{1}{2m}\,[m^2h^2 + k^2 - 2mhk]$

$A = \dfrac{1}{2}\,[-mh^2 - \dfrac{k^2}{m} + 2hk]$

$A = \dfrac{1}{2}\,[2hk - \dfrac{k^2}{m} - mh^2]$

Differentiating w.r.t.m, we get

$\dfrac{dA}{dm} = \dfrac{1}{2}\,[0 + \dfrac{k^2}{m^2} - h^2]$

Again differentiating w.r.t m, we get

$\dfrac{d^2A}{dm^2} = \dfrac{1}{2}\,[-\dfrac{2k^2}{m^3} - 0]$

$$\frac{d^2A}{dm^2} = -k^2/m^3$$

Now put, $\frac{dA}{dm} = 0 \Rightarrow \frac{k^2}{m^2} = h^2$

$\Rightarrow m^2 = (k/h)^2$

$\Rightarrow m = \pm k/h$

$$\left.\frac{d^2A}{dm^2}\right|_{\left(m=+\frac{k}{h}\right)} = -k^2/(k/h)^2$$

$= -$ ve

$$\left.\frac{d^2A}{dm^2}\right|_{\left(m=-\frac{k}{h}\right)} = -k^2/(k/h)^3$$

$= +$ ve

Hence, area will be minimum at $m = -k/h$

$$A_{min} = \frac{1}{2}\left[2hk + \frac{k^2}{k}\times h + \frac{k}{h}\times h^2\right] = 2hk$$

Q.17 The function $f(x) = |px - q| + r\,|x|$, $x \in (-\infty, \infty)$ where $p > 0$, $q > 0$, $r > 0$, assumes its minimum value only at one point if **[IIT 1995]**

(A) $p \neq q$ (B) $r \neq q$

(C) $r \neq p$ (D) $p = q = r$

Sol. **[C]**

$f(x) = |px - q| + r\,|x|$; $x \in R$

$p > 0, q > 0, r > 0$

when, $0 < x < q/p \Rightarrow |x| = x$

$|px - q| = -(px - q)$

$f(x) = -(px - q) + rx$

$f(x) = (r - p)x + q$

$f'(x) = (r - p)$

$(r - p)$ may be either positive or negative.

When $x > q/p \Rightarrow |px - q| = (px - q)$

$f(x) = (px - q) + rx$

$f(x) = (p + r)x - q$

$f'(x) = (p + r) = +$ve

$f''(x) = 0$

Hence, $f(x)$ would be minimum when $p + r = 0$

$\Rightarrow p = -r \Rightarrow p \neq r$

$\therefore$ option [C] is correct answer.

Q.18 Determine the points of maxima and minima of the function $f(x) = \frac{1}{8}\log x - bx + x^2$, $x > 0$, where $b \geq 0$ is a constant. **[IIT 1996]**

Sol. $f(x) = \frac{1}{8}\log x - bx + x^2$; $x > 0$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{8x} - b + 2x$$

Again differentiating w.r.t.x, we get

$$f''(x) = -\frac{1}{8x^2} - 0 + 2$$

$$f''(x) = 2 - \frac{1}{8x^2}$$

Now, put $f'(x) = 0 \Rightarrow \frac{1}{8x} - b + 2x = 0$

$\Rightarrow 1 - 8bx + 16x^2 = 0$

$\Rightarrow 16x^2 - 8bx + 1 = 0$

$$x = \frac{8b \pm \sqrt{64b^2 - b^4}}{2\times 16}$$

$$x = \frac{8b \pm 8\sqrt{b^2 - 1}}{2\times 16}$$

$$x = \frac{b \pm \sqrt{b^2-1}}{4};\ b \geq 0 \Rightarrow b > 1$$

$$\left.f''(x)\right|_{x=\frac{b+\sqrt{b^2-1}}{4}} = 2 - \frac{1}{8}\times\frac{16}{(b+\sqrt{b^2-1})^2}$$

