LIMIT
EXERCISE # 1

Question based on — Existence of limit

Q.1 The value of $\lim_{x\to 0}\frac{|x|}{x}$ is-

(A) 1 (B) 2
(C) 3 (D) Does not exist

Sol. [D] $\lim_{x\to 0}\frac{|x|}{x}$

R.H.L. = $\lim_{x\to 0^+}\frac{x}{x} = 1$

L.H.L. = $\lim_{x\to 0^-} -\frac{x}{x} = -1$

L.H.L. $\neq$ R.H.L.

Q.2 If $f(x) = \begin{cases} x^2+1 & , \ x\geq 1 \\ 3x-1 & , \ x<1\end{cases}$ then the value of $\lim_{x\to 1} f(x)$ is-

(A) 1 (B) 2
(C) 3 (D) Does not exist

Sol.[B] $f(x) = \begin{cases} x^2+1, & x\geq 1 \\ 3x-1, & x<1\end{cases}$

L.H.L. = $\lim_{x\to 1^-}(3x-1) = 2$

R.H.L. = $\lim_{x\to 1^+}(x^2+1) = 2$

L.H.L. = R.H.L. = 2

Q.3 $\text{Lim}_{x\to 1}$ (1 – x + [x – 1] + [1 – x]) where [x] denotes greatest integer but not greater than x

(A) 1 (B) –1
(C) 0 (D) Does not exist

Sol.[B] $\lim_{x\to 1}(1-x+[x-1]+[1-x])$

L.H.L. = $\lim_{x\to 1^-}(1-x-1) = -1$

R.H.L. = $\lim_{x\to 1^+}(1-x-1) = -1$

L.H.L. = R.H.L. = –1

Q.4 If $f(x) = 3+\frac{1}{1+7^{1/(1-x)}}$ then-

(A) $\text{Lim}_{x\to 1^+} f(x) = 3$ (B) $\text{Lim}_{x\to 1^-} f(x) = 4$

(C) $\text{Lim}_{x\to 1} f(x) = 4$ (D) $\text{Lim}_{x\to 1} f(x)$ does not exist

Sol.[D] $f(x) = 3+\frac{1}{1+7^{1/(1-x)}}$

$\lim_{x\to 1^+} 3+\frac{1}{1+7^{1/1-x}} = 4$

$\lim_{x\to 1^-} 3+\frac{1}{1+7^{1/1-x}} = 3$

R.H.L. $\neq$ L.H.L

Question based on — Substitution

Q.5 $\lim_{x\to 1}[\cos^{-1}(\cos x)]$, where $[\cdot]$ denotes greatest integer function

(A) 0 (B) 1
(C) Does not exist (D) None of these

Sol.[C] $\lim_{x\to 1}[\cos^{-1}(\cos x)]$

$\lim_{x\to 1}[x]$

L.H.L. = $\lim_{x\to 1^-}[x] = 0$

R.H.L. = $\lim_{x\to 1^+}[x] = 1$

R.H.L. $\neq$ L.H.L.

Q.6 $\lim_{\theta\to -\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{\theta+\frac{\pi}{4}} =$

(A) $\sqrt{2}$ (B) 1
(C) 2 (D) Does not exist

Sol.[A] $\lim_{\theta\to -\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{\theta+\frac{\pi}{4}}$

$\lim_{\theta\to -\frac{\pi}{4}}\frac{\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)}{\left(\theta+\frac{\pi}{4}\right)} = \sqrt{2}$

Q.7 $\lim_{x\to\infty}\frac{x^3+x^2+1}{x^3-x^2+1} =$

(A) 0 (B) 1
(C) 2 (D) Does not exist

Sol.[B] $\lim_{x\to\infty}\frac{x^3+x^2+1}{x^3-x^2+1}$

$$\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^3}}{1-\frac{1}{x}+\frac{1}{x^3}}=1$$

Q.8 $\lim_{x\to\infty}\frac{x^4+x^2+1}{x^5+x^2-1}=$

(A) 0 (B) 1

(C) 2 (D) Does not exist

Sol.[A] $\lim_{x\to\infty}\frac{x^4+x^2+1}{x^5+x^2-1}$

$$\lim_{x\to\infty}\frac{1+\frac{1}{x^2}+\frac{1}{x^4}}{x+\frac{1}{x^2}-\frac{1}{x^4}}=0$$

Q.9 $\lim_{x\to\infty}\frac{\sqrt{3}x^5+x^2+13}{x^4+7x^2-\sqrt{17}}=$

(A) 0 (B) 2

(C) infinite (D) None

Sol.[C] $\lim_{x\to\infty}\frac{\sqrt{3}x^5+x^2+13}{x^4+7x^2-\sqrt{17}}$

$$\lim_{x\to\infty}\frac{\sqrt{3}x+\frac{1}{x^3}+\frac{13}{x^4}}{1+\frac{7}{x^2}-\frac{\sqrt{17}}{x^4}}=\infty$$

Question based on Factoriosation

Q.10 $\lim_{x\to-1}\frac{x^3-2x-1}{x^5-2x-1}=$

(A) 2/3 (B) 1/3 (C) 4/3 (D) 5/3

Sol.[B] $\lim_{x\to-1}\frac{x^3-2x-1}{x^5-2x-1}$

$$\lim_{x\to-1}\frac{(x+1)(x^2-x-1)}{(x+1)(x^4-x^3+x^2-x-1)}=\frac{1}{3}$$

Q.11 $\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}=$

(A) 0 (B) 3 (C) –3 (D) 1

Sol.[C] $\lim_{x\to-\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

$$\lim_{x\to-\frac{\pi}{6}}\frac{\left(\sin x-\frac{1}{2}\right)(\sin x+1)}{\left(\sin x-\frac{1}{2}\right)(\sin x-1)}=\frac{3/2}{-1/2}=-3$$

Q.12 $\lim_{x\to3}\frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}=$

(A) $\frac{2}{3}$ (B) $\frac{2}{9}$ (C) $\frac{1}{9}$ (D) 1

Sol.[B] $\lim_{x\to3}\frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$

$$\lim_{x\to3}\frac{(x-3)(x^2-4x+3)}{(x-3)(x^3-2x^2-6x+9)}$$

$$\lim_{x\to3}\frac{(x-3)^2(x-1)}{(x-3)^2(x^2+x-3)}=\frac{2}{9}$$

Question based on Rationalization

Q.13 $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=$

(A) 0 (B) 1 (C) 2 (D) None

Sol.[A] $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\times\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$

$$\lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\frac{2}{\infty}=0$$

Q.14 $\lim_{x\to0}\frac{\sqrt{(1+x^2)}-\sqrt{(1+x)}}{\sqrt{(1+x^3)}-\sqrt{(1+x)}}=$

(A) 0 (B) 1

(C) 2 (D) 4

Sol.[B] $\lim_{x\to0}\frac{\sqrt{(1+x^2)}-\sqrt{(1+x)}}{\sqrt{(1+x^3)}-\sqrt{(1+x)}}\times\frac{\sqrt{(1+x^2)}+\sqrt{(1+x)}}{\sqrt{(1+x^2)}+\sqrt{(1+x)}}\times\frac{\sqrt{(1+x^3)}+\sqrt{(1+x)}}{\sqrt{(1+x^3)}+\sqrt{(1+x)}}$

$$\lim_{x\to0}\frac{x^2-x}{x^3-x}$$

$$\lim_{x\to0}\frac{x-1}{x^2-1}$$

$$\lim_{x\to0}\frac{1}{x+1}=1$$

Question based on Expansion of function

Q.15 $\lim_{x\to 0} \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x} =$

(A) 3 (B) $\frac{1}{3}$ (C) 0 (D) 1

Sol.[B] $\lim_{x\to 0} \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$

$$\lim_{x\to 0} \frac{2 - \frac{\sin^{-1} x}{x}}{2 + \frac{\tan^{-1} x}{x}} = \frac{1}{3}$$

Q.16 $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2} \log(1 + x) =$

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) 2

Sol.[B] $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2} \log(1 + x)$

using expansion of log (1 + x)

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2}\left(x - \frac{x^2}{2!} + \frac{2x^3}{3!} \ldots\ldots\right)$$

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x} + \frac{1}{2!} - \frac{2x}{3!} + \ldots\ldots\ldots$$

$$= \frac{1}{2}$$

Question based on Application of standard limits

Q.17 $\lim_{x\to a} \frac{\log_e\{1 + \tan(x - a)\}}{\tan(x - a)} =$

(A) 0 (B) 1 (C) 2 (D) 3

Sol.[B] $\lim_{x\to a} \frac{\lambda n\{1 + \tan(x - a)\}}{\tan(x - a)}$; as $\lim_{x\to 0} \frac{\lambda n(1 + x)}{x} = 1$

Q.18 $\lim_{x\to 1} \frac{\cos(\pi x/2)}{1 - x} =$

(A) 0 (B) π (C) π/2 (D) 2π

Sol.[C] $\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{1 - x}$

$$\lim_{x\to 1} \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}x\right)}{1 - x}$$

$$\lim_{x\to 1} \frac{\sin\frac{\pi}{2}(1 - x)}{\frac{\pi}{2}(1 - x)} \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

Q.19 $\lim_{x\to\infty} \left(\frac{x}{1 + x}\right)^x =$

(A) e (B) $\frac{1}{e}$ (C) 0 (D) 1

Sol.[B] $\lim_{x\to\infty} \left(\frac{x}{1 + x}\right)^x$ (1^∞)

$$\Rightarrow e^{\lim_{x\to\infty}(x)\left(\frac{x}{1+x} - 1\right)}$$

$$\Rightarrow e^{\lim_{x\to\infty}(x)\left(\frac{x-1-x}{1+x}\right)}$$

$$\Rightarrow e^{\lim_{x\to\infty} \frac{-1}{1+\frac{1}{x}}} = e^{-1}$$

Q.20 $\lim_{x\to 0} (1 + x)^{1/(13x)} =$

(A) e^{13} (B) $e^{\frac{1}{13}}$ (C) e (D) 1

Sol.[B] $\lim_{x\to 0} (1 + x)^{1/13x}$

$$e^{\lim_{x\to 0} \frac{1}{13x}(1 + x - 1)} = e^{1/13}$$

Q.21 $\lim_{x\to 1} \left(\frac{1 + x}{2 + x}\right)^{\frac{1-\sqrt{x}}{1-x^2}} =$

(A) $(2)^{1/3}$ (B) $(2/3)^{1/2}$

(C) $(2/3)^{1/4}$ (D) Does not exist

Sol.[C] $\lim_{x\to 1} \left(\frac{1 + x}{2 + x}\right)^{\frac{1-\sqrt{x}}{1-x^2}}$

$$\lim_{x\to 1} \left(\frac{1 + x}{2 + x}\right)^{\frac{1-\sqrt{x}}{(1+x)(1+\sqrt{x})(1-\sqrt{x})}} = \left(\frac{2}{3}\right)^{1/4}$$

Q.22 $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^3} =$

(A) 0 (B) 1

(C) e (D) Does not exist

Sol.[D] $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^3}$ (1^∞)

L.H.L. = $\lim_{x\to 0^-} \left(\frac{\tan x}{x}\right)^{-\infty}$

R.H.L. = $\lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^{+\infty}$

L.H.L. ≠ R.H.L.

$\Rightarrow$ limit does not exist

Q.23 $\lim\limits_{x\to1}(2-x)^{\tan\frac{\pi x}{2}}$ =

(A) $e^{-2/\pi}$ (B) $e^{1/\pi}$
(C) $e^{2/\pi}$ (D) $e^{-1/\pi}$

Sol.[C] $\lim\limits_{x\to1}(2-x)^{\tan\frac{\pi x}{2}}$ (1^{∞})

$\Rightarrow e^{\lim\limits_{x\to1}\tan\frac{\pi x}{2}(2-x-1)}$

$\Rightarrow e^{\lim\limits_{x\to1}\left(\tan\frac{\pi x}{2}\right)(1-x)}$

$\Rightarrow e^{\lim\limits_{x\to1}\frac{\left(\sin\frac{\pi x}{2}\right)(1-x)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}x\right)}}$

$\Rightarrow e^{\lim\limits_{x\to1}}\frac{\frac{\pi}{2}(1-x)}{\sin\frac{\pi}{2}(1+x)}\cdot\frac{2}{\pi} = e^{2/\pi}$

Question based on **L Hospital's Rule**

Q.24 $\lim\limits_{x\to0}\frac{\sin x+\log(1-x)}{x^2}$ =

(A) 0 (B) 1/2
(C) –1/2 (D) does not exist

Sol.[C] $\lim\limits_{x\to0}\frac{\sin x+\log(1-x)}{x^2}$ $\left(\frac{0}{0}\right)$

using L Hospital Rule

$\lim\limits_{x\to0}\frac{\cos x-\frac{1}{1-x}}{2x}$ $\left(\frac{0}{0}\right)$

$\lim\limits_{x\to0}\frac{-\sin x-\frac{1}{(1-x)^2}}{2} = -\frac{1}{2}$

Q.25 $\lim\limits_{x\to0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$ =

(A) 10/3 (B) 3/10
(C) 6/5 (D) 5/6

Sol.[A] $\lim\limits_{x\to0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$

$\lim\limits_{x\to0}\frac{2\sin^2 x.\frac{\sin 5x}{5x}.5}{x^2\frac{\sin 3x}{3x}.3} = \frac{10}{3}$

Q.26 Let f(a) = g(a) = k and their n^th derivatives exist and are not equal for some n.
Further if

$\lim\limits_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4$

then k is equal to-

(A) 0 (B) 4 (C) 2 (D) 1

Sol.[B] f(a) = g(a) = k

$\lim\limits_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4$

differentiating n times

$\lim\limits_{x\to a}\frac{f(a)g^n(x)-g(a)f^n(x)}{g^n(x)-f^n(x)} = 4$

since g(a) = f(a) = k

$\lim\limits_{x\to a}\frac{k(g^n(x)-f^n(x))}{g^n(x)-f^n(x)} = 4$

$\Rightarrow$ k = 4

Question based on **Sandwich Theorem**

Q.27 $\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{[x]}}$ = (where [x] is greatest integer function $\le$ x)

(A) 0 (B) 1
(C) 2 (D) Does not exist

Sol.[A] $\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{[x]}}$

$\Rightarrow x-1\le[x]\le x$

$\underbrace{\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{x-1}}}_{L_1}\le\underbrace{\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{[x]}}}_{L}\le\underbrace{\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{x}}}_{L_2}$

$L_1=\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{x-1}}$ $\left(\frac{\infty}{\infty}\right)$

$=\lim\limits_{x\to\infty}\frac{2x+1}{e^{x-1}}$ $\left(\frac{\infty}{\infty}\right)$

$=\lim\limits_{x\to\infty}\frac{2}{e^{x-1}} = 0$

$\Rightarrow L_2=\lim\limits_{x\to\infty}\frac{x^2+x+1}{e^{x}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{2x+1}{e^x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2}{e^x} = 0$

$\therefore L_1 = L_2 = L = 0$

Q.28 $\lim_{n\to\infty} \frac{[x]+[2x]+[3x]+.....+[nx]}{1+2+3+......+n} =$

(where [·] denotes the greatest integer function)

(A) 0 (B) $\frac{x}{2}$ (C) $\frac{x}{6}$ (D) x

Sol.[D] $\lim_{n\to\infty} \frac{[x]+[2x]+[3x]+.....+[nx]}{1+2+3+......+n}$

$x - 1 \le [x] \le x$

$2x - 1 \le [2x] \le 2x$

⋮

$nx - 1 \le [nx] \le nx$

$(x - 1) + (2x - 1) + ... + (nx - 1) \le [x] + [2x] + ... + [nx] \le x + 2x + ... + nx$

$\underbrace{\lim_{n\to\infty} \frac{(x+2x+3x+.....+nx)-n}{1+2+.....+n}}_{L_1} \le \lim_{n\to\infty} \underbrace{\frac{[x]+[2x]+....+[nx]}{1+2+......+n}}_{L} \le \underbrace{\lim_{n\to\infty} \frac{x+2x+.....+nx}{1+2+....+n}}_{L_2}$

$L_1 = \lim_{n\to\infty} \frac{(x+2x+.....+nx)-n}{1+2+.....+n}$

$= \lim_{n\to\infty} \frac{\frac{(x)(n)(n+1)}{2}-n}{\frac{n(n+1)}{2}}$

$\Rightarrow \lim_{n\to\infty} \frac{xn(n+1)-2n}{n(n+1)}$

$\Rightarrow \lim_{n\to\infty} \frac{x\left(1+\frac{1}{n}\right)-\frac{2}{n}}{\left(1+\frac{1}{n}\right)} = x$

$L_2 = \lim_{n\to\infty} \frac{x+2x+3x+....+nx}{1+2+3+....+n}$

$= \lim_{n\to\infty} \frac{(x)(n)\frac{(n+1)}{2}}{(n)\frac{(n+1)}{2}} = x$

$\Rightarrow L_1 = L_2 = L = x$

Question based on **Miscellaneous question**

Q.29 $\lim_{x\to\frac{\pi}{4}}$ [Max (sinx, cosx)]

(A) $\frac{1}{2}$ (B) 0

(C) – 1 (D) $1/\sqrt{2}$

Sol.[D] $\lim_{x\to\frac{\pi}{4}}$ [Max (sin x, cos x)]

