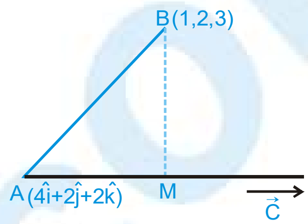
SOLVED EXAMPLES

Ex. 1 Find the distance of the point $B(\hat{i} + 2\hat{j} + 3\hat{k})$ from the line which is passing through $A(4\hat{i} + 2\hat{j} + 2\hat{k})$ and which is parallel to the vector $\vec{C} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

Sol. $AB = \sqrt{3^2 + 1^2} = \sqrt{10}$



$$AM = \overrightarrow{AB}.\hat{i} = (-3\hat{i} + \hat{k}).\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}$$

$$= -6 + 6 = 0$$

$BM^2 = AB^2 - AM^2$

So, $\quad BM = AB = \sqrt{10}$

Ex. 2 Find the direction cosines ℓ, m, n of a line which are connected by the relations $\ell + m + n = 0,\ 2mn + 2m\ell - n\ell = 0$

Sol. Given, $\ell + m + n = 0$ ..... **(i)**

$2mn + 2m\ell - n\ell = 0$ ..... **(ii)**

From (1), $n = -(\ell + m)$.

Putting $n = -(\ell + m)$ in equation (ii), we get,

$-2m(\ell + m) + 2m\ell + (\ell + m)\ell = 0$

or, $\quad -2m\ell - 2m^2 + 2m\ell + \ell^2 + m\ell = 0$

or, $\quad \ell^2 + m\ell - 2m^2 = 0$

or, $\quad \left(\frac{1}{m}\right)^2 + \left(\frac{1}{m}\right) - 2 = 0 \qquad$ [dividing by m^2]

or $\quad \frac{1}{m} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

Case I. when $\frac{1}{m} = 1$: In this case $m = \ell$

From (1), $2\ell + n = 0 \quad \Rightarrow \quad n = -2\ell$

$\therefore \quad \ell : m : n = 1 : 1 : -2$

$\therefore \quad$ Direction ratios of the line are 1, 1, – 2

$\therefore \quad$ Direction cosines are

$$\pm\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm\frac{-2}{\sqrt{1^2 + 1^2 + (-2)^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \quad \text{or} \quad -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

Case II. When $\frac{1}{m} = -2$: In this case ● $= -2m$

From (i), $-2m + m + n = 0 \quad \Rightarrow \quad n = m$

$\therefore$ ● $: m : n = -2m : m : m$

$= -2 : 1 : 1$

$\therefore$ Direction ratios of the line are $-2, 1, 1$.

$\therefore$ Direction cosines are

$$\pm\frac{-2}{\sqrt{(-2)^2+1^2+1^2}},\ \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}},\ \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}}$$

$\Rightarrow \quad \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$ or $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

Ex. 3 If $\vec{a}, \vec{b}, \vec{c}$ a re three non zero vectors such that $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$.

Sol. $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{a} = \vec{b}\times\vec{c}$

$\Rightarrow \quad \vec{c}\perp\vec{a},\ \vec{c}\perp\vec{b}$ and $\vec{a}\perp\vec{b},\ \vec{a}\perp\vec{c}$

$\Rightarrow \quad \vec{a}\perp\vec{b},\ \vec{b}\perp\vec{c}$ and $\vec{c}\perp\vec{a}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

Again, $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = |\vec{c}|$ and $|\vec{b}\times\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\frac{\pi}{2} = |\vec{c}|$ and $|\vec{b}||\vec{c}|\sin\frac{\pi}{2} = |\vec{a}| \quad \left(\because \vec{a}\perp\vec{b} \text{ and } \vec{b}\perp\vec{c}\right)$

$\Rightarrow \quad |\vec{a}||\vec{b}| = |\vec{c}|$ and $|\vec{b}||\vec{c}| = |\vec{a}| \quad \Rightarrow \quad |\vec{b}|^2\,|\vec{c}| = |\vec{c}|$

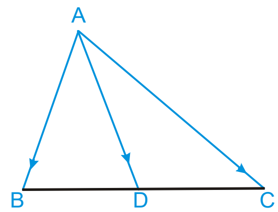
$\Rightarrow \quad |\vec{b}|^2 = 1 \quad \Rightarrow \quad |\vec{b}| = 1$

putting in $|\vec{a}||\vec{b}| = |\vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\vec{c}|$

Ex. 4 D is the mid point of the side BC of a ΔABC, show that $AB^2 + AC^2 = 2(AD^2 + BD^2)$

Sol. We have $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$



$\Rightarrow \quad AB^2 = (\overrightarrow{AD} + \overrightarrow{DB})^2$

$\Rightarrow \quad AB^2 = AD^2 + DB^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DB}$ ....(i)

Also we have $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$\Rightarrow \quad AC^2 = (\overrightarrow{AD} + \overrightarrow{DC})^2$

$\Rightarrow \quad AC^2 = AD^2 + DC^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DC}$ ....(ii)

Adding (i) and (ii), we get $AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\,\overrightarrow{AD}\cdot(\overrightarrow{DB} + \overrightarrow{DC})$

$\Rightarrow \quad AB^2 + AC^2 = 2(AD^2 + BD^2) \quad \because \ \overrightarrow{DB} + \overrightarrow{DC} = \vec{0}$

Ex. 5 For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2\,|\vec{a}|^2$

Sol. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$$\vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i} = a_1(\hat{i}\times\hat{i})+a_2(\hat{j}\times\hat{i})+a_3(\hat{k}\times\hat{i}) = -a_2\hat{k}+a_3\hat{j}$$

$\Rightarrow$ $|\vec{a}\times\hat{i}|^2 = a_2^2 + a_3^2$

$$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j} = a_1\hat{k}-a_3\hat{i}$$

$\Rightarrow$ $|\vec{a}\times\hat{j}|^2 = a^2{}_1 + a_3^2$

$$\vec{a}\times\hat{k} = (a_i\,\hat{i}+a_2\hat{j}+a_3\hat{k})\times\vec{k} = -\,a_i\hat{j}+a_2\hat{i}$$

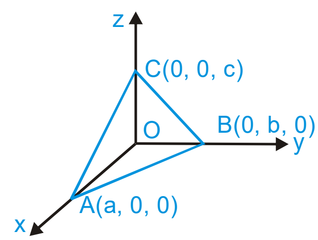
$\Rightarrow$ $|\vec{a}\times\hat{k}|^2 = a_1^2 + a_2^2$

$\therefore$ $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = a_2^2 + a_3^3 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$$= 2\,(a_1^2 + a_2^2 + a_3^2) = 2\,|\vec{a}|^2$$

Ex. 6 If a variable plane cuts the coordinate axes in A, B and C and is at a constant distance p from the origin, find the locus of the centroid of the tetrahedron OABC.

Sol. Let $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$ and $C \equiv (0, 0, c)$



$\therefore$ Equation of plane ABC is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$

Now p = length of perpendicular from O to plane **(i)**

$$= \frac{1}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}}} \quad \text{or} \quad p^2 = \frac{1}{\left(\dfrac{1}{a}\right)^2+\left(\dfrac{1}{b}\right)^2+\left(\dfrac{1}{c}\right)^2}$$

Let G(α, β, γ) be the centroid of the tetrahedron OABC, then

$$\alpha = \frac{a}{4},\ \beta = \frac{b}{4},\ \gamma = \frac{c}{4} \qquad \left[\because\ \alpha = \frac{a+0+0+0}{4} = \frac{a}{4}\right]$$

or, $a = 4\alpha,\ b = 4\beta,\ c = 4\gamma$

Putting these values of a, b, c in equation **(ii)**, we get

$$p^2 = \frac{16}{\left(\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}\right)} \quad \text{or} \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2}$$

$\therefore$ locus of (α, β, γ) is $x^{-2} + y^{-2} + z^{-2} = 16\,p^{-2}$

Ex. 7 Find the angle between the lines x – 3y – 4 = 0, 4y – z + 5 = 0 and x + 3y – 11 = 0, 2y – z + 6 = 0.

Sol. Given lines are $\left.\begin{aligned} x-3y-4=0 \\ 4y-z+5=0 \end{aligned}\right\}$ ..... **(1)**

and $\left.\begin{aligned} x+3y-11=0 \\ 2y-z+6=0 \end{aligned}\right\}$ ..... **(2)**

Let $\bullet_1, m_1, n_1$ and $\bullet_2, m_2, n_2$ be the direction cosines of lines **(1)** and **(2)** respectively

✈ line **(1)** is perpendicular to the normals of each of the planes

$x - 3y - 4 = 0$ and $4y - z + 5 = 0$

∴ $\bullet_1 - 3m_1 + 0.n_1 = 0$ ..... **(3)**

and $0\bullet_1 + 4m_1 - n_1 = 0$ ..... **(4)**

Solving equations **(3)** and **(4)**, we get $\dfrac{l_1}{3-0} = \dfrac{m_1}{0-(-1)} = \dfrac{n_1}{4-0}$

or, $\dfrac{l_1}{3} = \dfrac{m_1}{1} = \dfrac{n_1}{4} = k$ (let).

Since line **(2)** is perpendicular to the normals of each of the planes

$x + 3y - 11 = 0$ and $2y - z + 6 = 0$,

∴ $\bullet_2 + 3m_2 = 0$ ..... **(5)**

and $2m_2 - n_2 = 0$ ..... **(6)**

∴ $\bullet_2 = -3m_2$

or, $\dfrac{l_2}{-3} = m_2$

and $n_2 = 2m_2$ or, $\dfrac{n_2}{2} = m_2$.

∴ $\dfrac{l_2}{-3} = \dfrac{m_2}{1} = \dfrac{n_2}{2} = t$ (let).

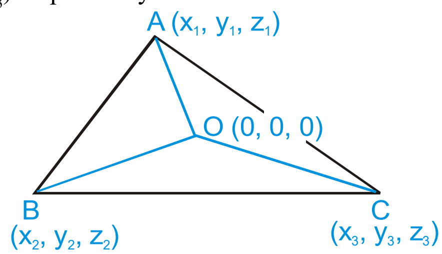
If θ be the angle between lines (1) and (2), then $\cos\theta = \bullet_1\bullet_2 + m_1m_2 + n_1n_2$

$= (3k)(-3t) + (k)(t) + (4k)(2t) = -9kt + kt + 8kt = 0$

∴ $\theta = 90°$.

Ex. 8 If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Sol. Let OABC be the tetrahedron, where O is the origin and co-ordinates of A, B, C are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3)$ respectively.



Let $OA \perp BC$ and $OB \perp CA$.

We have to prove that $OC \perp BA$.

Now, direction ratios of OA are $x_1 - 0, y_1 - 0, z_1 - 0$ or, x_1, y_1, z_1

direction ratios of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$.

$\Rightarrow$ $OA \perp BC$.

$\therefore$ $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$ ..... **(1)**

Similarly,

$\Rightarrow$ $OB \perp CA$

$\therefore$ $x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$ ..... **(2)**

Adding equations **(1)** and **(2)**, we get

$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ ($\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$)

Ex.9 If $\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ then shown that ; $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$

where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.

