

HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. angle = $\frac{\text{arc}}{\text{radius}}$ (i)

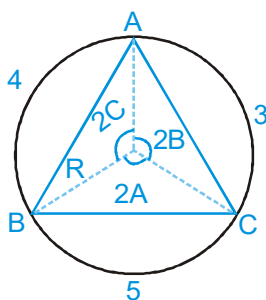
$$4 + 5 + 3 = 2\pi R$$

$$\Rightarrow R = 6/\pi$$

$$\rightarrow 2A = \frac{5}{R} = \frac{5\pi}{6},$$

$$2B = \frac{3}{R} = \frac{\pi}{2} \text{ and}$$

$$2C = \frac{4}{R} = \frac{2\pi}{3}$$



$$\text{Area of } \Delta ABC = \frac{1}{2} R^2 \left[\sin \frac{2\pi}{3} + \sin \frac{5\pi}{6} + \sin \frac{\pi}{2} \right] = \frac{R^2}{2}$$

$$\left[\frac{\sqrt{3}}{2} + \frac{1}{2} + 1 \right] = \frac{R^2}{2} \left[\frac{\sqrt{3}+3}{2} \right]$$

$$= \frac{\sqrt{3}(\sqrt{3}+1)}{4} \times \frac{36}{\pi^2} = \frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$$

4. $\left(\frac{4R^2 \sin^2 A}{\sin A} + \frac{4R^2 \sin^2 B}{\sin B} + \frac{4R^2 \sin^2 C}{\sin C} \right) \prod \sin \frac{A}{2}$

$$= 4R^2 (\sin A + \sin B + \sin C) \prod \sin \frac{A}{2}$$

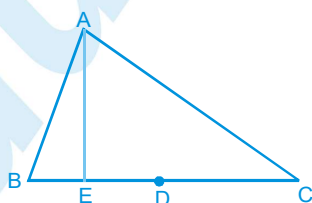
$$= 16R^2 \prod \cos \frac{A}{2} \cdot \prod \sin \frac{A}{2} = 2R^2 \sin A \cdot \sin B \cdot \sin C$$

$$= 2R^2 \frac{abc}{8R^3} = \frac{abc}{4R} = \Delta$$

5. $ED = \frac{a}{2} - c \cos B$

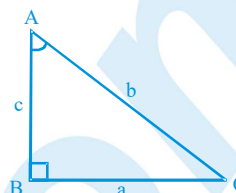
$$= \frac{a}{2} - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$= \frac{a}{2} - \left(\frac{a^2 + c^2 - b^2}{2a} \right) = \frac{a^2 - a^2 - c^2 + b^2}{2a} = \frac{b^2 - c^2}{2a}$$



6. $\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}$

$$= \frac{1 - \frac{c}{b}}{1 + \frac{c}{b}} = \frac{b - c}{b + c}$$



7. $a + b + c = 2s \Rightarrow s = 2a$

Applying half angle formulae.

$$\cot \frac{B}{2} \cdot \cot \frac{C}{2} = \sqrt{\frac{(s)(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{(s)(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s}{s-a} = 2$$

8. $r_1 + r_2 = \frac{\Delta c}{(s-a)(s-b)}$

$$\therefore \prod (r_1 + r_2) = \frac{\Delta^3 abc}{(s-a)^2 (s-b)^2 (s-c)^2} = \frac{\Delta^3 (abc)s^2}{\Delta^4}$$

$$= \frac{(abc)s^2}{\Delta} = \frac{4R\Delta s^2}{\Delta} = 4Rs^2$$

$$\therefore \frac{\prod (r_1 + r_2)}{Rs^2} = 4$$

9. $a = 1$

$$\rightarrow 2s = 6 \left(\frac{\sin A + \sin B + \sin C}{3} \right)$$

$$\Rightarrow 2s = 2 \left(\frac{a+b+c}{2R} \right)$$

$$R = 1$$

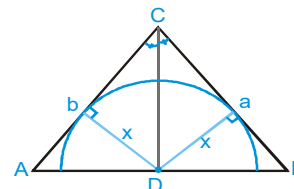
$$\rightarrow \frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{1}{2}$$

$$A = \frac{\pi}{6}$$

10. $\text{Area}(\Delta ADC) = \frac{1}{2} b \cdot x$, $\text{Area}(\Delta BCD) = \frac{1}{2} x \cdot a$

$$\Rightarrow \Delta = \frac{1}{2} x(b+a)$$

$$\Rightarrow x = \frac{2\Delta}{a+b}$$



12. Using $r_1 = \frac{\Delta}{(s-a)}$, $r_2 = \frac{\Delta}{(s-b)}$, $r_3 = \frac{\Delta}{(s-c)}$

we get $\frac{(2s-(a+b))(2s-(b+c))(2s-(c+a))}{\Delta^3}$

$$\Rightarrow \frac{abc}{\Delta^3} = \frac{KR^3}{(abc)^2} \Rightarrow \frac{64R^3}{(abc)^2} = \frac{KR^3}{(abc)^2}$$

hence $K = 64$

14.
$$\frac{\sum \frac{\Delta}{s-a}}{\sqrt{\sum \frac{\Delta^2}{(s-a)(s-b)}}} = \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{1}{(s-a)(s-b)}}}$$

$$= \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{s(s-c)}{\Delta^2}}} = \sum \frac{1}{(s-a)} \times \frac{\Delta}{s} = \sum \tan \frac{A}{2}$$

$$(\rightarrow) r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

15. $\cos C = \frac{(19)^2 + (17)^2 - c^2}{2 \cdot 19 \cdot 17} > 0$

$$\begin{aligned} \therefore c^2 &< (19)^2 + (17)^2 \\ c^2 &< (18+1)^2 + (18-1)^2 \\ c^2 &< 2(18^2 + 1) \\ c^2 &< 650 \end{aligned}$$

$$c > 2 \quad \text{and} \quad c < 26 \quad \dots(1)$$

$$\therefore 2 < c < 26 \quad \dots(2)$$

$$4 < c^2 < 676 \quad \dots(3)$$

$$\therefore c \text{ is } 3, 4, 5, \dots, 25$$

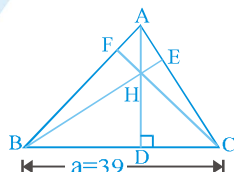
$$\therefore \text{number of integral values of } c \text{ is } 23$$

16. $AH = 2R \cos A$ (distance of orthocentre from the vertex)

given $\sin A = 3/5$; $\cos A = 4/5$
 $a = 39$

$$\therefore \frac{a}{\sin A} = 2R; \quad \frac{39 \cdot 5}{3} = 2R \Rightarrow 2R = 65$$

$$\therefore AH = 65 \cdot \frac{4}{5} = 52$$



17. $\Delta = \frac{6 \cdot h}{2} = 3h$; (where h is the altitude from A)

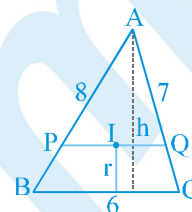
Also $\Delta = \frac{21 \cdot r}{2}$ (using $\Delta = r \cdot s$)

$$\frac{21 \cdot r}{2} = \frac{6 \cdot h}{2} = 3h \Rightarrow \frac{r}{h} = \frac{2}{7}$$

now APQ and ABC are similar

$$\frac{h-r}{h} = \frac{PQ}{6} \Rightarrow 1 - \frac{r}{h} = \frac{PQ}{6} \Rightarrow 1 - \frac{2}{7} = \frac{PQ}{6}$$

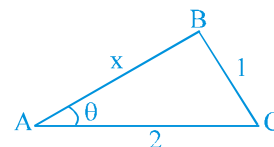
$$\Rightarrow \frac{5}{7} = \frac{PQ}{6} \Rightarrow PQ = \frac{30}{7}$$



18. Using cosine rule

$$\cos \theta = \frac{x^2 + 4 - 1}{4x} = \frac{x^2 + 3}{4x} = \frac{1}{4} \left[x + \frac{3}{x} \right]$$

$$= \frac{1}{4} \left[\left(\sqrt{x} - \sqrt{\frac{3}{x}} \right)^2 + 2\sqrt{3} \right]$$



hence $\cos \theta$ is minimum if $x = \sqrt{3}$

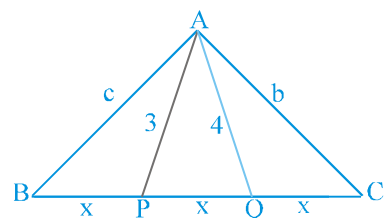
$$\therefore \text{minimum value of } \cos \theta = 2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\text{maximum value of } \theta = \frac{\pi}{6}$$

19. In $\triangle ABP$

$$9 = c^2 + x^2 - 2cx \cos B; \quad \text{but } \cos B = \frac{c}{3x}$$

$$9 = c^2 + x^2 - 2cx \cdot \frac{c}{3x}$$



$$9 = x^2 + \frac{c^2}{3} \quad \dots(1); \quad 16 = x^2 + \frac{b^2}{3} \quad \dots(2)$$

$$(1) + (2)$$

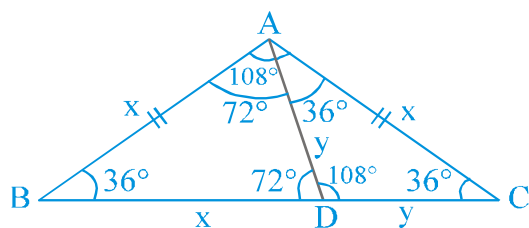
$$25 = 2x^2 + \frac{1}{3}(b^2 + c^2) = 2x^2 + 3x^2$$

$$\Rightarrow 5x^2 = 25 \Rightarrow x = \sqrt{5}; \quad \therefore BC = 3\sqrt{5} = \sqrt{45}$$

20. In $\triangle ABC$ $\left(\frac{x}{y} = ?\right)$

$$\frac{x+y}{\sin 108^\circ} = \frac{x}{\sin 36^\circ}$$

$$\frac{x+y}{x} = \frac{\sin 108^\circ}{\sin 36^\circ} \quad \dots (1)$$



$$1 + \frac{y}{x} = \frac{\sin 72^\circ}{\sin 36^\circ} = 2 \cos 36^\circ$$

$$\frac{y}{x} = \frac{\sqrt{5}+1}{2} - 1 = \frac{\sqrt{5}-1}{2}$$

$$\frac{x}{y} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{\sqrt{5}+1}{2}$$

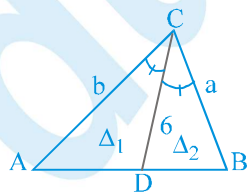
21. $\frac{\tan C}{\sin B} = \frac{\sin C}{\sin B} \cdot \frac{1}{\cos C}$

now using sine law $\frac{\sin C}{\sin B} = \frac{c}{b} = \frac{10}{9}$;

using cosine law $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5}{16}$

$$\therefore \frac{\tan C}{\sin B} = \frac{\sin C}{\sin B} \cdot \frac{1}{\cos C} = \frac{10}{9} \cdot \frac{16}{5} = \frac{32}{9}$$

22. $\Delta = \Delta_1 + \Delta_2 = \frac{1}{2} ab \sin C = ab \sin \frac{C}{2} \cos \frac{C}{2}$



$$= \frac{1}{2} 6b \sin \frac{C}{2} + \frac{1}{2} 6a \sin \frac{C}{2} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{9}$$