L.H.L = $\lim_{x\to\frac{\pi}{4}^-} \cos x = \frac{1}{\sqrt{2}}$

R.H.L. = $\lim_{x\to\frac{\pi}{4}} \sin x = \frac{1}{\sqrt{2}}$

L.H.L. = R.H.L. = $\frac{1}{\sqrt{2}}$

Q.30 $\lim_{x\to 0^+} (\text{cosec } x)^{1/\log x} =$

(A) e (B) e^{-1}

(C) e^2 (D) 1

Sol.[B] $\lim_{x\to 0^+} (\text{cosec } x)^{\frac{1}{\log x}} = y$

$\log y = \lim_{x\to 0^+} \frac{\log(\text{cosec } x)}{\log x} \left(\frac{\infty}{\infty}\right)$

$\log y = \lim_{x\to 0^+} -x \cot x$

$\log y = \lim_{x\to 0^+} -\frac{x}{\tan x} = -1 \Rightarrow y = e^{-1}$

Q.31 Which of the following statement is/are correct -

(A) $\lim_{x\to\pi^-}$ [sgn sin x] = 1

(B) $\lim_{x\to\pi^+}$ [sgn sin x] ≠ –1

(C) $\lim_{x\to\pi}$ [sgn sin x] = 1

(D) $\lim_{x\to\pi}$ [sgn sin x] does not exist

(Where [.] represent greatest integer function)

Sol.[A, D] $\lim_{x\to\pi^-}$ [sgn sin x] = 1

$\lim_{x\to\pi^+}$ [sgn sin x] = – 1

$\lim_{x\to\pi}$ [sgn sin x] = Does not exist

Q.32 The sum to infinity of the series :

$\frac{3}{1^3} + \frac{5}{1^3+2^3} + \frac{7}{1^3+2^3+3^3} + ...$ is-

(A) 3 (B) 4

(C) 5 (D) 6

Sol.[B] $S = \frac{3}{1^3} + \frac{5}{1^3+2^3} + \frac{7}{1^3+2^3+3^3} + \ldots\ldots$

$t_r = \frac{2r+1}{1^3+2^3+\ldots+r^3}$

$= \frac{2r+1}{\frac{r^2(r+1)^2}{4}}$

$= \frac{8r+4}{r^2(r+1)^2}$

$S = \lim_{n\to\infty} \sum_{r=1}^{n} t_r$

$t_r = \frac{8r+4}{r^2(r+1)^2}$

$= 4\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$

$S = \lim_{n\to\infty} 4\left[\frac{1}{1} - \frac{1}{4}\right.$

$+ \frac{1}{4} - \frac{1}{9}$

$+ \frac{1}{9} - \frac{1}{16}$

$\vdots$

$\left. + \frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$

$= \lim_{n\to\infty} 4\left[1 - \frac{1}{(n+1)^2}\right] = 4$

Q.33 Let

$f_k(\alpha) = \left(\cos\frac{\alpha}{k^2} + i\sin\frac{\alpha}{k^2}\right)\left(\cos\frac{2\alpha}{k^2} + i\sin\frac{2\alpha}{k^2}\right)\ldots$

$\ldots\left(\cos\frac{\alpha}{k} + i\sin\frac{\alpha}{k}\right)$ then $\lim_{n\to\infty} f_n(\alpha) =$

(A) $\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$ (B) $\cos\alpha + i\sin\alpha$

(C) $i\cos\alpha + \sin\alpha$ (D) $i\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}$

Sol.[A] $f_k(\alpha) = \left(\cos\frac{\alpha}{k^2} + i\sin\frac{\alpha}{k^2}\right)\left(\cos\frac{2\alpha}{k^2} + i\sin\frac{2\alpha}{k^2}\right)\ldots$

$\left(\cos\frac{k\alpha}{k^2} + i\sin\frac{k\alpha}{k^2}\right)$

$= e^{i\left(\frac{\alpha}{k^2}\right)} \cdot e^{i\left(\frac{2\alpha}{k^2}\right)} \cdot \ldots\ldots\ldots e^{i\left(\frac{k\alpha}{k^2}\right)}$

$f_n(\alpha) = e^{\frac{i\alpha}{n^2}(1+2+\ldots+n)}$

$= e^{\frac{i\alpha}{n^2}\frac{n(n+1)}{2}}$

$\lim_{n\to\infty} f_n(\alpha)$

$\Rightarrow \lim_{n\to\infty} e^{\frac{i\alpha}{n^2}\frac{n^2\left(1+\frac{1}{n}\right)}{2}} = e^{\frac{i\alpha}{2}}$

$= \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$

True or False type Questions

Q.34 $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(\pi-2x)^2} = \frac{1}{8}$

Sol. **[T]**

$\lim_{x\to\pi/2} \frac{1-\sin x}{(\pi-2x)^2} = \frac{1}{8}$

$\lim_{x\to\pi/2} \frac{1-\sin x}{(\pi-2x)^2}$ ($\frac{0}{0}$ form)

Use L–H Rule, we get

$\lim_{x\to\pi/2} \frac{0-\cos x}{2(\pi-2x)(-2)} = \lim_{x\to\pi/2} \frac{\cos x}{4(\pi-2x)}$

($\frac{0}{0}$ form)

Again use L-H Rule, we get

$\lim_{x\to\pi/2} \frac{-\sin x}{4(-2)} = \frac{1}{8} \lim_{x\to\pi/2} \sin x = \frac{1}{8}$

Q.35 $\lim_{x\to\sin 1^-} \{\sin^{-1}x\} = 1$ where $\{x\}$ is fractional part of x.

Sol. **[T]**

$\lim_{x\to\sin 1^-} \{\sin^{-1}x\} = 1$

$\{\sin^{-1}x\} = \sin^{-1}x - [\sin^{-1}x]$

$\lim_{x\to\sin 1^-} (\sin^{-1}x - [\sin^{-1}x])$

$= \lim_{h\to 0^-} (\sin^{-1}(\sin 1 - h) - [\sin^{-1}(\sin 1 - h)])$

$= \lim_{h\to 0^-} (1 - [\text{value less than } 1]) = 1$

Fill in the blanks type questions

Q.36 If $|x| < 1$, then

$\lim_{n\to\infty} (1+x)(1+x^2)\ldots\ldots(1+x^{2n}) \ldots\ldots\ldots$

Sol. Let A = $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$

$\ldots\ldots\ldots\ldots\ldots\ (1+x^{2^n})$; $|x| < 1$

$= \dfrac{(1+x)(1-x)}{(1-x)}(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$

$\ldots\ldots\ldots\ldots\ (1+x^{2^n})$

$= \dfrac{(1-x^2)(1+x^2)}{(1-x)}(1+x^4)(1+x^8)(1+x^{16})$

$\ldots\ldots\ldots\ (1+x^{2^n})$

$= \dfrac{(1-x^4)}{(1-x)}(1+x^4)(1+x^8)(1+x^{16})\ \ldots\ldots\ (1+x^{2^n})$

$= \dfrac{(1-x^8)}{(1-x)}(1+x^8)(1+x^{16})\ \ldots\ldots\ldots\ldots\ldots\ (1+x^{2^n})$

$= \dfrac{(1-x^{16})}{(1-x)}(1+x^{16})(1+x^{32})\ \ldots\ldots\ldots\ldots\ldots\ (1+x^{2^n})$

$= \dfrac{(1-x^{32})}{(1-x)}(1+x^{32})\ldots\ldots\ldots(1+x^{2^n})$

$= \dfrac{(1-x^{2^n})(1+x^{2^n})}{(1-x)} = \left[1-(x^{2^n})^2\right]\Big/(1-x)$

$= \dfrac{(1-x^{4^n})}{(1-x)}$

$\lim\limits_{n\to\infty} A = \lim\limits_{n\to\infty}\left(\dfrac{1-x^{4^n}}{1-x}\right) = \left(\dfrac{1}{1-x}\right)$;

As $\lim\limits_{n\to\infty} x^{4^n} \to 0$

EXERCISE # 2

Part-A Only single correct answer type questions

Q.1 $\lim_{x\to -1^+} \dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}} =$

(A) $\sqrt{\pi}$ (B) $\dfrac{1}{\sqrt{2\pi}}$ (C) $\sqrt{2\pi}$ (D) π

Sol.[B] $\lim_{x\to -1^+} \dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}$

Let x = cos θ

as $x \to -1^+$, $\theta \to \pi^-$

$\lim_{x\to\pi^-} \dfrac{\sqrt{\pi}-\sqrt{\theta}}{\sqrt{2}\cos\theta/2}\left(\dfrac{0}{0}\right)$

using L Hospital rule

$\lim_{x\to\pi^-} \dfrac{-\dfrac{1}{2\sqrt{\theta}}}{-\dfrac{\sqrt{2}}{2}\sin\dfrac{\theta}{2}} = \dfrac{1}{\sqrt{2\pi}}$

Q.2 If $\lim_{n\to\infty}\left(an - \dfrac{1+n^2}{1+n}\right)$ = b, a finite number then the ordered pair (a, b) is-

(A) (1, 1) (B) (–1, 1)
(C) (1, –1) (D) None of these

Sol. **[A]**

$\lim_{n\to\infty}\left(an - \dfrac{1+n^2}{1+n}\right) = b$

$\lim_{n\to\infty}\left(\dfrac{an+an^2-1-n^2}{1+n}\right) = b$

$\lim_{n\to\infty}\dfrac{a-1/n+n(a-1)}{1+1/n} = b$

Ordered pair must be (1,1)

∴ Option (A) is correct Answer

Q.3 $\lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$, a > 0 =

(A) $\sqrt{2a}$ (B) $2\sqrt{a}$
(C) $1/2\sqrt{a}$ (D) $\sqrt{a}$

Sol.[C] $\lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$, a > 0

$\lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$

$\lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}}{\sqrt{x-2a}.\sqrt{x+2a}} + \lim_{x\to 2a^+}\dfrac{\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}} \times \dfrac{\sqrt{x}+\sqrt{2a}}{\sqrt{x}+\sqrt{2a}}$

$\Rightarrow \dfrac{1}{2\sqrt{a}} + \lim_{x\to 2a^+}\dfrac{(x-2a)}{\sqrt{x-2a}\sqrt{x+2a}(\sqrt{x}+\sqrt{2a})}$

$\Rightarrow \dfrac{1}{2\sqrt{a}} + \lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}}{\sqrt{x+2a}(\sqrt{x}+\sqrt{2a})}$

$= \dfrac{1}{2\sqrt{a}}$

Q.4 $\text{Lim}_{x\to 0}\dfrac{e^{\sin x}-1-\sin x}{x^2} =$

(A) 1 (B) $\dfrac{1}{2}$ (C) $e^{1/2}$ (D) e

Sol.[B] $\lim_{x\to 0}\dfrac{e^{\sin x}-1-\sin x}{x^2}$

using expansion of e^x

$\lim_{x\to 0}\dfrac{(1+\sin x+\dfrac{\sin^2 x}{2!}+\dfrac{\sin^3 x}{3!}+......)-1-\sin x}{x^2}$

$\lim_{x\to 0}\dfrac{\sin^2 x}{x^2(2!)}+\dfrac{\sin^3 x}{x^2(3!)}+...... = \dfrac{1}{2}$

Q.5 $\text{Lim}_{n\to\infty}\dfrac{n^k\cos n!}{n+1}$; $0 < k < 1$

(A) 0 (B) 1!
(C) 2! (D) None of these

Sol. **[A]**

$\text{Lim}_{n\to\infty}\dfrac{n^k\cos n!}{n+1}\, 0 < k < 1$

$\text{Lim}_{n\to\infty}\dfrac{n^{k-1}\cos n!}{1+\dfrac{1}{n}}$

$$\lim_{n\to\infty} \frac{\cos n!}{(n^{1-k})\left(1+\frac{1}{n}\right)}$$

simple $0 < k < 1 \Rightarrow k - 1$ is negative

and $1 - k > 0$

as $n \to \infty$

$-1 \le \cos n! \le 1 \Rightarrow$ finite value

$n^{1-k} \to \infty$

$1 + \frac{1}{n} \to 1$

so $\lim_{n\to\infty} \frac{\cos n!}{(n^{1-k})\left(1+\frac{1}{n}\right)} = 0$

Q.6 If $S_n = a_1 + a_2 + \ldots\ldots\ldots a_n$ and $\lim_{n\to\infty} a_n = a$, then

$\lim_{n\to\infty} \frac{S_{n+1} - S_n}{\sqrt{\sum_{k=1}^{n} k}}$ is equal to-

(A) 0 (B) a

(C) $\sqrt{2}$ a (D) 2a

Sol. **[A]**

If $S_n = a_1 + a_2 + \ldots\ldots\ldots a_n$ and $\lim_{n\to\infty} a_n = a$, then

$S_{n+1} = a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_n + a_{n+1}$

$S_{n+1} - S_n = a_{n+1}$

$\lim_{n\to\infty} a_n = a \Rightarrow \lim_{(n+1)\to\infty} a_{n+1} = a = \lim_{n\to\infty} a_{n+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{\sqrt{\frac{n(n+1)}{2}}} = \lim_{n\to\infty} \frac{\sqrt{2}\, a_{n+1}}{n\sqrt{(1+1/n)}}$$

$$= \sqrt{2} \lim_{n\to\infty} a_{n+1} \cdot \lim_{n\to\infty} \frac{1}{n\sqrt{1+1/n}}$$

$$= \sqrt{2} \times a \times 0 = 0$$

$\therefore$ Option (A) is correct Answer.

Q.7 The sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ is equal to-

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$

Sol. **[C]**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

First we obtain following expression by partial fraction method

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{(n+1)} + \frac{1}{2(n+2)}$$

Sum of all n terms would be as follows

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n}\left[\frac{1}{2r} - \frac{1}{(r+1)} + \frac{1}{2(r+2)}\right]$$

$$= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right)$$

$$+ \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{6} + \frac{1}{14}\right) + \left(\frac{1}{12} - \frac{1}{7} + \frac{1}{16}\right)$$

$$+ \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)} + \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{(n+1)} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)}$$

$$\lim_{n\to\infty}\left[\frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)}\right] = \frac{1}{4}$$

Q.8 The continued product of $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\ldots\ldots\ldots\left(1-\frac{1}{n^2}\right)$ is P_n ; then $\lim_{n\to\infty} P_n$ is- (where $n \in N$)

(A) $-\frac{1}{2}$ (B) $\frac{n+1}{n}$ (C) $\frac{1}{2}$ (D) None

Sol. **[C]**

$$P_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\ldots\ldots\ldots\left(1-\frac{1}{n^2}\right)$$

$$= \left(\frac{1}{2}.\frac{3}{2}\right)\left(\frac{2}{3}.\frac{4}{3}\right).\left(\frac{3}{4}.\frac{5}{4}\right)\left(\frac{4}{5}.\frac{6}{8}\right)\ldots\ldots\left(\frac{n-1}{n}.\frac{n+1}{n}\right)$$

$$= \frac{n+1}{2n}$$

$$\lim_{n\to\infty} P_n$$

$$\Rightarrow \lim_{n\to\infty} \frac{n+1}{2n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}$$

Q.9 Inscribed in a circle of radius R is a square, a circle is inscribed in the square, a new square in the circle, and so on for n times. Find the limit of the sum of areas of all the circles and the limit of the sum of areas of all the squares as $n \to \infty$

(A) $2\pi R^2, R^2$ (B) $\pi R^2, 4R^2$

(C) $2\pi R^2, 4R^2$ (D) $4\pi R^2, 2R^2$

Sol. **[C]**

Let sides of first (or upper square) be a_1 & Radius of circle is R_1

Area of circle, $A_1 = \pi R_1^2$

$$R_1^2 = \left(\frac{a_1}{2}\right)^2 + \left(\frac{a_1}{2}\right)^2 = \frac{a_1^2}{2}$$

$a_1 = \sqrt{2}\, R_1$

Radius of 2nd upper circle, $R_2 = \frac{a_1}{2}$

$$= \frac{\sqrt{2}\, R_1}{2} = R_1/\sqrt{2}$$

Sides of 2nd upper square be a_2, then

$$R_2^2 = \left(\frac{a_2}{2}\right)^2 + \left(\frac{a_2}{2}\right)^2$$

$$= \frac{a_2^2}{2} = \frac{R_1^2}{2}$$

$\Rightarrow a_2 = R_1$

Radius of 3rd upper circle, $R_3 = a_2/2 = R_1/2$

Hence, Sum of Areas of circles, $(A_n)_{circle}$

$$= \pi R_1^2 + \frac{\pi R_1^2}{2} + \frac{\pi R_1^2}{4} + \ldots\ldots\ldots\ldots \text{ n terms}$$

$$= \pi R_1^2 \left[1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots + \text{n terms}\right]$$

$$= \pi R_1^2 \, \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

$$= 2\pi R_1^2 \left(1 - \frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} (A_n)_{circle} = \lim_{n\to\infty} 2\pi R_1^2 \left(1 - \frac{1}{2^n}\right)$$

$= 2\pi\, R_1^2$

Sum of areas of squares, $(A_n)_{square}$

$$= 2R_1^2 + R_1^2 + \frac{R_1^2}{2} + \ldots\ldots\ldots\ldots \text{ n terms}$$

$$= 2R_1^2 \left[1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\text{.n terms}\right]$$

$$(A_n)_{square} = 2R_1^2 \, \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - 1/2} = 4R_1^2 \,(1 - 1/2^n)$$

$$\lim_{n\to\infty} (A_n)_{square} = \lim_{n\to\infty} 4R_1^2 (1 - 1/2^n) = 4R_1^2$$

$\therefore$ Option (C) is correct Answer.

