HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

3. $\underset{x\to\infty}{\text{Limit}}\sqrt{x+1}-\sqrt{x}\ =0 \quad \Rightarrow \quad \cot^{-1}(0)=\pi/2$

$\underset{x\to\infty}{\text{Limit}}\left(\frac{2x+1}{x-1}\right)^x \to \infty \quad \Rightarrow \quad \sec^{-1}(\infty)=\pi/2$

$\therefore \quad l=1$

4. $\lim\limits_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+.....+\text{to n terms}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+.....+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}$

6. Let $\lim\limits_{x\to a} f(x)=L$ & $\lim\limits_{x\to a} g(x)=M$

$\Rightarrow \quad L+M=2$ & $L-M=1$

$\Rightarrow \quad L=\frac{3}{2}$ & $M=\frac{1}{2}$

So $\lim\limits_{x\to a} f(x)g(x)=L.M=\frac{3}{4}$

7. $\lim\limits_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$=\lim\limits_{x\to\alpha}\left(\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2(x-\beta)^2a^2}\right)(x-\beta)^2\ a^2$

$=\frac{1}{2}\ a^2(\alpha-\beta)^2$

8. $\lim\limits_{x\to 0}\frac{\sin 3x}{x^3}+\frac{a}{x^2}+b = \lim\limits_{x\to 0}\frac{\sin 3x+ax+bx^3}{x^3}$

$=\lim\limits_{x\to 0}\frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$

for existence of limit $3+a=0 \quad \Rightarrow \quad a=-3$

$\therefore \quad l = \lim\limits_{x\to 0}\frac{\sin 3x-3x+bx^3}{x^3} =$

9. $\lim\limits_{x\to 0}\frac{2\sin^2\frac{(ax^2+bx+c)}{2}}{\left(\frac{(ax^2+bx+c)}{2}\right)^2}\frac{(ax^2+bx+c)^2}{4(x-\alpha)^2}$

$\lim\limits_{x\to\alpha}\frac{2a^2\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}{4(x-\alpha)}\frac{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}{(x-\alpha)}$

$\lim\limits_{x\to\alpha}\frac{a^2\ (x-\alpha)^2\ \ (x-\beta)^2}{2(x-\alpha)^2}$;

$\lim\limits_{x\to\alpha}\frac{a^2}{2}(\alpha-\beta)^2 \quad \Rightarrow \quad C$

10. $l=e^2$ hence $\{e^2\}=e^2-7$

11. $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)=\lim\limits_{h\to 0}(1-1+h)\tan\left(\frac{\pi}{2}-\frac{\pi h}{2}\right)$

$=\lim\limits_{h\to 0} h\cot\frac{\pi}{2}h=\lim\limits_{h\to 0}\frac{h}{\tan\frac{\pi h}{2}}=\frac{2}{\pi}$

12. $\lim\limits_{x\to 0}\frac{(\tan(\{x\}-1))\sin\{x\}}{\{x\}(\{x\}-1)}$

$\text{LHL}=\lim\limits_{h\to 0^-}\frac{(\tan((1-h)-1))\sin(1-h)}{(1-h)((1-h)-1)}$

$[\Rightarrow \quad \{x\}=\{0-h\}=1-h]$

$=\lim\limits_{h\to 0^-}\frac{-\tan h\sin(1-h)}{(1-h)(-h)}=\sin 1$

$\text{RHL}=\lim\limits_{h\to 0^+}\frac{\tan(h-1)\sinh}{h(h-1)}=\tan 1$

$\Rightarrow \quad \text{LHL}\neq\text{RHL}$

$\therefore$ limit does not exist

13. $\lim\limits_{x\to\infty} \sin\sqrt{x+1} - \sin\sqrt{x}$

$= \lim\limits_{x\to\infty} 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)$

$= \lim\limits_{x\to\infty} \frac{2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\times\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)}$

$= \lim\limits_{x\to\infty} 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\times\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)} = 0$

14. $\lim\limits_{x\to\frac{\pi}{2}} \frac{(1-\tan x/2)}{(1+\tan x/2)}\frac{(1-\sin x)}{(\pi-2x)^3}$