$$\left.f''(x)\right|_{x=\frac{b+\sqrt{b^2-1}}{4}}$$

$$= 2 - \frac{1}{8}\,\frac{16}{(b^2 + b^2 - 1 + 2b\sqrt{b^2-1})}$$

$$= 2 - \frac{2}{(2b^2 + 2b\sqrt{b^2-1} - 1)}$$

$$= \frac{2[2b^2 + 2b\sqrt{b^2-1} - 1 - 1]}{(2b^2 + 2b\sqrt{b^2-1} - 1)}$$

$$= \frac{4[b^2 + b\sqrt{b^2-1} - 1]}{(2b^2 + 2b\sqrt{b^2-1} - 1)} = +\text{ve for } b > 1$$

$$\left.f''(x)\right|_{x=\frac{b-\sqrt{b^2-1}}{4}} = 2 - \frac{1}{8}\times\frac{16}{\left[b - \sqrt{b^2-1}\right]^2}$$

$$= 2 - \frac{2}{(b^2 + b^2 - 1 - 2bx\sqrt{b^2-1})}$$

$$= 2\left[1 - \frac{1}{2b^2 - 2b\sqrt{b^2-1} - 1}\right]$$

$$= 2\left[\frac{2b^2 - 2b\sqrt{b^2-1} - 1}{2b^2 - 2b\sqrt{b^2-1} - 1}\right]$$

$$= \frac{4[b^2 - b\sqrt{b^2-1} - 1]}{[2b^2 - 2b\sqrt{b^2-1} - 1]} = \text{–ve for } b > 1$$

Hence maxima at $x = \dfrac{b - \sqrt{b^2-1}}{4}$

minima at $x = \dfrac{b + \sqrt{b^2-1}}{4}$

Q.19 Find a point (α, β) on the ellipse $4x^2 + 3y^2 = 12$ in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum. **[IIT-1997]**

Sol. (3/2, 1)

Q.20 Suppose f(x) is a function satisfying the following conditions -

(i) f (0) = 2, f (1) = 1

(ii) f has a minimum value at x = 5/2 and

(iii) for all x

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a, b are some constants. Determine the constants a, b and the function f (x).

[IIT 1998]

Sol. Applying $R_3 \to R_3 - R_1 - 2R_2$ we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-a & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix}$$

$\Rightarrow$ f ′(x) = 2ax + b

Integrating, we get, $f(x) = ax^2 + bx + C$

where C is an arbitrary constant. Since f has a maximum at x = 5/2,

f ′(5/2) = 0 $\Rightarrow$ 5a + b = 0 ….(1)

Also f(0) = 2 $\Rightarrow$ C = 2 and f(1) = 1 $\Rightarrow$ a + b + c = 1

$\therefore$ a + b = –1

Solving (1)&(2) for a,b we get, a = 1/4, b = –5/4

….(2)

Thus, $f(x) = \dfrac{1}{4}x^2 - \dfrac{5}{4}x + 2.$

Q.21 The number of values of x where the function $f(x) = \cos x + \cos\left(\sqrt{2}x\right)$ attains its maximum is **[IIT 1998]**

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. **[B]**

$f(x) = \cos x + \cos(\sqrt{2}\, x)$

Differentiating w.r.t. x, we get

$f'(x) = -\sin x - \sqrt{2}\sin(\sqrt{2}\, x)$

Again differentiating w.r.t.x, we get

$f''(x) = -\cos x - 2\cos\sqrt{2}\, x$

$f''(x) = -[\cos x + 2\cos\sqrt{2}\, x]$

Now, put $f'(x) = 0$

$\Rightarrow -(\sin x + \sqrt{2}\sin(\sqrt{2}\, x)) = 0$

$\Rightarrow \sin x = -\sqrt{2}\sin(\sqrt{2}\, x)$

It holds only at x = 0

$f''(x)\big|_{x=0} = -[1 + 2.1] = -3$

Hence, f(x) would be maximum.

$\therefore$ option [B] is correct answer.