Q.10 $\lim_{x\to 1}\, , \frac{x \sin(x - [x])}{x - 1}$ where $[\cdot]$ denotes the greatest integer function, is equal to-

(A) 1 (B) –1

(C) ∞ (D) does not exist

Sol. **[D]**

$$\lim_{x\to 1} \frac{x\sin(x - [x])}{(x - 1)}$$

Put $x - 1 = h \Rightarrow x = h + 1$

As $(x - 1) \to 0 \Rightarrow h \to 0$

$$\lim_{h\to 0} \frac{(1+h)\sin(1 + h - [1 + h])}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)\sin(1 + h - 1 - [h])}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)\sin(h - [h])}{h}$$

$0 - h$ h $0 + h$

$$\text{L.H.L} = \lim_{h\to 0^-} f(0 - h)$$

$$= \lim_{h\to 0^-} \frac{(1 + 0 - h)\sin(0 - h - [0 - h])}{(0 - h)}$$

$$= \lim_{h\to 0^-} \frac{(1 - h)\sin(-h + h)}{(-h)} = -1$$

$$\text{R.H.L.} \quad \lim_{h\to 0^+} f(0 + h) =$$

$$\lim_{h\to 0^+} \frac{(1 + 0 + h)\sin(0 + h - [0 + h])}{h}$$

$$\lim_{h\to 0^+} \frac{(1+h)\sin(h-h)}{h} = 1$$

Since, L.H.L ≠ R.H.L

∴ Limit does not exist

Q.11 The value of $\lim_{x\to\pi/4} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x} =$

(A) $-\frac{1}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) None

Sol. **[A]** $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$

Rationalise Numerator and Denominator by $\sqrt{1+\sqrt{\sin 2x}}$

$$\lim_{x\to\pi/4} \frac{\sqrt{1-\sqrt{\sin 2x}}}{(\pi-4x)} \times \frac{\sqrt{1+\sqrt{\sin 2x}}}{\sqrt{1+\sqrt{\sin 2x}}}$$

$$= \lim_{x\to\pi/4} \frac{\sqrt{1-\sin 2x}}{(\pi-4x)\sqrt{1+\sqrt{\sin 2x}}}$$

$$= \lim_{x\to\pi/4} \frac{\sin x - \cos x}{(\pi-4x)\sqrt{1+\sqrt{\sin 2x}}} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L – H Rule, we get

$$= \lim_{x\to\pi/4} \frac{(\cos x + \sin x)}{(-4)\sqrt{1+\sqrt{\sin 2x}} + \frac{1}{2\sqrt{1+\sqrt{\sin 2x}}} \cdot \frac{1}{2\sqrt{\sin 2x}} \cdot \cos 2x . 2(\pi - 4x)}$$

$$= \frac{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{-4\sqrt{2}+0} = \frac{\sqrt{2}}{-4\sqrt{2}} = -\frac{1}{4}$$

Q.12 Evaluate :

$$\lim_{x\to 0-} \frac{[x]+[x^2]+[x^3]+......+[x^{2n+1}]+n+1}{1+[x]+|x|+2x}$$

(A) 1 (B) 0

(C) 2 (D) None of these

Sol. **[B]**

$$\lim_{x\to 0-} \frac{[x]+[x^2]+[x^3]+......+[x^{2n+1}]+n+1}{1+[x]+|x|+2x}$$

as $x \to 0^-$ $[x^{2n+1}] \to -1$ & $[x^{2n}] \to 0$

$$\lim_{x\to 0-} \frac{(-1)+(0)+(-1)+(0)....+(0)+(-1)+n+1}{1-1-x+2x}$$

$$\lim_{x\to 0-} \frac{(-1)(n+1)+n+1}{x} = \frac{0}{x} = 0$$

Q.13 $\lim_{x\to\infty} \left(\cos\sqrt{x+1} - \cos\sqrt{x}\right) =$

(A) 0 (B) 1

(C) 2 (C) None of these.

Sol. **[A]**

$$\lim_{x\to\infty} \left(\cos\sqrt{x+1} - \cos\sqrt{x}\right)$$

Apply : $\cos C - \cos D = 2\sin\frac{C+D}{2} . \sin\frac{D-C}{2}$

$$= \lim_{x\to\infty} 2\sin\frac{\sqrt{x+1}+\sqrt{x}}{2} . \sin\frac{\sqrt{x}-\sqrt{x+1}}{2}$$

$$= 2\lim_{x\to\infty} \sin\frac{\sqrt{x+1}+\sqrt{x}}{2} \times \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} \times \sin\frac{\sqrt{x}-\sqrt{x+1}}{2} \times \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}}$$

$$= 2\lim_{x\to\infty} \sin\frac{x+1-x}{2(\sqrt{x+1}-\sqrt{x})} \times \sin\frac{x-x-1}{(\sqrt{x}+\sqrt{x+1})}$$

$$= 2\lim_{x\to\infty} \frac{1}{2(\sqrt{x+1}-\sqrt{x})} \times \sin\frac{-1}{\sqrt{x}+\sqrt{x+1}} = 0$$

Q.14 $\lim_{x\to 2} \frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x-2}$, $x \in (0, \pi/2)$

(A) $\sin^2\alpha \, \lambda n (\sin \alpha)$

(B) $\cos^2 \alpha \, \lambda n (\cos \alpha)$

(C) $\cos^2 \alpha \, \lambda n (\cos \alpha) - \sin^2\alpha \, \lambda n (\sin \alpha)$

(D) $\cos^2 \alpha \, \lambda n (\cos \alpha) + \sin^2\alpha \, \lambda n (\sin \alpha)$

Sol.[D] $\lim_{x\to 2} \frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x-2}$

$$\lim_{x\to 2} \frac{(\cos\alpha)^x + (\sin\alpha)^x - \sin^2\alpha - \cos^2\alpha}{x-2}$$

$$\lim_{x\to 2} \cos^2\alpha \frac{(\cos\alpha)^{x-2} - 1}{x-2} +$$

$$\lim_{x\to 2} \frac{(\sin\alpha)^{x-2} - 1}{x-2} \sin^2\alpha$$

$= \cos^2\alpha \, \lambda n(\cos \alpha) + \sin^2\alpha \, \lambda n (\sin \alpha)$

Q.15 $\lim_{n\to\infty} \frac{[x] + \frac{1}{2}[2x] + \frac{1}{3}[3x] + ... + \frac{1}{n}[nx]}{1^2 + 2^2 + 3^2 + ... + n^2} =$

(where [·] denotes the greatest integer function)

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{6}$ (D) 1

Sol.[A] $\lim_{n\to\infty}\dfrac{[x]+\frac{1}{2}[2x]+\frac{1}{3}[3x]+....+\frac{1}{n}[nx]}{1^2+2^2+3^3+......+n^2}$

$x-1<[x]\le x$

$2x-1<[2x]\le 2x \Rightarrow x-\frac{1}{2} < \frac{1}{2}[2x]\le x$

$3x-1<[3x]\le 3x \Rightarrow x-\frac{1}{3} < \frac{1}{3}[3x]\le x$

$4x-1<[4x]\le 4x \Rightarrow x-\frac{1}{4} < \frac{1}{4}[4x]\le x$

$(nx-1)<[nx]\le nx \Rightarrow x-\frac{1}{n} < \frac{1}{n}[nx]\le x$

Adding all terms as :

$(x+x+x+\text{........n terms}) - (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+..................+\frac{1}{n}) <$

$[x]+\frac{1}{2}[2x]+\frac{1}{3}[3x]+\frac{1}{4}[4x]+........+\frac{1}{n}[nx]$

$\le (x+x+x+\text{.............}+n \text{ terms})$

$$\Rightarrow nx-\sum_{r=1}^{n}\frac{1}{r} < \sum_{r=1}^{n}\frac{1}{r}[rx]\le nx$$

$$\lim_{n\to\infty}\frac{nx}{\frac{n(n+1)(2n+1)}{6}}-\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\frac{1}{r}}{\frac{n(n+1)(2n+1)}{6}}$$

$$<\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\frac{1}{r}[rx]}{\frac{n(n+1)(2n+1)}{6}}\le\lim_{n\to\infty}\frac{nx}{\frac{n(n+1)(2n+1)}{6}}$$

$$\Rightarrow 0-0<\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\frac{1}{r}[rx]}{\frac{n(n+1)(2n+1)}{6}}\le 0$$

$$\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\frac{1}{r}[rx]}{\frac{n(n+1)(2n+1)}{6}}=0$$

Q.16 $\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1}=$

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{1}{2}$ (D) None

Sol. **[C]**

$\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1}$

$$=\lim_{x\to\infty}\frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+1/x^2}}=\frac{1}{2}$$

Q.17 The value of

$$\lim_{x\to 0^+}\left(\frac{e^{x\lambda n(2^x-1)}-(2^x-1)^x\sin x}{e^{x\lambda nx}}\right)^{1/x}=$$

(A) e (B) $\frac{1}{e}\ln 2$ (C) e ln 2 (D) None

Sol. **[B]**

$$\lim_{x\to 0^+}\left(\frac{e^{x\lambda n(2^x-1)}-(2^x-1)^x\sin x}{e^{x\lambda nx}}\right)^{1/x}$$

$$\Rightarrow \lim_{x\to 0^+}\left[\frac{(2^x-1)^x-(2^x-1)^x\sin x}{x^x}\right]^{1/x}$$

$$\Rightarrow \lim_{x\to 0^+}\frac{(2^x-1)(1-\sin x)^{1/x}}{x}$$

$\Rightarrow \lambda n2\ .\ \lim_{x\to 0^+}(1-\sin x)^{1/x}\ (1^\infty)$

$\Rightarrow \lambda n2\ .\ e^{\lim_{x\to 0^+}\frac{1}{x}(1-\sin x-1)}$

$\Rightarrow \lambda n2\ .\ e^{\lim_{x\to 0^+}\frac{-\sin x}{x}} \Rightarrow \frac{1}{e}\ \lambda n2$

Q.18 Let f(x) = $\lim_{n\to\infty}\{\sin x+2\sin^2x+3\sin^3x+...$ $....+n\sin^nx\}$ If $\sin x\ne n\pi+\pi/2,\ n\in I$:

Evaluate : $\lim_{x\to\pi/2}\left[(1-\sin x)^2 f(x)\right]^{\frac{1}{\sin x-1}}=$

(A) 1 (B) 0

(C) e (D) None of these

Sol. **[C]**

$f(x)=\lim_{n\to\infty}\{\sin x+2\sin^2x+3\sin^3x+.......+n\sin^nx\}$

f(x) = sum of infinite A.G.P.

$f(x)=\lim_{n\to\infty}\ \sin x$

Let $S=\{1+2\sin x+3\sin^2x+.....+n\sin^{n-1}x\}$

$S=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$

$= \frac{1}{1-\sin x} + \frac{\sin x}{(1-\sin x)^2}$

$= \frac{1-\sin x+\sin x}{(1-\sin x)^2}$

$= \frac{1}{(1-\sin x)^2}$

$\Rightarrow F(x) = \frac{\sin x}{(1-\sin x)^2}$

$\Rightarrow (1-\sin x)^2 F(x) = \sin x$

$\Rightarrow \lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\frac{1}{\sin x-1}} (1^\infty)$

$\Rightarrow e^{\lim\limits_{x\to\frac{\pi}{2}} \frac{1}{(\sin x-1)}.(\sin x-1)} \quad \Rightarrow e^1$

Q.19 $\lim\limits_{x\to 0} \frac{\cos(\sin x)-\cos x}{x^4} =$

(A) 1 (B) 6 (C) $-\frac{1}{6}$ (D) $\frac{1}{6}$

Sol. **[D]**

$\lim\limits_{x\to 0} \frac{\cos(\sin x)-\cos x}{x^4}$

Using expansion of cosx

$\lim\limits_{x\to 0} \frac{\left(1-\frac{\sin^2 x}{2!}+\frac{\sin^4 x}{4!}\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)}{x^4}$

using expansion of sinx

$\lim\limits_{x\to 0} \frac{-\frac{\left(x-\frac{x^3}{3!}\right)^2}{2!}+\frac{x^2}{2!}}{x^4} + \lim\limits_{x\to 0} \frac{\sin^4 x}{4!\,x^4}$

$- \lim\limits_{x\to 0} \frac{x^4}{4!\,x^4}$

$\lim\limits_{x\to 0} \frac{-\frac{x^2}{2!}-\frac{x^6}{(3!)^2(2!)}+\frac{2x^4}{2!3!}+\frac{x^2}{2!}}{x^4} + \frac{1}{4!} - \frac{1}{4!}$

$\lim\limits_{x\to 0} \frac{-x^2}{(3!)^2(2!)} + \frac{2}{2!3!} = \frac{1}{6}$

Q.20 Let $a = \min\{x^2 + 2x + 3, x \in R\}$ and $b = \lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}$. Then $\sum_{r=0}^{n} a^r.b^{n-r} =$

(A) $2^{n+1} + 1$ (B) $2^{n+1} - 1$

(C) $\frac{4^{n+1}-1}{3.2^n}$ (D) $\frac{4^{n+1}+1}{3.2^n}$

Sol. [C]

Q.21 $\lim\limits_{x\to a^-} \sqrt{a^2-x^2}\ \cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$ is-

(A) $\frac{a}{\pi}$ (B) $\frac{2a}{\pi}$ (C) $-\frac{a}{\pi}$ (D) $\frac{4a}{\pi}$

Sol. **[D]**

$\lim\limits_{x\to a^-} \sqrt{a^2-x^2}\ \cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$

Let $x = a\cos\theta$

as $x \to a^-$

$\theta \to 0^+$

$\lim\limits_{\theta\to 0^+} \sqrt{a^2-a^2\cos^2\theta}\ \cot\left(\frac{\pi}{2}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\right)$

$\lim\limits_{\theta\to 0^+} a\sin\theta \cot\left(\frac{\pi}{2}\tan\frac{\theta}{2}\right)$

$\lim\limits_{\theta\to 0^+} a\sin\theta \frac{\cos\left(\frac{\pi}{2}\tan\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{2}\tan\frac{\theta}{2}\right)}$

$\lim\limits_{\theta\to 0^+} \frac{a\sin\theta}{\theta} . \frac{\frac{\pi}{2}\tan\frac{\theta}{2}}{\sin\left(\frac{\pi}{2}\tan\frac{\theta}{2}\right)} . \frac{\frac{\theta}{2}.2}{\frac{\pi}{2}\tan\frac{\theta}{2}}$

$a . 1 . \frac{4}{\pi} = \frac{4a}{\pi}$

Q.22 $\lim\limits_{n\to\infty} n\lambda n\left(e\left(1+\frac{1}{n}\right)^{1-n}\right) =$

(A) 1 (B) $\frac{3}{2}$

(C) $\frac{2}{3}$ (D) None

Sol. **[B]**

$\lim\limits_{n\to\infty} n\ \lambda\ n\left(e\left(1+\frac{1}{n}\right)^{1-n}\right)$

$\lim\limits_{n\to\infty} n\ [1 + (1-n)\ \lambda n\ (1+\frac{1}{n})]$

$\lim\limits_{n\to\infty} [n + (n-n^2)]\lambda n(1+\frac{1}{n})]$

(using expansion of $\lambda n\ (1+x)$)

$$\lim_{n\to\infty}\left[n+(n-n^2)\left(\frac{1}{n}-\frac{1}{2n^2}+.....\right)\right]$$

$$\lim_{n\to\infty}\left[n+\left(1-n-\frac{1}{2n}+\frac{1}{2}+.....\right)\right]$$

$$1-0+\frac{1}{2}+0\,.........$$

$$1+\frac{1}{2}=\frac{3}{2}$$

Q.23 The value of $\lim_{x\to 0}\left(1-\frac{1}{2^x}\right)\left(\frac{1}{\sqrt{\tan x+4}-2}\right)$ is-

(A) $\log_a 16$ (B) Does not exist

(C) 3 ln 2 (D) 4 ln 2

Sol. **[D]**

$$\lim_{x\to 0}\left(1-\frac{1}{2^x}\right)\left(\frac{1}{\sqrt{\tan x+4}-2}\right)\ \left(\frac{0}{0}\text{ form}\right)$$

Apply L – H Rule, we get

$$=\lim_{x\to 0}\ 2^{-x}\log 2\Big/\frac{1}{2\sqrt{\tan x+4}}\ \sec^2 x$$

$$=\log 2/\frac{1}{4}=4\log 2.$$

∴ Option (D) is correct Answer.

Q.24 $\lim_{x\to\infty}\frac{\lambda n\ x-[x]}{[x]}=$ ([.] → G. I. F.)

(A) 0 (B) –1

(C) ∞ (D) None of these

Sol. **[B]**

$$\lim_{x\to\infty}\frac{\lambda nx-[x]}{[x]}$$

since $[x]\le x$

$$\lim_{x\to\infty}\frac{\lambda nx-x}{x}\quad(\infty/\infty\text{ form})$$

Apply L – H Rule, we get

$$=\lim_{x\to\infty}\frac{\frac{1}{x}.1-1}{1}=\frac{0-1}{1}=-1$$

∴ Option (B) is correct Answer.

Part-B One or more than one correct answer type questions

Q.25 Which of the following limits tends to unity?