$= \lim\limits_{x\to\frac{\pi}{2}} \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\frac{(1-\sin x)}{(\pi-2x)^3}$

$= \lim\limits_{h\to 0} \frac{\tan\left(\frac{\pi}{4}-\frac{\pi}{4}+\frac{h}{2}\right)(1-\cosh)}{(2h)^3}$

$= \lim\limits_{h\to 0} \frac{2\tan\frac{h}{2}\sin^2\frac{h}{2}}{8h^3} = \frac{1}{32}$

15. $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2 \Rightarrow e^{\lim\limits_{x\to\infty} 2x\left(1+\frac{a}{x}+\frac{b}{x^2}-1\right)} = e^2$

$\Rightarrow e^{2a} = e^2 \quad \Rightarrow a = 1$ & $b \in R$

16. $\lim\limits_{x\to 0}\frac{(1+3x)^{1/3}-1-x}{(1+x)^{101}-1-101x}$

$= \lim\limits_{x\to 0}\frac{\frac{3}{3}(1+3x)^{-2/3}-1}{(101)(1+x)^{100}-101}$ (By L' Hospital rule)

$= \lim\limits_{x\to 0}\frac{-\frac{2}{3}(1+3x)^{-5/3}.3}{(101)(100)(1+x)^{99}} = -\frac{1}{5050}$

17. $\lim\limits_{x\to 0}\frac{e^{x^3}-\tan x+\sin x-1}{x^n}$

$= \lim\limits_{x\to 0}\frac{e^{x^3}-1}{x^n}+\frac{-\tan x(1-\cos x)}{x^n}$

Now for existence of limit n should be 3

18. Let $y = \sqrt{\alpha+\sqrt{\alpha+\sqrt{\alpha+.........\infty}}}$

where $\alpha = 1-\cos x$;as $x\to 0, \alpha\to 0$

$y = \sqrt{\alpha+y}$; $y^2 = a+y \quad \Rightarrow y^2-y-\alpha = 0$

$y = \frac{1\pm\sqrt{1+4\alpha}}{2}$

(neglecting – ve sign as y can not be – ve)

$y = \frac{1+\sqrt{1+4\alpha}}{2}$

now $l = \lim\limits_{\substack{x\to 0\\ \alpha\to 0}}\frac{\left[\frac{1+\sqrt{1+4\alpha}}{2}-1\right]}{\frac{x^2}{1-\cos x}\cdot(1-\cos x)}$

$= \lim\limits_{\alpha\to 0}\frac{\sqrt{1+4\alpha}-1}{2\cdot 2\cdot\alpha}$ (as $\lim\limits_{x\to 0}\frac{x^2}{1-\cos x} = 2$)

$= \lim\limits_{\alpha\to 0}\frac{(1+4\alpha)-1}{4\alpha(\sqrt{1+4\alpha}+1)} = 2 = \frac{4}{2}$

(rationalising the D^r) $= \lim\limits_{\alpha\to 0}\frac{1}{\sqrt{1+4\alpha}+1} = \frac{1}{2}$]

19. $\lim\limits_{x\to\infty} [(x+a)(x+b)(x+c)]^{1/3}-x$

(using $a-b = \frac{a^3-b^3}{a^2+ab+b^2}$)

$= \frac{[x^3+(a+b+c)x^2+(ab+bc+ca)x+abc]-x^3}{[(x+a)(x+b)(x+c)]^{2/3}+x^2+x[(x+a)(x+b)(x+c)]^{1/3}}$

$\lim\limits_{x\to\infty} x^2\left[(a+b+c)+\frac{ab+bc+ca}{x}+\frac{abc}{x^3}\right]/$

$x^2\left[\left(1+\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^2}\right)+1+\left(1+\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^3}\right)^{1/3}\right]^{1/3}$