23. $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$

Note: $b^2 + c^2 - a^2 = 2bc \cos A$ (from cosine rule)

let $f(x) = b^2x^2 + (2bc \cos A)x + c^2 = 0$ also $A \in (0, \pi)$ in a triangle

$$\therefore \cos A \in (-1, 1)$$

$$2bc \cos A \in (-2bc, 2bc)$$

$$\Rightarrow D = (2bc \cos A)^2 - 4b^2c^2 = 4b^2c^2(\cos^2 A - 1)$$

$$\Rightarrow D < 0 \Rightarrow (A) \text{ is correct}$$

24. Put $a = 2R \sin A$ etc.

$$E = R[\sin 2B + \sin 2C + \sin 2C + \sin 2A + \sin 2A + \sin 2B]$$

$$= 2R(\sin 2A + \sin 2B + \sin 2C)$$

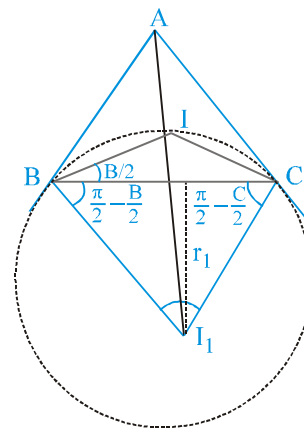
$$= 8R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{R^2}$$

25. $BICI_1$ is a cyclic quadrilateral with II_1 as the diameter

also $\angle BI_1C = \frac{\pi}{2} - \frac{A}{2}$

applying sine law in BCI_1

$$\frac{a}{\cos \frac{A}{2}} = II_1$$



$$\therefore II_1 = \frac{2R \cdot 2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}}{\cos \frac{A}{2}} = 4R \sin \frac{A}{2}$$

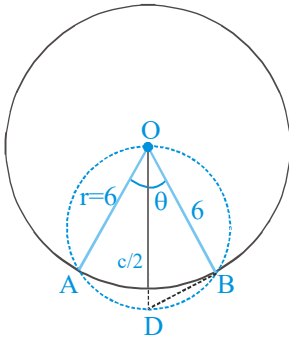
$$\therefore \prod II_1 = 64R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 16R^2 r$$

$$(\text{using } 4R \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = r)$$

$$26. R = \frac{abc}{4\Delta}; \quad \Delta = \frac{1}{2} \cdot 6 \cdot 6 \sin \theta = 18 \sin \theta$$

$$a = b = 6$$

$$\sin \frac{\theta}{2} = \frac{c}{12} \Rightarrow c = 12 \sin \frac{\theta}{2}$$



$$R = \frac{36 \cdot 12 \sin(\theta/2)}{4 \cdot 18 \cdot \sin \theta} = 3 \sec \frac{\theta}{2}$$

$$27. a^2 + b^2 < c^2 \quad a^2 + b^2 < a^2 + b^2 - 2ab \cos C$$

$$\therefore \cos C < 0$$

$$\Rightarrow C \text{ is obtuse} \Rightarrow (B)$$

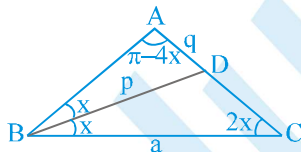
$$28. BC = BD + AD \quad a = p + q$$

$$\text{or } \frac{a}{p} = 1 + \frac{q}{p}$$

using Sine law in ΔBDC and in ΔABD

$$\frac{\sin 3x}{\sin 2x} = 1 + \frac{\sin x}{\sin 4x}$$

$$\Rightarrow \frac{\sin 3x \cdot \sin 4x - \sin x \cdot \sin 2x}{\sin 2x \cdot \sin 4x} = 1$$



$$2 \sin 3x \cdot \sin 4x - 2 \sin x \cdot \sin 2x = 2 \sin 2x \cdot \sin 4x$$

$$(\cos x - \cos 7x) - (\cos x - \cos 3x) = \cos 2x - \cos 6x$$

$$\cos 3x - \cos 7x = \cos 2x - \cos 6x$$

$$2 \sin 5x \cdot \sin 2x = 2 \sin 4x \cdot \sin 2x$$

$$\text{as } \sin 2x \neq 0$$

$$\text{hence } \sin 5x = \sin 4x$$

$$5x + 4x = 180^\circ$$

$$\Rightarrow x = 20^\circ; \text{ hence } \angle A = 180^\circ - 80^\circ = 100^\circ$$

$$29. \text{LHS} \quad \frac{R [\sin 2A + \sin 2B + \sin 2C]}{2R [\sin A + \sin B + \sin C]}$$

$$= \frac{4 \sin A \sin B \sin C}{2 \cdot 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{R}$$

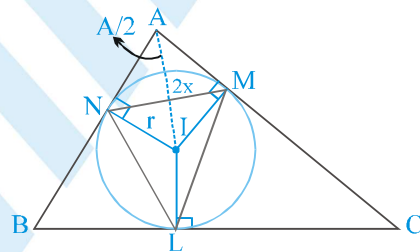
$$30. \text{Numerator of RHS} = ks^2 \cdot 4R \text{ and denominator of}$$

$$\text{RHS} = s^2 \quad (\text{using } r_1 = s \tan \frac{A}{2} \text{ etc.})$$

$$\Rightarrow \frac{\text{numerator}}{\text{denominator}} \text{ of RHS} = k \cdot 4R = \text{LHS} = R \Rightarrow k = \frac{1}{4}$$

$$31. \text{note that ANIM is a cyclic quadrilateral.}$$

$$\Rightarrow \operatorname{cosec} \frac{A}{2} = \frac{2x}{r} \Rightarrow 2x = r \operatorname{cosec} \frac{A}{2}$$

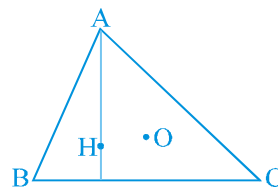


$$x = \frac{r}{2 \sin \frac{A}{2}} \Rightarrow xyz = \frac{r^3}{2 \cdot 4 \prod \sin \frac{A}{2}} = \frac{r^3 \cdot R}{2 \cdot r} = \frac{r^2 R}{2}$$

$$32. OA = HA$$

$$R = 2R \cos A \quad (\text{distance of orthocentre from the vertex A is } 2R \cos A)$$

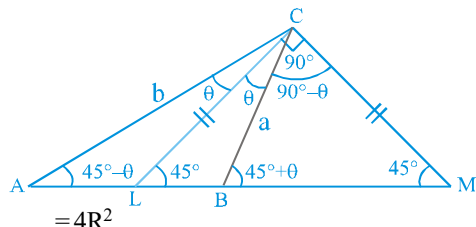
$$\Rightarrow \cos A = \frac{1}{2}$$



$$\Rightarrow A = \frac{\pi}{3} \Rightarrow (C)$$

$$33. a^2 + b^2 = 4R^2[\sin^2(45^\circ - \theta) + \sin^2(135^\circ - \theta)]$$

$$= 4R^2[\sin^2(45^\circ - \theta) + \cos^2(45^\circ - \theta)]$$



$$34. \cot \frac{B}{2} \cdot \cot \frac{C}{2} = \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta} \cdot \frac{(s-a)}{s-a}$$

$$= \frac{s}{s-a} = \frac{2s}{2s-2a}$$

but given that $a + b + c = 4a \Rightarrow 2s = 4a$

Hence $\cot \frac{B}{2} \cdot \cot \frac{C}{2} = \frac{4a}{2a} = 2$

$$35. \tan A = \frac{a}{2f} \Rightarrow \frac{1}{2} \Sigma \tan A = \frac{1}{2} \Pi \tan A$$

$$= \frac{1}{4} \left(\frac{a}{f} \cdot \frac{b}{g} \cdot \frac{c}{h} \right) \Rightarrow A$$

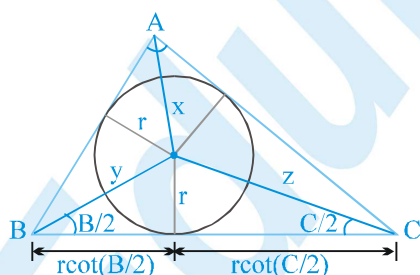
$$36. \text{ use } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} \text{ to get } A-B=60^\circ \text{ and}$$

$$A+B=150^\circ \text{ (given)}$$

$$\Rightarrow A=105^\circ$$

$$37. x = r \operatorname{cosec} \frac{A}{2}$$

$$a = r \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)$$



$$\frac{a}{x} = \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) \cdot \sin \frac{A}{2} = \frac{\sin \frac{A}{2} \cdot \cos \frac{A}{2}}{\sin \frac{B}{2} \cdot \sin \frac{C}{2}}$$

$$\therefore \frac{abc}{xyz} = \frac{\cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}}{\sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}} = \cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}$$

In a triangle $\prod \cot \frac{A}{2} = \sum \cot \frac{A}{2}$

$$38. \sin 2C = \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1. \text{ Now proceed } A=B=\frac{3\pi}{8}$$

and $C = \frac{\pi}{4}$

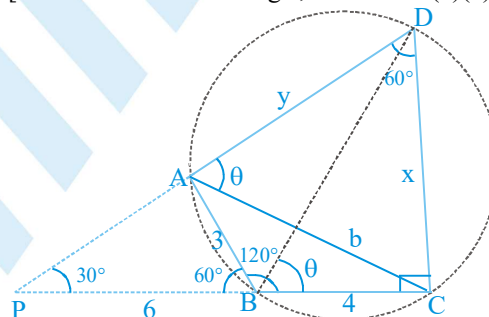
$1 \leq \cos(A-B) \Rightarrow \cos(A-B) = 1 \Rightarrow A=B$

39. Extend CB and DA to meet at P.

note that ΔPCD is right angle as shown.

now $\tan 30^\circ = \frac{x}{10}$

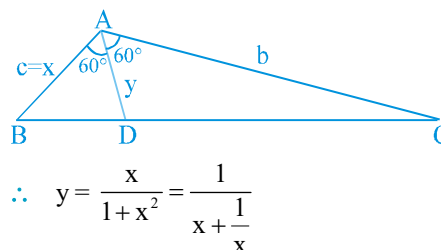
[PAB is $30^\circ-60^\circ-90^\circ$ triangle, hence $PB = (2)(3) = 6$]



$$\therefore x = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

$$40. AD = y = \frac{2bc}{b+c} \cos \frac{A}{2} = \frac{bx}{b+x} \text{ (as } c=x)$$

but $bx = 1 \Rightarrow b = \frac{1}{x}$



$$\therefore y = \frac{x}{1+x^2} = \frac{1}{x + \frac{1}{x}}$$

$y_{\max} = \frac{1}{2}$ since minimum value of the denominator is 2

if $x > 0 \Rightarrow (B)$

EXERCISE - 2

Part # I : Multiple Choice

1. (A) $\rightarrow \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right) \cot \frac{C}{2}$ (i)

$$\rightarrow \tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$$

$$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}} \rightarrow a = 5 \text{ and } b = 4$$

$\therefore$ from equation (i), we get

$$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right) \cot \frac{C}{2} \Rightarrow \frac{1}{3\sqrt{7}} = \frac{1}{9} \cot \frac{C}{2}$$

$$\Rightarrow \cot \frac{C}{2} = \frac{3}{\sqrt{7}}$$

$$\rightarrow \cos C = \frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$$

$$\rightarrow \cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

$$\Rightarrow c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c = 6$$

(B), (C) $\rightarrow$ Area = $\frac{1}{2} ab \sin C \rightarrow \cos C = \frac{1}{8}$

$$\Rightarrow \sin C = \sqrt{1 - \frac{1}{64}} = \frac{3\sqrt{7}}{8}$$

$$\text{Area} = \frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$$

$$\text{Area} = \frac{15\sqrt{7}}{4} \text{ sq. unit.}$$

$\rightarrow$ From Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \sin A = \frac{a \sin C}{c} = \frac{5 \times 3\sqrt{7}}{6 \times 8}$$

$$\therefore \sin A = \frac{5\sqrt{7}}{16}$$

3. $\cos A + \cos C = 4 \sin^2 \frac{B}{2}$

$$\Rightarrow \cos\left(\frac{A+C}{2}\right) \cdot \cos\left(\frac{A-C}{2}\right) = 2 \sin^2 \frac{B}{2}$$

$$= 2 \cos^2\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\cos\frac{A+C}{2}} = \frac{2}{1} \Rightarrow \cot \frac{A}{2} \cdot \cot \frac{C}{2} = \frac{2+1}{2-1}$$