(A) $\lim_{t\to 0}\frac{\sin(\tan t)}{\sin t}$

(B) $\lim_{x\to\pi/2}\frac{\sin(\cos x)}{\cos x}$

(C) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(D) $\lim_{x\to 0}\frac{\sqrt{x^2}}{x}$

Sol. **[A, B, C]**

(A) $\lim_{t\to 0}\frac{\sin(\tan t)}{\sin t}$

$$\lim_{t\to 0}\frac{\sin(\tan t)}{\tan t}\cdot\frac{\tan t}{t.}\cdot\frac{t}{\sin t}=1$$

(B) $\lim_{x\to\pi/2}\frac{\sin(\cos x)}{\cos x}=1$

(C) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$\lim_{x\to 0}\frac{(1+x)-(1-x)}{x(2)}\Rightarrow\lim_{x\to 0}\frac{2x}{2x}=1$$

(D) $\lim_{x\to 0}\frac{\sqrt{x^2}}{x}=\lim_{x\to 0}\frac{|x|}{x}$

L.H.L. = – 1, R.H.L. = +1

Limit does not exist

Q.26 Consider the function $f(x)=\left(\frac{ax+1}{bx+2}\right)^x$ where $a^2+b^2\ne 0$ and $a>0$ & $b>0$ then $\lim_{x\to\infty}f(x)$

(A) exists for all values of a and b

(B) is zero for $0<a<b$

(C) is non existent for $a>b$

(D) is $e^{-\left(\frac{1}{a}\right)}$ or $e^{-\left(\frac{1}{b}\right)}$ if $a=b$

Sol. **[B, C, D]**

$$f(x)=\left(\frac{ax+1}{bx+2}\right)^x\quad a^2+b^2\ne 0$$

$$\lim_{x\to\infty}f(x)$$

$$=\lim_{x\to\infty}\left(\frac{ax+1}{bx+2}\right)^x$$

$$=\lim_{x\to\infty}\left(\frac{a+1/x}{b+2/x}\right)^x$$

if $a<b$

$$\lim_{x\to\infty}\left(\frac{a+1/x}{b+2/x}\right)^x = \left(\frac{a}{b}\right)^{\infty} = 0$$

if a > b

$$\lim_{x\to\infty}\left(\frac{a+1/x}{b+2/x}\right)^x = \left(\frac{a}{b}\right)^{\infty} \to \infty$$

$\Rightarrow$ non existent

if a = b

$$\lim_{x\to\infty}\left(\frac{a+1/x}{b+2/x}\right)^x \quad (1^{\infty})$$

$$e^{\lim_{x\to\infty} x\left(\frac{ax+1}{bx+2}-1\right)}$$

$$= e^{\lim_{x\to\infty} x\left(\frac{(a-b)x-1}{bx+2}-1\right)}$$

$$= e^{\lim_{x\to\infty} \frac{-1}{b+2/x}}$$

$= e^{-1/b} = e^{-1/a}$ ($\because$ a = b)

Q.27 $\lim_{x\to c} f(x)$ does not exist when-

(A) f(x) = [[x]] – [2x –1], c = 3

(B) f(x) = [x] – x, c = 1

(C) f(x) = $\{x\}^2 - \{-x\}^2$, c = 0

(D) $f(x) = \dfrac{\tan(\text{sgn } x)}{\text{sgn } x}$, c = 0

Where [.] denotes greatest integer function & {x} fractional part function.

Sol. **[B, C]**

(A) $\lim_{x\to 3} [[x]] - [2x-1]$

L.H.L.

$\lim_{x\to 3^-} [[x]] - [2x] + 1$

2 – 5 + 1 = – 2

R.H.L.

$\lim_{x\to 3^+} [[x]] - [2x] + 1$

3 – 6 + 1 = – 2

L.H.L. = R.H.L. $\Rightarrow$ Limit exists

(B) $\lim_{x\to 1} [x] - x$

L.H.L.

$\lim_{x\to 1^-} [x] - x$

0 – 1 = – 1

R.H.L.

$\lim_{x\to 1^+} [x] - x$

1 – 1 = 0

L.H.L. $\neq$ R.H.L. $\Rightarrow$ Limit does not exist

(C) $\lim_{x\to 0} \{x\}^2 - \{-x\}^2$

L.H.L.

$\lim_{x\to 0^-} \{x\}^2 - \{-x\}^2$

1 – 0 = 1

R.H.L.

$\lim_{x\to 0^+} \{x\}^2 - \{-x\}^2 = 0 - 1 = -1$

R.H.L. $\neq$ L.H.L. $\Rightarrow$ Limit does not exist.

(D) $\lim_{x\to 0} \dfrac{\tan(\text{sgn } x)}{\text{sgn}(x)}$ L.H.L. = $\lim_{x\to 0^-} \dfrac{\tan(\text{sgn } x)}{\text{sgn}(x)}$

= tan 1

R.H.L. = $\lim_{x\to 0^+} \dfrac{\tan(\text{sgn } x)}{\text{sgn}(x)}$ = tan 1

L.H.L. = R.H.L.

$\Rightarrow$ Limit exists

Q.28 Identify the true statement(s).

(A) $\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right] = 1$, where [.] denotes the greatest integer function

(B) If f(x) = (x –1) {x}, then limit of f(x) does not exist at all integers except {1}

(C) $\lim_{x\to 0^+}\left[\frac{\tan x}{x}\right] = 1$, where [.] denotes the greatest integer function.

(D) $\left[\lim_{x\to 0^+}\frac{\tan x}{x}\right] = 1$, where [.] denote the greatest integer function

Sol. **[B, C, D]**

(A) $\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+........+\frac{1}{2^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]$$

$$= \lim_{n\to\infty}\left[1-\frac{1}{2^n}\right] = 0$$

(B) $f(x) = (x-1)\{x\}$

$\lim_{x\to 1} \{x\}(x-1)$

L.H.L. = $\lim_{x\to 1^-} (1)(0) = 0$

$\Rightarrow$ Limit exists

R.H.L. = $\lim_{x\to 1^+} (0)(0) = 0$

for any other integer except x = 1

L.H.L. ≠ R.H.L.

so limit does not exist

(C) $\lim_{x\to 0^+} \left[\frac{\tan x}{x}\right] = 1$ (as tan x > x)

(D) $\left[\lim_{x\to 0^+} \frac{\tan x}{x}\right] = 1$

Q.29 For a > 0, let $\lambda = \lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$ and

$m = \lim_{x\to -\infty}\left(\sqrt{x^2+ax} - \sqrt{x^2-ax}\right)$ then-

(A) λ is always greater than 'm' for all values of a > 0

(B) λ is always greater than 'm' only when a ≥ 1

(C) λ is always greater than 'm' for all values of 'a' satisfying $a > e^{-a}$

(D) $e^{\lambda} + m = 0$

Sol. a > 0

$$\lambda = \lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

$$\lambda = \lim_{x\to\frac{\pi}{2}} a^{\cos x}\left[\frac{a^{\cot x-\cos x}-1}{\cot x - \cos x}\right] = \lambda n\, a$$

$$m = \lim_{x\to\infty}\left(\sqrt{x^2+ax} - \sqrt{x^2-ax}\right) \times \frac{\left(\sqrt{x^2+ax}+\sqrt{x^2-ax}\right)}{\left(\sqrt{x^2+ax}+\sqrt{x^2-ax}\right)}$$

$$= \lim_{x\to\infty} \frac{(x^2+ax)-(x^2-ax)}{\sqrt{x^2+ax}+\sqrt{x^2-ax}}$$

$$= \lim_{x\to\infty} \frac{2ax}{x\left(\sqrt{1+\frac{a}{x}}+\sqrt{1-\frac{a}{x}}\right)} = \frac{2a}{2} = a$$

λ = λn a, m = a

λ > m

λn a > a

$a > e^a$

this is never true for a > 0

so m > λ always

Q.30 If f(x) = sinx + cosx, [x] is the greatest integer function, then

(A) $\lim_{x\to 0^-} [f(x)] = 0$

(B) $\lim_{x\to(2n\pi+\pi/2)^-} [f(x)] = 1$ (n ∈ I)

(C) $\lim_{x\to 2n\pi} [f(x)] = 0$, (n ∈ I)

(D) Range of f(x) is {–2, –1, 0, 1}

Sol. **[A, B]**

f(x) = sinx + cosx

sinx + cosx = $\sqrt{2}\sin(x+\pi/4)$

we have to check for every options as :

For A : $\lim_{x\to 0^-} [f(x)]$ means L.H.L

0–h 0 0+h

$$\text{L.H.L} = \lim_{h\to 0^-}\left[\sqrt{2}\sin\left(\frac{\pi}{4}+0-h\right)\right]$$

$$= \lim_{h\to 0^-}\left[\sqrt{2}\sin\left(\frac{\pi}{4}-h\right)\right]$$

(∵ f(x) is equal to 1 at x = 1)

$= \lim_{h\to 0^-} [\sqrt{2} \times \text{value less than } 1/\sqrt{2}\,]$

$= \lim_{h\to 0^-}$ [Value less than 1]

= 0

∴ Option (A) is correct Answer.

For Option (B) :

$\lim_{x\to\left(2n\pi+\frac{\pi}{2}\right)^-} [f(x)]$ (n ∈ I)

$2n\pi + \frac{\pi}{2} - h$ $2n\pi + \frac{\pi}{2}$ $2n\pi + \frac{\pi}{2} + h$

$$\text{L.H.L} = \lim_{h\to 0^-}\left[\sqrt{2}\sin\left(\frac{\pi}{4}+2n\pi+\frac{\pi}{2}-h\right)\right];\ n\in I$$

$$= \lim_{h\to 0^-}\left[\sqrt{2}\sin\left(\frac{\pi}{2}+2n\pi+\frac{\pi}{4}-h\right)\right];\ n\in I$$

$$= \lim_{h\to 0^-}\left[\sqrt{2}\cos\left(2n\pi+\frac{\pi}{4}-h\right)\right];\ n\in I$$

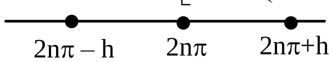
$$= \lim_{h\to 0^-}\left[\sqrt{2}\cos\left(\frac{\pi}{4}-h\right)\right]$$

$$= \lim_{h\to 0^-}\ [\sqrt{2}\ \text{value greater than } \frac{1}{\sqrt{2}}]$$

$$= \lim_{h\to 0^-}\ [\text{Value greater than 1}] = 1$$

∴ Option (B) is correct Answer.

Option (C) : $\lim_{x\to 2n\pi}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]$;



$$\text{L.H.L} = \lim_{h\to 0^-}\left[\sqrt{2}\sin(2n\pi - h + \pi/4)\right]$$

; n ∈ Integer

$$= \lim_{h\to 0^-}\left[\sqrt{2}\sin\left(\frac{\pi}{4}-h\right)\right]$$

$$= \lim_{h\to 0^-}\ [\sqrt{2}\ \times \text{value less than } \frac{1}{\sqrt{2}}]$$

= [Value less than 1] = 0

$$\text{R.H.L} = \lim_{h\to 0^+}\left[\sqrt{2}\sin\left(2n\pi+\frac{\pi}{4}+h\right)\right]$$

; n ∈ Integer

$$= \lim_{h\to 0^+}\left[\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)\right]$$

$$= \left[\sqrt{2}\times \text{value greater than } \frac{1}{\sqrt{2}}\right]$$

= [Value greater than 1] = 1

Hence, limit does not exist

∴ Option (C) is not correct

For D : $y = \sqrt{2}\sin(x+\pi/4)$

$\Rightarrow \sin(x + \pi/4) = y/\sqrt{2}$

$\Rightarrow x + \pi/4 = \sin^{-1}y/\sqrt{2}$

$\Rightarrow x = \sin^{-1}y/\sqrt{2} - \pi/4$

x to be defined if $-1 \le y/\sqrt{2} \le 1$

$\Rightarrow -\sqrt{2} \le y \le \sqrt{2}$

Q.31 If $A_i = \frac{x-a_i}{|x-a_i|}$, i = 1, 2, 3,....n and if $a_1 < a_2 < a_3 < ...< a_n$ and 1 < m < n then -

(A) $\lim_{x\to a_m^+} (A_1 A_2... A_n) = (-1)^{n-m+1}$

(B) $\lim_{x\to a_m^+} (A_1 A_2... A_n) = (-1)^{n-m}$

(C) $\lim_{x\to a_m^-} (A_1 A_2... A_n) = (-1)^{n-m+1}$

(D) $\lim_{x\to a_m^-} (A_1 A_2... A_n) = (-1)^{n-m}$

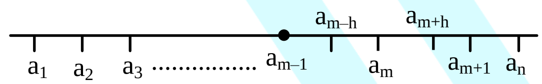
Sol. **[B, C]**

$A_i = \frac{x-a_i}{|x-a_i|}$; i = 1, 2, 3 ...., n

and $a_1 < a_2 < a_2 < a_3 <$ .... $< a_n$

$A_1A_2A_3$ .... A_n =

$$\frac{(x-a_1)(x-a_2)(x-a_3)........(x-a_n)}{|(x-a_1)(x-a_2)(x-a_3)......(x-a_n)|}$$



L.H.L

$$= \lim_{h\to 0^-} f(0-h) = \lim_{h\to 0^-}$$

$$\frac{(0-h-a_1)(0-h-a_2)(0-h-a_3)..........(0-h-a_n)}{|(0-h-a_1)(0-h-a_2)(0-h-a_3).........(0-h-a_n)|}$$

$$= \lim_{h\to 0^-}\frac{(h+a_1)(h+a_2)(h+a_3)........(h+a_n)(-1)^n}{|(h+a_1)(h+a_2)(h+a_3)............(h+a_n)|}$$

$$= \lim_{h\to 0^-}\frac{(h+a_1)(h+a_2)(h+a_3)........(h+a_n)(-1)^n}{(h+a_1)(h+a_2)(h+a_3)............(h+a_n)(-1)^{m-1}(+1)^{n-m+1}}$$

$= (-1)^{n-m+1}$

$$\text{R.H.L} = \lim_{h\to 0^+} f(0+h)$$

$$= \lim_{h\to 0^+}\frac{(0+h-a_1)(0+h-a_2).........(0+h-a_n)}{|(0+h-a_1)(0+h-a_2)........(0+h-a_n)|}$$

$$= \lim_{h\to 0^+}\frac{(h-a_1)(h-a_2)..........(h-a_n)}{|(h-a_1)(h-a_2)........(h-a_n)|}$$

$$= \lim_{h\to 0^+}\frac{(h-a_1)(h-a_2)..........(h-a_n)}{(h-a_1)(h-a_2)........(h-a_n)(+1)^m(-1)^{n-m}}$$

$= (-1)^{n-m}$

Q.32 If f(x) = |x – 1|– [x], where [x] is the greatest integer less than or equal to x, then-

(A) f(1 + 0) = –1, f(1 – 0) = 0

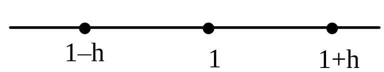
(B) f(1 + 0) = 0 = f(1 – 0)

(C) $\lim_{x\to 1} f(x)$ exists

(D) $\lim_{x\to 1} f(x)$ does not exist

Sol. **[A, D]**

f(x) = |x – 1| – [x]



L.H.L = $\lim_{h\to 0^-} f(1-h)$

$= \lim_{h\to 0^-} \{|1-h-1| - [1-h]\}$

$= \lim_{h\to 0^-} \{|-h| - [\text{Value less than } 1]\}$

$= 0 - 0$

$= 0$

R.H.L = $\lim_{h\to 0^+} f(1+h)$

$= \lim_{h\to 0^+} \{|1+h-1| - [1+h]\}$

$= \lim_{h\to 0^+} \{|h| - [\text{Value greater than } 1]\}$

$= \lim_{h\to 0^+} \{h-1\} = -1$

Since limit does not exist.

$\therefore$ Option (A) & (D) are correct Answers.

Part-C Assertion-Reason type Questions

The following questions 33 to 34 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.33 **Assertion** : The value of $\lim_{x\to\pi/2} (\sin x)^{\tan x}$ is e.

Reason : $\lim_{x\to a} (1+f(x))^{g(x)}$ is $e^{\lim_{x\to a} g(x) f(x)}$.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$.

Sol. **[D]**

Assertion : $\lim_{x\to\pi/2} (\sin x)^{\tan x}$. It is $(1)^\infty$ Type

$= e^{\lim_{x\to\pi/2} (\sin x - 1)\times\tan x}$

$= e^{\lim_{x\to\pi/2} \frac{\sin x - 1}{\cot x}}$

$= e^{\lim_{x\to\pi/2} \frac{\cos x}{-\mathrm{cosec}^2 x}}$

$= e^0 = 1$

Hence, Assertion is False.

Reason : $\lim_{x\to a} (1+f(x))^{g(x)} = e^{\lim_{x\to a} f(x).g(x)}$

as $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) \to \infty$

It is true. Hence, Reason is correct

$\therefore$ Option (D) is correct

Q.34 **Assertion** : $\lim_{x\to 0^+} x\sin\frac{1}{x} = 1.$

Reason : $\lim_{y\to\infty} y\sin\frac{1}{y} = 1.$

Sol. **[D]**

Assertion : $\lim_{x\to 0^+} x\sin\frac{1}{x} = 1$

$\sin\frac{1}{x} = \frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \ldots\ldots\ldots$

$x\sin\frac{1}{x} = x\left(\frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \ldots\ldots\right)$

$= 1 - \frac{1}{3!\,x^2} + \frac{1}{5!x^5} - \ldots\ldots\ldots.$

$\therefore \lim_{x\to 0^+} x\sin\frac{1}{x}$ does not exist

$\therefore$ Assertion is wrong

Reason : $\lim_{y\to\infty} y\sin\frac{1}{y} = 1$

Put $x = \frac{1}{y}$

As $y\to\infty,\ x\to 0$

$\lim_{y\to\infty} y\sin\frac{1}{y} = \lim_{x\to 0}\frac{\sin x}{x} = 1$

$\therefore$ Reason is correct

$\therefore$ Option (D) is correct Answer.