Sol. Here $\vec{a}\times\vec{a}' = \frac{\vec{a}\times(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\vec{a}\times\vec{a}' = \frac{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly $\vec{b}\times\vec{b}' = \frac{(\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ & $\vec{c}\times\vec{c}' = \frac{(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = \frac{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]} \quad [\because \vec{a}.\vec{b} = \vec{b}.\vec{a} \text{ etc.}]$$

$= 0.$

Ex.10 Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value(s) of α, if any, such that $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\times\left(\vec{c}\times\vec{a}\right) = \vec{0}$.

Sol. $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\times\left(\vec{c}\times\vec{a}\right) = \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}\times\left(\vec{c}\times\vec{a}\right)$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\left(\vec{a}.\vec{b}\right)\vec{c} - \left(\vec{b}.\vec{c}\right)\vec{a}\right\}$

which vanishes if **(i)** $\left(\vec{a}.\vec{b}\right)\vec{c} = \left(\vec{b}.\vec{c}\right)\vec{a}$ **(ii)** $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

(i) $(\vec{a}.\vec{b})\,\vec{c} = (\vec{b}.\vec{c})\,\vec{a}$ leads to the equation $2\alpha^3 + 10\alpha + 12 = 0$, $\alpha^2 + 6\alpha = 0$ and $6\alpha - 6 = 0$, which do not have a common solution.

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0 \qquad \Rightarrow \qquad 3\alpha = 2 \qquad \Rightarrow \qquad \alpha = \frac{2}{3}$$

Ex. 11 If $\vec{x}\times\vec{a} + k\vec{x} = \vec{b}$, where k is a scalar and $\vec{a}, \vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a}, \vec{b}$ and k.

Sol. $\vec{x}\times\vec{a} + k\vec{x} = \vec{b}$ ....**(i)**

Premultiply the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a}) + k\,(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$

$\Rightarrow$ $(\vec{a}.\vec{a})\,\vec{x} - (\vec{a}.\vec{x})\,\vec{a} + k(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$ ....**(ii)**

Premultiply **(i)** scalarly by $\vec{a}$

$[\vec{a}\ \vec{x}\ \vec{a}] + k\,(\vec{a}.\vec{x}) = \vec{a}.\vec{b}$

$k(\vec{a}.\vec{x}) = \vec{a}.\vec{b}$ ....**(iii)**

Substituting $\vec{x}\times\vec{a}$ from **(i)** and $\vec{a}.\vec{x}$ from **(iii)** in **(ii)** we get

$$\vec{x} = \frac{1}{a^2+k^2}\left[k\vec{b} + (\vec{a}\times\vec{b}) + \frac{(\vec{a}.\vec{b})}{k}\vec{a}\right]$$

Ex. 12 Forces of magnitudes 5, 4, 3 units act on a particle in the directions $2\hat{i}-2\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+2\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$ respectively, and the particle gets displaced from the point A whose position vector is $6\hat{i}+2\hat{j}+3\hat{k}$, to the point B whose position vector is $9\hat{i}+7\hat{j}+5\hat{k}$. Find the work done.

Sol. If the forces are $\vec{F}_1, \vec{F}_2, \vec{F}_3$ then $\vec{F}_1 = \frac{5}{3}(2\hat{i}-2\hat{j}+\hat{k})$; $\vec{F}_2 = \frac{4}{3}(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{F}_3 = \frac{3}{3}(-2\hat{i}+\hat{j}-2\hat{k})$ and hence the sum force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{1}{3}(8\hat{i}+\hat{j}+7\hat{k})$

Displacement vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 9\hat{i}+7\hat{j}+5\hat{k} - (6\hat{i}+2\hat{j}+3\hat{k}) = 3\hat{i}+5\hat{j}+2\hat{k}$

Work done $= \frac{1}{3}(8\hat{i}+\hat{j}+7\hat{k}).(3\hat{i}+5\hat{j}+2\hat{k}) = \frac{1}{3}(24+5+14) = \frac{43}{3}$ units.

Ex. 13 Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

Sol. We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

and, $\overrightarrow{CA}$ = Position vector of A – Position vector of C

$= (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

Since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = (-\hat{i}-2\hat{j}-6\hat{k}) + (2\hat{i}-\hat{j}+\hat{k}) + (-\hat{i}+3\hat{j}+5\hat{k}) = \vec{0}$

So A, B and C are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i}-\hat{j}+\hat{k}) \cdot (-\hat{i}+3\hat{j}+5\hat{k}) = -2-3+5 = 0$

$\Rightarrow \quad \overrightarrow{BC} \perp \overrightarrow{CA} \qquad \Rightarrow \qquad \angle BCA = \dfrac{\pi}{2}$

Hence ABC is a right angled triangle.

Since A is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore

$$\cos A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(-\hat{i}-2\hat{j}-6\hat{k})\cdot(\hat{i}-3\hat{j}-5\hat{k})}{\sqrt{(-1)^2+(-2)^2+(-6)^2}\sqrt{1^2+(-3)^2+(-5)^2}}$$

$$= \frac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}}$$

$$A4 = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\cos B = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(\hat{i}+2\hat{j}+6\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+2^2+6^2}\sqrt{2^2+(-1)^2+(1)^2}}$$

$$\Rightarrow \quad \cos B = \frac{2-2+6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}} \qquad \Rightarrow \qquad B = \cos^{-1}\sqrt{\frac{6}{41}}$$

Ex.14 If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Sol. Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ ....**(i)**

Now, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$ [Using **(i)**]

$= 3\lambda^2$ [$\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$]

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ....(ii)

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}.(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}}$$ [Using (ii)]

$\therefore \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly, $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \quad \theta_1 = \theta_2 = \theta_3$.

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclineded with $\vec{a}, \vec{b}$ and $\vec{c}$

Ex. 15 Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \times \vec{v}).\vec{w}| \le \frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

Sol. $\vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$ ......(i)

$\Rightarrow \quad \vec{w}\times\vec{u} = \vec{v} - \vec{w} \quad \Rightarrow \quad (\vec{w}\times\vec{u})^2 = v^2 + w^2 - 2\vec{v}.\vec{w}$

$\Rightarrow \quad 2\vec{v}.\vec{w} = 1 + w^2 - (\vec{u}\times\vec{w})^2$ ......(ii)

also taking dot product of (i) with $\vec{v}$, we get

$\vec{w}.\vec{v} + (\vec{w}\times\vec{u}).\vec{v} = \vec{v}.\vec{v}$

$\Rightarrow \quad \vec{v}.(\vec{w}\times\vec{u}) = 1 - \vec{w}.\vec{v}$ ....(iii) $\{\because \vec{v}.\vec{v} = |\vec{v}|^2 = 1\}$

Now ; $\vec{v}.(\vec{w}\times\vec{u}) = 1 - \frac{1}{2}(1 + w^2 - (\vec{u}\times\vec{w})^2)$ (using (ii) and (iii))

$= \frac{1}{2} - \frac{w^2}{2} + \frac{(\vec{u}\times\vec{w})^2}{2}$ $(\because 0 \le \cos^2\theta \le 1)$

$= \frac{1}{2}(1 - w^2 + w^2\sin^2\theta)$ ......(iv)

as we know ; $0 \le w^2\cos^2\theta \le w^2$

$\therefore \quad \frac{1}{2} \ge \frac{1 - w^2\cos^2\theta}{2} \ge \frac{1 - w^2}{2}$

$\Rightarrow \quad \frac{1 - w^2\cos^2\theta}{2} \le \frac{1}{2}$ ......(v)

from (iv) and (v)

$|\vec{v}.(\vec{w}\times\vec{u})| \le \frac{1}{2}$

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Equality holds only when $\cos^2\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

i.e., $\vec{u} \perp \vec{w} \Rightarrow \vec{u}.\vec{w} = 0 \quad \Rightarrow \quad \vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$

$\Rightarrow \quad \vec{u}\cdot\vec{w} + \vec{u}.(\vec{w} \times \vec{u}) = \vec{u}.\vec{v}$ (taking dot with $\vec{u}$)

$\Rightarrow \quad 0 + 0 = \vec{u}\cdot\vec{v} \quad \Rightarrow \quad \vec{u}\cdot\vec{v} = 0 \quad \Rightarrow \quad \vec{u} \perp \vec{v}$

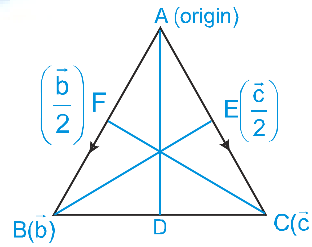
Ex.16 Prove using vectors : If two medians of a triangle are equal, then it is isosceles.

Sol. Let ABC be a triangle and let BE and CF be two equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

P.V. of E = $\frac{1}{2}\vec{c}$ and P.V. of F = $\frac{1}{2}\vec{b}$

$\therefore \quad \overrightarrow{BE} = \frac{1}{2}(\vec{c} - 2\vec{b})$

$\overrightarrow{CF} = \frac{1}{2}(\vec{b} - 2\vec{c})$



Now, $BE = CF \quad \Rightarrow \quad |\overrightarrow{BE}| = |\overrightarrow{CF}|$

$\Rightarrow \quad |\overrightarrow{BE}|^2 = |\overrightarrow{CF}|^2 \quad \Rightarrow \quad \left|\frac{1}{2}(\vec{c} - 2\vec{b})\right|^2 = \left|\frac{1}{2}(\vec{b} - 2\vec{c})\right|^2$

$\Rightarrow \quad \frac{1}{4}|\vec{c} - 2\vec{b}|^2 = \frac{1}{4}|\vec{b} - 2\vec{c}|^2 \Rightarrow \quad |\vec{c} - 2\vec{b}|^2 = |\vec{b} - 2\vec{c}|^2$

$\Rightarrow \quad (\vec{c} - 2\vec{b}) \cdot (\vec{c} - 2\vec{b}) = (\vec{b} - 2\vec{c}) \cdot (\vec{b} - 2\vec{c})$

$\Rightarrow \quad \vec{c}.\vec{c} - 4\vec{b}.\vec{c} + 4\vec{b}.\vec{b} = \vec{b}.\vec{b} - 4\vec{b}.\vec{c} + 4\vec{c}.\vec{c}$

$\Rightarrow \quad |\vec{c}|^2 - 4\vec{b}.\vec{c} + 4|\vec{b}|^2 = |\vec{b}|^2 - 4\vec{b}.\vec{c} + 4|\vec{c}|^2$

$\Rightarrow \quad 3|\vec{b}|^2 = 3|\vec{c}|^2 \quad \Rightarrow \quad |\vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow \quad AB = AC$

Hence triangle ABC is an isosceles triangle.

Ex.17 Using vectors : Prove that cos (A + B) = cos A cos B – sin A sin B

Sol. Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively. Let $\angle XOP = A$ and $\angle XOQ = B$. Drawn $PL \perp OX$ and $QM \perp OX$.

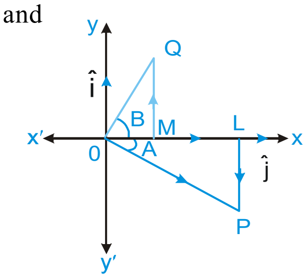
Clearly angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is A + B

In ΔOLP, OL = OP cos A and LP = OP sin A. Therefore $\overrightarrow{OL} = (OP\cos A)\,\hat{i}$ and

$\overrightarrow{LP} = (OP\sin A)\left(-\hat{j}\right)$

Now, $\overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OP}$

$\Rightarrow \quad \overrightarrow{OP} = OP\,[(\cos A)\,\hat{i} - (\sin A)\,\hat{j}\,]$ ......(i)



In ΔOMQ, OM = OQ cos B and MQ = OQ sin B.