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$\Rightarrow \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$$

4. $\beta_a = \frac{2bc}{b+c} \cos \frac{A}{2}$

(A) correct
(B) incorrect

(C) $\frac{abc \cos \frac{A}{2}}{2R(b+c)} = \frac{abc \cos \frac{A}{2}}{\frac{a}{\sin A} \cdot (b+c)}$

$$= \frac{bc \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2} \cdot (b+c)} = \frac{2bc}{(b+c)} \cos \frac{A}{2}$$

(D) $\rightarrow \frac{2\Delta}{(b+c)} \operatorname{cosec} \frac{A}{2} = \frac{bc \sin A}{(b+c)} \cdot \frac{1}{\sin \frac{A}{2}}$

$$= \frac{2bc \sin \frac{A}{2} \cos \frac{A}{2}}{(b+c)} \cdot \frac{1}{\sin \frac{A}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2}$$

6. $(r_1 - r)(r_2 - r)(r_3 - r)$

$$= \left(\frac{\Delta}{s-a} - \frac{\Delta}{s}\right) \left(\frac{\Delta}{s-b} - \frac{\Delta}{s}\right) \left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right)$$

$$= \frac{\Delta^3 abc}{s^2 \cdot \Delta^2} \left(R = \frac{abc}{4\Delta}\right)$$

$$= \frac{\Delta}{s^2} \cdot \frac{abc}{4\Delta} \cdot 4\Delta = 4r^2 R$$

9. Product of distances of incenter from angular points

$$= \frac{r^3}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r^3}{r/4R}$$

$$= 4Rr^2$$

$$= \frac{abc}{\Delta} r^2 = \frac{(abc)(r)}{\frac{\Delta}{r}} = \frac{(abc)(r)}{s}$$

10. (A) $\frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{\Delta}{s-a} - \frac{\Delta}{s}$

$$\Rightarrow \frac{1}{(s-b)(s-c)} = \frac{1}{s(s-a)}$$

$$\Rightarrow \tan^2 \frac{A}{2} = 1 \Rightarrow A = 90^\circ$$

(B) $4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = 8R^2$

$$1 - (\cos^2 A - \sin^2 B) + \sin^2 C = 2$$

$$1 - \cos(A+B) \cos(A-B) + 1 - \cos^2 C = 2$$

$$\cos C \cos(A-B) - \cos^2 C = 0$$

$$\cos C [\cos(A-B) - \cos C] = 0$$

$$\cos C [\cos(A-B) + \cos(A+B)] = 0$$

$$2 \cos A \cos B \cos C = 0 \Rightarrow A = 90^\circ$$

or $B = 90^\circ$ or $C = 90^\circ$

(C) $r_1 = s$

$$\tan A/2 = s \Rightarrow \tan A/2 = 1 \Rightarrow A = 90^\circ$$

(D) $\frac{a}{\sin A} = \frac{a\Delta}{s(s-a)} \Rightarrow \frac{1}{\sin A} = \tan A/2$

$$\Rightarrow 2\sin^2 A/2 = 1$$

$$\Rightarrow 1 - \cos A = 1 \Rightarrow \cos A = 0 \Rightarrow A = 90^\circ$$

11. $\rightarrow r_1 - r = \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta a}{s(s-a)} = a \tan \frac{A}{2}$

$$\therefore \Pi(r_1 - r) = abctan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$$

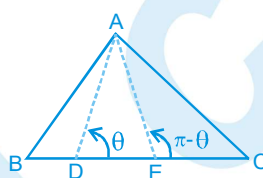
$$= abc \Pi \tan \frac{A}{2}$$

$$= abc \frac{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \frac{(abc)r}{4R \cdot \frac{(\sin A + \sin B + \sin C)}{4}} = \frac{(abc)r}{R \left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \right)}$$

$$= \frac{2(abc)r}{2s} = \frac{4R\Delta r}{s} = 4Rr^2$$

13.



if we apply m-n Rule in ΔABE , we get

$$(1+1) \cot \theta = 1 \cdot \cot B - 1 \cdot \cot \theta$$

$$2 \cdot \cot \theta = \cot B - \cot \theta$$

$$3 \cdot \cot \theta = \cot B$$

$$\tan \theta = 3 \tan B \quad \dots\dots(i)$$

Similarly, if we apply m-n Rule in ΔACD , we get

$$(1+1) \cot(\pi - \theta) = 1 \cdot \cot \theta - 1 \cdot \cot C$$

$$\cot C = 3 \cot \theta \Rightarrow \tan \theta = 3 \tan C \quad \dots\dots(ii)$$

from (1) and (2) we can say that $\tan B = \tan C$

$$\Rightarrow B = C$$

$$\rightarrow A + B + C = \pi$$

$$\therefore A = \pi - (B + C) = \pi - 2B$$

$$\rightarrow B = C$$

$$\therefore \tan A = -\tan 2B = -\left(\frac{2 \tan B}{1 - \tan^2 B} \right) = -\frac{2 \tan \theta}{1 - \frac{\tan^2 \theta}{9}}$$

$$\Rightarrow \tan A = \frac{6 \tan \theta}{\tan^2 \theta - 9}$$

$$14. \frac{1}{2}ra + \frac{1}{2}rb = \frac{1}{2}ab \sin C$$

$$r(a+b) = 2\Delta$$

$$r = \frac{2\Delta}{a+b} \quad \dots(1)$$

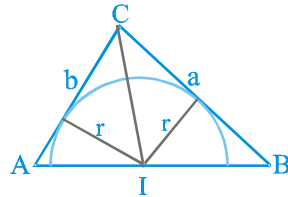
$$\therefore r = \frac{2abc}{4R(2R \sin A + 2R \sin B)} = \frac{abc}{4R^2(\sin A + \sin B)}$$

$\Rightarrow$ (A)

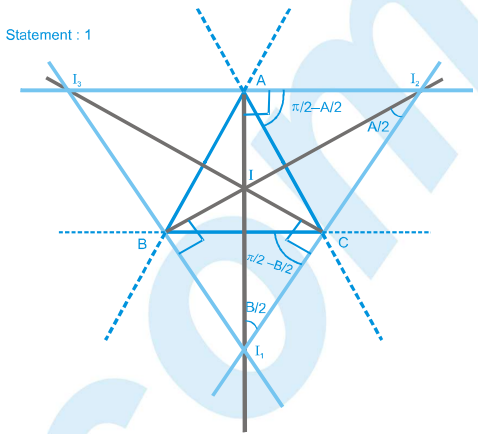
$$\text{also } x = \frac{2ab}{a+b} \cos \frac{C}{2}$$

$$\text{from (1) } r = \frac{2 \cdot \frac{1}{2} ab \sin C}{a+b} = \frac{2ab \sin \frac{C}{2} \cos \frac{C}{2}}{a+b}$$

$$= \frac{2ab \cos \frac{C}{2}}{a+b} \cdot \sin \frac{C}{2} = x \sin \frac{C}{2} \Rightarrow (C)$$



Statement : 1



$$2R_{\text{ex}} = 4R \quad R_{\text{ex}} = 2R$$

$\therefore \Delta ABC$ is pedal triangle of $\Delta I_1 I_2 I_3$

$\therefore$ statement - 1 and statement - 2 both are correct

and statement - 2 also explains Statement - 1

$$15. r_1 = 2r_2 = 3r_3$$

$$\Rightarrow \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c} \Rightarrow \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c}$$

$$\text{From (i) and (ii) we get } a-b = c/3 \quad \dots(1)$$

$$\text{From (i) and (iii), we get } 2a-b = 2c \quad \dots(2)$$

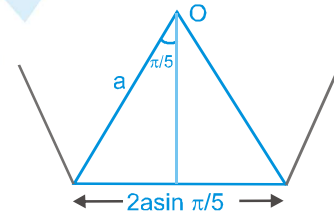
$$\text{From (ii) and (iii), we get } a-5b = -5c \quad \dots(3)$$

let $c = k$, then from (1) and (2), we get

$$a = \frac{5k}{3} \quad \text{and} \quad b = \frac{4k}{3}$$

$$\therefore \frac{a}{b} = \frac{5}{4}; \quad \frac{a}{c} = \frac{5}{3}$$

$$3. \text{ Perimeter} = 10a \sin \frac{\pi}{5}$$



$$\text{For } n \text{ sided polygon, perimeter} = \left(2a \sin \frac{\pi}{n} \right) \times n$$

Hence statement II is false

$$4. \text{ Angles of } \Delta DEF \text{ are } \pi - 2A, \pi - 2B, \pi - 2C$$

Incentre of ΔDEF is the orthocentre of ΔABC

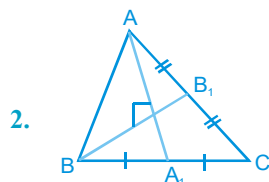
Part # II : Assertion & Reason

1. $I_1 I_2 = 4R \cos \frac{C}{2}$ if we apply Sine-Rule in $\Delta I_1 I_2 I_3$, then

$$2R_{\text{ex}} = \frac{I_1 I_2}{\sin \left(\frac{A}{2} + \frac{B}{2} \right)} = \frac{4R \cos \frac{C}{2}}{\sin \left(\frac{A+B}{2} \right)} = \frac{4R \cos \frac{C}{2}}{\cos \frac{C}{2}}$$

EXERCISE - 3

Part # I : Matrix Match Type



(A) $\rightarrow$ AA_1 and BB_1 are perpendicular

$$\therefore a^2 + b^2 = 5c^2$$

$$\therefore c^2 = \frac{a^2 + b^2}{5} = 5 \Rightarrow c = \sqrt{5}$$

$$\rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 9 - 5}{2 \times 4 \times 3} = \frac{5}{6}$$

$$\rightarrow \sin C = \frac{\sqrt{11}}{6}$$

$$\therefore \Delta = \frac{1}{2} ab \sin C = \sqrt{11}$$

$$\therefore \Delta^2 = 11$$

(B) G.M. $\geq$ H.M.

$$(r_1 r_2 r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$$

$$\Rightarrow (r_1 r_2 r_3)^{1/3} \geq 3r$$

$$\Rightarrow \frac{r_1 r_2 r_3}{r^3} \geq 27$$

(C) $\rightarrow \tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)} \rightarrow a=5, b=4$

$$2s = 9 + c$$

$$= \frac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \frac{c^2-1}{81-c^2}$$

$$\Rightarrow \frac{7}{9} = \frac{c^2-1}{81-c^2} \Rightarrow c^2 = 36 \Rightarrow c = 6$$

(D) $\rightarrow 2a^2 + 4b^2 + c^2 = 4ab + 2ac$

$$\Rightarrow (a-2b)^2 + (a-c)^2 = 0 \Rightarrow a = 2b = c$$

$$\rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7}{8}$$

$$\therefore 8 \cos B = 7$$

3. Use $p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

(A) $\frac{3}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \leq \sqrt[3]{p_1 p_2 p_3}$ (HM $\leq$ GM)

(B) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{a \cos A + b \cos B + c \cos C}{2\Delta} = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2\Delta}$
 $(\because a = 2R \sin A)$
 $= \frac{R \cdot 4 \cdot \sin A \cdot \sin B \cdot \sin C}{2\Delta} = \frac{4R}{2\Delta} \cdot \frac{abc}{8R^3} = \frac{1}{R}$

(C) $\frac{b^2}{c} \cdot \frac{2\Delta}{a} + \frac{c^2}{a} \cdot \frac{2\Delta}{b} + \frac{a^2}{b} \cdot \frac{2\Delta}{c} = 2\Delta \left(\frac{a^3 + b^3 + c^3}{abc} \right)$

Now, $\frac{a^3 + b^3 + c^3}{3} \geq abc$ (AM $\geq$ GM)

$$\frac{a^3 + b^3 + c^3}{abc} \geq 3$$

$$\Rightarrow 2\Delta \cdot \left(\frac{a^3 + b^3 + c^3}{abc} \right) \geq 6\Delta$$

(D) $\Sigma p_1^{-2} = \frac{\Sigma a^2}{4\Delta^2}$

Part # II : Comprehension

Comprehension # 1

Angles BEC, ABD, ABE and BAC are in A.P.