Part-D Column Matching type questions

Match the entry in Column 1 with the entry in Column 2.

Q.35

Column 1	**Column 2**
(A) $\lim_{x\to\frac{\pi}{2}} [\sin^{-1}(\sin x)]$	(P) –2

(B) $\lim_{x\to -\infty} [\tan^{-1}x]$ (Q) 0

(C) $\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$ (R) 1

(D) $\lim_{x\to 0^+} \left[\frac{\sin|x|}{x}\right]$ (S) does not exist

Sol. **A – R, B – P, C – S, D – Q**

(A) $\lim_{x\to \pi/2} [\sin^{-1}(\sin x)]$

$$\text{L.H.L} = \lim_{h\to 0^-}\left[\sin^{-1}\left(\sin\left(\frac{\pi}{2}-h\right)\right)\right]$$

$$= \lim_{h\to 0^-}\left[\frac{\pi}{2}-h\right] = 1$$

$$\text{R.H.L} = \lim_{h\to 0^+}\left[\sin^{-1}\left(\sin\left(\frac{\pi}{2}+h\right)\right)\right]$$

$$= \lim_{h\to 0^+}\left[\frac{\pi}{2}+h\right] = 1$$

(B) $\lim_{x\to -\infty} [\tan^{-1}x] = [-1.57] = -2$

(C) $\lim_{x\to \pi/4} \frac{\sqrt{1-\sqrt{\sin 2x}}}{(\pi-4x)} = \frac{\sqrt{1\pm 1}}{0}$

$$= \infty \text{ or } \frac{0}{0}$$

$\therefore$ Limit does not exist.

or other method

$$\lim_{x\to \pi/4} \frac{\sqrt{1-\sqrt{\sin 2x}}}{(\pi-4x)} \times \frac{\sqrt{1+\sqrt{\sin 2x}}}{\sqrt{1+\sqrt{\sin 2x}}}$$

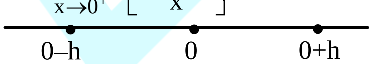
$$= \lim_{x\to \pi/4} \frac{\sqrt{1-\sin 2x}}{(\pi-4x)\sqrt{1+\sqrt{\sin 2x}}}$$

$$= \lim_{x\to \pi/4} \frac{|\sin x - \cos x|}{(\pi-4x)\sqrt{1+\sqrt{\sin 2x}}}$$

$$= \sqrt{2} \times \lim_{x\to \pi/4} \frac{|\sin(x-\pi/4)|}{(\pi-4x)\sqrt{1+\sqrt{\sin 2x}}}$$

Hence, from above, limit does not exit.

(D) $\lim_{x\to 0^+} \left[\frac{\sin|x|}{x}\right]$



$$= \lim_{h\to 0^+}\left[\frac{\sin|0+h|}{0+h}\right]$$

$$= \lim_{h\to 0^+}\left[\frac{\sin h}{h}\right]$$

$\sin h = h - h^3/3! + h^5/5! - \ldots\ldots\ldots$

$\frac{\sin h}{h} = 1 - h^2/3! + h^4/5! - \ldots\ldots$

$\lim_{h\to 0^+} \frac{\sin h}{h}$ = Value less than 1

Hence $\lim_{h\to 0^+}\left[\frac{\sin h}{h}\right]$ = [Value less than 1] = 0

Q.36 Find (a, b, c) if

Column 1

(A) $\lim_{x\to 0} \frac{a\sin x - bx + cx^2 + x^3}{2x^2\ln(1+x) - 2x^3 + x^4}$ = Finite

(B) $\lim_{x\to 0} \frac{\sin x + ae^x + be^{-x} + c\ln(1+x)}{x^3}$ = Finite

(C) $\lim_{x\to 0} \frac{axe^x - b\log(1+x) + c\,x\,e^{-x}}{x^2\sin x} = 2$

Column 2

(P) (3, 12, 9)
(Q) (–4, 3, any real no.)
(R) (6, 6, 0)
(S) $\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$

Sol. **A – R, B – S, C – P**

(A) $\lim_{x\to 0} \frac{a\sin x - bx + cx^2 + x^3}{2x^2\ln(1+x) - 2x^3 + x^4}$ = finite

$$\lim_{x\to 0} \frac{a(x - x^3/6 + x^5/120 - \ldots\ldots) - bx + cx^2 + x^3}{2x^2(x - x^2/2 + x^3/3\ldots\ldots) - 2x^3 + x^4}$$

= finite

$$\lim_{x\to 0} \frac{x(a-b) + cx^2 + x^3\left(-\frac{a}{6}+1\right) + \frac{a}{120}x^5\ldots.}{\frac{2}{3}x^5}$$

= finite

If limit be finite, then

coefficient of $x = 0 \Rightarrow$ a = b

coefficient of $x^2 = 0 \Rightarrow c = 0$

coefficient of $x^3 = 0 \Rightarrow a = 6$

then limit will be finite.

$\because \frac{a/120}{2/3}$ = finite $\Rightarrow a = k = 6$

(k, k, Any real number) or (6, 6 Any real No.)

(B) $\lim_{x\to 0} \frac{\sin x + ae^x + be^{-x} + c\ln(1+x)}{x^3}$ = finite

$$\lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+.....\right)+a\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+...\right)+b\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+...\right)+c\left(x-\frac{x^3}{3}+\frac{x^5}{5}-........\right)}{x^3}$$

= finite

$$\lim_{x\to 0}\frac{x(1+a-b+c)+(a+b)+x^2\left(\frac{a}{2}+\frac{b}{2}\right)+x^3\left(-\frac{1}{6}+\frac{a}{6}-\frac{b}{6}-\frac{c}{3}\right)+........}{x^3}$$

= finite

If limit be finite then

Coefficient of constant = 0 i.e.

a + b = 0

coefficient of x = 0 ⇒ 1 + a – b + c = 0

coefficient of x^2 = 0 ⇒ a + b = 0

then (–1/6 + a/6 – b/6 – c/3) = finite

⇒ – 1 + a – b – 2c = finite

⇒ 1 – 1 + a – b – 1 + c – c – 2c = finite

⇒ 1 + a – b + c – 2 – 3c = finite

$\Rightarrow\ 0-2-3c = \text{finite} \Rightarrow c = \frac{\text{finite}}{-3}$

c = finite

a – b = finite

a + b = 0

$\Rightarrow\ 2a = \text{finite} \Rightarrow a = \frac{\text{finite}}{2}$

$b = -\frac{\text{finite}}{2}$

(C) $\lim_{x\to 0}\frac{axe^x - b\log(1+x)+cxe^{-x}}{x^2\sin x} = 2$

$$\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+....\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....\right)+cx\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+.....\right)}{x^2\left(x-x^3/6+......\right)}$$

= 2

$$\lim_{x\to 0}\frac{x(a-b+c)+x^2(a+b/2-c)+x^3\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)+.....}{x^3} = 2$$

If limit be finite it must be

coefficient of x= 0 ⇒a – b + c = 0

coefficient of x^2=0 ⇒ a + b/2 – c = 0

and $\frac{a}{2}-\frac{b}{3}+\frac{c}{2} = 2$

⇒ 3a – 2b + 3c = 12

a + b/2 = b – a ⇒ 2a = b/2 ⇒ b = 4a

c = b – a = 4a – a = 3a

3a – 8a + 9a = 12 ⇒ 4a = 12

⇒ a = 3

b = 12

c = 9

Q.37 **Column 1**

(A) $\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)...\left(n+\frac{1}{2^{n-1}}\right)\right\}^n$

(B) $\lim_{x\to\infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x$

(C) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

(D) $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$

Column 2

(P) $e^{-1/6}$

(Q) e

(R) e^6

(S) e^2

Sol. **A –S, B – R, C – P, D – Q**

(B) $\lim_{x\to\infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x$

$\frac{x^2+4x-3}{x^2-2x+5} = \left(1+\frac{6x-8}{x^2-2x+5}\right)$

$\therefore\ \lim_{x\to\infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x =$

$\lim_{x\to\infty}\left(1+\frac{6x-8}{x^2-2x+5}\right)^x$ it is $(1)^\infty$ type

$= e^{\lim_{x\to\infty}\frac{6x-8}{x^2-2x+5}x}$

$= e^{\lim_{x\to\infty}\frac{(6-8/x)x^2}{x^2(1-2/x+5/x^2)}} = e^6$

(C) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$ it is $(1)^\infty$ type

$\lim_{x\to 0}\left(\frac{\sin x}{x}-1+1\right)^{1/x^2}$

$= e^{\lim_{x\to 0}\frac{\sin x - x}{x}\times\frac{1}{x^2}}$

$$e^{\lim_{x\to 0}\frac{\left(x-x^3/6+....-x\right)}{x^3}} = e^{-1/6}$$

(D) $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$

Put $x = \frac{1}{y}$ so that $x \to \infty$ as $y \to 0$

$\lim_{y\to 0}(\sin y+\cos y)^{1/y}$. It is $(1)^{\infty}$ type

$\therefore \lim_{y\to 0}(\sin y + \cos y - 1 + 1)^{1/y}$

$= e^{\lim_{y\to 0}(\sin y+\cos y-1)\times\frac{1}{y}}$

$= e^{\lim_{y\to 0}(\cos y-\sin y).\frac{1}{1}}$

$= e$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Evaluate :

$$\lim_{n\to\infty} \frac{1.n + 2(n-1) + 3(n-2) + \ldots\ldots + n.1}{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}$$

Sol. $$\lim_{n\to\infty} \frac{1.n + 2(n-1) + 3(n-2) + \ldots\ldots + n.1}{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} r.(n-r+1) \Big/ \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} n.r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 1}{\frac{n(n+1)(2n+1)}{6}}$$

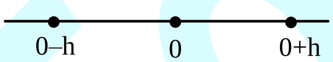
$$= \lim_{n\to\infty} \frac{\frac{n.n.(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + n}{\frac{n(n+1)(2n+1)}{6}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^3}{2}\left(1+\frac{1}{n}\right) - \frac{n(n+1)(2n+1)}{6} + n}{\frac{n^3(1+1/n)(2+1/n)}{6}}$$

$$= \frac{6}{2} \times \frac{1}{2} - 1 = \frac{3}{2} - 1 = \frac{1}{2} \text{ Ans.}$$

Q. 2 Evaluate : $\lim_{x\to 0} x.\frac{e^{[x]+|x|} - 2}{[x]+|x|} = ?$

Sol. L.H.L $= \lim_{h\to 0^-} (0-h)\frac{e^{[0-h]+|0-h|} - 2}{[0-h]+|0-h|}$



$$= \lim_{h\to 0^-}(-h)\frac{e^{-1-h} - 2}{-h-h} = \frac{e^{-1} - 2}{2}$$

R.H.L $= \lim_{h\to 0^+} (0+h)\frac{e^{[0+h]+|0+h|} - 2}{[0+h]+h}$

$$= \lim_{h\to 0^+} h\frac{e^{0+h} - 2}{2h} = \frac{1-2}{2} = -1/2.$$

Since. R.H.L ≠ L.H.L
Hence, limit does not exist.

Q. 3 $\lim_{x\to 0} \frac{8}{x^8}\left[1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right] =$

Sol. $$\lim_{x\to 0} \frac{8}{x^8}\left[1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\cos\frac{x^2}{2} = 1 - \frac{x^4}{8} + \frac{x^8}{16\times 24} \ldots\ldots\ldots\ldots\ldots$$

$$\cos\frac{x^2}{4} = 1 - \frac{x^4}{16\times 2} + \frac{x^8}{16\times 16\times 24} \ldots\ldots$$

$$1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} = 1 - 2 + \frac{5}{32}x^4 - \frac{17}{16\times 16\times 24}x^8 + \ldots\ldots$$

$$\cos\frac{x^2}{2}.\cos\frac{x^2}{4} = \left(1 - \frac{x^4}{8} + \frac{x^8}{16\times 24}\ldots\ldots\right)\left(1 - \frac{x^4}{32} + \frac{x^8}{16\times 16\times 24}\ldots\ldots\right)$$

$$= 1 - \frac{x^4}{8} + \frac{x^8}{16\times 24} - \frac{x^4}{32} + \frac{x^8}{8\times 32} + \frac{x^8}{16\times 16\times 24}$$

$$= 1 - \frac{5}{32}x^4 + x^8 \times \frac{17}{16\times 16\times 24} + \frac{x^8}{8\times 32}$$

$$\Rightarrow 1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4} = \frac{x^8}{8\times 32} + \ldots\ldots$$

$$\lim_{x\to 0} \frac{8}{x^8} \times \left(\frac{x^8}{8\times 32} + \ldots\ldots\right) = \frac{1}{32} \text{ Ans.}$$

Q. 4 Let $f(x) = \frac{\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$, then find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$, where {x} denotes the fractional part of x.

Sol. $$f(x) = \frac{\sin^{-1}(1-x+[x])\cos^{-1}(1-x+[x])}{\sqrt{2(x-[x])}(1-x+[x])}$$

$\lim_{x\to0^+} f(x) =$

$$\lim_{h\to0^+}\frac{\sin^{-1}(1-(0+h)+[0+h])\cos^{-1}(1-(0+h)+[0+h])}{\sqrt{2(0+h-[0+h])}(1-(0+h)+[0+h])}$$

$$= \lim_{h\to0^+}\frac{\sin^{-1}(1-h+0)\cos^{-1}(1-h+0)}{\sqrt{2(h-0)}(1-h+0)}$$

$$= \lim_{h\to0^+}\frac{\sin^{-1}(1-h)\times\cos^{-1}(1-h)}{\sqrt{2h}\times(1-h)}$$

$$= \lim_{h\to0^+}\frac{\sin^{-1}(1-h)}{(1-h)}\times\lim_{h\to0^+}\frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

$$= \pi/2\times\lim_{h\to0^+}\frac{\cos^{-1}(1-h)}{\sqrt{2h}} \qquad \left(\frac{0}{0}\text{form}\right)$$

Apply L – H Rule, we get

$$= \pi/2\times\lim_{h\to0^+}\frac{-1}{\sqrt{1-(1-h)^2}}\times(-1)\times\frac{1}{\frac{1}{2\sqrt{h}}}$$

$$= \frac{\pi}{2}\times\lim_{h\to0^+}\frac{1}{\sqrt{1-h^2+2h-1}}\times\frac{\sqrt{2h}}{1}$$

$$= \pi/2\times\lim_{h\to0^+}\frac{\sqrt{2}\times\sqrt{h}}{\sqrt{h}\sqrt{2-h}} = \pi/2 \text{ Ans.}$$

$\lim_{x\to0^-} f(x)$

$$\lim_{h\to0^-}\frac{\sin^{-1}(1-(0-h)+[0-h])\cos^{-1}(1-(0-h)+[0-h])}{\sqrt{2(0-h-[0-h])}\,(1-(0-h)+[0-h])} =$$

$$\lim_{h\to0^-}\frac{\sin^{-1}(1+h-1)\cos^{-1}(1+h-1)}{\sqrt{2(-h+1)}\,(1+h-1)}$$

$$= \lim_{h\to0^-}\frac{\sin^{-1}(h)\times\cos^{-1}(h)}{\sqrt{2(1-h)}\times(h)}$$

$$= \lim_{h\to0^-}\frac{\sinh}{h}\times\frac{1}{\sqrt{2}}\times\lim_{h\to0^-}\frac{\cos^{-1}h}{\sqrt{1-h}}$$

$$= 1\times\frac{1}{\sqrt{2}}\times\pi/2 = \pi/2\sqrt{2} \text{ Ans.}$$

Q. 5 $\lim_{n\to\infty}\prod_{r=2}^{n}\frac{r^3-1}{r^3+1} =$

(where Π stands for the product)

Sol. $$\lim_{n\to\infty}\prod_{r=2}^{n}\frac{r^3-1}{r^3+1} = \lim_{n\to\infty}\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdot\frac{5^3-1}{5^3+1}\cdot\frac{6^3-1}{6^3+1}\ \dots\dots\dots\ \frac{n^3-1}{n^3+1}$$

Since $\frac{2^3-1}{2^3+1} = \frac{(2-1)}{2+1}\cdot\frac{2^2+2.1+1^2}{2^2-2.1+1^2} = \frac{1}{3}\cdot\frac{7}{3}$

$$\frac{3^3-1}{3^3+1} = \frac{3-1}{3+1}\cdot\frac{3^2+3.1+1^2}{3^2-3.1+1^2} = \frac{2}{4}\cdot\frac{13}{7}$$

$$\frac{4^3-1}{4^3+1} = \frac{4-1}{4+1}\cdot\frac{4^2+4.1+1^2}{4^2-4.1+1} = \frac{3}{5}\cdot\frac{21}{13}$$

$$\frac{5^3-1}{5^3+1} = \frac{5-1}{5+1}\cdot\frac{5^2+5.1+1^2}{5^2-5.1+12} = \frac{4}{6}\cdot\frac{31}{21}$$

$$\frac{6^3-1}{6^3+1} = \frac{6-1}{6+1}\cdot\frac{6^2+6.1+1^2}{6^2-6.1+1^2} = \frac{5}{7}\times\frac{43}{31}$$

$$\frac{n^3-1}{n^3+1} = \frac{n-1}{n+1}\cdot\frac{n^2+n.1+1^2}{n^2-n.1+1^2} \text{ ------------}$$

Hence, $\lim_{n\to\infty}\prod_{r=2}^{n}\frac{r^3-1}{r^3+1}$

$$= \lim_{n\to\infty}\frac{1.2.3.4\dots\dots..n-1}{3.4.5.6.7\dots\dots.n+1}\times\frac{7.13.21.31.43\dots.n^2+n+1}{3.7.13.21.31\dots.n^2-n+1}$$

$$= \lim_{n\to\infty}\frac{1.2.3.4\dots\dots\dots(n-1).n(n+1)}{3.4.5.6.7\dots.(n+1)\,n(n+1)}\times\lim_{n\to\infty}\frac{7.13.21.31.43\dots..(n^2+n+1)}{3.7.13.21.31\dots.(n^2-n+1)}$$

$$= \lim_{n\to\infty}\frac{2}{n(n+1)}\times\lim_{n\to\infty}\frac{(n^2+n+1)}{3}$$

$$= \lim_{n\to\infty}\frac{2}{3}\times\frac{n^2(1+1/n+1/n^2)}{n^2(1+1/n)} = 2/3.$$

Q.6 If $S_k = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{(k+1)^i}$, then find the value of $\sum_{k=1}^{n}k\,S_k$.