$= \frac{a+b+c}{3}$

20. Let $f(n)=\frac{1}{n^2+n+1}+\frac{2}{n^2+n+2}+.....+\frac{n}{n^2+n+n}$

consider g (n)

$$=\frac{1}{n^2+n+n}+\frac{2}{n^2+n+n}+.....\frac{n}{n^2+n+n}$$

$$=\frac{1+2+3+....+n}{n^2+2n}=\frac{n(n+1)}{2(n^2+2n)}$$

$g(n)<f(n)$ ....(1)

|||ly h(n)

$$=\frac{1}{n^2+n+1}+\frac{2}{n^2+n+1}+.......+\frac{n}{n^2+n+1}$$

$$=\frac{n(n+1)}{2(n^2+n+1)}$$

$\therefore$ $f(n)<h(n)$ ....(2)

from (1) and (2)

$g(n)<f(n)<h(n)$

but $\lim_{n\to\infty} g(n)=\lim_{n\to\infty} h(n)=\frac{1}{2}$;

Hence using Sandwich theorem

$\therefore$ $\lim_{n\to\infty} f(n)=\frac{1}{2}$

21. $n<\sqrt{n^2+n+1}<n+1$

Hence $\left[\sqrt{n^2+n+1}\right]=n$

$\therefore$ $l=\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-n\right)=\frac{1}{2}$

23. α, β are the roots of the equation $ax^2+bx+c=0$

$\Rightarrow$ $ax^2+bx+c=a(x-\alpha)(x-\beta)$

$$=e^{\lim_{x\to\alpha}\frac{ax^2+bx+c}{x-\alpha}}=e^{\lim_{x\to\alpha}\frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}}=e^{a(\alpha-\beta)}$$

24. R.H.L. $=\lim_{x\to0^+}\left[(1-e^x)\frac{\sin x}{x}\right]$

when $x\in(0,h)$ and $h\to0$ then $(1-e^x)\in(-1,0)$

and $\frac{\sin x}{x}<1$

So $-1<(1-e^x)\frac{\sin x}{x}<0$,

$$\lim_{x\to0^+}\left[(1-e^x)\frac{\sin x}{x}\right]=-1$$

L.H.L. $=\lim_{x\to0^-}\left[(1-e^x)\frac{\sin x}{-x}\right]=$

$$\lim_{x\to0^-}\left[(e^x-1)\frac{\sin x}{x}\right]$$

when $x\in(-h,0)$ and $h\to0$, then $e^x-1\in(-1,0)$

and $\frac{\sin x}{x}<1$

So $-1<(e^x-1)\frac{\sin x}{x}<0$ So

$$\lim_{x\to0^-}\left[(e^x-1)\frac{\sin x}{x}\right]=-1$$

L.H.L. = R.H.L. = –1

26.

$$\lim_{y\to0}\left(\lim_{x\to\infty}\frac{\exp\left(x\ln\left(1+\frac{ay}{x}\right)\right)-\exp\left(x\ln\left(1+\frac{by}{x}\right)\right)}{y}\right)$$

$$=\lim_{y\to0}\left(\lim_{x\to\infty}\frac{\left(1+\frac{ay}{x}\right)^x-\left(1+\frac{by}{x}\right)^x}{y}\right)$$ by expansion

$$=\lim_{y\to0}$$

$$\left(\lim_{x\to\infty}\frac{\left(1+ay+\frac{x(x-1)}{2!}.\frac{a^2y^2}{x^2}+.....\right)-\left(1+by+\frac{x(x-1)}{2!}.\frac{b^2y^2}{x^2}+....\right)}{y}\right)$$

$$=\lim_{y\to0}\left[\frac{y(a-b)+\frac{y^2}{2}(a^2-b^2)+.....}{y}\right]$$

$=a-b$

27. $\lim_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right)\right)$

$\Rightarrow$ $\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}+\sin\left(\frac{1}{x^2}\right)\right)=1+0=1$

EXERCISE - 2

Part # I : Multiple Choice

1. $f(x) = \lim_{n\to\infty} \dfrac{2x^{2n}\sin\frac{1}{x} + x}{1 + x^{2n}}$

$$= \begin{cases} 2\sin\frac{1}{x} & \text{for } x > 1 \text{ or } x < -1 \\ \dfrac{2\sin(1/x) + x}{2} & \text{for } x = 1 \text{ or } -1 \\ x & \text{if } -1 \le x < 1 \end{cases}$$