Q.22 If $f(x) = \dfrac{x^2-1}{x^2+1}$, for every real number x, then the minimum value of f : **[IIT 1998]**

(A) does not exist because f is unbounded

(B) is not attained even though f is bounded

(C) is equal to 1

(D) is equal to –1

Sol. **[D]**

$f(x) = \dfrac{x^2-1}{x^2+1}$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{2x(x^2+1) - (x^2-1).2x}{(1+x^2)^2}$$

$$= \frac{2x^3 + 2x^2 - 2x^3 + 2x}{(1+x^2)^2} \Rightarrow f'(x) = \frac{4x}{(1+x^2)^2}$$

Again differentiating w.r.t. x, we get

$$f''(x) = \frac{4.1(1+x^2)^2 - 4x.2(1+x^2).2x}{(1+x^2)^4}$$

$$= \frac{4(1+x^2)^2 - 16x^2(1+x^2)}{(1+x^2)^4}$$

Now, put $f'(x) = 0$

$$\Rightarrow \frac{4x}{(1+x^2)^2} = 0 \Rightarrow x = 0$$

$$f''(x)\big|_{x=0} = \frac{4.-0}{1} = 4$$

Hence, f(x) would be minimum

$$f(x)\big|_{(x=0)} = f_{min} = \left(\frac{x^2-1}{x^2+1}\right)\Bigg|_{(x=0)} = -1$$

Q.23 Let $f(x) = \begin{cases} |x| & \text{for} \quad 0 < |x| \le 2 \\ 1 & \text{for} \quad x = 0 \end{cases}$, then at x = 0, f has - **[IIT Scr. 2000]**

(A) a local maximum (B) no local maximum

(C) a local minimum (D) no extremum

Sol. **[A]**

$$f(x) = \begin{cases} |x| \text{for} & 0 < |x| \le 2 \\ 1 \text{for} & x = 0 \end{cases}$$

$$= \begin{cases} x & \text{for} \quad 0 \le x \le 2 \\ -x & \text{for} \quad -2 \le x < 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$$

For local maximum, f(a) ≥ f(a – h)

f(a) ≥ f(a + h)

⇒ f(0) ≥ f(0 – h)

f(0) ≥ f(0 + h)

⇒ 1 ≥ 0 which is true.

1 ≥ 0

For local minimum,

f(a) ≤ f(a – h)

f(a) ≤ f(a + h)

⇒ f(0) ≤ f(a – h)

f(0) ≤ f(a + h)

⇒ 1 ≤ 0 which is not true

1 ≤ 0

Hence, option [A] is correct answer.

Q.24 If 'f' is differentiable and 'g' is a double differentiable function such that f (x) ∈ [–1, 1] and f ' (x) = g(x). If $f^2(0) + g^2(0) = 9$ then prove that there exist some c ∈ (–3, 3) such that g(c) g "(c) < 0. **[IIT 2005]**

Sol. Given that f(x) is a differentiable function such that

f ′(x) = g(x), then

$$\int_0^3 g(x)\,dx = \int_0^3 f'(x)\,dx = [f(x)]_0^3 = f(3) - f(0)$$

But | f(x) | < 1 ⇒ –1 < f(x) < 1, ∀ x ∈ R

∴ f(3) = f(0) ∈ (–2, 2)

Similarly $\int_{-3}^0 g(x)\,dx = \int_{-3}^0 f'(x)\,dx = [f(0) - f(3)]$ ∈ (–2, 2)

Also given $[f(0)]^2 + [g(0)]^2 = 9$

⇒ $[g(0)]^2 = 9 - [f(0)]^2$

⇒ $[g(0)]^2 > 9 - 1$ [Θ | f(x) | < 1]

⇒ $|g(0)| > 2\sqrt{2}$

⇒ $g(0) > 2\sqrt{2}$ or $g(0) < -2\sqrt{2}$

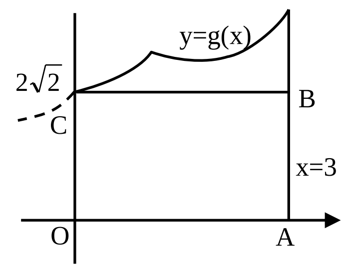
First let us consider $g(0) > 2\sqrt{2}$

Let us suppose that g ″ (x) be positive for all x ∈ (–3, 2)

Then g ″(x) > 0 ⇒ the curve y = g(x) is open upwards.

Now one of the two situations are possible.