Sol. $$S_k = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{(k+1)^i}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{(k+1)}+\frac{1}{(k+1)^2}+\frac{1}{(k+1)^3}+\dots.+\frac{1}{(k+1)^n}\right)$$

{It is a G.P. with common ratio $\left(\frac{1}{k+1}\right)$}

$$= \lim_{n\to\infty} 1.\frac{1-\frac{1}{(k+1)^{n+1}}}{1-\frac{1}{k+1}}$$

$$= \frac{k+1}{k}.\left[1-\frac{1}{(k+1)^{n+1}}\right] = \frac{k+1}{k}$$

$$\sum_{k=1}^{n} k.S_k = \sum_{k=1}^{n} k.\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n}(k+1)$$

$$= \frac{n(n+1)}{2}+n = \frac{n}{2}(n+3)$$

Q.7 $\lim_{x\to\infty} \frac{2x^{1/2}+3x^{1/3}+4x^{1/4}+........+nx^{1/n}}{(2x-3)^{1/2}+(2x-3)^{1/3}+.......+(2x-3)^{1/n}}$

Sol. $\lim_{x\to\infty} = \frac{2x^{1/2}+3x^{1/3}+4x^{1/4}+........+nx^{1/n}}{(2x-3)^{1/2}+(2x-3)^{1/3}+.......+(2x-3)^{1/n}}$

$$\frac{x^{1/2}\left[2+3/x^{1/6}+4/x^{1/4}+......+n/x^{\frac{n-2}{2n}}\right]}{x^{1/2}\left[(2-3/x)^{1/2}+\frac{(2-3/x)^{1/3}}{x^{1/6}}+.....+\frac{(2-3/x)^{1/n}}{x^{\frac{n-2}{2n}}}\right]}$$

$$= \frac{[2+0+0+.......+0]}{[\sqrt{2}+0+0+......+0]} = \sqrt{2} \text{ Ans.}$$

Q.8 $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$ p, q $\in$ N

Sol. $\frac{p-q}{2}$

Q.9 Given f(x) = $\lim_{n\to\infty} \tan^{-1}(nx)$; g(x) = $\lim_{n\to\infty} \sin^{2n} x$

and sin (h(x)) = $\frac{1}{2}$ [cos π(g(x)+ cos (2f(x))].

Find the domain and range of h(x).

Sol. Domain, x ∈ R, Range x = $\frac{n\pi}{2}$; n ∈ I

Q.10 Evaluate:

$$\lim_{n\to\infty}\frac{n.1+(n-1)(1+2)+(n-2)(1+2+3)+....+1.\sum_{n=1}^{n} r}{n^4}$$

Sol.

$$\lim_{n\to\infty}\frac{n.1+(n-1)(1+2)+(n-2)(1+2+3)+....+1.\sum_{n=1}^{n} r}{n^4}$$

Its r^{th} term would be

$$(n+1-r)\sum_{k=1}^{r} k$$

Hence $\lim_{n\to\infty}\sum_{r=1}^{n}(n+1-r)\sum_{k=1}^{r} k/n^4$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}(n+1-r).\frac{r(r+1)}{2}/n^4$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left\{\frac{(n+1)(r^2+r)}{2}-\frac{1}{2}(r^3+r^2)\right\}/n^4$$

$$= \lim_{n\to\infty}\left\{\sum_{r=1}^{n}\left(\frac{n+1}{2}\right)(r^2+r)-\lim_{n\to\infty}\frac{1}{2}\sum_{r=1}^{n}(r^3+r^2)\right\}/n^4$$

$$= \lim_{n\to\infty}\frac{\left(\frac{n+1}{2}\right)\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]-\lim_{n\to\infty}\frac{1}{2}\left[\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}\right]}{n^4}$$

$$= \lim_{n\to\infty}\frac{n^4}{2}\frac{(1+1/n)(1+1/n)(2+1/n)}{6\times n^4}+\lim_{n\to\infty}\frac{n^3(1+1/n)^2}{n^4\times 4}$$

$$- \lim_{n\to\infty}\frac{1}{2}\times\frac{1}{4}\frac{n^4(1+1/n)^2}{n^4} -$$

$$\lim_{n\to\infty}\frac{1}{2\times 6}\frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{n^4}$$

$$= \frac{1}{6}+0-\frac{1}{8}-0 = \frac{1}{6}-\frac{1}{8} = \frac{1}{24} \text{ Ans.}$$

Q.11 Find the value of

$$\lim_{x\to 1}\frac{(1-x)(1-x^2).........(1-x^{2n})}{[(1-x)(1-x^2).........(1-x^n)]^2}$$

Sol. $\frac{2n!}{(n!)^2}$

Q.12 Find the value of $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$

Sol. $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$

$$\lim_{x\to 2}\frac{2^x+8.2^{-x}-6}{2^{-x/2}-2^{1-x}}\quad\left(\frac{0}{0}\text{form}\right)$$

use L– H Rule, we get

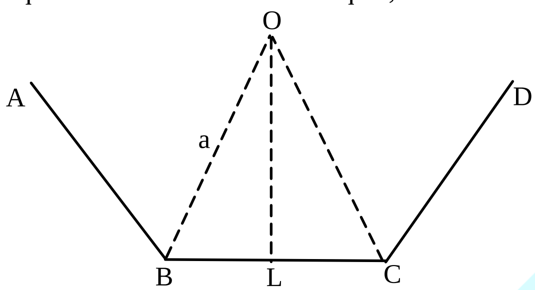
$$\lim_{x\to 2}\frac{2^x\log 2+8.2^{-x}\log 2(-1)-0}{2^{-\frac{x}{2}}\left(-\frac{1}{2}\right)\log 2-2.2^{-x}\log 2(-1)}$$

$$\lim_{x\to 2}\frac{2^x-8.2^{-x}}{-2^{-\frac{x}{2}-1}+2^{1-x}} = \frac{2^2-8/4}{-2^{-2}+2^{-1}}$$

$$= \frac{4-2}{-\frac{1}{4}+\frac{1}{2}} = \frac{2}{1/4} = 8 \text{ Ans}$$

Q.13 Find the area of a regular n-sided polygon inscribed in a circle of radius 'a'. Utilize the result to determine the area of the circle.

Sol. A regular polygon is that which have all angles equal as well as all sides are equal,



Each angle = $\frac{2n-4}{n}$ × right angle

$$= \frac{2n-4}{n} \times \frac{\pi}{2}$$

where n is Number of angles of Regular Polygon

$\therefore\ \angle BOC = \frac{\text{Four right angles}}{n} = (2\pi/n)$

$\angle BOL = \frac{1}{2}\angle BOC = \pi/n$

$\therefore \sin(\pi/n) = BL/a \Rightarrow BL = a\sin(\pi/n)$

$\Rightarrow BC = 2BL$

$= 2a \sin \pi/n$

$\cos\pi/n = \frac{OL}{a} \Rightarrow OL = a\cos\pi/n$

$\therefore$ Area of Δ BOC = $\frac{1}{2}$.OL.BC

$$= \frac{1}{2}\ a\cos\frac{\pi}{n}.2a\sin\pi/n$$

$$= \frac{a^2}{2}\sin 2\pi/n$$

$\therefore$ Area of Regular Polygon = n × Area of $\Delta BOC = n \times \frac{a^2}{2} \times \sin 2\pi/n$

Area of circle = $\lim_{n\to\infty} n \times \frac{a^2}{2} \times \sin 2\pi/n$

$$= \lim_{n\to\infty}\frac{a^2}{2}\times\frac{n}{2\pi}\times 2\pi\times\sin 2\pi/n$$

$$= \frac{a^2}{2}\times 2\pi\times\lim_{n\to\infty}\frac{\sin 2\pi/n}{(2\pi/n)}$$

$= \pi a^2 \times 1 = \pi a^2$ Ans.

Q.14 If $Z_n = \cos\frac{\pi}{(2n+1)(2n+3)} + i\sin\frac{\pi}{(2n+1)(2n+3)}$, then find the value of $\lim_{n\to\infty}(Z_1 Z_2 \ldots\ldots\ldots Z_n)$.

Sol. $Z_n = \cos\frac{\pi}{(2n+1)(2n+3)} + i\sin\frac{\pi}{(2n+1)(2n+3)}$

Then, $Z_1.Z_2.Z_3 \ldots\ldots Z_n$

$$= \left(\cos\frac{\pi}{3.5}+i\sin\frac{\pi}{3.5}\right)\left(\cos\frac{\pi}{5.7}+i\sin\frac{\pi}{5.7}\right)$$

$$\left(\cos\left(\frac{\pi}{7.9}\right)+i\sin\frac{\pi}{7.9}\right)\ldots\ldots\ldots$$

$$\ldots.\left(\cos\frac{\pi}{(2n+1)(2n+3)}+i\sin\frac{\pi}{(2n+1)(2n+3)}\right)$$

we can use $\cos\theta + i\sin\theta = e^{i\theta}$

Hence, $Z_1\ Z_2\ Z_3 \ldots\ldots\ldots\ldots\ldots\ldots Z_n$ =

$$= e^{i\frac{\pi}{3.5}}e^{i\frac{\pi}{5.7}}e^{i\frac{\pi}{7.9}}e^{i\frac{\pi}{9.11}}\ldots\ldots\ldots\ldots\ldots\ldots e^{i\pi/(2n+1)(2n+3)}$$

$$= e^{i\pi\left(\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\ldots..+\frac{1}{(2n+1)(2n+3)}\right)}$$

$$= e^{i\pi\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}}$$

Since, $\frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\left[\frac{1}{2r+1}-\frac{1}{(2r+3)}\right]$

$$= e^{i\pi\sum_{r=1}^{n}\frac{1}{2}\left[\frac{1}{2r+1}-\frac{1}{2r+3}\right]}$$

$T_n = \frac{1}{2}\left[\frac{1}{2n+1}-\frac{1}{2n+3}\right]$

$T_1 = \frac{1}{2}\left[\frac{1}{3}-\frac{1}{5}\right]$

$T_2 = \frac{1}{2}\left[\frac{1}{5}-\frac{1}{7}\right]$

$T_3 = \frac{1}{2}\left[\frac{1}{7}-\frac{1}{9}\right]$

$T_4 = \frac{1}{2}\left[\frac{1}{9}-\frac{1}{11}\right]$

....

....

$T_n = \frac{1}{2}\left[\frac{1}{2n+1}-\frac{1}{2n+3}\right]$

$T_1 + T_2 + T_3 + T_4 + \ldots\ldots\ldots\ldots + T_n$

$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right]$

$= \frac{n}{3(2n+3)}$

Therefore, $\lim\limits_{n\to\infty} (z_1 z_2 z_3 z_4 \ldots\ldots z_n) =$

$\lim\limits_{n\to\infty} e^{i\pi \frac{n}{3n(2+3/n)}}$

$= \lim\limits_{n\to\infty} e^{i\pi/6}$

$= \cos\pi/6 + i \sin\pi/6 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ Ans.

Q.15 If $\lim\limits_{x\to 0} (1 + ax + bx^2)^{2/x} = e^3$, then find all possible value of a & b.

Sol. $\lim\limits_{x\to 0} (1 + ax + bx^2)^{2/x} = e^3$

It is $(1)^\infty$ Type

Hence $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (ax + bx^2) = 0$

$f(x) = ax + bx^2$

& $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} (2/x) \to \infty$

$g(x) = 2/x$

$\therefore \lim\limits_{x\to 0} (1 + f(x))^{g(x)} = e^{\lim\limits_{x\to 0} f(x)\times g(x)}$

$= e^{\lim\limits_{x\to 0} (ax+bx^2)\times\frac{2}{x}} = e^3$

$= e^{\lim\limits_{x\to 0} 2(a+bx)} = e^3$

This holds for all $b \in R$

$e^{2a} = e^3$

$\Rightarrow a = 3/2$ & $b \in R$

Q.16 Evaluate : $\lim\limits_{x\to\infty}\left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots\ldots n^{1/x}}{n}\right)^{nx}$

Sol. $\lim\limits_{x\to\infty}\left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots\ldots + n^{1/x}}{n}\right)^{nx}$

It is the type of $(1)^\infty$

Here, $f(x) = \frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots\ldots + n^{1/x}}{n} - 1$

So that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} (nx) \to \infty$

$\therefore \lim\limits_{x\to\infty}\left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots\ldots + n^{1/x}}{n} - 1 + 1\right)^{nx}$

$= e^{\lim\limits_{x\to\infty}\left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots . n^{1/x}}{n} - 1\right)\times nx}$

$= e^{\lim\limits_{x\to\infty}\left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots . n^{1/x}}{n} - 1\right)\times \frac{1}{\frac{1}{nx}}}$

Use L-H Rule, we get

$$= e^{\lim\limits_{x\to\infty} \frac{(0 + 2^{1/x}\log 2\left(-\frac{1}{x^2}\right) + 3^{1/x}\log 3\left(-\frac{1}{x^2}\right) \ldots\ldots\ldots + n^{1/x}\log n\left(-\frac{1}{x^2}\right) - 0)}{n} \times \frac{1}{\frac{1}{-nx^2}}}$$

$= e^{\lim\limits_{x\to 0} -\frac{1}{nx^2}\left[2^{1/x}\log 2 + 3^{1/3}\log 3 + \ldots\ldots . n^{1/x}\log n\right]} \times (-nx^2)$

$= e^{(\log 2 + \log 3 + \ldots\ldots\ldots + \log n)} = e^{\log 2.3.4.5\ldots\ldots\ldots\ldots\ldots n}$

$= e^{\log(n!)} = n!$ Ans.

Q.17 Prove that $\lim\limits_{\theta\to 0}\left(\left[\frac{n\sin\theta}{\theta}\right] + \left[\frac{n\tan\theta}{\theta}\right]\right) =$ odd integer. Where [x] denotes the greatest integer function less than or equal to x and $n \in I$.

Sol. $\lim\limits_{\theta\to 0}\left(\left[\frac{n\sin\theta}{\theta}\right] + \left[\frac{n\tan\theta}{\theta}\right]\right)$

We know that,

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{\sin\theta}{\theta} = 1 - \theta^2/3! + \theta^4/5! - \ldots\ldots\ldots\ldots\ldots\ldots$

$\lim\limits_{\theta\to 0}\left(\frac{n\sin\theta}{\theta}\right) = \lim\limits_{\theta\to 0} n(1 - \theta^2/3! + \theta^4/5! - \ldots\ldots .)$

= approaches to n but less than n

Hence, $\lim\limits_{\theta\to 0}\left[\frac{n\sin\theta}{\theta}\right] \to$ n but less than n

$n - 1 \le \left[\frac{n\sin\theta}{\theta}\right] < n$

$\therefore \lim\limits_{\theta\to 0}\left[\frac{n\sin\theta}{\theta}\right] = n - 1$

Similarly, $\tan\theta = \theta + \theta^3/3 + \frac{2}{15}\theta^5 + \ldots\ldots\ldots$

$\frac{\tan\theta}{\theta} = 1 + \theta^2/3 + \frac{2}{15}\theta^4 + \ldots\ldots\ldots$

$\lim\limits_{\theta\to 0}\left(\frac{n\tan\theta}{\theta}\right) = \lim\limits_{\theta\to 0} n\left(1 + \frac{\theta^2}{3} + \frac{2}{15}\theta^4 + \ldots\ldots\right)$

= approaches to n but greater than n.

Hence, $\lim_{\theta\to 0}\left[\frac{n\tan\theta}{\theta}\right]$ = [n × value greater than 1]

= n

$n \le \left[\frac{n\tan\theta}{\theta}\right] < n+1$

Therefore, $\lim_{\theta\to 0}\left(\left[\frac{n\sin\theta}{\theta}+\frac{n\tan\theta}{\theta}\right]\right)$

= n – 1 + n

= 2n – 1

= odd Integer Ans.