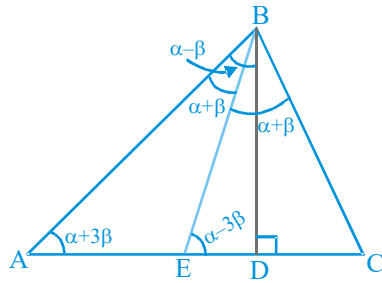
let $BEC = \alpha - 3\beta$

$ABD = \alpha - \beta$

$ABE = \alpha + \beta$

and $BAC = \alpha + 3\beta$

now, $\alpha - 3\beta = (\alpha + 3\beta) + (\alpha + \beta)$
 [using exterior angle theorem]



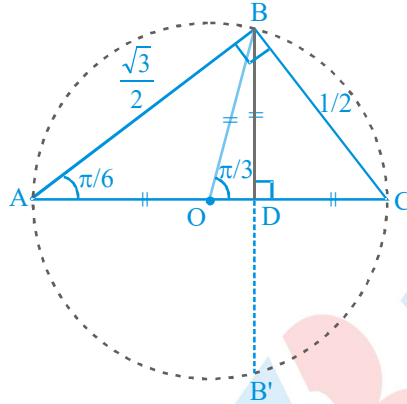
$$\Rightarrow \alpha = -7\beta$$

$$\therefore \beta = -\frac{\pi}{24}, \alpha = \frac{7\pi}{24}$$

and From $\triangle ABD$

$$\alpha - \beta + \alpha + 3\beta = \frac{\pi}{2}$$

$$2\beta + 2\beta = \frac{\pi}{2} \Rightarrow \alpha + \beta = \frac{\pi}{4}$$



$$\therefore \angle B = 2(\alpha + \beta) = \frac{\pi}{2}, \angle A = \frac{\pi}{6}, \angle C = \frac{\pi}{3}$$

$\Rightarrow$ ABC is $30^\circ-90^\circ-60^\circ$ triangle

(i) Area of circle circumscribing $\triangle ABC = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4}$

(ii) $\triangle BOC$ is equilateral $\Rightarrow r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4} \left(\frac{1}{2} \right)^2}{\frac{1}{2} \left(\frac{3}{2} \right)} = \frac{1}{4\sqrt{3}}$

(iii) $BD = OB \sin \frac{\pi}{3} = \frac{1}{2} \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4}$

$$\therefore BB' = 2BD = \frac{\sqrt{3}}{2}$$

Comprehension # 2

1. Let $\angle I_3 I_1 I_2 = \theta$

Then angle of pedal triangle $= \pi - 2\theta = A$

$$\theta = \frac{\pi}{2} - \frac{A}{2}$$

2. Side of pedal triangle $= I_2 I_3 \cos \theta = BC$

$$I_2 I_3 = \frac{a}{\cos \left(\frac{\pi}{2} - \frac{A}{2} \right)}$$

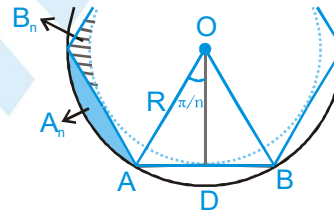
$$I_2 I_3 = 4R \cos \left(\frac{A}{2} \right)$$

3. $I I_1 = 4R \sin \frac{A}{2}$

$$I_2 I_3 = 4R \cos \frac{A}{2}$$

$$\therefore I I_1^2 + I_2 I_3^2 = 16R^2$$

Comprehension # 3



In $\triangle OAD$

$$OD = R \cos \frac{\pi}{n}, AD = R \sin \frac{\pi}{n}$$

A_n = Area of circle (circumscribing polygon) - Area of polygon

$$A_n = \pi R^2 - \frac{n}{2} R^2 \sin \left(\frac{2\pi}{n} \right)$$

B_n = Area of polygon - Area of circle (Inscribed)

$$B_n = \frac{n}{2} R^2 \sin \left(\frac{2\pi}{n} \right) - \pi R^2 \cos^2 \left(\frac{\pi}{n} \right)$$

1. If $n = 6$ then

$$A_n = \pi R^2 - \frac{3\sqrt{3}}{2} R^2$$

2. If $n = 4$ then value of

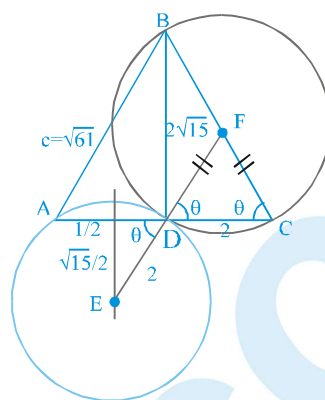
$$B_n = 2R^2 - \frac{\pi R^2}{2} = R^2 \left(\frac{4 - \pi}{2} \right)$$

$$3. \frac{A_n}{B_n} = \frac{\pi - \frac{n}{2} \sin \frac{2\pi}{n}}{\frac{n}{2} \sin \left(\frac{2\pi}{n} \right) - \pi \cdot \cos^2 \frac{\pi}{n}}$$

put $\pi = n\theta$

$$\text{we get } \frac{2\theta - \sin 2\theta}{\sin 2\theta - 2\theta \cos^2 \theta}$$

$$= \frac{\theta - \sin \theta \cdot \cos \theta}{\sin \theta \cdot \cos \theta - \theta \cdot \cos^2 \theta} = \frac{\theta - \sin \theta \cdot \cos \theta}{\cos \theta (\sin \theta - \theta \cos \theta)}$$



$$\therefore \frac{\text{area } \triangle ADE}{\text{area } \triangle DBC} = \frac{\sqrt{15}/4}{2\sqrt{15}} = \frac{1}{8} \Rightarrow \text{(D) Ans.}$$

Comprehension #5

$b = AC = 3$ and ratio is $1 : 2$, Hence
 $AD = 1$ and $DC = 2$

(iii) In $\triangle ABC$, $\frac{c}{\sin \theta} = 2R$;

$$R = \frac{c}{2\sin \theta} = \frac{\sqrt{61} \cdot 4}{2\sqrt{15}} = \frac{2\sqrt{61}}{\sqrt{15}}$$

Ans.

(i) Now $\sin \theta = \frac{\sqrt{15}}{4} \Rightarrow \theta = \sin^{-1} \frac{\sqrt{15}}{4}$ Ans.

(ii) Obviously $\angle FDC = \theta = \cos^{-1} \frac{1}{4} = \sin^{-1} \frac{\sqrt{15}}{4} = \angle FCD$

hence A, B, C are correct.

$$\text{now area } \triangle ADE = 1 \cdot \frac{\sqrt{15}}{2} \cdot \frac{1}{2} = \frac{\sqrt{15}}{4}$$

$$\text{now } \tan \theta = \frac{BD}{2} \quad (\text{in } \triangle BDC)$$

$$\therefore \sqrt{15} = \frac{BD}{2} \Rightarrow BD = 2\sqrt{15}$$

$$\tan A = \frac{2\sqrt{15}}{1} = 2\sqrt{15}$$

$$\sin A = \frac{2\sqrt{15}}{\sqrt{61}} \quad \dots (1)$$

$$\text{also } \sin A = \frac{2\sqrt{15}}{c} = \frac{2\sqrt{15}}{\sqrt{61}} \Rightarrow c = \sqrt{61}$$

$$\text{Area of } \triangle DBC = \frac{2 \cdot 2\sqrt{15}}{2} = 2\sqrt{15}$$

EXERCISE - 4

Subjective Type

$$\begin{aligned}
 1. \text{ LHS} &= \frac{1}{2}(a^2 + (b+c-a) + b^2(c+a-b)) \\
 &= \frac{1}{2}(a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) \\
 &\quad + c(a^2 + b^2 - c^2)) \\
 &= \frac{1}{2}(2abc \cos A + 2abc \cos B + 2abc \cos C) \\
 &= abc \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right) \\
 &= 4R\Delta \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)
 \end{aligned}$$

2. Since $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$, we have

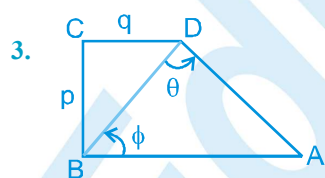
$$\begin{aligned}
 (b^2 - c^2) \cot A &= 4R^2(\sin^2 B - \sin^2 C) \cot A \\
 &= 4R^2 \sin(B+C) \sin(B-C) \cot A \\
 &= 4R^2 \sin A \sin(B-C) \frac{\cos A}{\sin A} \\
 &= -4R^2 \sin(B-C) \cos(B+C) \\
 &\quad (\rightarrow \cos A = -\cos(B+C)) \\
 &= -2R^2[2 \sin(B-C) \cos(B+C)] \\
 &= -2R^2[\sin 2B - \sin 2C] \quad \dots(i)
 \end{aligned}$$

$$\begin{aligned}
 \text{Similarly, } (c^2 - a^2) \cot B &= -2R^2[\sin 2C - \sin 2A] \quad \dots(ii)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } (a^2 - b^2) \cot C &= -2R^2[\sin 2A - \sin 2B] \quad \dots(iii)
 \end{aligned}$$

Adding eq. (i), (ii), and (iii), we get

$$\begin{aligned}
 (b^2 - c^2) \cot A + (c^2 - a^2) \cot B \\
 + (a^2 - b^2) \cot C &= 0
 \end{aligned}$$



If we apply Sine-Rule in ΔABD , we get

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin(\pi - (\theta + \phi))}$$

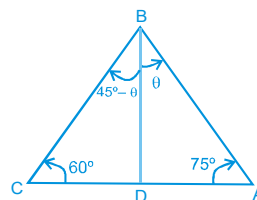
$$\Rightarrow AB = \frac{BD \sin \theta}{\sin(\theta + \phi)} = \frac{BD \sin \theta}{\sin \theta \cos \phi + \cos \theta \sin \phi} \quad \dots(i)$$

$$\sin \phi = \frac{p}{\sqrt{p^2 + q^2}} \quad \text{and} \quad \cos \phi = \frac{q}{\sqrt{p^2 + q^2}}$$

$$\therefore \text{ from equation (i), we get } AB = \frac{(\sqrt{p^2 + q^2}) \sin \theta}{\frac{q \sin \theta}{\sqrt{p^2 + q^2}} + \frac{p \cos \theta}{\sqrt{p^2 + q^2}}}$$

$$\therefore AB = \frac{(p^2 + q^2) \sin \theta}{q \sin \theta + p \cos \theta}$$

$$\begin{aligned}
 4. \cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) &= 0 \\
 \Rightarrow \cos A(\sin B - \sin C) + 2 \cos(B+C) \sin(B-C) &= 0 \\
 \Rightarrow B+C &= \pi - A \\
 \Rightarrow \cos A(\sin B - \sin C) - 2 \cos A \sin(B-C) &= 0 \\
 \Rightarrow \cos A[(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C)] &= 0 \\
 \Rightarrow \text{either } \cos A = 0 \Rightarrow A = 90^\circ \Rightarrow \text{right angled} \\
 \text{or } (\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C) &= 0 \\
 \Rightarrow (b-c) - 2\left(b \cdot \frac{a^2 + b^2 - c^2}{2ab} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right) &= 0 \\
 \Rightarrow a(b-c) - 2(b^2 - c^2) &= 0 \\
 (b-c)[a - 2(b+c)] &= 0 \\
 \therefore b-c = 0 \Rightarrow b=c \\
 \Rightarrow \text{isosceles}
 \end{aligned}$$