Therefore, $\overrightarrow{OM} = (OQ\cos B)\,\hat{i}$, $\overrightarrow{MQ} = (OQ\sin B)\,\hat{j}$

Now, $\quad \vec{OM} + \vec{MQ} = \vec{OQ}$

$\Rightarrow \quad \vec{OQ} = OQ[(\cos B)\hat{i} + (\sin B)\hat{j}]$ ......**(ii)**

From **(i)** and **(ii)**, we get

$$\vec{OP} \cdot \vec{OQ} = OP\,[(\cos A)\ \hat{i} - (\sin A)\ \hat{j}]\,.\,OQ\,[(\cos B)\ \hat{i} + (\sin B)\ \hat{j}]$$

$$= OP\,.\,OQ\,[\cos A \cos B - \sin A \sin B]$$

But, $\vec{OP} \cdot \vec{OQ} = |\vec{OP}|\ |\vec{OQ}|\ \cos(A+B) = OP\,.\,OQ \cos(A+B)$

$\therefore \quad OP\,.\,OQ \cos(A+B) = OP\,.\,OQ\,[\cos A \cos B - \sin A \sin B]$

$\Rightarrow \quad \cos(A+B) = \cos A \cos B - \sin A \sin B$

Ex. 18 A point $A(x_1, y_1)$ with abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with ordinate $y_2 = 11$ are given in a rectangular cartesian system of co-ordinates OXY on the part of the curve $y = x^2 - 2x + 3$ which lies in the first quadrant. Find the scalar product of $\vec{OA}$ and $\vec{OB}$.

Sol. Since (x_1, y_1) and (x_2, y_2) lies on $y = x^2 - 2x + 3$.

$\therefore \quad y_1 = x_1^2 - 2x_1 + 3$

$y_1 = 1^2 - 2(1) + 3 \qquad$ (as ; $x_1 = 1$)

$y_1 = 2$

so the co-ordinates of A(1, 2)

Also, $\quad y_2 = x_2^2 - 2x_2 + 3$

$11 = x_2^2 - 2x_2 + 3 \Rightarrow x_2 = 4,\ x_2 \neq -2$ (as B lie in 1st quadrant)

$\therefore \quad$ co-ordinates of B (4, 11).

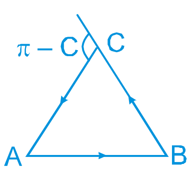
Hence, $\quad \vec{OA} = \hat{i} + 2\hat{j} \quad$ and $\quad \vec{OB} = 4\hat{i} + 11\hat{j}$

$\Rightarrow \quad \vec{OA} \cdot \vec{OB} = 4 + 22 = 26.$

Ex. 19 Prove that in any triangle ABC

(i) $c^2 = a^2 + b^2 - 2ab \cos C \qquad$ **(ii)** $c = b\cos A + a\cos B$.

Sol. **(i)** In ΔABC, $\vec{AB} + \vec{BC} + \vec{CA} = 0$



$\Rightarrow \quad \vec{BC} + \vec{CA} = -\vec{AB}$ ......**(i)**

Squaring both sides

$(\vec{BC})^2 + (\vec{CA})^2 + 2(\vec{BC}).\vec{CA} = (\vec{AB})^2$

$\Rightarrow \quad a^2 + b^2 + 2(\vec{BC}.\vec{CA}) = c^2$

$\Rightarrow \quad c^2 = a^2 + b^2 + 2\,ab \cos(\pi - C)$

$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab \cos C$

(ii) $(\vec{BC} + \vec{CA}).\vec{AB} = -\vec{AB}.\vec{AB}$

$\vec{BC}.\vec{AB} + \vec{CA}.\vec{AB} = -c^2$

$-ac \cos B - bc \cos A = -c^2$

$a\cos B + b\cos A = c.$

Ex. 20 Through a point P(h, k, l) a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\frac{p^5}{2|hkl|}$.

Sol. $OP = \sqrt{h^2+k^2+l^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+l^2}}, \frac{k}{\sqrt{h^2+k^2+l^2}}, \frac{l}{\sqrt{h^2+k^2+l^2}}$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+l^2}}x + \frac{k}{\sqrt{h^2+k^2+l^2}}y + \frac{l}{\sqrt{h^2+k^2+l^2}}z = \sqrt{h^2+k^2+l^2}$$

or, $hx + ky + lz = h^2 + k^2 + l^2 = p^2$

$\therefore$ $A \equiv \left(\frac{p^2}{h}, 0, 0\right), B \equiv \left(0, \frac{p^2}{k}, 0\right), C \equiv \left(0, 0, \frac{p^2}{l}\right)$

Now area of ΔABC, $\Delta^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \frac{1}{2}\frac{p^4}{|kl|}$ and $A_{zx} = \frac{1}{2}\frac{p^4}{|lh|}$

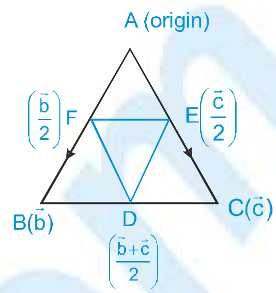
$\therefore$ $\Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2l^2} + \frac{1}{4}\frac{p^8}{h^2l^2} = \frac{p^{10}}{4h^2k^2l^2}$

or $\Delta = \frac{p^5}{2|hkl|}$

Ex.21 If D, E, F are the mid-points of the sides of a triangle ABC, prove by vector method that area of $\Delta DEF = \frac{1}{4}$ (area of ΔABC)

Sol. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then the position vectors of D, E and F are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

Now, $\overrightarrow{DE} = \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{b}}{2}$



and $\quad \overrightarrow{DF} = \frac{1}{2}\vec b - \frac{1}{2}\left((\vec b+\vec c)\right) = \frac{-\vec c}{2}$

$\therefore$ Vector area of $\Delta DEF = \frac{1}{2}(\overrightarrow{DE}\times\overrightarrow{DF}) = \frac{1}{2}\left(\frac{-\vec b}{2}\times\frac{-\vec c}{2}\right)$

$= \frac{1}{8}(\vec b\times\vec c) = \frac{1}{4}\left\{\frac{1}{2}(\overrightarrow{AB}\times\overrightarrow{AC})\right\} = \frac{1}{4}$(vector area of ΔABC)

Hence area of $\Delta DEF = \frac{1}{4}$ area of ΔABC.

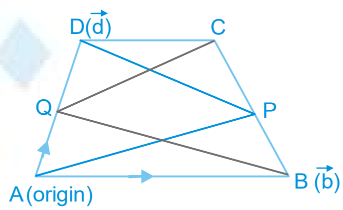
Ex. 22 P, Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD. Show that $\Delta APD = \Delta CQB$.

Sol. Let $\overrightarrow{AB} = \vec b$ and $\overrightarrow{AD} = \vec d$

Now DC is parallel to AB $\Rightarrow$ there exists a scalar t such that $\overrightarrow{DC} = t\,\overrightarrow{AB} = t\,\vec b$

$\therefore \quad \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec d + t\,\vec b$

The position vectors of P and Q are $\frac{1}{2}(\vec b+\vec d+t\,\vec b)$ and $\frac{1}{2}\vec d$ respectively.



Now $\quad 2\Delta\,\overrightarrow{APD} = \overrightarrow{AP}\times\overrightarrow{AD}$

$= \frac{1}{2}(\vec b+\vec d+t\,\vec b)\times\vec d = \frac{1}{2}(1+t)(\vec b\times\vec d)$

Also $\quad 2\Delta\,\overrightarrow{CQB} = \overrightarrow{CQ}\times\overrightarrow{CB} = \left[\frac{1}{2}\vec d-(\vec d+t\vec b)\right]\times[\vec b-(\vec d+t\,\vec b)]$

$= \left[-\frac{1}{2}\vec d-t\,\vec b\right]\times\left[-\vec d+(1-t)\,\vec b\right] = -\frac{1}{2}(1-t)(\vec d\times\vec b)+t\,(\vec b\times\vec d)$

$= \frac{1}{2}(1-t+2t)(\vec b\times\vec d) = \frac{1}{2}(1+t)(\vec b\times\vec d) = 2\Delta\overrightarrow{APD}$ Hence Prove.

Ex. 23 If 'a' is real constant and A, B, C are variable angles and $\sqrt{a^2-4}\,\tan A + a\tan B + \sqrt{a^2+4}\,\tan C = 6a$ then find the least value of $\tan^2 A + \tan^2 B + \tan^2 C$

Sol. The given relation can be re-written as ;

$(\sqrt{a^2-4}\,\hat i + a\hat j + \sqrt{a^2+4}\,\hat k)\cdot(\tan A\hat i + \tan B\hat j + \tan C\hat k) = 6a$

$\Rightarrow \quad \sqrt{(a^2-4)+a^2+(a^2+4)}.\ \sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\ .\cos\theta = 6a$

(as, a.b = |a| |b| cos θ)

$\Rightarrow \quad \sqrt{3}\,a\ .\ \sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\,\cos\theta = 6a$

$\Rightarrow \quad \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta$ ......(i)

also, $\quad 12\sec^2\theta \geq 12$ (as, $\sec^2\theta \geq 1$) ......(ii)

from (i) and (ii),

$\tan^2 A + \tan^2 B + \tan^2 C \geq 12$

$\therefore$ least value of $\tan^2 A + \tan^2 B + \tan^2 C = 12$.

Ex.24 Let $\vec{u}$ and $\vec{v}$ are unit vectors and $\vec{w}$ is a vector such that $(\vec{u}\times\vec{v})+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$ then find the value of $[\vec{u}\ \vec{v}\ \vec{w}]$.

Sol. Given $(\vec{u}\times\vec{v})+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$

$\Rightarrow$ $(\vec{u}\times\vec{v})+\vec{u}\times\vec{u}=\vec{w}\times\vec{u}$

$\Rightarrow$ $(\vec{u}\times\vec{v})\times\vec{u}+\vec{u}\times\vec{u}=\vec{v}$ (as $\vec{w}\times\vec{u}=\vec{v}$)

$\Rightarrow$ $(\vec{u}.\vec{u})\,\vec{v}-(\vec{v}.\vec{u})\,\vec{u}+\vec{u}\times\vec{u}=\vec{v}$ (using $\vec{u}\,.\,\vec{u}=1$ and $\vec{u}\times\vec{u}=0$, since unit vector)

$\Rightarrow$ $\vec{v}-(\vec{v}.\vec{u})\,\vec{u}=\vec{v}$ $\Rightarrow$ $(\vec{u}.\vec{v})\,\vec{u}=\vec{0}$

$\Rightarrow$ $\vec{u}.\vec{v}=0$ (as $\vec{u}\neq 0$) ....**(i)**

Now $\vec{u}.(\vec{v}\times\vec{w})$

$=\vec{u}.(\vec{v}\times((\vec{u}\times\vec{v})+\vec{u}))$ (given $\vec{w}=(\vec{u}\times\vec{v})+u$)

$=\vec{u}.(\vec{v}\times(\vec{u}\times\vec{v})+\vec{v}\times\vec{u}) = \vec{u}.((\vec{v}.\vec{v})\,\vec{u}-(\vec{v}.\vec{u})\,\vec{v}+\vec{v}\times\vec{u})$

$=\vec{u}.(|\vec{v}|^2\,u-0+\vec{v}\times\vec{u})$ (as $\vec{u}.\vec{v}=0$ from **(i)**)

$=|\vec{v}|^2\ (\vec{u}.\vec{u})-\vec{u}\,.\,(\vec{v}\times\vec{u})$

$=|\vec{v}|^2\ |\vec{u}|^2-0$ (as $[\vec{u}\ \vec{v}\ \vec{u}]=0$)

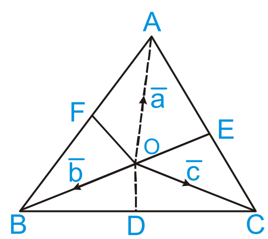
$=1$ (as $|\vec{u}|=|\vec{v}|=1$)

$\therefore$ $[\vec{u}\ \vec{v}\ \vec{w}]=1$

Ex.25 In any triangle, show that the perpendicular bisectors of the sides are concurrent.