Q. 18 If $\lambda = \lim_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1} - r\sin\frac{\pi}{r}\right)$ then find {λ}. (where { } denotes the fractional part function).

Sol. π – 3

Q.19 $\lim_{h\to 0}\left[\frac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}\right]$

Sol. $\lim_{h\to 0}\left[\frac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}\right]$ $\left(\frac{0}{0}\text{form}\right)$

Apply L – H Rule, we get

$\lim_{h\to 0}\left[\frac{\cos(a+3h).3-3\cos(a+2h)2+3\cos(a+h)-0}{3h^2}\right]$ $\left(\frac{0}{0}\text{form}\right)$

Apply L – H Rule, we get

$\lim_{h\to 0}\left[\frac{-9\sin(a+3h)+12\sin(a+2h)-3\sin(a+h)}{6h}\right]$ $\left(\frac{0}{0}\text{form}\right)$

$\lim_{h\to 0}\left[\frac{-27\cos(a+3h)+24\cos(a+2h)-3\cos(a+h)}{6}\right]$

$= \frac{-27\cos a+24\cos a-3\cos a}{6} = \frac{-6}{6}\cos a = -\cos a$ **Ans.**

Q.20 $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right]$

Sol. $\lim_{x\to\infty}\frac{\left[\tan^{-1}\frac{x+1}{x+2}-\pi/4\right]}{1/x}$ $\left(\frac{0}{0}\text{form}\right)$

Apply L-H Rule, we get

$= \lim_{x\to\infty}\frac{\frac{1}{1+\left(\frac{x+1}{x+2}\right)^2}\times\frac{(x+2).1-(x+1).1}{(x+2)^2}-0}{(-1/x^2)}$

$= \lim_{x\to\infty}\frac{(x+2)^2}{(x+2)^2+(x+1)^2}\times\frac{1}{(x+2)^2}\times(-x^2)$

$= \lim_{x\to\infty}\frac{1}{x^2\left[(1+2/x)^2+(1+1/x)^2\right]}\times(-x^2)$

= –1/2 Ans.

Q.21 $\lim_{x\to\pi/2}\tan^2 x\,[(2\sin^2 x+3\sin x+4)^{1/2} - (\sin^2 x+6\sin x+2)^{1/2}]$

Sol. $\lim_{x\to\pi/2}\tan^2 x\left[(2\sin^2 x+3\sin x+4)^{1/2}-(\sin^2 x+6\sin x+2)^{1/2}\right]$

$\lim_{x\to\pi/2}\frac{\tan^2 x[2\sin^2 x+3\sin x+4-\sin^2 x-6\sin x-2]}{[(2\sin^2 x+3\sin x+4)^{1/2}+(\sin^2 x+6\sin x+2)^{1/2}]}$

$= \lim_{x\to\pi/2}\frac{\tan^2 x[\sin^2 x-3\sin x+2]}{6}$

$= \lim_{x\to\pi/2}\frac{[\sin^2 x-3\sin x+2]}{6\times\cot^2 x}$ $\left(\frac{0}{0}\text{form}\right)$

Apply L – H Rule

$= \lim_{x\to\pi/2}\frac{[2\sin x\cos x-3\cos x+0]}{6\times 2\cot x(-\text{cosec}^2 x)}$

$= \lim_{x\to\pi/2}\left(\frac{1}{-12}\right)\times[2\sin x-3]\times\sin^3 x$

$= -\frac{1}{12}\times(2-3)\times 1$

$= \frac{1}{12}$ **Ans.**

Q.22 Prove that: $\lim_{m\to\infty}\lim_{n\to\infty}\left(1+\cos^{2m}(n!\pi x)\right) = 1$ where x is irrational and $\lim_{m\to\infty}\lim_{n\to\infty}\left(1+\cos^{2m}(n!\pi x)\right) = 2$, where x is rational.

Sol. $\lim_{m\to\infty}\lim_{n\to\infty}\left(1+\cos^{2m}(n!\pi x)\right)$

Case 1 : When x is Rational number i.e. $x \in Q$

as 0, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...1

then πx(n!) will be integral multiple of π

cos(n! πx) = ± 1.

$\lim_{m\to\infty}\lim_{n\to\infty}\left(1+\cos^{2m}(n!\pi x)\right) = 1+1 = 2$

Case II : When x is Irrational number i.e.,

$x \in Q^C$ as $\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$, ... ; 0 < x < 1

then (n!) πx will not be integral multiple of π.

cos (n! πx) → Value lies between –1 and 1.

$$\lim_{m\to\infty} \lim_{n\to\infty} \left\{1+\cos^{2m}(n!)\pi x\right\} = 1 + 0$$

$$= 1 \qquad \left\{\begin{array}{l}\because \quad x^{\infty} = 0 \\ \text{as } 0 < x < 1\end{array}\right.$$

Q.23 If f (x + y) = f(x) + f (y) for x, y ∈ R and f (1) = 1, then find, $\lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$

Sol. f(x + y) = f(x) + f(y) for x, y ∈ R and f(1) = 1

f(1 + 1) = f(1) + f(1) = 1 + 1 = 2 ⇒ f(2) = 2

f(1 + 2) = f(1) + f(2) = 1 + 2 = 3 ⇒ f(3) = 3

f(1+ 3) = f(1) + f(3) = 1 +3 = 4 ⇒ f(4) = 4

f(1 + 4) = f(1) + f(4) = 1 + 4 = 5 ⇒ f(5) = 5

....

....

f(x) = x for x ∈ R

⇒ f(tanx) = tanx and f(sinx) = sinx

$$2^{\tan x} = 1 + \tan x \log 2 + \frac{\tan^2 x(\log 2)^2}{2!} + ...$$

$$2^{\sin x} = 1 + \sin x \log 2 + \sin^2 x \frac{(\log 2)^2}{2!} + ...$$

$$2^{\tan x} - 2^{\sin x} = \log 2(\tan x - \sin x) + \frac{(\log 2)^2}{2!}(\tan^2 x - \sin^2 x) + ...$$

$$\frac{2^{\tan x} - 2^{\sin x}}{x^2 \sin x} = \frac{\log 2}{x^2} \times (\sec x - 1) + \frac{(\log 2)^2}{2!} \times \frac{1}{x^2} (\sin x \sec^2 x - \sin x) + ...$$

$$= \frac{\log 2}{x^2}(\sec x - 1) + \frac{(\log 2)^2}{2!} \times \frac{1}{x^2} \sin x (\sec^2 x - 1) + ...$$

$$\lim_{x\to 0} \log 2 \times \frac{(\sec x - 1)}{x^2} \left(\frac{0}{0} \text{ form}\right)$$

Apply L –H Rule, we get

$$= \lim_{x\to 0} \log 2 \times \frac{(\sec x \tan x - 0)}{2x} \left(\frac{0}{0} \text{ form}\right)$$

Again Apply L – H Rule, we get

$$= \lim_{x\to 0} \log 2 \times \left(\frac{\sec x \tan^2 x + \sec^3 x}{2 \times 1}\right)$$

$$= \frac{1}{2} \log 2$$

$$= \lim_{x\to 0} \frac{(\log 2)^2}{2!} \times \frac{\sin x(\sec^2 x - 1)}{x^2} \left(\frac{0}{0} \text{form}\right)$$

Apply L – H Rule, we get

$$\lim_{x\to 0} \frac{(\log 2)^2}{2} \times \frac{\cos x(\sec^2 x - 1) + \sin x(2\sec^2 x \tan x)}{2x}$$

$$\lim_{x\to 0} \frac{(\log 2)^2}{2} \times \frac{(\sec x - \cos x) + 2\sin^2 x \sec^3 x}{2x} \left(\frac{0}{0} \text{form}\right)$$

Apply L – H Rule, we get

$$\lim_{x\to 0} \frac{(\log 2)^2}{2} \times \frac{(\sec x \tan x + \sin x) + 4\sin x \sec^2 x + 2\sin^2 x \, 3\sec^2 x \sec x \tan x}{2 \times 1}$$

$$= \frac{(\log 2)^2}{2} \times \frac{0+0+0}{2} = 0$$

Hence, $\lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$

$$= \lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \sin x} = \frac{1}{2} \log 2 \quad \text{Ans.}$$

Q.24 If $h(x) = \text{Lim}_{n\to\infty} \frac{x^{2n} f(x) + g(x)}{1 + x^{2n}}$, then find h(x) in terms of f(x) and g(x) in the interval (– ∞, ∞).

Sol. $h(x) = \lim_{n\to\infty} \frac{x^{2n} f(x) + g(x)}{1 + x^{2n}}$

= g(x) if $\lim_{n\to\infty} x^{2n} \to 0$ i.e., |x| < 1

= $\frac{1}{2}$ {f(x) + g(x)} if |x| = 1

= f(x) if $\lim_{x\to\infty} 1/x^{2n} \to 0$

i.e. | x | > 1

Part-B Passage based objective questions

Passage: I (Q. No. 25 to 27)

Let there are two functions defined here.

$$f(x) = \begin{cases} \sin x & x \neq n\pi \\ 2 & x = n\pi \end{cases}$$

$$g(x) = \begin{cases} x^2+1 & x \neq 0,\ x \neq 2 \\ 4 & x = 0 \\ 6 & x = 2 \end{cases}$$

Q.25 Find $\lim_{x\to 0} g(f(x))$-

(A) 0 (B) 1

(C) –1 (D) None

Sol.[B] $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} (\sin^2 x + 1) = 1.$

$\therefore$ Option (B) is correct Answer.

Q.26 Evaluate $\lim_{x\to 1} f(f(x))$-

(A) sin 1 (B) sin sin1

(C) 1 (D) None of these

Sol.[B] $\lim_{x\to 1} f(f(x)) = \lim_{x\to 1} \sin(\sin x)$

$= \sin(\sin 1)$

$\therefore$ Option (B) is correct Anwer.

Q.27 Evaluate $\sum_{i=1}^{n} \lim_{x\to i^+} \{g(x)-1\}$, where $\{x\}$ is the fractional part of real x-

(A) 0 (B) n, $n \in N$

(C) 1 (D) None of these.

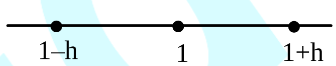
Sol.[A] $\sum_{i=1}^{i=n} \lim_{x\to i^+} \{g(x)-1\}$

$\{g(x) - 1\} = g(x) - 1 - [g(x) - 1]$

$= g(x) - 1 - [g(x)] + 1$

$= g(x) - [g(x)]$

$\sum_{i=1}^{i=n} \lim_{x\to i^+} (g(x)-[g(x)])$



$= \lim_{x\to 1^+} (g(x) - [g(x)]) + \lim_{x\to 2^+} (g(x) - [g(x)]) + \lim_{x\to 3^+} (g(x) - [g(x)]) + \ldots\ldots\ldots\ldots$

$+ \lim_{x\to n^+} (g(x) - [g(x)])$

$\lim_{x\to 1^+} (g(x) - [g(x)])$

= use the definition of $g(x) - 1 < [g(x)] \le g(x)$

$\lim_{x\to 1^+} (g(x) - g(x)) = 0$

$\lim_{x\to 2^+} (g(x) - g(x)) = 0$ .... $\lim_{x\to n^+} (g(x)-g(x))=0$

Hence

$\sum_{i=1}^{i=n} \lim_{x\to i^+} \{g(x)-1\} = 0$

$\therefore$ Option (A) is correct Answer.

Passage II (Q. No. 28 to 30)

f(x) is polynomial function of degree six. Let consider f(x) : $a_0 + a_1x + a_2x^2 + \ldots.. + a_6x^6$. Such that f(1) = 2, f(–1) = 0 and satisfying

$\lim_{x\to 0} \left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e^2.$

Another function

$g(x) = \lim_{m\to\infty} \frac{x^m A(x) + B(x) + 1}{2x^m + 3x + 3}$ and also satisfy the condition

$\lim_{x\to 1} g(x) = \lim_{x\to 0} \left\{1 + \frac{f(x)}{x^3}\right\}^{1/x}$

Q.28 Find the value of a_2 and a_3

(A) 0, 1 (B) 1, 0 (C) 0, 0 (D) None

Sol.[C] $\lim_{x\to 0} \left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$

i.e. $\lim_{x\to 0} \frac{f(x)}{x^3} = 0$

$\Rightarrow \lim_{x\to 0}$

$\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6}{x^3}$

= 0

For existence of limit

$a_0 = 0$

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$\therefore$ Option (C) is correct Answer.

Q.29 Find the $\lim_{x\to 1} f(x)$

(A) 2 (B) 3 (C) 4 (D) None

Sol.[A] $\lim_{x\to 1} f(x) = 2$

$\therefore$ Option (A) is correct Answer.

Q.30 Find the value of B(1)

(A) $6e^2$ (B) $6e^2 + 1$ (C) $6e^2 - 1$ (D) None

Sol.[C] $g(x) = \lim_{m\to\infty} \frac{x^m A(x) + B(x) + 1}{2x^m + 3x + 3}$

$$\lim_{x\to1} g(x) = \lim_{x\to1}\lim_{m\to\infty} \frac{A(x) + B(x)/x^m + 1/x^m}{2 + 3x/x^m + 3/x^m}$$

$$= \lim_{x\to0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x}$$

$$\lim_{x\to1} \frac{A(x)}{2} = e^2$$

$\Rightarrow A(1) = 2e^2$

Also

$$\lim_{x\to1} g(x) = \lim_{m\to\infty}\lim_{x\to1} \frac{x^m A(x) + B(x) + 1}{2x^m + 3x + 3} = e^2$$

$$\Rightarrow \lim_{m\to\infty} \frac{(1)^m A(1) + B(1) + 1}{2(1)^m + 3 + 3} = e^2$$

$\Rightarrow A(1) + B(1) = 8e^2 - 1$

$B(1) = 8e^2 - 2e^2 - 1 = 6e^2 - 1$

$B(1) = 6e^2 - 1$

$\therefore$ Option (C) is correct Answer.

Passage : III (Q.No.31 to 32)

Let $f(x) = \begin{cases} x - [x] \; ; & x \notin I \\ 1 \; ; & x \in I \end{cases}$; where I is the set of integers & [x] represents greatest integer $\le$ x.

If $g(x) = \lim_{n\to\infty} \frac{(f(x))^{2n} - 1}{(f(x))^{2n} + 1}$, then :

Q.31 Period of f(2x) is-

(A) not defined (B) 1 (C) 1/2 (D) 2

Sol.[C] Since f(x) = x – [x] is periodic with period 1.

$\therefore$ f(2x) = 2x – [2x] is periodic with period $\frac{1}{2}$

Q.32 $f(x) = |g(x)|$ is satisfied by-

(A) no real x

(B) all integer values of x

(C) x = 0 only (D) x = 1 only

Sol.[A] f(x) = |g(x)| = 1

$$= \begin{cases} x - [x] \; ; & x \notin I \\ 1 \; ; & x \in I \end{cases}$$

x – [x] = 1

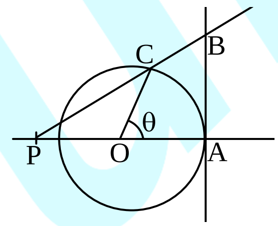
x = [x] + 1 ; $x \notin I$

i.e., no real value of x

$\therefore$ Option (A) is correct Answer.

Passage : IV (Q.No.33 to 35)

A tangent line is drawn to a circle of radius unity at point A and a segment AB is laid off whose length is equal to that of the arc AC. A straight line BC is drawn to intersect the extension of diameter AO at point P as shown in figure.



Q.33 The length PA in terms of 'θ' is given by-

(A) $\frac{4(1-\cos\theta)}{\theta\sin\theta}$ (B) $\frac{\theta^2\sin\theta}{(\theta-\tan\theta)}$

(C) $\frac{2(\theta-\sin\theta)}{(\theta-\tan\theta)}$ (D) $\frac{\theta(1-\cos\theta)}{\theta-\sin\theta}$

Sol. **[D]**

Q.34 $\lim_{\theta\to0}$ (PA) is equal to-

(A) 1/3 (B) 1 (C) 2 (D) 3

Sol. [D]

Q.35 $\lim_{\theta\to0} \theta^2\left(\frac{PC}{BC}\right)$ is equal to-

(A) 1/3 (B) 1 (C) 3 (D) 6

Sol. [D]

EXERCISE # 4

Old IIT-JEE Questions

Q.1 $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2} =$ **[IIT Scr. 2001]**

(A) $-\pi$ (B) π (C) $\dfrac{\pi}{2}$ (D) 1

Sol.[C] $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Apply L – H Rule, we get

$$= \lim_{x\to 0}\frac{\cos(\pi\cos^2 x)\pi.2\cos x(-\sin x)}{2x}$$

$$= \lim_{x\to 0}\frac{\cos(\pi\cos^2 x)\pi.(-\sin 2x)}{2x}\ \left(\frac{0}{0}\text{form}\right)$$

$$= \frac{\pi}{2}\lim_{x\to 0}\frac{-\cos 2x\,2\cos(\pi\cos^2 x) + \sin 2x\sin(\pi\cos^2 x)\pi 2\cos x(-\sin x)}{2}$$

$$= \frac{\pi}{2}\lim_{x\to 0}\frac{-2\cos 2x\cos(\pi\cos^2 x) - \pi(\sin 2x)^2\sin(\pi\cos^2 x)}{2}$$

$$= \frac{\pi}{2}\,\frac{2-0}{2} = \frac{\pi}{2}$$

$\therefore$ Option (C) is correct Answer.