5.

$$\text{Area of } \Delta BAD = \sqrt{3} \times \text{Area of } \Delta BCD$$

$$\Rightarrow \frac{1}{2} BD \times BA \sin \theta = \sqrt{3} \times \frac{1}{2} BC \times BD \sin(45^\circ - \theta)$$

$$\frac{BA}{BC} = \sqrt{3} \frac{\sin(45^\circ - \theta)}{\sin \theta} \quad \dots(i)$$

From Sine-Rule

$$\frac{BC}{\sin 75^\circ} = \frac{AB}{\sin 60^\circ}$$

$$\therefore \frac{BA}{BC} = \frac{\sin 60^\circ}{\sin 75^\circ} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}+1}$$

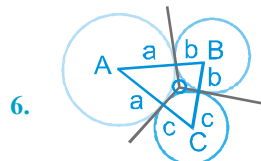
From equation (i)

$$\frac{\sqrt{3} \cdot \sqrt{2}}{(\sqrt{3}+1)} = \sqrt{3} \left[\frac{1}{\sqrt{2}} \cot \theta - \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow \frac{2}{(\sqrt{3}+1)} = \cot \theta - 1 \Rightarrow \frac{2(\sqrt{3}-1)}{2} = \cot \theta - 1$$

$$\Rightarrow \cot \theta = \sqrt{3} \Rightarrow \theta = 30^\circ$$

$$\Rightarrow \angle ABD = 30^\circ$$



required distance = inradius of ΔABC

$$\begin{aligned} \rightarrow 2s &= a + b + b + c + c + a \\ &= 2(a + b + c) \end{aligned}$$

$$s = a + b + c$$

$$\begin{aligned} \rightarrow \Delta &= \sqrt{s(s-(a+b))(s-(b+c))(s-(c+a))} \\ &= \sqrt{(a+b+c)(abc)} \end{aligned}$$

$$\therefore \text{required distance} = \frac{\Delta}{s} = \frac{\sqrt{(a+b+c)(abc)}}{(a+b+c)}$$

$$= \sqrt{\frac{abc}{a+b+c}} = \left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}$$

7. (i) L.H.S. = $(r_3 + r_1)(r_3 + r_2) \sin C$

$$= \frac{\Delta b}{(s-a)(s-c)} \cdot \frac{\Delta a}{(s-c)(s-b)} \sin C$$

$$= \frac{ab\Delta^2}{(s-a)(s-b)(s-c)(s-c)} \sin C$$

$$= \frac{abs(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)(s-c)} \sin C = \frac{abs \sin C}{(s-c)}$$

$$= \frac{2\Delta s}{(s-c)} = 2sr_3$$

$$\text{R.H.S.} = 2r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2} = 2r_3 \sqrt{s^2} = 2sr_3$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

(ii) L.H.S. = $-\frac{1}{\Delta}$

$$\left[\frac{(s-b)(s-c)}{(a-b)(c-a)} + \frac{(s-a)(s-c)}{(a-b)(b-c)} + \frac{(s-a)(s-b)}{(c-a)(b-c)} \right]$$

$$= -\frac{1}{\Delta}$$

$$\left[\frac{(s-b)(s-c)(b-c) + (s-a)(s-c)(c-a) + (s-a)(s-b)(a-b)}{(a-b)(b-c)(c-a)} \right]$$

$$= \frac{1}{\Delta} = \text{R.H.S.}$$

(iii) First term = $(r + r_1) \tan \frac{B-C}{2}$

$$= \left(\frac{\Delta}{s} + \frac{\Delta}{s-a} \right) \left(\frac{b-c}{b+c} \right) \cot \frac{A}{2} = \frac{\Delta(2s-a)}{s(s-a)}$$

$$\left(\frac{b-c}{b+c} \right) \cdot \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= b-c$$

similarly second term = $c-a$

& third term = $a-b$

$$\therefore \text{L.H.S.} = b-c + c-a + a-b = 0 = \text{R.H.S.}$$

(iv) $r_1 + r_2 + r_3 - r = 4R$

$$\therefore (r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1 r_2 + r_2 r_3 + r_3 r_1) \quad \dots\dots\dots (i)$$

$$\rightarrow r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$$

$$\text{and } r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$$

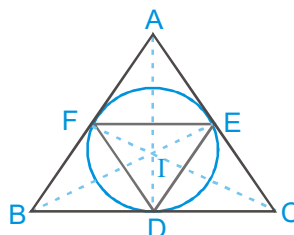
$$\therefore \text{from equation (i)}$$

$$16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 - 2(ab + bc + ca - s^2) + 2s^2$$

$$\begin{aligned} \therefore r^2 + r_1^2 + r_2^2 + r_3^2 &= 16R^2 - 4s^2 + 2(ab + bc + ca) \\ &= 16R^2 - (a+b+c)^2 + 2(ab + bc + ca) \\ &= 16R^2 - a^2 - b^2 - c^2 \end{aligned}$$

8. (i) EIFA is a cyclic quadrilateral

$$\therefore \frac{EF}{\sin A} = AI$$



$$\rightarrow AI = r \operatorname{cosec} \frac{A}{2}$$

$$\begin{aligned} \therefore EF &= r \operatorname{cosec} \frac{A}{2} \cdot \sin A \\ &= 2r \cos \frac{A}{2} \end{aligned}$$

$$\text{similarly } DF = 2r \cos \frac{B}{2} \quad \text{and} \quad DE = 2r \cos \frac{C}{2}.$$

(ii) IECD is a cyclic quadrilateral

$$\therefore \angle ICE = \angle IDE = \frac{C}{2}$$

$$\text{similarly } \angle IDF = \angle IBF = \frac{B}{2}$$

$$\therefore \angle FDE = \frac{B}{2} + \frac{C}{2} = \frac{\pi - A}{2} = \frac{\pi}{2} - \frac{A}{2}$$

(iii) area of $\triangle DEF = \frac{1}{2} FD \cdot DE \sin \angle FDE$

$$= \frac{1}{2} \left(2r \cos \frac{B}{2} \right) \left(2r \cos \frac{C}{2} \right) \sin \left(\frac{\pi}{2} - \frac{A}{2} \right)$$

$$= 2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 2r^2 \left(\frac{\sin A + \sin B + \sin C}{4} \right) = \frac{r^2}{2} \left(\frac{2\Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab} \right)$$

$$= \frac{r^2}{2} \left[\frac{2\Delta(a+b+c)}{abc} \right] = \frac{r^2 \Delta \cdot 2s}{abc} = \frac{2r^2 \cdot \Delta s^2}{(abc)s}$$

$$= \frac{2\Delta(rs)^2}{(abc)s} \Rightarrow = \frac{2\Delta^3}{(abc)s} = \frac{1}{2} \frac{r\Delta}{R}$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

$$1. \tan \left(\frac{\pi}{n} \right) = \frac{a}{2r}; \sin \left(\frac{\pi}{n} \right) = \frac{a}{2R}$$

$$\Rightarrow r + R = \frac{a}{2} \left[\cot \frac{\pi}{n} + \operatorname{cosec} \frac{\pi}{n} \right]$$

$$\Rightarrow r + R = \frac{a}{2} \cdot \cot \left(\frac{\pi}{2n} \right)$$

$$2. a \frac{s(s-c)}{ab} + c \frac{s(s-a)}{bc} = \frac{3b}{2}$$

$$\Rightarrow \frac{s}{b} (s-c+s-a) = \frac{3b}{2}$$

$$\Rightarrow a+b+c = 3b.$$

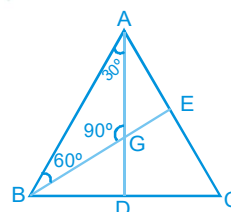
$$\Rightarrow a, b, c \text{ are in A.P.}$$

$$\Rightarrow a+c=2b$$

$$3. AD=4$$

$$\rightarrow AG = \frac{2}{3} \times 4 = \frac{8}{3}$$

$$\therefore \text{Area of } \triangle ABG = \frac{1}{2} \times AB \times AG \sin 30^\circ$$



$$\rightarrow = \frac{1}{2} \times \frac{16}{3\sqrt{3}} \times \frac{8}{3} \times \frac{1}{2} = \frac{32}{9\sqrt{3}} \quad \rightarrow \sin 60^\circ = \frac{AG}{AB}$$

$$\Rightarrow AB = \frac{2AG}{\sqrt{3}} = \frac{16}{3\sqrt{3}}$$

$$\therefore \text{Area of } \triangle ABC = 3(\text{Area of } \triangle ABG) = \frac{32}{3\sqrt{3}}$$

$$4. \cos \beta = \frac{\sin^2 \alpha + \cos^2 \alpha - 1 - \sin \alpha \cos \alpha}{2 \sin \alpha \cos \alpha} = -\frac{1}{2}$$

$$\Rightarrow \beta = 120^\circ$$

5. $\angle C = \pi/2$

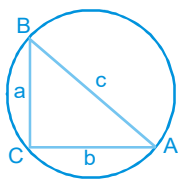
$$r = (s - c) \tan \frac{C}{2} \quad \therefore C = 90^\circ$$

$$r = s - 2R$$

$$\rightarrow 2r + 2R = 2(s - 2R) + 2R \\ = 2s - 2R$$

$$= (a + b + c) - \frac{c}{\sin C}$$

$$= a + b + c - c = a + b$$



6. $\frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c}$ are in H.P.

$$\Rightarrow \frac{a}{2\Delta}, \frac{b}{2\Delta}, \frac{c}{2\Delta} \text{ are in A.P.}$$

$$\Rightarrow a, b, c \text{ are in A.P.}$$

7. $\frac{r}{R} = \cos\left(\frac{\pi}{n}\right)$

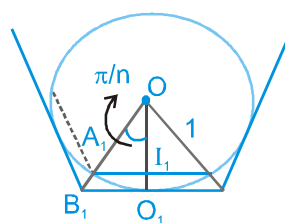
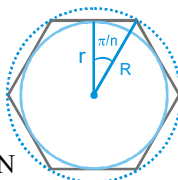
$$\text{Let } \cos \frac{\pi}{n} = \frac{2}{3} \text{ for some } n \geq 3, n \in \mathbb{N}$$

$$\text{As } \frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}} \Rightarrow \cos \frac{\pi}{3} < \cos \frac{\pi}{n} < \cos \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{3} > \frac{\pi}{n} > \frac{\pi}{4}$$

$\Rightarrow 3 < n < 4$, which is not possible
so option (2) is the false statement
so it will be the right choice

Hence correct option is (2)



$$O_n = 2n \times \text{area of } \triangle OB_1O_1$$

$$\Rightarrow O_n = 2n \times \frac{1}{2} \times B_1O_1 \times O_1O$$

$$= n \times \tan \frac{\pi}{n} \times 1 = n \tan \frac{\pi}{n}$$

$$\Rightarrow O_n = n \tan \frac{\pi}{n} \quad \dots(ii)$$

Now

$$\text{R.H.S.} = \frac{O_n}{2} \left[1 + \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} \right] = \frac{O_n}{2} \left[1 + \cos \frac{2\pi}{n} \right]$$

$$= \frac{O_n}{2} \times 2 \cos^2 \frac{\pi}{n} = O_n \cdot \cos^2 \frac{\pi}{n} = n \tan \frac{\pi}{n} \cdot \cos^2 \frac{\pi}{n}$$

$$= \frac{n}{2} \sin \frac{2\pi}{n} = I_n = \text{L.H.S.}$$

2. Let angle of the triangle be $4x, x$ and x .

$$\text{Then } 4x + x + x = 180^\circ$$

$$\Rightarrow x = 30^\circ$$

Longest side is opposite to the largest angle.