$\lim_{x\to\infty} x\,f(x) = \lim_{x\to\infty} 2x\sin\frac{1}{x} = 2$

$\lim_{x\to 1^+} f(x) = 2\sin 1$; $\lim_{x\to 1^-} f(x) = 1$

$\Rightarrow \lim_{x\to 1} f(x)$ does not exist. $\Rightarrow$ **(B)**

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} 2\sin\frac{1}{x} = 0$]

5. $f(x) = \dfrac{|x + \pi|}{\sin x}$

(A) $f(-\pi^+) = \lim_{h\to 0} \dfrac{|-\pi + h + \pi|}{\sin(-\pi + h)} = \lim_{h\to 0} \dfrac{|h|}{-\sin h} = -1$

(B) $f(-\pi^-) = \lim_{h\to 0} \dfrac{|-\pi - h + \pi|}{\sin(-\pi - h)} = \lim_{h\to 0} \dfrac{|h|}{\sin h} = 1$

(C) $f(-\pi^+) \ne f(-\pi^-)$ So $\lim_{x\to -\pi} f(x)$ does not exist

(D) for $\lim_{x\to\pi} f(x)$

LHL $= \lim_{x\to\pi^-} \dfrac{|x + \pi|}{\sin x} = \lim_{h\to 0} \dfrac{2\pi - h}{\sinh} = \dfrac{2\pi}{0} = \infty$

RHL $= \lim_{x\to\pi^+} \dfrac{|x + \pi|}{\sin x} = \lim_{h\to 0} \dfrac{2\pi + h}{-\sinh} = -\dfrac{2\pi}{0} = -\infty$

LHL $\ne$ RHL

So $\lim_{x\to\pi} f(x)$ does not exist.

9. put $\theta = -1$; $\dfrac{1-1-2}{2} \le f(-1) \le \dfrac{1-2-1}{2}$

$-1 \le f(-1) \le -1$

$\Rightarrow f(-1) = -1$

$\lim_{\theta\to -1} \dfrac{\theta^2 + \theta - 2}{\theta + 3} = -1 = \lim_{\theta\to -1} \dfrac{\theta^2 + 2\theta - 1}{\theta + 3}$

using sequeeze play theorem

$\lim_{\theta\to -1} \dfrac{f(\theta)}{\theta^2} = -1$; $\lim_{\theta\to -1} f(\theta) = -1$

12. Put $x = \dfrac{\pi}{2} - h$

● $= \lim_{h\to 0} \dfrac{a^{\tanh} - a^{\sinh}}{\tanh - \sinh}$

$= \lim_{h\to 0} a^{\sinh}\left(\dfrac{a^{\tanh - \sinh} - 1}{\tanh - \sinh}\right)$

$= \log_e a$

$m = \lim_{x\to -\infty} \dfrac{x^2 + ax - x^2 + ax}{\sqrt{x^2 + ax} + \sqrt{x^2 - ax}}$

$= \lim_{x\to -\infty} \dfrac{2ax}{|x|\left(\sqrt{1 + \frac{a}{x}} + \sqrt{1 - \frac{a}{x}}\right)}$

$= \dfrac{2ax}{-x(2)} = -a$

Part # II : Assertion & Reason

1. Statement - 1

$\lim_{x\to 0} \dfrac{\sqrt{\dfrac{1 - \cos 2x}{2}}}{x} = \lim_{x\to 0} \dfrac{|\sin x|}{x}$

$\Rightarrow$ L.H.L. $= -1$ & R.H.L. $= 1$

Statement - 2 is true

2. For all x, $x-1<[x]\le x$, where [.] denotes greatest integer function.

$$\Rightarrow x^n-1<[x^n]\le x^n \Rightarrow \frac{1}{x^n}\le\frac{1}{[x^n]}<\frac{1}{x^n-1}$$

Multiplying the inequation by $x^n+nx^{n-1}+1$ and taking the limit as $x\to\infty$,

We get,

$$\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{x^n}\le\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{[x^n]}<\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{x^n-1}$$

Evaluating the limits on the left and right side of the inequality, we obtain

$$\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{x^n}=\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{x^n-1}=1$$

And hence by sandwich theorem,

$$\lim_{x\to\infty}\frac{x^n+nx^{n-1}+1}{[x^n]}=1$$

$\Rightarrow$ Statement 1 is false.