Q.2 Evaluate : $\lim\limits_{x\to 0}\dfrac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$ $a > 0$ **[REE 2001]**

Sol. $\lim\limits_{x\to 0}\dfrac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$; $a > 0$

We know, $a^x = 1 + \dfrac{x\log a}{1!} + \dfrac{x^2(\log a)^2}{2!} + \ldots.$

$$\lim_{x\to 0}\frac{\left(1+\frac{\tan x\log a}{1!}+\frac{\tan^2 x(\log a)^2}{2!}+\ldots\right) - \left(1+\frac{\sin x\log a}{1!}+\frac{\sin^2 x(\log a)^2}{2!}+\ldots.\right)}{\tan x - \sin x}$$

$$= \lim_{x\to 0}\frac{\log a(\tan x - \sin x) + \frac{(\log a)^2}{2}(\tan x - \sin x)(\tan x + \sin x) + \ldots.}{(\tan x - \sin x)}$$

$= \log a + 0 = \log a$ **Ans.**

Q.3 The value of Integer n ; for which $\lim\limits_{x\to 0}\dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non zero number- **[IIT Scr. 2002]**

(A) 1 (B) 2 (C) 3 (D) 4

Sol.[C] $\lim\limits_{x\to 0}\dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots\ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots$$

$$= \lim_{x\to 0}\frac{\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\ldots\ldots-1\right]\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots..\right)-\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.\right)\right]}{x^n}$$

$$= \lim_{x\to 0}\frac{\left[-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\ldots\ldots\right]\left[-x-x^2-\frac{x^3}{6}-\ldots\right]}{x^n}$$

therefore, for finite limit, n must be 3.

$\therefore$ Option (C) is correct Answer.

Q.4 If $\lim\limits_{x\to 0}\dfrac{(\sin nx)[(a-n)nx - \tan x]}{x^2} = 0$ then the value of a is-

(A) $\dfrac{1}{n+1}$ (B) $\dfrac{n}{n+1}$ (C) $n+\dfrac{1}{n}$ (D) n

Sol.[C] $\lim\limits_{x\to 0}\dfrac{(\sin nx)[(a-n)nx - \tan x]}{x^2} = 0$

$\dfrac{0}{0}$ form, apply L–H rule, we get

$$\lim_{x\to 0}\frac{(\cos nx.n)[(a-n)nx - \tan x] + (\sin nx)[(a-n)n - \sec^2 x]}{2x}$$

$= 0$ $\left(\dfrac{0}{0}\text{form}\right)$

$$\lim_{x\to 0}\frac{(-n^2\sin nx)[(a-n)nx - \tan x] + n\cos nx[(a-n).n - \sec^2 x] + \cos nx.n[(a-n)n - \sec^2 x] + \sin nx(-2\sec x\sec x\tan x)}{2} = 0$$

$$= \frac{0 + n.1[(a-n)n - 1] + n.1.[(a-n)n - 1] + 0}{2} = 0$$

$\Rightarrow an^2 - n^3 - n + an^2 - n^3 - n = 0$

$\Rightarrow 2(n^2a - n^3 - n) = 0$

$\Rightarrow an^2 = n(1 + n^2)$

$$a = n + \frac{1}{n}$$

$\therefore$ Option (C) is correct Answer.

Q.5 $\lim_{x\to 0}\left((\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right)$, for x > 0

[IIT 2006]

(A) 0 (B) –1 (C) 2 (D) 1

Sol.[D] $\lim_{x\to 0} (\sin x)^{1/x} + \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$; for x > 0

Let A = $\lim_{x\to 0} (\sin x)^{1/x}$

lnA = $\lim_{x\to 0} \frac{1}{x}\, ln \sin x = \infty \times (-\infty)$

$= -\infty$

$\Rightarrow$ A = $e^{-\infty} \to 0$

B = $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$

ln B = $\lim_{x\to 0} \sin x\ ln\frac{1}{x}$

$= -\lim_{x\to 0} \sin x \times \lambda nx$

$= -\lim_{x\to 0} \frac{lnx}{\text{cosec}x}$ ($\frac{\infty}{\infty}$ form)

Apply L – H Rule, we get

$= -\lim_{x\to 0} \frac{1/x}{-\text{cosec}x\cot x}$

$= \lim_{x\to 0} \frac{\sin x\tan x}{x}$ $\left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to 0} \frac{\sin x\sec^2 x + \tan x\cos x}{1}$

= 0

$\Rightarrow$ B = e^0 = 1

Hence, $\lim_{x\to 0}\left((\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right) = 1$

$\therefore$ Option (D) is correct Answer.

Q.6 Let $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, a > 0

If L is finite, then **[IIT 2009]**

(A) a = 2 (B) a = 1

(C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

Sol. **[A, C]**

$$L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4},\ a > 0$$

L is finite

$$= \lim_{x\to 0} \frac{\left(a - \frac{x^2}{4} - \sqrt{a^2 - x^2}\right)}{x^4} \times \frac{\left(a - \frac{x^2}{4}\right) + \left(\sqrt{a^2 - x^2}\right)}{\left(a - \frac{x^2}{4}\right) + \left(\sqrt{a^2 - x^2}\right)}$$

$$= \lim_{x\to 0} \frac{\left(a - \frac{x^2}{4}\right)^2 - (a^2 - x^2)}{x^4\left[a - \frac{x^2}{4} + \sqrt{a^2 - x^2}\right]}$$

$$= \lim_{x\to 0} \frac{a^2 + \frac{x^4}{16} - \frac{9x^2}{2} - a^2 + x^2}{2a\ x^4}$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{16} + \left(1 - \frac{a}{2}\right)x^2}{2a\,x^2} = \lim_{x\to 0} \frac{\frac{x^2}{16} + \left(1 - \frac{a}{2}\right)}{2ax^2}$$

for L to be finite

$$1 - \frac{a}{2} = 0 \Rightarrow a = 2$$

substituting in limit we get

$$L = \lim_{x\to 0} \frac{\frac{x^2}{16} + 0}{4x^2} = \frac{1}{64}$$

Q.7 If $\lim_{x\to 0}[1 + x\ln(1 + b^2)]^{1/x} = 2b\sin^2\theta$, $b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is - **[IIT 2011]**

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

Sol.[D] $\lim_{x\to 0} (1 + x\ \lambda n(1 + b^2))^{1/x} = 2b\sin^2\theta$ b > 0;

$\theta \in (-\pi, \pi)$

$$\lim_{x\to 0}\left([1+x\lambda n(1+b^2)]^{\frac{1}{x\lambda n(1+b^2)}}\right)^{\lambda n(1+b^2)} = 2b$$

$\sin^2\theta$

$e^{\lambda n(1+b^2)} = 2b\sin^2\theta$

$1 + b^2 = 2b\sin^2\theta$

$2\sin^2\theta = b + \frac{1}{b}$

RHS $= b + \frac{1}{b} \ge 2$ as $b > 0$

But LHS $= 2\sin^2\theta \le 2$

Only possibility

$2\sin^2\theta = 2$

$\sin^2\theta = 1$

$\theta = \pm\frac{\pi}{2}$

Q.8 If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then **[IIT 2012]**

(A) a = 1, b = 4 (B) a = 1, b = – 4

(C) a = 2, b = –3 (D) a = 2, b = 3

Sol. [B] $\lim_{x\to\infty}\left(\frac{x^2+x+1-ax^2-ax-bx-b}{x+1}\right) = 4$

$$\lim_{x\to\infty}\left(\frac{x^2(1-a)+x(1-a-b)+1-b}{x+1}\right) = 4$$

As limit is finite so $1 - a = 0$

$\Rightarrow a = 1$

Now $\lim_{x\to\infty}\left(\frac{(1-a-b)+\frac{1-b}{x}}{1+\frac{1}{x}}\right) = 4$

$\Rightarrow 1 - a - b = 4$

as $a = 1 \Rightarrow b = -4$

Q.9 Let α (a) and β(a) be the roots of the equation

$$\left(\sqrt[3]{1+a}-1\right)x^2 + \left(\sqrt{1+a}-1\right)x + \left(\sqrt[6]{1+a}-1\right) = 0$$

where a > – 1.

Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[IIT 2012]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and – 1

(C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

Sol. [A] $(1 + a) = t^6$

$(t^2-1)x^2 + (t^3-1)x + (t-1) = 0$

$$x = \frac{-(t^3-1)\pm\sqrt{(t^3-1)^2-4(t-1)(t^2-1)}}{2(t^2-1)}$$

$$x = \frac{-(t^3-1)\pm(t-1)\sqrt{(t^2+t+1)^2-4(t+1)}}{2(t-1)(t+1)}$$

$$x = \frac{-(t^2+t+1)\pm\sqrt{(t^2+t+1)^2-4(t+1)}}{2(t+1)}$$

$a\to 0^+ \Rightarrow t\to 1^+$

$$x = \frac{-3\pm\sqrt{9-8}}{2(2)} \Rightarrow x = \frac{-3\pm 1}{4}$$

$\Rightarrow x = -1, -\frac{1}{2}$

EXERCISE # 5

Q.1 $\lim\limits_{x\to-\infty}\left[\dfrac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right] = \ldots\ldots\ldots\ldots$

[IIT-1987]

Sol. **[– 1]**

$$\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$$

as $x \to -\infty$

$|x| = -x$

$$\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1-x^3}\right]$$

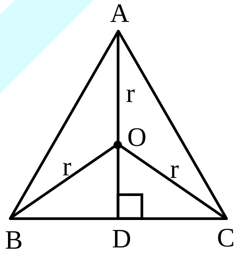
$$\lim_{x\to-\infty}\frac{x\sin\frac{1}{x}+\frac{1}{x}}{\frac{1}{x^3}-1}$$

$$\lim_{x\to-\infty}-x\sin\frac{1}{x}$$

$$\lim_{x\to-\infty}-\frac{\sin\frac{1}{x}}{\frac{1}{x}} = -1$$

Q.2 ABC is an isosceles triangle inscribed in a circle of radius r. If AB = AC and h is the altitude from A to BC then the triangle ABC has perimeter $P = 2(\sqrt{2hr-h^2}+\sqrt{2\,hr})$ and area A= ...... also $\lim\limits_{h\to0}\dfrac{A}{P^3} = \ldots\ldots$ **[IIT-1989]**

Sol.



In ΔABC, AB = AC

$AD \perp BC$ (D is mid pt of BC)

Let AD = h

r = radius of circumcircle

$\therefore$ OA = OB = OC = r

Now $BD = \sqrt{BO^2-OD^2} = \sqrt{r^2-(h-r)^2}$

$= \sqrt{2rh-h^2}$ $\quad\therefore BC = 2\sqrt{2rh-h^2}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\times BC\times AD = h\sqrt{2rh-h^2}$

Also $\lim\limits_{h\to0}\dfrac{A}{p^3} = \dfrac{h\sqrt{2rh-h^2}}{8(\sqrt{2rh-h^2}+\sqrt{2hr})^3}$

$$= \lim_{h\to0}\frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}(\sqrt{2r-h}+\sqrt{2r})^3}$$

$$= \lim_{h\to0}\frac{\sqrt{2r-h}}{8[\sqrt{2r-h}+\sqrt{2r}]^3}$$

$$= \frac{\sqrt{2r}}{8(\sqrt{2r}+\sqrt{2r})^3} = \frac{\sqrt{2r}}{8.8.2r.\sqrt{2r}} = \frac{1}{128r}$$

Q.3 $\lim\limits_{x\to0}\dfrac{x\tan2x-2x\tan x}{(1-\cos2x)^2}$ is **[IIT 1999]**

(A) $\frac{1}{2}$ (B) –2 (C) 2 (D) $-\frac{1}{2}$

Sol.[A] $\lim\limits_{x\to0}\dfrac{x\tan2x-2x\tan x}{(1-\cos2x)^2}$

We know, $\tan x = x + \dfrac{x^3}{3}+\dfrac{2}{15}x^5+\ldots\ldots\ldots\ldots$

$\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}\ldots\ldots\ldots\ldots$

$\therefore$

$$\lim_{x\to0}\frac{x\left[2x+\frac{8}{3}x^3+\frac{2}{15}\times32x^5+\ldots..\right]-2x\left[x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots.\right]}{\left[1-1+\frac{4x^2}{2!}-\frac{16x^4}{4!}\right]^2}$$

$$\lim_{x\to 0}\frac{x^4\left(\frac{8}{3}-\frac{2}{3}\right)+\frac{2}{15}x^6(32-2)+\ldots\ldots\ldots}{x^4\left[\frac{4}{2}-\frac{16}{24}x^2+\ldots..\right]^2}$$

$$=\frac{2+0}{4-0}=\frac{1}{2}\text{ Ans.}$$

Q.4 $\lim_{x\to a}\frac{1}{(a^2-x^2)^2}\left(\frac{a^2+x^2}{ax}-2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right)$

where a is an odd integer.

Sol. $\frac{\pi^2a^2+4}{16a^4}$

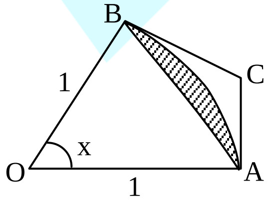
Q.5 If a sequence of numbers $\{x_n\}$, determine by the equality $x_n = \frac{x_{n-1}+x_{n-2}}{2}$ and the values x_0 and x_1. Prove that $\lim_{n\to\infty} x_n = \frac{x_0+2x_1}{3}$.

Q.6 Let $x_0 = 2\cos\frac{\pi}{6}$ & $x_n = \sqrt{2+x_{n-1}}$, n = 1, 2, 3, ……., find $\lim_{n\to\infty} 2^{(n+1)}.\sqrt{2-x_n}$.

Sol. $\pi/3$

Q.7 Let $L = \prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right)$; $M = \prod_{n=2}^{\infty}\left(\frac{n^3-1}{n^3+1}\right)$ and $N = \prod_{n=1}^{\infty}\frac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1} + M^{-1} + N^{-1}$.

Q.8 A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B, Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:



(a) T(x)

(b) S(x) &

(c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$

Sol. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$. or $\tan\frac{x}{2}-\frac{\sin x}{2}$;

$S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit = $\frac{3}{2}$

Q.9 Through a point A on a circle, a chord AP is drawn and on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

Q.10 If $L = \lim_{x\to 0}\left(\frac{1}{\lambda n(1+x)}-\frac{1}{\lambda n(x+\sqrt{1+x^2})}\right)$ then find the value of $\frac{L+153}{L}$.

Sol. 307

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	B	B	D	C	A	B	A	C	B	C	B	A	B	B	B	B	C	B	B
Q.No.	21	22	23	24	25	26	27	28	29	30	31	32	33							
Ans.	C	D	C	C	C	B	A	D	D	B	A, D	B	A							

34. True **35.** True **36.** $\frac{1}{1-x}$

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12
Ans.	C	A	C	B	A	A	C	C	C	D	D	B
Q.No.	13	14	15	16	17	18	19	20	21	22	23	24
Ans.	A	D	A	C	B	C	D	C	D	B	D	B

(PART-B)

Q.No.	25	26	27	28	29	30	31	32
Ans.	A,B,C	B,C,D	B,C	B,C,D	C,D	A,B	B,C	A,D

(PART-C)

Q.No.	33	34
Ans.	D	D

(PART-D)

35. (A) $\to$ R (B) $\to$ P (C) $\to$ S (D) $\to$ Q **36.** (A) $\to$ R (B) $\to$ S (C) $\to$ P

37. (A) $\to$ S (B) $\to$ R (C) $\to$ P (D) $\to$ Q

EXERCISE # 3

(1) $\frac{1}{2}$ **(2)** Does not exist **(3)** 1/32 **(4)** $\frac{\pi}{2}$, $\frac{\pi}{2\sqrt{2}}$

(5) 2/3 **(6)** n/2 (n+3) **(7)** $\sqrt{2}$ **(8)** (p – q) / 2

(9) Domain, $x \in R$, Range $x = \frac{n\pi}{2}$; $n \in I$ **(10)** 1/24 **(11)** $\frac{2n!}{(n!)^2}$ **(12)** 8

(13) $(n/2)a^2 \sin(2\pi/n)$, πa^2 **(14)** $\frac{1}{2}$ $(\sqrt{3} + i)$ **(15)** a = 3/2 , $b \in R$ **(16)** n!

(18) $\pi - 3$ **(19)** – cos a **(20)** –1/2 **(21)** 1/12

(23) 1/2. λn 2

(24) f(x) ; |x| > 1 , g(x) ; |x| < 1 , [f(x) + g(x)] / 2 ; | x | = 1

Q.No.	25	26	27	28	29	30	31	32	33	34	35
Ans.	B	B	A	C	A	C	C	A	D	D	D

EXERCISE # 4

1. (B) **2.** λn a **3.** (C) **4.** (C) **5.** (D) **6.** (A, C) **7.** (D)

8. (B) **9.** (B)

EXERCISE # 5

(1) –1 **(2)** $\frac{1}{128r}$ **(3)** (A) **(4)** $\frac{\pi^2 a^2 + 4}{16a^4}$ **(6)** $\frac{\pi}{3}$ **(7)** B

(8) $T(x) = \frac{1}{2}\tan^2\frac{x}{2}$. sin x. or $\tan\frac{x}{2} - \frac{\sin x}{2}$; $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit = $\frac{3}{2}$ **(10)** 307