Sol. Let ABC be the triangle and D, E and F are respectively middle points of sides BC, CA and AB. Let the perpendicular bisectors of BC and CA meet at O. Join OF. We are required to prove that OF is $\perp$ to AB. Let the position vectors of A, B, C with O as origin of reference be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore$ $\overrightarrow{OD}=\frac{1}{2}(\vec{b}+\vec{c})$, $\overrightarrow{OE}=\frac{1}{2}(\vec{c}+\vec{a})$ and $\overrightarrow{OF}=\frac{1}{2}(\vec{a}+\vec{b})$



Also $\overrightarrow{BC}=\vec{c}-\vec{b}$, $\overrightarrow{CA}=\vec{a}-\vec{c}$ and $\overrightarrow{AB}=\vec{b}-\vec{a}$

Since OD $\perp$ BC

$\Rightarrow$ $\frac{1}{2}(\vec{b}+\vec{c}).(\vec{c}-\vec{b})=0$

$\Rightarrow$ $b^2=c^2$ ....**(i)**

Similarly OE $\perp$ CA

$\Rightarrow$ $\frac{1}{2}(\vec{c}+\vec{a}).(\vec{a}-\vec{c})=0$

$\Rightarrow$ $a^2=c^2$ ....**(ii)**

from **(i)** and **(ii)** we have $a^2-b^2=0$

$\Rightarrow$ $(\vec{a}+\vec{b}).(\vec{b}-\vec{a})=0$ $\Rightarrow$ $\frac{1}{2}(\vec{b}+\vec{a}).(\vec{b}-\vec{a})=0$ $\Rightarrow$ $\overrightarrow{OF}\perp\overrightarrow{AB}$

Ex. 26 Find the locus of a point, the sum of squares of whose distances from the planes : $x - z = 0$, $x - 2y + z = 0$ and $x + y + z = 0$ is 36

Sol. Given planes are $x - z = 0$, $x - 2y + z = 0$ and, $x + y + z = 0$

Let the point whose locus is required be $P(\alpha,\beta,\gamma)$. According to question

$$\frac{|\alpha-\gamma|^2}{2}+\frac{|\alpha-2\beta+\gamma|^2}{6}+\frac{|\alpha+\beta+\gamma|^2}{3}=36$$

or $3(\alpha^2+\gamma^2-2\alpha\gamma)+\alpha^2+4\beta^2+\gamma^2-4\alpha\beta-4\beta\gamma+2\alpha\gamma+2(\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma)=36\times 6$

or $6\alpha^2+6\beta^2+6\gamma^2=36\times 6$

or $\alpha^2+\beta^2+\gamma^2=36$

Hence, the required equation of locus is $x^2+y^2+z^2=36$

Ex. 27 A, B, C, D are four points in space. using vector methods, prove that $AC^2+BD^2+AD^2+BC^2 \ge AB^2+CD^2$ what is the implication of the sign of equality.

Sol. Let the position vector of A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively then

$AC^2+BD^2+AD^2+BC^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a})+(\vec{d}-\vec{b}).(\vec{d}-\vec{b})+(\vec{d}-\vec{a}).(\vec{d}-\vec{a})+(\vec{c}-\vec{b}).(\vec{c}-\vec{b})$

$= |\vec{c}|^2+|\vec{a}|^2-2\vec{a}.\vec{c}+|\vec{d}|^2+|\vec{b}|^2-2\vec{d}.\vec{b}+|\vec{d}|^2+|\vec{a}|^2-2\vec{a}.\vec{d}+|\vec{c}|^2+|\vec{b}|^2-2\vec{b}.\vec{c}$

$= |\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}+|\vec{c}|^2+|\vec{d}|^2-2\vec{c}.\vec{d}+|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+|\vec{d}|^2$

$+2\vec{a}.\vec{b}+2\vec{c}.\vec{d}-2\vec{a}.\vec{c}-2\vec{b}.\vec{d}-2\vec{a}.\vec{d}-2\vec{b}.\vec{c}$

$= (\vec{a}-\vec{b}).(\vec{a}-\vec{b})+(\vec{c}-\vec{d}).(\vec{c}-\vec{d})+(\vec{a}+\vec{b}-\vec{c}-\vec{d})^2$

$= AB^2+CD^2+(\vec{a}+\vec{b}-\vec{c}-\vec{d}).(\vec{a}+\vec{b}-\vec{c}-\vec{d}) \ge AB^2+CD^2$

$\Rightarrow$ $AC^2+BD^2+AD^2+BC^2 \ge AB^2+CD^2$

for the sign of equality to hold, $\vec{a}+\vec{b}-\vec{c}-\vec{d}=0$

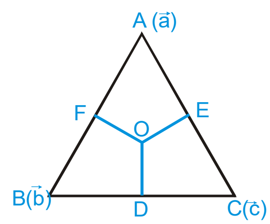
$\vec{a}-\vec{c}=\vec{d}-\vec{b}$

$\Rightarrow$ $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are collinear, the four points A, B, C, D are collinear

Ex. 28 Prove that the right bisectors of the sides of a triangle are concurrent.

Sol. Let the right bisectors of sides BC and CA meet at O and taking O as origin, let the position vectors of A, B and C be taken as $\vec{a}, \vec{b}, \vec{c}$ respectively. Hence the mid-points D, E, F are

$$\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}, \frac{\vec{a}+\vec{b}}{2}$$



$\because$ $\overrightarrow{OD} \perp \overrightarrow{BC}$, $\therefore$ $\frac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b})=0$

i.e. $b^2=c^2$

Again since $\overrightarrow{OE} \perp \overrightarrow{CA}$, $\therefore$ $\frac{\vec{c}+\vec{a}}{2}.(\vec{a}-\vec{c})=0$

or $a^2=c^2$ $\therefore$ $a^2=b^2=c^2$ .... **(i)**

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Now we have to prove that $\overrightarrow{OF}$ is also $\perp$ to $\overrightarrow{AB}$ which will be true if $\frac{\vec{a}+\vec{b}}{2}.(\vec{b}-\vec{a})=0$

i.e. $b^2 = a^2$

which is true by **(i)**

Ex. 29 If P be any point on the plane $\bullet x + my + nz = p$ and Q be a point on the line OP such that $OP . OQ = p^2$, show that the locus of the point Q is $p(\bullet x + my + nz) = x^2 + y^2 + z^2$.

Sol. Let $P \equiv (\alpha, \beta, \gamma), Q \equiv (x_1, y_1, z_1)$

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have

$$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k \text{ (say)} \qquad \text{..... (1)}$$

As P (α, β, γ) lies on the plane $\bullet x + my + nz = p$,

$$\bullet\alpha + m\beta + n\gamma = p \text{ or } \qquad k(\bullet x_1 + my_1 + nz_1) = p \qquad \text{..... (2)}$$

Given $OP . OQ = p^2$

$$\therefore \qquad \sqrt{\alpha^2+\beta^2+\gamma^2}\,\sqrt{x_1^2+y_1^2+z_1^2} = p^2$$

$$\text{or,} \qquad \sqrt{k^2(x_1^2+y_1^2+z_1^2)}\,\sqrt{x_1^2+y_1^2+z_1^2} = p^2$$

$$\text{or,} \qquad k(x_1^2+y_1^2+z_1^2) = p^2 \qquad \text{..... (3)}$$

On dividing **(2)** by **(3)**, we get $\frac{1x_1+my_1+nz_1}{x_1^2+y_1^2+z_1^2} = \frac{1}{p}$

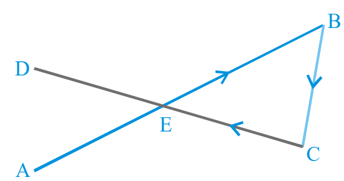
$$\text{or,} \qquad p(\bullet x_1 + my_1 + nz_1) = x_1^2+y_1^2+z_1^2$$

Hence the locus of point Q is $p(\bullet x + my + nz) = x^2 + y^2 + z^2$.

Ex. 30 A, B, C and D are four points such that $\overrightarrow{AB} = m(2\hat{i}-6\hat{j}+2\hat{k})$, $\overrightarrow{BC} = (\hat{i}-2\hat{j})$ and $\overrightarrow{CD} = n(-6\hat{i}+15\hat{j}-3\hat{k})$. Find the conditions on the scalars m and n so that $\overrightarrow{CD}$ intersects $\overrightarrow{AB}$ at some point E. Also find the area of the triangle BCE.

Sol. $\overrightarrow{AB} = m(2\hat{i}-6\hat{j}+2\hat{k})$, $\overrightarrow{BC} = (\hat{i}-2\hat{j})$

$\overrightarrow{CD} = n(-6\hat{i}+15\hat{j}-3\hat{k})$



If AB and CD intersect at E, then $\overrightarrow{EB} = p\overrightarrow{AB}$, $\overrightarrow{CE} = q\overrightarrow{CD}$

where both p and q are positive quantities less than 1

Now we know that $\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CE} = \overrightarrow{EE} = 0$

$$\therefore \qquad p\overrightarrow{AB} + \overrightarrow{BC} + q\overrightarrow{CD} = 0 \qquad \{\text{by (i)}\}$$

$$\text{or} \qquad pm(2\hat{i}-6\hat{j}+2\hat{k}) + (\hat{i}-2\hat{j}) + q.n(-6\hat{i}+15\hat{j}-3\hat{k}) = 0$$

Since $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar, the above relation implies that if $x\hat{i}+y\hat{j}+z\hat{k}=0$, then $x = 0, y = 0$ and $z = 0$

$$\therefore \qquad 2mp + 1 - 6qn = 0, \ -6pm - 2 + 15qn = 0$$

$$2pm - 3qn = 0$$

Solving these for pm and qn, we get

$$pm = \frac{1}{2}, \; qn = \frac{1}{3} \qquad \therefore \qquad p = \frac{1}{2m}, \; q = \frac{1}{3n}$$

$$\therefore \qquad 0 < \frac{1}{2m} \le 1, \; 0 \le \frac{1}{3n} \le 1 \quad \text{or} \quad m \ge \frac{1}{2}, \; n \ge \frac{1}{3}$$

Again area of $\Delta BCE = \frac{1}{2}\left|\overrightarrow{EC} \times \overrightarrow{EB}\right| = \frac{1}{2}\left|-q\overrightarrow{CD} \times p\,\overrightarrow{AB}\right| = \frac{1}{2} pqnm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 15 & -3 \\ 2 & -6 & 2 \end{vmatrix}$

Put $\quad pm = \frac{1}{2}, \; qn = \frac{1}{3}$

$$= \frac{1}{2}.\frac{1}{2}.\frac{1}{3}\left|\,12\hat{i} + 6\hat{j} - 6\hat{k}\right| = \frac{1}{12}.6\sqrt{6} = \frac{1}{2}\sqrt{6}$$

Ex. 31 $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar unit vectors such that angle between any two is α. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = \bullet\,\vec{a} + m\vec{b} + n\vec{c}$, then determine $\bullet$, m, n in terms of α.