Using the law of sines

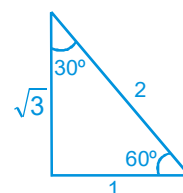
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\therefore a = R, b = R, c = \sqrt{3}R$$

$$\therefore 2S = (2 + \sqrt{3})R$$

$$\therefore \frac{c}{2S} = \frac{\sqrt{3}}{2 + \sqrt{3}}$$

3. Clearly the triangle is right angled. Hence angles are $30^\circ, 60^\circ$ and 90° are in ratio $1 : 2 : 3$



Part # II : IIT-JEE ADVANCED

1. $I_n = 2n \times \text{area of } \triangle OA_1I_1$

$$\Rightarrow I_n = 2n \times \frac{1}{2} \times A_1I_1 \times OI_1$$

$$\Rightarrow I_n = n \times \sin \frac{\pi}{n} \times \cos \frac{\pi}{n}$$

$$\Rightarrow I_n = \frac{n}{2} \sin \frac{2\pi}{n} \quad \dots(i)$$

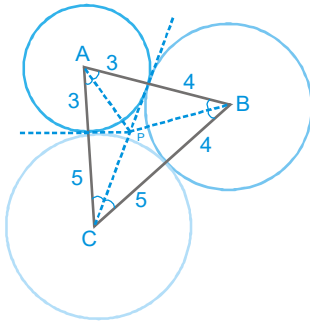
4. Consider $\frac{b-c}{a} = \frac{k(\sin B - \sin C)}{k \sin A}$

$$= \frac{2 \cos\left(\frac{B+C}{2}\right) \sin\left(\frac{B-C}{2}\right)}{2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right) \sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

5. Clearly P is the incentre of triangle ABC.

$$r = \frac{\Delta}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$



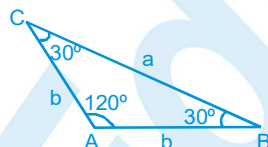
Here $2s = 7 + 8 + 9 \Rightarrow s = 12$

Here $r = \sqrt{\frac{5 \cdot 4 \cdot 3}{12}} = \sqrt{5}$

6. $\Delta = \frac{1}{2} \cdot b \cdot b \cdot \sin 120^\circ = \frac{\sqrt{3}}{4} b^2$ (i)

Also $\frac{\sin 120^\circ}{a} = \frac{\sin 30^\circ}{b} \Rightarrow a = \sqrt{3}b$ (ii)

and $\Delta = \sqrt{3}s$ and $s = \frac{1}{2}(a + 2b)$

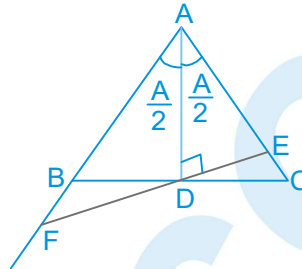


$$\Rightarrow \Delta = \frac{\sqrt{3}}{2}(a + 2b) \text{(3)}$$

From (1), (2) and (3), we get $\Delta = (12 + 7\sqrt{3})$

7. We have $\Delta ABC = \Delta ABD + \Delta ACD$

$$\Rightarrow \frac{1}{2} b c \sin A = \frac{1}{2} c A D \sin \frac{A}{2} + \frac{1}{2} b \times A D \sin \frac{A}{2}$$



$$\Rightarrow AD = \frac{2bc}{b+c} \cos \frac{A}{2}$$

Again $AE = AD \sec \frac{A}{2}$

$$= \frac{2bc}{b+c} \Rightarrow AE \text{ is HM of } b \text{ and } c.$$

$$EF = ED + DF = 2DE = 2 \times AD \tan \frac{A}{2} = \frac{2 \times 2bc}{b+c} \times \cos \frac{A}{2} \times \tan \frac{A}{2} = \frac{4bc}{b+c} \sin \frac{A}{2}$$

As $AD \perp EF$ and $DE = DF$ and AD is bisector

$\Rightarrow \Delta AEF$ is isosceles.

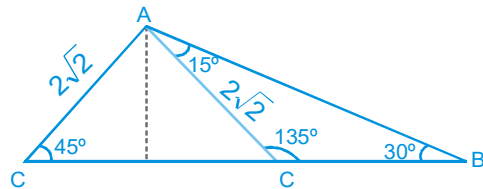
Hence A, B, C and D are correct answers.

8. In ΔABC , by sine rule

$$\frac{a}{\sin A} = \frac{2\sqrt{2}}{\sin 30^\circ} = \frac{4}{\sin C} \Rightarrow C = 45^\circ, C' = 135^\circ$$

When $C = 45^\circ \Rightarrow A = 180^\circ - (45^\circ + 30^\circ) = 105^\circ$

When $C' = 135^\circ \Rightarrow A = 180^\circ - (135^\circ + 30^\circ) = 15^\circ$



$$\text{Area of } \Delta ABC' = \frac{1}{2} AB \cdot AC' \cdot \sin \angle BAC' = \frac{1}{2} \times 4 \times$$

$$2\sqrt{2} \sin(15^\circ) = 4\sqrt{2} \times \frac{\sqrt{3}-1}{2\sqrt{2}} = 2(\sqrt{3}-1)$$

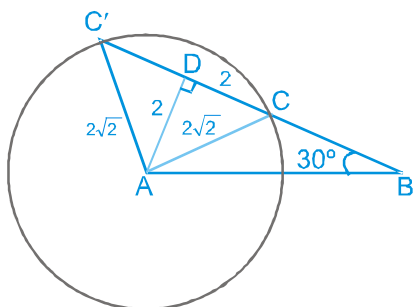
Area of $\triangle ABC$

$$= \frac{1}{2} AB \cdot AC \cdot \sin A = \frac{1}{2} \times 4 \times 2\sqrt{2} \sin(105^\circ) = 2(\sqrt{3} + 1)$$

Absolute difference of areas of triangles

$$= |2(\sqrt{3} + 1) - 2(\sqrt{3} - 1)| = 4$$

Aliter



$AD = 2$, $DC = 2$

Difference of Areas of triangle ABC and $ABC' =$ Area of triangle ACC'

$$= \frac{1}{2} AD \times CC' = \frac{1}{2} \times 2 \times 4 = 4$$

9. $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$

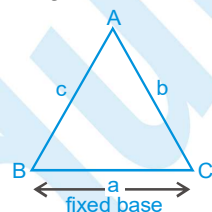
$$\Rightarrow 2 \cos \frac{B+C}{2} \cos \frac{B-C}{2} = 4 \sin^2 \frac{A}{2}$$

$$\Rightarrow 2 \sin \frac{A}{2} \left[\cos \frac{B-C}{2} - 2 \sin \frac{A}{2} \right] = 0$$

$$\Rightarrow \cos \left(\frac{B-C}{2} \right) - 2 \cos \left(\frac{B+C}{2} \right) = 0 \text{ as } \sin \frac{A}{2} \neq 0$$

$$\Rightarrow -\cos \frac{B}{2} \cos \frac{C}{2} + 3 \sin \frac{B}{2} \sin \frac{C}{2} = 0$$

$$\Rightarrow \tan \frac{B}{2} \tan \frac{C}{2} = \frac{1}{3}$$



$$\Rightarrow \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \cdot \frac{(s-b)(s-a)}{s(s-c)} = \frac{1}{3}$$

$$\Rightarrow \frac{s-a}{s} = \frac{1}{3} \Rightarrow 2s = 3a \Rightarrow b+c = 2a$$

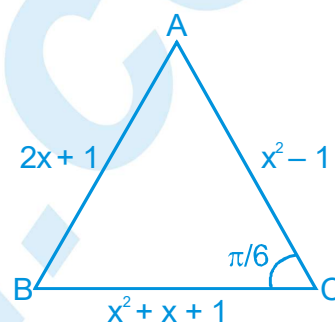
$\therefore$ Locus of A is an ellipse

10. $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A = \frac{2}{2R} (a \cos C + c \cos A)$
 $= \frac{b}{R} = 2 \sin B = 2 \sin 60^\circ = \sqrt{3}$

11. $\cos \frac{\pi}{6} = \frac{(x^2-1)^2 + (x^2+x+1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$$

$$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$$



$$\Rightarrow \sqrt{3} = \frac{x^2-1+x(x+2)}{x^2+x+1}$$

$$\Rightarrow \sqrt{3}(x^2+x+1) = 2x^2+2x-1$$

$$\Rightarrow (\sqrt{3}-2)x^2 + (\sqrt{3}-2)x + (\sqrt{3}+1) = 0$$

on solving

$$x^2+x-(3\sqrt{3}+5) = 0 \text{ we get } x = \sqrt{3}+1, -(2+\sqrt{3})$$

At $x = -(2+\sqrt{3})$, Side c becomes negative.

$$\therefore x = \sqrt{3}+1$$

12. Area of triangle $= \frac{1}{2} ab \sin C = 15\sqrt{3}$

$$\Rightarrow \frac{1}{2} \cdot 6 \cdot 10 \sin C = 15\sqrt{3} \Rightarrow \sin C = \frac{\sqrt{3}}{2}$$

$$\Rightarrow C = \frac{2\pi}{3} \text{ (C is obtuse angle)}$$

Now $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow -\frac{1}{2} = \frac{36+100-c^2}{2 \cdot 6 \cdot 10} \Rightarrow c = 14$$

$$\therefore r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{\frac{6+10+14}{2}} = \sqrt{3} \Rightarrow r^2 = 3$$

13. $a = 2 = QR$

$$b = \frac{7}{2} = PR$$

$$c = \frac{5}{2} = PQ$$

$$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$$

$$\frac{2 \sin P - 2 \sin P \cos P}{2 \sin P + 2 \sin P \cos P} = \frac{2 \sin P(1 - \cos P)}{2 \sin P(1 + \cos P)} = \frac{1 - \cos P}{1 + \cos P}$$

$$= \frac{2 \sin^2 \frac{P}{2}}{2 \cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2}$$

$$= \frac{\left(4 - \frac{7}{2}\right)^2 \left(4 - \frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

14. $\cos P = \frac{(2n+2)^2 + (2n+4)^2 - (2n+6)^2}{2(2n+2)(2n+4)} = \frac{1}{3}$

$$\Rightarrow \frac{4n^2 - 16}{8(n+1)(n+2)} = \frac{1}{3}$$

$$= \frac{n^2 - 4}{2(n+1)(n+2)} = \frac{1}{3}$$

$$\Rightarrow \frac{n-2}{2(n+1)} = \frac{1}{3}$$

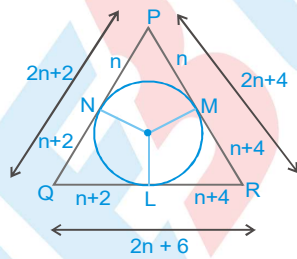
$$= 3n - 6 = 2n + 2$$

$$\Rightarrow n = 8$$

$$\Rightarrow 2n + 2 = 18$$

$$\Rightarrow 2n + 4 = 20$$

$$\Rightarrow 2n + 6 = 22$$



MOCK TEST

1. Using properties of pedal triangle, we have

$$\angle MLN = 180^\circ - 2A$$

$$\Rightarrow \angle LMN = 180^\circ - 2B$$

$$\Rightarrow \angle MNL = 180^\circ - 2C$$

Hence the required sum

$$= \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

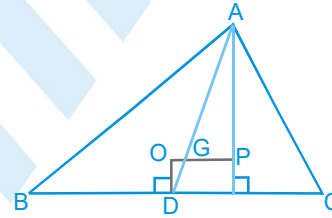
2. (D)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 81 - 49}{2 \cdot 8 \cdot 9} = \frac{145 - 49}{144} = \frac{96}{144}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49 + 81 - 64}{2 \cdot 7 \cdot 9} = \frac{66}{126} = \frac{11}{21}$$

$$\therefore \cos A : \cos B = \frac{14}{11}$$

- 3.