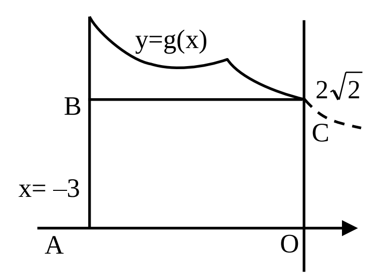
(i) g(x) is increasing



∴ $\left|\int_0^3 g(x)\,dx\right|$ > area of rect. OABC

i.e. $\left|\int_0^3 g(x)\,dx\right| > 6\sqrt{2} > 2$

a contradiction as $\int_0^1 g(x)\;dx \in (-2, 2)$



∴ at least at one of the pt. C ∈ (–3, 3)

g ′(x) < 0. But g(x) > 0 on (–3, 3)

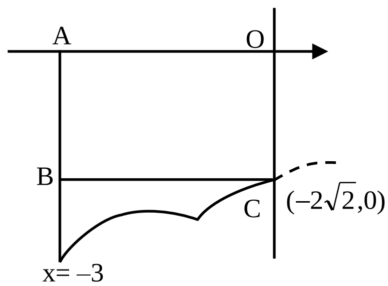
Hence g(x) g ″(x) < 0 at some x ∈ (–3, 3).

(ii) g(x) is decreasing

∴ $\left|\int_{-3}^0 g(x)\,dx\right|$ > ar of rect. OABC i.e. $\left|\int_{-3}^{0} g(x)\,dx\right| > 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{-3}^{0} g(x)\ dx \in (-2, 2)$

$\therefore$ at least at one of the pt C $\in$ (–3, 3)
$g''(x)$ should be –ve. But g(x) > 0 on (–3, 3).
Hence g(x) $g''(x) < 0$ at some x $\in$ (–3, 3)
Secondly let us consider $g(0) < -\ 2\sqrt{2}$.
Let us suppose that $g''()$ be –ve on (–3, 3).
then $g''(x) < 0 \Rightarrow$ the curve y = g(x) is open downward



Again one of the two situations are possible
(i) g(x) is decreasing then

$\left|\int_{0}^{3} g(x)\,dx\right|$ > At of rect. OABC =

$3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{0}^{3} g(x)\,dx \in (-2, 2)$

$\therefore$ At least at one of the pt. C $\in$ (–3, 3) $g''(x)$ is +ve. But g(x) < 0 on (–3, 3).
Hence g(x) $g''(x) < 0$ for some x $\in$ (–3, 3).
(ii) g(x) is increasing then

$\left|\int_{0}^{-3} g(x)\,dx\right|$ > Ar of rect. OABC .

$3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{-3}^{0} g(x)\,dx \in (-2, 2)$

$\therefore$ At least at one of the pt. C $\in$ (–3, 3)

$g''(x)$ is +ve. But g(x) < 0 on (–3, 3).

Hence g(x) $g''(x) < 0$ for some x $\in$ (–3, 3).

Combining all the cases, discussed above, we can conclude that at least at one point in (–3, 3) g(x) $g''(x) < 0$.

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	A	A	D	B	A	B	A	A	C	A	D	A	A	B
Q.No.	16	17	18	19	20	21	22	23	24	25					
Ans.	D	A	A	D	D	D	B	A	C	A					

26. True **27.** True **28.** False **29.** (2, – 4) **30.** 5 **31.** (3a, 27a/4) **32.** $\lambda \in (-\infty, 1] \cup [2,\infty)$

EXERCISE # 2

PART-A

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	B	A	A	A	B	C	A	A	D	D	D	C	A	D	C

PART-B

Q.No.	16	17	18	19	20	21	22	23	24	25	26
Ans.	A,B,C	A,B	B	B	A,B,C	A,C,D	A,B	A,C	B,C	A,B,C	A,B,C

PART-C

Q.No.	27	28	29
Ans.	A	A	A

PART-D

30. $A \to R$; $B \to P$; $C \to S$; $D \to Q$ **31.** $A \to S$; $B \to R$; $C \to P$; $D \to Q$