Sol. $a^2 = b^2 = c^2 = 1$, $[abc] \ne 0$

$$\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{c} = \vec{c}\,.\,\vec{a} = \cos\alpha \qquad \text{.........(i)}$$

Multiply both sides of given relation scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$$0 + [\vec{a}\,\vec{b}\,\vec{c}\,] = \bullet.1 + (m+n)\cos\alpha \qquad \text{.........(ii)}$$

$$0 = m + (n + \bullet)\cos\alpha \qquad \text{.........(iii)}$$

$$[\vec{a}\,\vec{b}\,\vec{c}\,] + 0 = (\bullet + m)\cos\alpha + n \qquad \text{.........(iv)}$$

Adding, we get

$$2[\vec{a}\,\vec{b}\,\vec{c}\,] = (\bullet + m + n) + 2(\bullet + m + n)\cos\alpha$$

or $\quad 2[\vec{a}\,\vec{b}\,\vec{c}\,] = (\bullet + m + n)(1 + 2\cos\alpha) \qquad$ ....(v)

From (ii), $(m+n) = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}\,] - 1}{\cos\alpha}$

Putting in (v), we get $\quad 2[\vec{a}\,\vec{b}\,\vec{c}\,] = \left\{1 + \dfrac{[\vec{a}\,\vec{b}\,\vec{c}\,] - 1}{\cos\alpha}\right\}(1 + 2\cos\alpha)$

or $\quad [\vec{a}\,\vec{b}\,\vec{c}\,]\left\{2 - \dfrac{1 + 2\cos\alpha}{\cos\alpha}\right\} = 1\left(1 - \dfrac{1}{\cos\alpha}\right)(1 + 2\cos\alpha)$

$\therefore \quad 1 = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}\,]}{(1 + 2\cos\alpha)(1 - \cos\alpha)} = n \qquad$ {as above}

and $\quad m = -(n + \bullet)\cos\alpha = \dfrac{-2[\vec{a}\,\vec{b}\,\vec{c}\,]\cos\alpha}{(1 + 2\cos\alpha)(1 - \cos\alpha)}$

Thus the values of ●, m, n depend on $[\vec{a}\,\vec{b}\,\vec{c}\,]$

Hence we now find the value of scalar $[\vec{a}\,\vec{b}\,\vec{c}\,]$ in terms of α.

$$\text{Now } [\vec{a}\,\vec{b}\,\vec{c}\,]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ \cos\alpha & 1 & \cos\alpha \\ \cos\alpha & \cos\alpha & 1 \end{vmatrix} \qquad (\text{Apply } C_1 + C_2 + C_3)$$

$$= (1 + 2\cos\alpha)\begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ 1 & 1 & \cos\alpha \\ 1 & \cos\alpha & 1 \end{vmatrix} \qquad (\text{Apply } R_2 - R_1 \text{ and } R_3 - R_1)$$

$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}\,]^2 = (1 + 2\cos\alpha)(1 - \cos\alpha)^2$

$\therefore \quad \dfrac{[\vec{a}\,\vec{b}\,\vec{c}\,]}{1 - \cos\alpha} = \sqrt{1 + 2\cos\alpha}$

Putting in the value of ●, m, n we have $1 = \dfrac{1}{\sqrt{(1+2\cos\alpha)}} = n, m = \dfrac{-2\cos\alpha}{\sqrt{(1+2\cos\alpha)}}$

Ex.32 Find the image of the point P (3, 5, 7) in the plane $2x + y + z = 0$.

Sol. Given plane is $2x + y + z = 0$ ..... **(1)**

$P \equiv (3, 5, 7)$

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane **(1)**. Let PQ meet plane (1) in R

then PQ $\perp$ plane (1)

Let $R \equiv (2r + 3, r + 5, r + 7)$

Since R lies on plane (1)

$\therefore \quad 2(2r + 3) + r + 5 + r + 7 = 0$

or, $\quad 6r + 18 = 0 \qquad \therefore \quad r = -3$

$\therefore \quad R \equiv (-3, 2, 4)$

Let $\quad Q \equiv (\alpha, \beta, \gamma)$

Since R is the middle point of PQ

$\therefore \quad -3 = \dfrac{\alpha + 3}{2} \Rightarrow \alpha = -9$

$2 = \dfrac{\beta + 5}{2} \Rightarrow \beta = -1$

$4 = \dfrac{\gamma + 7}{2} \Rightarrow \gamma = 1 \qquad \therefore \qquad Q = (-9, -1, 1).$

Ex.33 Vectors $\vec{x}, \vec{y}$ and $\vec{z}$ each of magnitude $\sqrt{2}$, make angles of 60° with each other. If $\vec{x} \times (\vec{y} \times \vec{z}) = \vec{a}$, $\vec{y} \times (\vec{z} \times \vec{x}) = \vec{b}$ and $\vec{x} \times \vec{y} = \vec{c}$, then find $\vec{x}, \vec{y}$ and $\vec{z}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

Sol. $\vec{x}.\vec{y} = \sqrt{2}\sqrt{2}\cos 60° = 1 = \vec{y}.\vec{z} = \vec{z}.\vec{x}$ ...... **(i)**

Also $x^2 = y^2 = z^2 = 2$

Again $\vec{a} = (\vec{x}.\vec{z})\vec{y} - (\vec{x}.\vec{y})\vec{z} = \vec{y} - \vec{z}$ {by **(i)**}

$\therefore$ $\vec{a} = \vec{y} - \vec{z},\ \vec{b} = \vec{z} - \vec{x}$ ....**(ii)**

Now $\vec{a} \times \vec{c} = (\vec{y} - \vec{z}) \times (\vec{x} \times \vec{y}) = \vec{y} \times (\vec{x} \times \vec{y}) - \vec{z} \times (\vec{x} \times \vec{y})$

$= [(\vec{y}\,.\,\vec{y})\vec{x} - (\vec{y}\,.\,\vec{x})\vec{y}] - [(\vec{z}\,.\,\vec{y})\vec{x} - (\vec{z}\,.\,\vec{x})\vec{y}] = (2\vec{x} - \vec{y}) - (\vec{x} - \vec{y})$ {by **(i)**}

or $\vec{a} \times \vec{c} = \vec{x}$

Similarly, $\vec{b} \times \vec{c} = \vec{y}$

Now $\vec{z} = \vec{y} - \vec{a}$ or $\vec{z} = \vec{b} + \vec{x}$ {by **(ii)**}

$\therefore$ $\vec{z} = (\vec{b} \times \vec{c} - \vec{a})$ or $\vec{b} + (\vec{a} \times \vec{c})$

Ex.34 The plane $x - y - z = 4$ is rotated through 90° about its line of intersection with the plane $x + y + 2z = 4$. Find its equation in the new position.

Sol. Given planes are $x - y - z = 4$ ..... **(1)**

and $x + y + 2z = 4$ ..... **(2)**

Since the required plane passes through the line of intersection of planes **(1)** and **(2)**

$\therefore$ its equation may be taken as

$x + y + 2z - 4 + k\,(x - y - z - 4) = 0$

or $(1 + k)x + (1 - k)y + (2 - k)z - 4 - 4k = 0$ ..... **(3)**

Since planes (1) and (3) are mutually perpendicular,

$\therefore$ $(1 + k) - (1 - k) - (2 - k) = 0$

or, $1 + k - 1 + k - 2 + k = 0$ or $k = \dfrac{2}{3}$

Putting $k = \dfrac{2}{3}$ in equation (3), we get $5x + y + 4z = 20$

Ex.35 If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Sol. Given planes are $x - cy - bz = 0$ ..... **(1)**

$cx - y + az = 0$ ..... **(2)**

$bx + ay - z = 0$ ..... **(3)**

Equation of any plane passing through the line of intersection of planes **(1)** and **(2)** may be taken as $x - cy - bz + \lambda\,(cx - y + az) = 0$

or, $x\,(1 + \lambda c) - y\,(c + \lambda) + z\,(-b + a\lambda) = 0$ ..... **(4)**

If planes **(3)** and **(4)** are the same, then equations **(3)** and **(4)** will be identical.

$\therefore \quad \dfrac{1+c\lambda}{b} = \dfrac{-(c+\lambda)}{a} = \dfrac{-b+a\lambda}{-1}$

(i) **(ii)** **(iii)**

From **(i)** and **(ii)**, $a + ac\lambda = -bc - b\lambda$

or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... **(5)**

From **(ii)** and **(iii)**,

$c + \lambda = -ab + a^2\lambda$ or $\lambda = \dfrac{-(ab+c)}{1-a^2}$ ..... **(6)**

From **(5)** and **(6)**, we have $\dfrac{-(a+bc)}{ac+b} = \dfrac{-(ab+c)}{(1-a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$

or, $a^2 + b^2 + c^2 + 2abc = 1$.

This is the equation of the required plane.

Ex. 36 Find direction ratios of normal to the plane which passes through the point (1, 0, 0) and (0, 1, 0) which makes angle $\pi/4$ with $x + y = 3$.

Sol. The plane by intercept form is $\dfrac{x}{1} + \dfrac{y}{1} + \dfrac{z}{c} = 1$

d.r.'s of normal are $1, 1, \dfrac{1}{c}$ and of given plane are 1, 1, 0.

$$\therefore \quad \cos\frac{\pi}{4} = \frac{1\,.\,1 + 1\,.\,1 + 0 \cdot \frac{1}{c}}{\sqrt{1+1+\frac{1}{c^2}}\ \sqrt{1+1+0}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2+\frac{1}{c^2}}\ \sqrt{2}} \quad \Rightarrow \quad 2 + \frac{1}{c^2} = 4 \quad \Rightarrow \quad c = \frac{1}{\sqrt{2}}$$

$\therefore$ d.r.'s are $1, 1, \sqrt{2}$

Ex. 37 Find the equation of the plane passing through (1, 2, 0) which contains the line

$\dfrac{x+3}{3} = \dfrac{y-1}{4} = \dfrac{z-2}{-2}$.