From figure, we can observe that $\triangle OGD$ is directly similar to $\triangle PGA$

4. (A)

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$$

$$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$$

$$= \frac{s}{\Delta} [3s - (a+b+c)] = \frac{s[3s-2s]}{\Delta} = \frac{s^2}{\Delta}$$

$$= \left(\frac{a+b+c}{2}\right)^2 \times \frac{4R}{abc} = \frac{(a+b+c)^2 R}{abc} \quad [\rightarrow \Delta = \frac{abc}{4R}]$$

$$\text{also } \frac{s^2}{\Delta} = \frac{\Delta^2}{r^2 \Delta} = \frac{\Delta}{r^2}$$

5. $\tan \frac{C}{2} (a \tan A + b \tan B) = a + b$

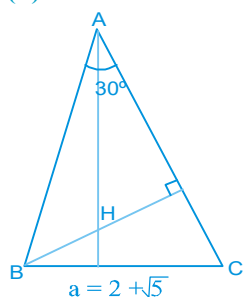
$$\Rightarrow a \cos B \cos \left(A + \frac{C}{2} \right) + b \cos A \cos \left(B + \frac{C}{2} \right) = 0$$

$$\Rightarrow (a \cos B - b \cos A) \cos \left(A + \frac{C}{2} \right) = 0,$$

as $\cos \left(B + \frac{C}{2} \right) = \cos \left(\pi - A - \frac{C}{2} \right)$

$$\Rightarrow A = B, \text{ in either case}$$

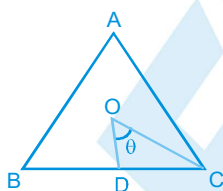
6. (B)



$$R = \frac{a}{2 \sin A} = \frac{2 + \sqrt{5}}{2 \sin 30^\circ} = \frac{2 + \sqrt{5}}{2 \times \frac{1}{2}} = (2 + \sqrt{5})$$

Now $AH = 2R \cos A = 2(2 + \sqrt{5}) \cos 30^\circ = (2 + \sqrt{5}) \sqrt{3}$

7. (A) In the adjacent figure we have $\angle OCB = \frac{\pi}{2} - A$



$$\Rightarrow \angle ODC = \pi - \left(\frac{\pi}{2} - A + \theta \right) = \frac{\pi}{2} + (A - \theta)$$

if R_1 be the circumradius of $\triangle OCD$, then

$$\frac{OC}{\sin \left(\frac{\pi}{2} + (A - \theta) \right)} = 2R_1$$

$$\Rightarrow 2R_1 = \frac{R}{\cos(A - \theta)}$$

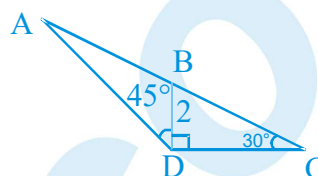
As $D \rightarrow C \Rightarrow \theta \rightarrow 0$

$$\Rightarrow 2R_1 \rightarrow \frac{R}{\cos A}$$

$$\therefore R_1 \rightarrow \frac{R}{2 \cos A}$$

8. In $\triangle ABD$, $\frac{AD}{\sin 120^\circ} = \frac{BD}{\sin 15^\circ}$

$$\Rightarrow AD = \frac{2\sqrt{3}}{\sqrt{3}-1} = 3\sqrt{2} + \sqrt{6}$$



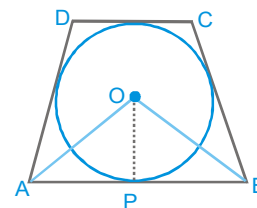
9. (B)

Let 'O' be the centre of circle and 'P' be its point of contact with side AB. We have

$$AP = OP \cdot \cot \frac{A}{2} = \cot \frac{A}{2} \text{ and}$$

$$PB = OP \cdot \cot \frac{B}{2} = \cot \frac{B}{2}$$

$$\Rightarrow AP + PB = \cot \frac{A}{2} + \cot \frac{B}{2}$$



$$\frac{\sin \left(\frac{A+B}{2} \right)}{\sin \frac{A}{2} \cdot \sin \frac{B}{2}} = AB$$

Similarly

$$CD = \frac{\sin \left(\frac{C+D}{2} \right)}{\sin \frac{C}{2} \cdot \sin \frac{D}{2}}$$

Since $A + B + C + D = 2\pi \Rightarrow \frac{A+B}{2} = \pi - \frac{C+D}{2}$

$$\Rightarrow \sin \left(\frac{A+B}{2} \right) = \sin \left(\frac{C+D}{2} \right)$$

$$\Rightarrow AB \cdot \sin \frac{A}{2} \cdot \sin \frac{B}{2} = \sin \frac{C}{2} \cdot \sin \frac{D}{2} \cdot CD$$

10. (D)

 $S_1 :$

$$\frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{\Delta}{s-a} - \frac{\Delta}{s} \Rightarrow \frac{a}{(s-b)(s-c)} = \frac{a}{s(s-a)}$$

$$\text{or } \frac{s(s-a)}{(s-b)(s-c)} = 1 \Rightarrow \cot \frac{A}{2} = 1 \Rightarrow A = 90^\circ$$

$$S_2 : 4R^2 (\sum \sin^2 A) = 8R^2$$

$$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$$

$$\Rightarrow 2 \cos A \cos B \cos C = 0$$

$$\Rightarrow A \text{ or } B \text{ or } C = 90^\circ$$

$$S_3 : 2r_1 = 2s \Rightarrow s \tan \frac{A}{2} = s \Rightarrow \tan \frac{A}{2} = 1 \Rightarrow A = 90^\circ$$

$$S_4 : 2R = 4R \left[\left(\sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \right) \right.$$

$$\left. - \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right]$$

$$1 = 2 \sin \frac{A}{2} \cos \left(\frac{B+C}{2} \right) \text{ or } 1 = 2 \sin^2 \frac{A}{2}$$

$$\Rightarrow \cos A = 0 \Rightarrow A = 90^\circ$$

11. (A, C)

$$\Delta = \Delta_1 + \Delta_2$$

$$\frac{1}{2} bc \sin A = \frac{1}{2} cx \sin \frac{A}{2} + \frac{1}{2} xb \sin \frac{A}{2}$$

12. (A, C, D)

$$p = \frac{2\Delta}{a}, q = \frac{2\Delta}{b}, r = \frac{2\Delta}{c}$$

$$(A) (\Sigma p) \left(\Sigma \frac{1}{p} \right) = 2\Delta \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(\frac{a+b+c}{2\Delta} \right) = (\Sigma a) \left(\Sigma \frac{1}{a} \right)$$

 (C) $(\Sigma p) (\Sigma pq) (\Pi a)$

$$= \left(\frac{2\Delta}{a} + \frac{2\Delta}{b} + \frac{2\Delta}{c} \right) \left(\frac{4\Delta^2}{ab} + \frac{4\Delta^2}{bc} + \frac{4\Delta^2}{ca} \right) abc$$

$$= \frac{2\Delta(ab+bc+ca)}{abc} \cdot 4\Delta^2 \frac{(a+b+c)}{abc} \cdot abc$$

$$= (\Sigma a) (\Sigma ab) (\Pi p)$$

$$\begin{aligned} (D) & \left(\Sigma \frac{1}{p} \right) \Pi \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \Pi a^2 \\ &= \left(\frac{a+b+c}{2\Delta} \right) \left(\frac{a+b-c}{2\Delta} \right) \left(\frac{a-b+c}{2\Delta} \right) \\ & \quad \left(\frac{b+c-a}{2\Delta} \right) \cdot a^2 b^2 c^2 \\ &= \frac{s}{\Delta} \cdot \frac{(s-c)}{\Delta} \cdot \frac{(s-b)}{\Delta} \cdot \frac{(s-a)}{\Delta} (abc)^2 = \left(\frac{abc}{\Delta} \right)^2 \\ &= (4R)^2 = 16R^2. \end{aligned}$$

13. (A, C, D)

$$\text{given expression } (a-c)^2 + (a-2b)^2 = 0$$

$$\Rightarrow a = 2b \text{ and } c = a. \text{ Sides are } 2b, b, 2b$$

$$\Rightarrow \text{isosceles,}$$

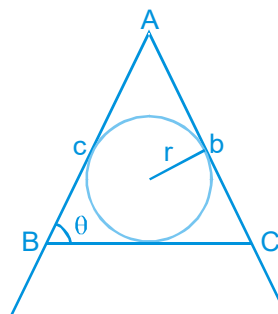
$$\rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7b^2}{8b^2} = \frac{7}{8}$$

$$\text{and } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{4}$$

14. (A)

$$r = \frac{\Delta}{s}; \rho = \frac{\Delta}{s-a}$$

$$\rho r = \frac{\Delta^2}{s(s-a)} = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$$



$$= (s-b)(s-c) = (s-b)^2 \quad (\rightarrow b=c)$$

$$= \frac{(2s-2b)^2}{4} = \frac{(a+b+c-2b)^2}{4} \quad (\rightarrow b=c)$$

$$= \frac{a^2}{4} = \frac{4R^2 \sin^2 A}{4} = R^2 \sin^2 A$$

$$\text{Also if } \angle B = \theta \Rightarrow \angle A = \pi - 2\theta$$

$$\therefore \rho r = R^2 \sin^2 (\pi - 2\theta) = R^2 \sin^2 2\theta = R^2 \sin^2 2B$$

15. We have $\frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c}$

$\Rightarrow a : b : c = 5 : 4 : 3$

16. (A)

Statement-II is true.

Statement-I $\tan A = \tan B = \tan C$

(By using statement-1)

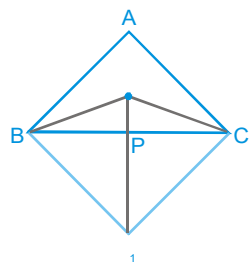
$A = B = C$ i.e. $a = b = c$

$r_1 = r_2 = r_3$

$\therefore \frac{r_1 + r_2 + r_3}{r} = 3 \cdot \frac{r_1}{r} = 3 \cdot \frac{s-a}{\frac{\Delta}{s}} = 3 \left(\frac{a+b+c}{b+c-a} \right) = 9$

17. (C)

$\angle ICI_1 = \frac{\pi}{2}, \angle IBI_1 = \frac{\pi}{2}$



$\therefore BICI_1$ is cyclic quadrilateral

$\therefore BP \cdot PC = IP \cdot I_1P$

18. (C)

$\frac{\cos^2 \frac{A}{2}}{a} = \frac{s(s-a)}{abc}$

$\therefore \sum \frac{\cos^2 \frac{A}{2}}{a} = \frac{s^2}{abc}$

19. (A)

$s = 21$

$\Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{3 \cdot 7 \cdot 2^4 \cdot 7 \cdot 3} = 3 \cdot 7 \cdot 4 = 84$

$r = \frac{\Delta}{s} = \frac{84}{21} = 4$

20. (B)

Statement-I :

$a < b < c$

$s - a > s - b > s - c$

$s > s - a > s - b > s - c$

$\frac{\Delta}{s} < \frac{\Delta}{s-a} < \frac{\Delta}{s-b} < \frac{\Delta}{s-c}$

$r < r_1 < r_2 < r_3$

Statement-II :

$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}, r = \frac{\Delta}{s}$

$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$

21. (A) $\rightarrow$ (p), (B) $\rightarrow$ (q), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)

(A) $(b+c)^2 - a^2 = \lambda bc$

or $b^2 + c^2 - a^2 = (\lambda - 2) bc$

$\frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$

$\cos A = \frac{\lambda - 2}{2} < 1$

or $\lambda - 2 < 2$

$\lambda < 4$

$\therefore$ greatest value of λ is 3.