32. $A \to P,R,T$; $B \to P, S$; $C \to Q,R$, $D \to Q,S$

EXERCISE # 3

1. $a \in \left[-\sqrt{\frac{7}{3}}, -5^{\frac{1}{4}}\right) \cup \left(5^{\frac{1}{4}}, \sqrt{\frac{7}{3}}\right]$ **2.** $k \in (2\sqrt{3}, 3\sqrt{3})$ **3.** a = –3, b = –9, $c \in R$

4. The least value of the difference is 32/9. **5.** $\frac{(b-a)^3}{4}$ **6.** $\left[\ell n\left(1+\left(\frac{1}{e}\right)^{1/e}\right), \ell n\ 2\right)$

7. Max. at t = –1 or x = 31 ; y max. = 14 ; Min at t = 3/2 or x = –1033/32 ; y min = –69/4

8. (a) Local maxima at x = 1; Local minima at x = 6 , (b) Local maxima at x = –1/5; Local minima at x = 1

(c) Local minima at x = 1/e ; No local maxima

9. $b \in (0, e]$

10. (a) maximum = 8; minimum = –10 (b) maximum = 25; minimum = –39

(c) maximum = 3/4 at x = $\pi/6$; minimum = 1/2 at x = 0 & $\pi/2$

(d) Not defined (e) maximum f(–1) = 18, minimum f(1/8) = –9/4

13. Dimensions are 10, 10, 10, ft **14.** 10 cm **16.** $3\sqrt{3}/4$ m **18.** 0

19. Increasing in $(0, \pi/6] \cup [\pi/2, 5\pi/6] \cup [3\pi/2, \pi]$ and decreasing in $[\pi/6, \pi/2] \cup [5\pi/6, 3\pi/2]$ **20.** $\pi/4$

21. 32 sq units **22.** (a) $x = y = \frac{d}{\sqrt{2}}$, (b) $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}\,d$ **23.** 320

24. (a) 0, 3 (c) 3/4, $t = \lambda n\, 4$

27. (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \pi/2$ and $-3\pi/2$ and no minima,
(d) $x = 3\pi/2$ and $x = -\pi/2$

28. 74

29. (i) I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$;
(ii) L.Min. at x = 1 and x = 8; L.Max. x = 6
(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$;
(iv) x = 2, 3, 5, 7

30. (a) increasing in (0, 2) and decreasing in $(-\infty, 0) \cup (2, \infty)$, local min. value = 0 and local max. value = 2
(b) concave up for $(-\infty, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, \infty)$ and concave down in $(2 - \sqrt{2}), (2 + \sqrt{2})$
(c) $f(x) = \frac{1}{2} e^{2-x}.x^2$

Q.No.	32	33	34	35	36	37	38	39	40
Ans.	D	A	B	C	A	B	B	D	D

EXERCISE # 4

1. (D) **2.** (A) **3.** 18 **4.** (A) **5.** (2, 1) **6.** 5 units

7. Point of local max. (3, –22) ; point minima (–1, 10), distance = $4\sqrt{65}$ unit **8.** (A, B, C) **9.** (C) **10.** 7

11. D **12.** 1 **13.** 5 **14.** 9

EXERCISE # 5

1. (A, D) **2.** (A, B, C, D) **3.** Radius = 7m, length = 7m **4.** $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$ **5.** (C)

7. (C) **8.** (0, 2) **9.** $\frac{3\sqrt{3}}{4} r^2$ **10.** $6 + \pi : 6$ **11.** $x + \sqrt{2}\, y = \sqrt{2}$ or $x - \sqrt{2}\, y = -\sqrt{2}$

12. $b \in (-2, -1) \cup (1, \infty)$ **13.** (A, B, C, D) **14.** $\frac{4\sqrt{3}}{9}$ sq. units **15.** abe **16.** 2 hk **17.** (C)

18. Minima at $x = \frac{1}{4}\left[b + \sqrt{b^2 - 1}\right]$ and maxima at $x = 1/4\,[b - \sqrt{b^2 - 1}]$; b > 1

19. $\left(\frac{3}{2}, 1\right)$ **20.** $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$ **21.** (B) **22.** (D) **23.** (A)