Sol. Equation of any plane passing through (1, 2, 0) may be taken as

$a(x-1) + b(y-2) + c(z-0) = 0$ ..... **(i)**

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\dfrac{x+3}{3} = \dfrac{y-1}{4} = \dfrac{z-2}{-2}$ ..... **(ii)**

If plane **(1)** contains the given line, then

$3a + 4b - 2c = 0$ ..... **(iii)**

Also point (– 3, 1, 2) on line **(2)** lies in plane **(1)**

$\therefore$ $a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... **(iv)**

Solving equations **(iii)** and **(iv)**, we get $\dfrac{a}{8-2} = \dfrac{b}{8-6} = \dfrac{c}{-3+16}$

or, $\dfrac{a}{6} = \dfrac{b}{2} = \dfrac{c}{13} = k$ (say). ..... **(v)**

Substituting the values of a, b and c in equation (1), we get

$6(x-1) + 2(y-2) + 13(z-0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Ex. 38 If $\vec{x} \times \vec{y} = \vec{a}$, $\vec{y} \times \vec{z} = \vec{b}$, $\vec{x} \cdot \vec{b} = \gamma$, $\vec{x} \cdot \vec{y} = 1$ and $\vec{y} \cdot \vec{z} = 1$, then find $\vec{x}$, $\vec{y}$ and $\vec{z}$ in terms of $\vec{a}$, $\vec{b}$ and γ.

Sol. $\vec{x} \times \vec{y} = \vec{a}$ .... **(i)**

$\vec{y} \times \vec{z} = \vec{b}$ .... **(ii)**

Also $\vec{x} \cdot \vec{b} = \gamma$, $\vec{x} \cdot \vec{y} = 1$, $\vec{y} \cdot \vec{z} = 1$ .... **(iii)**

We have to make use of the relations given above.

From **(i)**

$\vec{x} \cdot (\vec{x} \times \vec{y}) = \vec{x} \cdot \vec{a}$

$\therefore$ $\vec{x} \cdot \vec{a} = 0$ $\Rightarrow$ $[\vec{x}\ \vec{x}\ \vec{y}] = 0$

Similarly $\vec{y} \cdot \vec{a} = 0$, $\vec{y} \cdot \vec{b} = 0$, $\vec{z} \cdot \vec{b} = 0$ .... **(iv)**

Multiplying **(i)** vectorially by $\vec{b}$,

$\vec{b} \times (\vec{x} \times \vec{y}) = \vec{b} \times \vec{a}$ or $(\vec{b} \cdot \vec{y})\vec{x} - (\vec{b} \cdot \vec{x})\vec{y} = \vec{b} \times \vec{a}$

or $0 - \gamma\vec{y} = -(\vec{a} \times \vec{b})$ $\therefore$ $\vec{y} = \dfrac{(\vec{a} \times \vec{b})}{\gamma}$ .... **(v)**

by using relations is **(iii)** and **(iv)**.

Again multiplying **(i)** vectorially by $\vec{y}$,

$(\vec{x} \times \vec{y}) \times \vec{y} = \vec{a} \times \vec{y}$ or $(\vec{x} \cdot \vec{y})\vec{y} - (\vec{y} \cdot \vec{y})\vec{x} = \vec{a} \times \vec{y}$

$\vec{y} - \vec{a} \times \vec{y} = |\vec{y}|^2 \vec{x}$ {by **(iii)**}

$\therefore$ $\vec{x} = \dfrac{1}{|\vec{y}|^2}[\vec{y} - \vec{a} \times \vec{y}]$

where $\vec{y} = \dfrac{\vec{a} \times \vec{b}}{\gamma}$ {by **(v)**}

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+91-9350679141

Hence x is known in terms of $\vec{a}, \vec{b}$ and γ.

Again multiplying **(ii)** vectorially by $\vec{y}$, we get

$$(\vec{y} \times \vec{z}) \times \vec{y} = \vec{b} \times \vec{y} \quad \text{or} \quad |\vec{y}|^2 \vec{z} - (\vec{y} \cdot \vec{z})\vec{y} = \vec{b} \times \vec{y} \quad \text{or} \quad |\vec{y}|^2 \vec{z} = \vec{b} \times \vec{y} + \vec{y} \quad \{\text{by (iii)}\}$$

or $$z = \frac{1}{|\vec{y}|^2}[\vec{b} \times \vec{y} + \vec{y}]$$

where $\vec{y}$ is given by **(v)** .... **(vi)**

Results **(v)** and **(vi)** give the values of $\vec{x}, \vec{y}$ and $\vec{z}$ in terms of $\vec{a}, \vec{b}$ and γ.

Ex.39 Find the equation of the sphere if it touches the plane $\vec{r}.(2\hat{i} - 2\hat{j} - \hat{k}) = 0$ and the position vector of its centre is $3\hat{i} + 6\hat{j} - 4\hat{k}$

Sol. Given plane is $\vec{r}.(2\hat{i} - 2\hat{j} - \hat{k}) = 0$ ..... **(1)**

Let H be the centre of the sphere, then $\overrightarrow{OH} = 3\hat{i} + 6\hat{j} - 4\hat{k} = \vec{c}$ (say)

Radius of the sphere = length of perpendicular from H to plane (1)

$$= \frac{|\vec{c}.(2\hat{i} - 2\hat{j} - \hat{k})|}{|2\hat{i} - 2\hat{j} - \hat{k}|} = \frac{|(3\hat{i} + 6\hat{j} - 4\hat{k}).(2\hat{i} - 2\hat{j} - \hat{k})|}{|2\hat{i} - 2\hat{j} - \hat{k}|} = \frac{|6 - 12 + 4|}{3} = \frac{2}{3} = a \text{ (say)}$$

Equation of the required sphere is $|\vec{r} - \vec{c}| = a$

or $$|x\hat{i} + y\hat{j} + z\hat{k} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = \frac{2}{3}$$

or $$|(x-3)\vec{i} + (y-6)\vec{j} + (z+4)\vec{k}|^2 = \frac{4}{9}$$

or $$(x-3)^2 + (y-6)^2 + (z+4)^2 = \frac{4}{9}$$

or $$9(x^2 + y^2 + z^2 - 6x - 12y + 8z + 61) = 4$$

or $$9x^2 + 9y^2 + 9z^2 - 54x - 108y + 72z + 545 = 0$$

Ex.40 Find the equation of the sphere passing through the points (3, 0, 0), (0, – 1, 0), (0, 0, – 2) and whose centre lies on the plane $3x + 2y + 4z = 1$

Sol. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ..... **(i)**

Let $A \equiv (3, 0, 0), B \equiv (0, -1, 0), C \equiv (0, 0, -2)$

Since sphere **(i)** passes through A, B and C,

$\therefore$ $9 + 6u + d = 0$ ..... **(ii)**

$1 - 2v + d = 0$ ..... **(iii)**

$4 - 4w + d = 0$ ..... **(iv)**

Since centre (– u, – v, – w) of the sphere lies on plane

$3x + 2y + 4z = 1$

$\therefore$ $-3u - 2v - 4w = 1$ ..... **(v)**

(ii) – **(iii)** $\Rightarrow$ $6u + 2v = -8$ ..... **(vi)**

(iii) – **(iv)** $\Rightarrow$ $-2v + 4w = 3$ ..... **(vii)**

From **(vi)**, $u = \dfrac{-2v-8}{6}$ ..... **(viii)**

From **(vii)**, $4w = 3 + 2v$ ..... **(ix)**

Putting the values of u, v and w in **(v)**, we get $\dfrac{2v+8}{2} - 2v - 3 - 2v = 1$

$\Rightarrow$ $2v + 8 - 4v - 6 - 4v = 2$ $\Rightarrow$ $v = 0$

From **(viii)**, $u = \dfrac{0-8}{6} = -\dfrac{4}{3}$

From **(ix)**, $4w = 3$ $\therefore$ $w = \dfrac{3}{4}$

From **(iii)**, $d = 2v - 1 = 0 - 1 = -1$

From **(i)**, equation of required sphere is $x^2 + y^2 + z^2 - \dfrac{8}{3}x + \dfrac{3}{2}z - 1 = 0$

or $6x^2 + 6y^2 + 6z^2 - 16x + 9z - 6 = 0$

Exercise # 1

[Single Correct Choice Type Questions]

1. If $\vec{a} + \vec{b}$ is along the angle bisector of $\vec{a}$ & $\vec{b}$ then -

(A) $\vec{a}$ & $\vec{b}$ are perpendicular **(B)** $|\vec{a}| = |\vec{b}|$

(C) angle between $\vec{a}$ & $\vec{b}$ is 60° **(D)** $|\vec{a}| \neq |\vec{b}|$

2. If ABCDEF is a regular hexagon and if $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = \lambda\overrightarrow{AD}$, then λ is -

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

3. If the vector $\vec{b}$ is collinear with the vector $\vec{a} = (2\sqrt{2}, -1, 4)$ and $|\vec{b}| = 10$, then:

(A) $\vec{a} \pm \vec{b} = 0$ **(B)** $\vec{a} \pm 2\vec{b} = 0$ **(C)** $2\vec{a} \pm \vec{b} = 0$ **(D)** none

4. The plane XOZ divides the join of (1, –1, 5) and (2, 3, 4) in the ratio λ : 1, then λ is -

(A) –3 **(B)** –1/3 **(C)** 3 **(D)** 1/3

5. Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is :

(A) $-\hat{i} + \hat{j} + 2\hat{k}$ **(B)** $3\hat{i} - \hat{j} + \hat{k}$ **(C)** $3\hat{i} + \hat{j} - \hat{k}$ **(D)** $\hat{i} - \hat{j} - \hat{k}$

6. The vectors $\overrightarrow{AB} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{BC} = -\hat{i} + 2\hat{k}$ are the adjacent sides of a parallelogram ABCD then the angle between the diagonals is -

(A) $\cos^{-1}\left(\sqrt{\frac{1}{85}}\right)$ **(B)** $\pi - \cos^{-1}\left(\sqrt{\frac{49}{85}}\right)$ **(C)** $\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ **(D)** $\cos^{-1}\left(\sqrt{\frac{3}{10}}\right)$

7. The value of $\left[(\vec{a} + 2\vec{b} - \vec{c})\ (\vec{a} - \vec{b})(\vec{a} - \vec{b} - \vec{c})\right]$ is equal to the box product:

(A) $[\vec{a}\ \vec{b}\ \vec{c}]$ **(B)** $2[\vec{a}\ \vec{b}\ \vec{c}]$ **(C)** $3[\vec{a}\ \vec{b}\ \vec{c}]$ **(D)** $4[\vec{a}\ \vec{b}\ \vec{c}]$

8. Let ABCD be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of triangles ABC, ACD and ADB be 3, 4 and 5 sq. units respectively. Then the area of the triangle BCD, is -

(A) $5\sqrt{2}$ **(B)** 5 **(C)** $\frac{5}{\sqrt{2}}$ **(D)** $\frac{5}{2}$

9. If $|\vec{a}| = 5$, $|\vec{a} - \vec{b}| = 8$ and $|\vec{a} + \vec{b}| = 10$, then $|\vec{b}|$ is equal to :

(A) 1 **(B)** $\sqrt{57}$ **(C)** 3 **(D)** none of these

10. The values of a, for which the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $a\hat{i} - 3\hat{j} + \hat{k}$ respectively are the vertices of a right angled triangle with $C = \frac{\pi}{2}$ are -

(A) –2 and 1 **(B)** 2 and –1 **(C)** 2 and 1 **(D)** –2 and –1

11. The co-ordinates of the centre and the radius of the circle $x + 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z = 11$ are

(A) $(4, 3, 1), \sqrt{5}$ **(B)** $(3, 4, 1), \sqrt{6}$ **(C)** $(1, 3, 4), \sqrt{7}$ **(D)** none of these

12. Which one of the following statement is INCORRECT ?

(A) If $\vec{n}\cdot\vec{a} = 0$, $\vec{n}\cdot\vec{b} = 0$ and $\vec{n}\cdot\vec{c} = 0$ for some non zero vector $\vec{n}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(B) there exist a vector having direction angles $\alpha = 30°$ and $\beta = 45°$

(C) locus of point in space for which $x = 3$ and $y = 4$ is a line parallel to the z-axis whose distance from the z-axis is 5

(D) In a regular tetrahedron OABC where 'O' is the origin, the vector $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ is perpendicular to the plane ABC.