(B) $\tan A + \tan B + \tan C = 9$

in any triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\frac{\tan^2 A + \tan^2 B + \tan^2 C}{3} \geq (\tan A \tan B \tan C)^{2/3}$

$k \geq 3(9)^{2/3}$

$k \geq 9 \cdot 3^{1/3}$

(C) Since the line joining the circumcenter to the incentre is parallel to BC

$\therefore r = R \cos A$

$\therefore 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = R \cos A$

$\therefore -1 + \cos A + \cos B + \cos C = \cos A$

$\therefore \cos B + \cos C = 1$

(D) $a = 5, b = 4$

$$\cos(A - B) = \frac{31}{32}$$

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2} = \frac{1}{9} \cot \frac{C}{2}$$

$$\rightarrow \cos(A - B) = \frac{1 - \tan^2 \frac{A - B}{2}}{1 + \tan^2 \frac{A - B}{2}}$$

$$\frac{31}{32} = \frac{1 - \frac{1}{81} \cot^2 \frac{C}{2}}{1 + \frac{1}{81} \cot^2 \frac{C}{2}}$$

$$31 + \frac{31}{81} \cot^2 \frac{C}{2} = 32 - \frac{32}{81} \cot^2 \frac{C}{2}$$

$$\frac{7}{9} \cot^2 \frac{C}{2} = 1$$

$$\cot^2 \frac{C}{2} = \frac{9}{7}$$

$$\rightarrow \cos C = \frac{1 - \tan^2 \frac{C}{2}}{1 + \tan^2 \frac{C}{2}} = \frac{1 - \frac{7}{9}}{1 + \frac{7}{9}} = \frac{2}{16} = \frac{1}{8}$$

$$\rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \frac{25 + 16 - c^2}{2 \times 20} = \frac{1}{8}$$

$$25 + 16 - c^2 = 5$$

$$c^2 = 36$$

$$c = 6$$

22. (A) $\rightarrow$ (p), (B) $\rightarrow$ (p, r), (C) $\rightarrow$ (r), (D) $\rightarrow$ (p, r)

(A) Since $\cos A = \frac{\sin B}{2 \sin C}$, we have $\frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2c}$

or $b^2 + c^2 - a^2 = b^2$ or $c^2 = a^2$

Hence $c = a$ and so the $\triangle ABC$ is isosceles

(B) $\cos A (\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

or $\cos A (\sin B - \sin C) + 2 \sin(B - C) \cos(B + C) = 0$

or $\cos A (\sin B - \sin C) - 2 \cos A \sin(B - C) = 0$

$\therefore$ either $\cos A = 0 \Rightarrow A = 90^\circ$

or $(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C) = 0$

$$\therefore (b - c) - 2 \left[b \cdot \frac{a^2 + b^2 - c^2}{2ab} - c \cdot \frac{c^2 + a^2 - b^2}{2ca} \right] = 0$$

or $a(b - c) - 2(b^2 - c^2) = 0$

$\Rightarrow (b - c)[a - 2(b + c)] = 0$

$\therefore b - c = 0$ or $b = c$

$\therefore$ isosceles

(C) Combine first and third terms and put the value of $\cos B$, we get

$$\therefore \frac{2}{ac} \cdot (B) + \frac{1}{b} \cdot \frac{c^2 + a^2 - b^2}{2ca} = \frac{a^2 + b^2}{abc}$$

or $4b^2 + c^2 + a^2 - b^2 = 2a^2 + 2b^2$

$\therefore b^2 + c^2 = a^2$

$\therefore \angle A = 90^\circ$

(D) $\frac{\sin(A - B)}{\sin(A + B)} = \frac{k^2(\sin^2 A - \sin^2 B)}{k^2(\sin^2 A + \sin^2 B)}$ by sine formula

or $\frac{\sin(A - B)}{\sin C} = \frac{\sin(A - B)\sin(A + B)}{\sin^2 A + \sin^2 B}$

or $\sin(A - B) \left[\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B} \right] = 0$

$\therefore$ either $\sin(A - B) = 0$

$\therefore A = B$ i.e. $\triangle$ is isosceles

or $\sin^2 A + \sin^2 B = \sin^2 C$ or $a^2 + b^2 = c^2$

$\therefore \triangle$ is right angled

23.

1. $PG = \frac{1}{3} AD = \frac{1}{3} \cdot \frac{2\Delta}{a} = \frac{2}{3a} \cdot \frac{1}{2} \cdot ab \sin C$ or $= \frac{1}{3} b \sin C$

($\rightarrow \Delta = \frac{1}{2} ac \sin B$)

$\therefore PG = \frac{2}{3a} \cdot \frac{1}{2} ac \sin B = \frac{1}{3} c \sin B$

2. Area of $\triangle GPL = \frac{1}{2} (PL) (PG)$ and Area of $\triangle ALD$

$= \frac{1}{2} (DL) (AD)$

$\rightarrow PL = \frac{1}{3} DL$ and $PG = \frac{1}{3} AD$

$$\therefore \frac{\text{Area of } \triangle GPL}{\text{Area of } \triangle ALD} = \frac{\frac{1}{2} (PL) (PG)}{\frac{1}{2} (DL) (AD)}$$

$$= \frac{\frac{1}{3} (DL) \times \frac{1}{3} (AD)}{(DL) (AD)} = \frac{1}{9}$$

3. Area of ΔPQR = Area of ΔPGQ + Area of ΔQGR +
Area of ΔRGP (i)

→ Area of $\Delta PGQ = \frac{1}{2} PG \cdot GQ \cdot \sin(\angle PGQ)$

$= \frac{1}{2} \times \frac{1}{3} AD \times \frac{1}{3} BE \sin(\pi - C)$

$= \frac{1}{18} \times \frac{2\Delta}{a} \times \frac{2\Delta}{b} \sin C$

$= \frac{2}{9ab} \times \frac{1}{2} bc \sin A \times \frac{1}{2} ac \sin B \times \sin C$

$= \frac{c^2}{18} \sin A \cdot \sin B \cdot \sin C$

Similarly Area of $\Delta QGR = \frac{a^2}{18} \sin A \cdot \sin B \cdot \sin C$ and

Area of $\Delta RGP = \frac{b^2}{18} \sin A \cdot \sin B \cdot \sin C$

∴ From equation (i), we get Area of $\Delta PQR = \frac{1}{18} (a^2 + b^2 + c^2) \sin A \cdot \sin B \cdot \sin C$

24.

1. $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = b \sin B + c \sin C + a \sin A = \frac{b^2 + c^2 + a^2}{2R}$

∴ $k = 2R$

2. $\cot A + \cot B + \cot C = \frac{R}{abc} (b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$

$= \frac{R}{abc} (b^2 + c^2 + a^2) = \frac{R}{abc} \left(\frac{4\Delta^2}{x^2} + \frac{4\Delta^2}{y^2} + \frac{4\Delta^2}{z^2} \right)$

$= \frac{4\Delta^2 R}{abc} \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right)$

$= \frac{4\Delta R}{abc} \cdot \Delta \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right)$

$= \Delta \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right)$

∴ $k = \Delta$

3. $\sum \frac{c \sin B + b \sin C}{x} = \sum \frac{x+x}{x} = 6$

25. (B, B, B)

$\cos(A - B) = \frac{4}{5} \Rightarrow \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} = \frac{4}{5}$

$\Rightarrow \frac{2 \tan^2 \frac{A-B}{2}}{2} = \frac{1}{9}$

$\Rightarrow \tan \frac{A-B}{2} = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{6-3}{6+3} \cot \frac{C}{2}$

$\Rightarrow \cot \frac{C}{2} = 1 \Rightarrow C = 90^\circ$

Area of triangle = $\frac{1}{2} ab \sin C$

$\Rightarrow \text{Area} = \frac{1}{2} \times 6 \times 3 \times 1 = 9$

$\frac{a}{\sin A} = \frac{\sqrt{a^2 + b^2}}{1}$

$\Rightarrow \frac{6}{\sin A} = \sqrt{45}$

$\Rightarrow \sin A = \frac{2}{\sqrt{5}}$

26. (8)

$r = \frac{\Delta}{s}$

$s = 5$ or $a + b + c = 10$

$\Delta = \frac{abc}{4R}$ or $abc = 60$

Now $\Delta^2 = s(s-a)(s-b)(s-c)$

or $5 = (5-a)(5-b)(5-c)$

$= 125 - 25(a+b+c) + 5(ab+bc+ca) - abc$

∴ $ab+bc+ca = 38$

or $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 24$

27. (3)

$a + b - c = 2$

and $2ab - c^2 = 4$

$\Rightarrow a^2 + b^2 + c^2 + 2ab - 2bc - 2ca = 4 = 2ab - c^2$

$\Rightarrow (b-c)^2 + (a-c)^2 = 0$

$\Rightarrow a = b = c$

Triangle is equilateral ;

hence $a = 2$

$\Rightarrow \Delta = \sqrt{3}$

28. (2)

We have $\cos^2 \frac{C}{2} = \frac{s(s-c)}{ab}$

So that

$$\frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2} = \frac{2ab}{2s\Delta} \cdot \frac{s(s-c)}{ab} = \frac{s-c}{\Delta}$$

Now, the area of triangle ABC is $\Delta = \frac{1}{2} ap_1$,

i.e., $p_1 = 2\Delta/a$. Similarly, $p_2 = 2\Delta/b$ and $p_3 = 2\Delta/c$.

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta} = \frac{2s-2c}{2\Delta} = \frac{s-c}{\Delta}$$

29. (2)

$$\begin{aligned} \frac{a+c}{\sqrt{a^2-ac+c^2}} &= \frac{\sin A + \sin C}{\sqrt{\sin^2 A - \sin A \sin C + \sin^2 C}} \\ &= \frac{2\sin \frac{A+C}{2} \cos \frac{A-C}{2}}{\sqrt{\frac{1-\cos 2A}{2} - \frac{\cos(A-C) - \cos(A+C)}{2} + \frac{1-\cos 2C}{2}}} \\ &= \frac{2\sqrt{2} \cdot \frac{\sqrt{3}}{2} \cos \frac{A-C}{2}}{\sqrt{2 - (\cos 2A + \cos 2C) - \cos(A-C) + \cos(A+C)}} \\ &= \frac{\sqrt{6} \cos \frac{A-C}{2}}{\sqrt{\frac{3}{2} - 2\cos(A+C)\cos(A-C) - \cos(A-C)}} \\ &= \frac{\sqrt{6} \cos \frac{A-C}{2}}{\sqrt{\frac{3}{2} + \cos(A-C) - \cos(A-C)}} = 2\cos \frac{A-C}{2} \end{aligned}$$

30. (6)

$$\frac{h_1+r}{h_1-r} + \frac{h_2+r}{h_2-r} + \frac{h_3+r}{h_3-r} \geq 6$$

$$\Delta = \frac{1}{2} ah \Rightarrow h_1 = \frac{2\Delta}{a}$$

$$\text{Similarly } h_2 = \frac{2\Delta}{b}, h_3 = \frac{2\Delta}{c}$$

$$\begin{aligned} \text{So } \frac{h_1+r}{h_1-r} + \frac{h_2+r}{h_2-r} + \frac{h_3+r}{h_3-r} &= \frac{2\Delta/a + \Delta/s}{2\Delta/a - \Delta/s} + \frac{2\Delta/b + \Delta/s}{2\Delta/b - \Delta/s} + \frac{2\Delta/c + \Delta/s}{2\Delta/c - \Delta/s} \\ &= \frac{2s+a}{2s-a} + \frac{2s+b}{2s-b} + \frac{2s+c}{2s-c} = \frac{4s}{2s-a} + \frac{4s}{2s-b} + \frac{4s}{2s-c} - 3 \\ &= 3 \left[\frac{1}{3} \left\{ \frac{4s}{2s-a} + \frac{4s}{2s-b} + \frac{4s}{2s-c} \right\} \right] - 3 \end{aligned}$$

$$\geq 3 \left[\frac{3}{\frac{2s-a}{4s} + \frac{2s-b}{4s} + \frac{2s-c}{4s}} \right] - 3,$$

Since $(AM \geq HM) \geq 6$