13. OABCDE is a regular hexagon of side 2 units in the XY–plane in the Ist quadrant . O being the origin and OA taken along the X–axis. A point P is taken on a line parallel to Z–axis through the centre of the hexagon at a distance of 3 units from O in the positive Z direction. Then vector $\overrightarrow{AP}$ is:

(A) $-\hat{i} + 3\hat{j} + \sqrt{5}\,\hat{k}$ **(B)** $\hat{i} - \sqrt{3}\,\hat{j} + 5\hat{k}$ **(C)** $-\hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$ **(D)** $\hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$

14. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, then the value of $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} =$

(A) 2 **(B)** 4 **(C)** 16 **(D)** 64

15. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b} + \vec{c}$, $\vec{b}$ to $\vec{c} + \vec{a}$ and $\vec{c}$ to $\vec{a} + \vec{b}$. Then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to :

(A) $2\sqrt{5}$ **(B)** $2\sqrt{2}$ **(C)** $10\sqrt{5}$ **(D)** $5\sqrt{2}$

16. Consider the following 5 statements

(I) There exists a plane containing the points (1, 2, 3) and (2, 3, 4) and perpendicular to the vector $\vec{V}_1 = \hat{i} + \hat{j} - \hat{k}$

(II) There exist no plane containing the point (1, 0, 0); (0, 1, 0); (0, 0, 1) and (1, 1, 1)

(III) If a plane with normal vector $\vec{N}$ is perpendicular to a vector $\vec{V}$ then $\vec{N}\cdot\vec{V} = 0$

(IV) If two planes are perpendicular then every line in one plane is perpendicular to every line on the other plane

(v) Let P_1 and P_2 are two perpendicular planes. If a third plane P_3 is perpendicular to P_1 then it must be either parallel or perpendicular or at an angle of 45° to P_2.

Choose the correct alternative.

(A) exactly one is false **(B)** exactly 2 are false **(C)** exactly 3 are false **(D)** exactly four are false

17. Taken on side $\overrightarrow{AC}$ of a triangle ABC, a point M such that $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$. A point N is taken on the side $\overrightarrow{CB}$ such that $\overrightarrow{BN} = \overrightarrow{CB}$, then for the point of intersection X of $\overrightarrow{AB}$ and $\overrightarrow{MN}$ which of the following holds good?

(A) $\overrightarrow{XB} = \frac{1}{3}\overrightarrow{AB}$ **(B)** $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AB}$ **(C)** $\overrightarrow{XN} = \frac{3}{4}\overrightarrow{MN}$ **(D)** $\overrightarrow{XM} = 3\overrightarrow{XN}$

18. Let $\vec{a} = \hat{i} + \hat{j}$ & $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ & $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is -

(A) $-\hat{i} + \hat{j} + \hat{k}$ **(B)** $3\hat{i} - \hat{j} + \hat{k}$ **(C)** $3\hat{i} + \hat{j} - \hat{k}$ **(D)** $\hat{i} - \hat{j} - \hat{k}$

19. Consider a tetrahedron with faces f_1, f_2, f_3, f_4. Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the vectors whose magnitudes are respectively equal to the areas of f_1, f_2, f_3, f_4 and whose directions are perpendicular to these faces in the outward direction. Then,

(A) $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{0}$ **(B)** $\vec{a}_1 + \vec{a}_3 = \vec{a}_2 + \vec{a}_4$

(C) $\vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ **(D)** none

20. Let L_1 be the line $\vec{r}_1 = 2\hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} + 2\hat{k})$ and let L_2 be the line $\vec{r}_2 = 3\hat{i} + \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$.
Let Π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane Π from the origin is -

(A) $\sqrt{2/7}$ **(B)** $1/7$ **(C)** $\sqrt{6}$ **(D)** none of these

21. A plane meets the coordinate axes in A, B, C and (α, β, γ) is the centroid of the triangle ABC, then the equation of the plane is

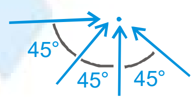
(A) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$ **(B)** $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$ **(C)** $\frac{3x}{\alpha} + \frac{3y}{\beta} + \frac{3z}{\gamma} = 1$ **(D)** $\alpha x + \beta y + \gamma z = 1$

22. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and λ is a real number then $\left[\lambda(\vec{a} + \vec{b})\ \ \lambda^2\vec{b}\ \ \lambda\vec{c}\right] = \left[\vec{a}\ \ \vec{b} + \vec{c}\ \ \vec{b}\right]$ for -

(A) exactly two values of λ **(B)** exactly three values of λ

(C) no value of λ **(D)** exactly one value of λ

23. Four coplanar forces are applied at a point O. Each of them is equal to k and the angle between two consecutive forces equals 45º as shown in the figure. Then the resultant has the magnitude equal to :



(A) $k\sqrt{2 + 2\sqrt{2}}$ **(B)** $k\sqrt{3 + 2\sqrt{2}}$ **(C)** $k\sqrt{4 + 2\sqrt{2}}$ **(D)** none

24. The intercept made by the plane $\vec{r} \cdot \vec{n} = q$ on the x-axis is -

(A) $\frac{q}{\hat{i} \cdot \vec{n}}$ **(B)** $\frac{\hat{i} \cdot \vec{n}}{q}$ **(C)** $(\hat{i} \cdot \vec{n})q$ **(D)** $\frac{q}{|\vec{n}|}$

25. If $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$, then find value of $|3\vec{a} + 4\vec{b} + 12\vec{c}|$ if $\vec{a}, \vec{b}, \vec{c}$ are vectors of same magnitude.

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14

26. Volume of the tetrahedron whose vertices are represented by the position vectors, A (0, 1, 2); B (3, 0, 1); C (4, 3, 6) & D (2, 3, 2) is -

(A) 3 **(B)** 6 **(C)** 36 **(D)** none

27. The equation of the plane passing through the point (1, – 3, – 2) and perpendicular to planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$, is

(A) $2x - 4y + 3z - 8 = 0$ **(B)** $2x - 4y - 3z + 8 = 0$

(C) $2x + 4y + 3z + 8 = 0$ **(D)** None of these

28. If from the point P(f, g, h) perpendiculars PL, PM be drawn to yz and zx planes then the equation to the plane OLM is

(A) $\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$ **(B)** $\frac{x}{f} + \frac{y}{g} - \frac{z}{h} = 0$

(C) $\frac{x}{f} - \frac{y}{g} + \frac{z}{h} = 0$ **(D)** $-\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$

29. If $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors, then which one of the following set of vectors is linearly dependent ?

(A) $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ **(B)** $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ **(C)** $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ **(D)** none

30. The sine of angle formed by the lateral face ADC and plane of the base ABC of the tetrahedron ABCD where $A \equiv (3, -2, 1)$; $B \equiv (3, 1, 5)$; $C \equiv (4, 0, 3)$ and $D \equiv (1, 0, 0)$ is -

(A) $\frac{2}{\sqrt{29}}$ **(B)** $\frac{5}{\sqrt{29}}$ **(C)** $\frac{3\sqrt{3}}{\sqrt{29}}$ **(D)** $\frac{-2}{\sqrt{29}}$

31. Given the points $A(-2, 3, -4)$, $B(3, 2, 5)$, $C(1, -1, 2)$ & $D(3, 2, -4)$. The projection of the vector $\vec{AB}$ on the vector $\vec{CD}$ is -

(A) $\frac{22}{3}$ **(B)** $-\frac{21}{4}$ **(C)** $-\frac{47}{7}$ **(D)** –47

32. Given the vertices A (2, 3, 1), B (4, 1, –2), C (6, 3, 7) & D (–5, –4, 8) of a tetrahedron. The length of the altitude drawn from the vertex D is -

(A) 7 **(B)** 9 **(C)** 11 **(D)** none

33. Equation of the angle bisector of the angle between the lines $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1}$ & $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{-1}$ is :

(A) $\frac{x-1}{2} = \frac{y-2}{2}$; $z - 3 = 0$ **(B)** $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$

(C) $x - 1 = 0$; $\frac{y-2}{1} = \frac{z-3}{1}$ **(D)** None of these

34. The distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line, $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$, is:

(A) 1 **(B)** 6/7 **(C)** 7/6 **(D)** None of these

Add. 41-42A, Ashok Park Main, New Rohtak Road, New Delhi-110035
+91-9350679141

35. The line which contains all points (x, y, z) which are of the form $(x, y, z) = (2, -2, 5) + \lambda(1, -3, 2)$ intersects the plane $2x - 3y + 4z = 163$ at P and intersects the YZ plane at Q. If the distance PQ is $a\sqrt{b}$, where $a, b \in N$ and $a > 3$ then $(a + b)$ equals -

(A) 23 **(B)** 95 **(C)** 27 **(D)** none of these

36. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is:

(A) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$ **(B)** $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$

(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$ **(D)** $x^2 + y^2 + z^2 + x + 2y + 3z = 0$

37. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-zero vectors such that $\vec{a}$ and $\vec{b}$ are non-collinear & satisfies $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$.

If θ is the angle between the vectors $\vec{b}$ and $\vec{c}$ then $\sin\theta$ equals -

(A) $\frac{2}{3}$ **(B)** $\sqrt{\frac{2}{3}}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2\sqrt{2}}{3}$

38. A, B, C & D are four points in a plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ respectively such that $(\vec{a} - \vec{d}).(\vec{b} - \vec{c}) = (\vec{b} - \vec{d}).(\vec{c} - \vec{a}) = 0$. Then for the triangle ABC, D is its:

(A) incentre **(B)** circumcentre **(C)** orthocentre **(D)** centroid

39. A plane passes through the point P(4, 0, 0) and Q(0, 0, 4) and is parallel to the y-axis. The distance of the plane from the origin is -

(A) 2 **(B)** 4 **(C)** $\sqrt{2}$ **(D)** $2\sqrt{2}$

40. Let $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{r}_1 = \vec{a} - \vec{b} + \vec{c}$, $\vec{r}_2 = \vec{b} + \vec{c} - \vec{a}$, $\vec{r}_3 = \vec{c} + \vec{a} + \vec{b}$, $\vec{r} = 2\vec{a} - 3\vec{b} + 4\vec{c}$.

If $\vec{r} = \lambda_1\vec{r}_1 + \lambda_2\vec{r}_2 + \lambda_3\vec{r}_3$, then the values of λ_1, λ_2 and λ_3 respectively are

(A) 7, 1, –4 **(B)** 7 / 2, 1, –1 / 2 **(C)** 5 / 2, 1, 1/2 **(D)** –1 / 2, 1, 7 / 2

41. The vertices of a triangle are A (1, 1, 2), B(4, 3, 1) and C(2, 3, 5). A vector representing the internal bisector of the angle A is :

(A) $\hat{i} + \hat{j} + 2\hat{k}$ **(B)** $2\hat{i} - 2\hat{j} + \hat{k}$ **(C)** $2\hat{i} + 2\hat{j} - \hat{k}$ **(D)** $2\hat{i} + 2\hat{j} + \hat{k}$

42. A, B, C, D be four points in a space and if, $|\vec{AB} \times \vec{CD} + \vec{BC} \times \vec{AD} + \vec{CA} \times \vec{BD}| = \lambda$ (area of triangle ABC)

then the value of λ is -

(A) 4 **(B)** 2 **(C)** 1 **(D)** none of these

43. For a non zero vector $\vec{A}$ if the equations $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ hold simultaneously, then:

(A) $\vec{A}$ is perpendicular to $\vec{B} - \vec{C}$ **(B)** $\vec{A} = \vec{B}$

(C) $\vec{B} = \vec{C}$ **(D)** $\vec{C} = \vec{A}$

44. The distance between the parallel planes given by the equations, $\vec{r}\,.\,(2\hat{i} - 2\hat{j} + \hat{k}) + 3 = 0$ and $\vec{r}\,.\,(4\hat{i} - 4\hat{j} + 2\hat{k}) + 5 = 0$ is -

(A) 1/2 **(B)** 1/3 **(C)** 1/4 **(D)** 1/6

45. If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a} \parallel (\vec{b} \times \vec{c})$, then $(\vec{a} \times \vec{b})\,.\,(\vec{a} \times \vec{c})$ is equal to

(A) $\vec{a}^2\,(\vec{b}\,.\,\vec{c})$ **(B)** $\vec{b}^2\,(\vec{a}\,.\,\vec{c})$ **(C)** $\vec{c}^2\,(\vec{a}\,.\,\vec{b})$ **(D)** none of these

46. Unit vector perpendicular to the plane of the triangle ABC with position vectors $\vec{a}, \vec{b}, \vec{c}$ of the vertices A, B, C, is (where Δ is the area of the triangle ABC).

(A) $\dfrac{\left(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right)}{\Delta}$ **(B)** $\dfrac{\left(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right)}{2\Delta}$

(C) $\dfrac{\left(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right)}{4\Delta}$ **(D)** none of these

47. If the volume of the parallelopiped whose conterminous edges are represented by $-12\hat{i} + \lambda\hat{k},\ 3\hat{j} - \hat{k},\ 2\hat{i} + \hat{j} - 15\hat{k}$ is 546, then λ equals-

(A) 3 **(B)** 2 **(C)** –3 **(D)** –2

48. The reflection of the point (2, –1, 3) in the plane $3x - 2y - z = 9$ is :

(A) $\left(\dfrac{26}{7}, \dfrac{15}{7}, \dfrac{17}{7}\right)$ **(B)** $\left(\dfrac{26}{7}, \dfrac{-15}{7}, \dfrac{17}{7}\right)$ **(C)** $\left(\dfrac{15}{7}, \dfrac{26}{7}, \dfrac{-17}{7}\right)$ **(D)** $\left(\dfrac{26}{7}, \dfrac{17}{7}, \dfrac{-15}{7}\right)$

49. If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(k)$ with x-axis, then k is equal to -

(A) $\dfrac{\sqrt{3}}{2}$ **(B)** $\dfrac{2}{7}$ **(C)** $\dfrac{\sqrt{2}}{3}$ **(D)** 1

50. A line makes angles α, β, γ with the coordinate axes. If $\alpha + \beta = 90^\circ$, then $\gamma =$

(A) 0 **(B)** 90º **(C)** 180º **(D)** None of these

51. Given the vertices A (2, 3, 1), B (4, 1, –2), C (6, 3, 7) & D (–5, –4, 8) of a tetrahedron. The length of the altitude drawn from the vertex D is:

(A) 7 **(B)** 9 **(C)** 11 **(D)** none of these

52. If $\vec{a} + 5\vec{b} = \vec{c}$ and $\vec{a} - 7\vec{b} = 2\vec{c}$, then-

(A) $\vec{a}$ and $\vec{c}$ are like but $\vec{b}$ and $\vec{c}$ are unlike vectors

(B) $\vec{a}$ and $\vec{b}$ are unlike vectors and so also $\vec{a}$ and $\vec{c}$

(C) $\vec{b}$ and $\vec{c}$ are like but $\vec{a}$ and $\vec{b}$ are unlike vectors

(D) $\vec{a}$ and $\vec{c}$ are unlike vectors and so also $\vec{b}$ and $\vec{c}$

53. The straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$ are

(A) Parallel lines **(B)** Intersecting at 60º

(C) Skew lines **(D)** Intersecting at right angle

54. A variable plane forms a tetrahedron of constant volume $64K^3$ with the coordinate planes and the origin, then locus of the centroid of the tetrahedron is -

(A) $x^3+y^3+z^3=6K^2$ **(B)** $xyz=6k^3$ **(C)** $x^2+y^2+z^2=4K^2$ **(D)** $x^{-2}+y^{-2}+z^{-2}=4k^{-2}$

55. The locus represented by $xy + yz = 0$ is

(A) A pair of perpendicular lines **(B)** A pair of parallel lines

(C) A pair of parallel planes **(D)** A pair of perpendicular planes

56. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar and $\vec{p}$, $\vec{q}$, $\vec{r}$ are reciprocal vectors to $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, then

$(\bullet\vec{a}+m\vec{b}+n\vec{c}).(\bullet\vec{p}+m\vec{q}+n\vec{r})$ is equal to : (where $\bullet$, m, n are scalars)

(A) $\bullet^2+m^2+n^2$ **(B)** $\bullet m+mn+n\bullet$ **(C)** 0 **(D)** none of these

57. Let $\vec{a}=x\hat{i}+12\hat{j}-\hat{k}$, $\vec{b}=2\hat{i}+2x\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{k}$. If the ordered set $\left[\vec{b}\ \vec{c}\ \vec{a}\right]$ is left handed, then :

(A) $x\in(2,\infty)$ **(B)** $x\in(-\infty,-3)$ **(C)** $x\in(-3,2)$ **(D)** $x\in\{-3,2\}$

58. The expression in the vector form for the point $\vec{r}_1$ of intersection of the plane $\vec{r}\cdot\vec{n}=d$ and the perpendicular line $\vec{r}=\vec{r}_0+t\vec{n}$ where t is a parameter given by -

(A) $\vec{r}_1=\vec{r}_0+\left(\frac{d-\vec{r}_0\cdot\vec{n}}{\vec{n}^2}\right)\vec{n}$ **(B)** $\vec{r}_1=\vec{r}_0-\left(\frac{\vec{r}_0\cdot\vec{n}}{\vec{n}^2}\right)\vec{n}$

(C) $\vec{r}_1=\vec{r}_0-\left(\frac{\vec{r}_0\cdot\vec{n}-d}{|\vec{n}|}\right)\vec{n}$ **(D)** $\vec{r}_1=\vec{r}_0+\left(\frac{\vec{r}_0\cdot\vec{n}}{|\vec{n}|}\right)\vec{n}$

59. If 3 non zero vectors $\vec{a},\vec{b},\vec{c}$ are such that $\vec{a}\times\vec{b}=2(\vec{a}\times\vec{c})$, $|\vec{a}|=|\vec{c}|=1$; $|\vec{b}|=4$ the angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\frac{1}{4}$ then $\vec{b}=1\vec{c}+\mu\vec{a}$ where $|\bullet|+|\mu|$ is -

(A) 6 **(B)** 5 **(C)** 4 **(D)** 0

60. If $\vec{x}$ & $\vec{y}$ are two non collinear vectors and a, b, c represent the sides of a ΔABC satisfying $(a-b)\vec{x}+(b-c)\vec{y}+(c-a)(\vec{x}\times\vec{y})=0$ then ΔABC is -

(A) an acute angle triangle **(B)** an obtuse angle triangle

(C) a right angle triangle **(D)** a scalene triangle

61. If a plane cuts off intercepts OA = a, OB = b, OC = c from the coordinate axes (where 'O' is the origin), then the area of the triangle ABC is equal to

(A) $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$ **(B)** $\frac{1}{2}(bc+ca+ab)$

(C) $\frac{1}{2}abc$ **(D)** $\frac{1}{2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}$

62. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three non-coplanar vectors then $(\vec{A}+\vec{B}+\vec{C}).[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]$ equals -

(A) 0 **(B)** $[\vec{A}\ \vec{B}\ \vec{C}]$ **(C)** $2[\vec{A}\ \vec{B}\ \vec{C}]$ **(D)** $-[\vec{A}\ \vec{B}\ \vec{C}]$

63. If $\vec{a} = i+j-k$, $\vec{b} = i-j+k$, $\vec{c}$ is a unit vector such that $\vec{c}.\vec{a} = 0$, $[\vec{c}\,\vec{a}\,\vec{b}] = 0$ then a unit vector $\vec{d}$ perpendicular to both $\vec{a}$ and $\vec{c}$ is

(A) $\frac{1}{\sqrt{6}}(2i-j+k)$ **(B)** $\frac{1}{2}(j+k)$ **(C)** $\frac{1}{\sqrt{2}}(i+j)$ **(D)** $\frac{1}{\sqrt{2}}(i+k)$

64. The equation of a plane which passes through (2, – 3, 1) & is perpendicular to the line joining the points (3, 4, – 1) & (2, – 1, 5) is given by:

(A) $x+5y-6z+19=0$ **(B)** $x-5y+6z-19=0$

(C) $x+5y+6z+19=0$ **(D)** $x-5y-6z-19=0$

65. The equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point (0, 7, –7) is -

(A) $x+y+z=1$ **(B)** $x+y+z=2$ **(C)** $x+y+z=0$ **(D)** none of these

66. Equation of plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$ and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at greatest distance from the point (0, 0, 0) is :

(A) $4x+3y+5z=25$ **(B)** $4x+3y+5z=50$

(C) $3x+4y+5z=49$ **(D)** $x+7y-5z=2$

67. If the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$ and $\frac{x+k}{3} = \frac{y-1}{2} = \frac{z-2}{h}$ are concurrent then

(A) $h=-2, k=-6$ **(B)** $h=\frac{1}{2}, k=2$ **(C)** $h=6, k=2$ **(D)** $h=2, k=\frac{1}{2}$

68. Consider the lines $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, then the equation of the line which

(A) bisects the angle between the lines is $\frac{x}{3} = \frac{y}{3} = \frac{z}{8}$

(B) bisects the angle between the lines is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

(C) passes through origin and is perpendicular to the given lines is $x = y = -z$

(D) none of these

69. The coplanar points A, B, C, D are $(2-x, 2, 2)$, $(2, 2-y, 2)$, $(2, 2, 2-z)$ and $(1, 1, 1)$ respectively, then

(A) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

(B) $x+y+z=1$

(C) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=1$

(D) none of these

70. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors equally inclined to one another at an acute angle θ. Then $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$ in terms of θ is equal to:

(A) $(1+\cos\theta)\sqrt{\cos 2\theta}$

(B) $(1+\cos\theta)\sqrt{1-2\cos 2\theta}$

(C) $(1-\cos\theta)\sqrt{1+2\cos\theta}$

(D) none of these