EXERCISE - 3

Part # I : Matrix Match Type

3. (A) $\lim\limits_{x\to0}\dfrac{\tan[e^2]x^2-\tan[-e^2]x^2}{\sin^2 x}$

$$=\lim_{x\to0}\frac{[e^2]x^2\dfrac{\tan[e^2]x^2}{[e^2]x^2}-[-e^2]x^2\dfrac{\tan[-e^2]x^2}{[-e^2]x^2}}{x^2\dfrac{\sin^2 x}{x^2}}$$

$=[e^2]-[-e^2]=15$

(B) $\lim\limits_{x\to0}\left[\left(\min(t^2+4t+6)\right)\dfrac{\sin x}{x}\right]=\lim\limits_{x\to0}\left[\dfrac{2\sin x}{x}\right]$

$\because$ $\sin x<x$

$\Rightarrow \dfrac{2\sin x}{x}<2 \quad\Rightarrow\quad \left[\dfrac{2\sin x}{x}\right]=1$

So $\lim\limits_{x\to0}\left[\dfrac{2\sin x}{x}\right]=1$

(C) $\lim\limits_{x\to0}\dfrac{(1+x^2)^{1/3}-(1-2x)^{\frac{1}{4}}}{x+x^2}$

$$=\lim_{x\to0}\frac{\left(1+\dfrac{x^2}{3}+....\right)-\left(1-\dfrac{x}{2}+\dfrac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}4x^2+....\right)}{x+x^2}$$

$=\dfrac{1}{2}$

(D) $\lim\limits_{x\to0}=\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}=\lim\limits_{x\to0}\dfrac{2\sqrt{2}\sin^2\frac{x}{4}}{\sin^2 x}$

$$=\lim_{x\to0}\frac{2\sqrt{2}\sin^2\frac{x}{4}}{16.\dfrac{x^2}{16}.\dfrac{\sin^2 x}{x^2}}$$

$=\dfrac{\sqrt{2}}{8}$

4. (A) $L = \lim_{h\to 0} \dfrac{\cos\left(\tan^{-1}\left(\tan\left(\frac{\pi}{2}+h\right)\right)\right)}{\frac{\pi}{2}+h-\frac{\pi}{2}}$

$= \dfrac{\cos\left(-\frac{\pi}{2}+h\right)}{h} = \dfrac{\sin h}{h} = 1$

Now $\cos[2\pi(1)] = 1$

(B) $k = \lim_{n\to\infty} \prod_{r=2}^{n} \dfrac{r^3-1}{r^3+1} = \lim_{n\to\infty} \prod_{r=2}^{n} \dfrac{(r-1)}{(r+1)}\dfrac{(r^2+r+1)}{(r^2-r+1)}$

$= \lim_{n\to\infty} \dfrac{2}{n(n+1)} \times \dfrac{n^2+n+1}{3} = \dfrac{2}{3}$

so $\operatorname{cosec}\theta = \dfrac{2}{3}$ $\Rightarrow$ No. of solution is zero

(C) $\lim_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^x = 4$

$\Rightarrow e^{\lim_{x\to\infty} x\left(\frac{x+c-x+c}{x-c}\right)} = 4 \Rightarrow e^{\lim_{x\to\infty} x\left(\frac{2c}{x-c}\right)} = 4$

$\Rightarrow e^{2c} = 4$ $\Rightarrow e^c = 2$ (only positive value)

$\Rightarrow \dfrac{-e^c}{2} = -1$

(D) $\lim_{x\to-\infty} \dfrac{(3x^4+2x^2)\sin(1/x) - x^3 + 5}{(-x)^3 + (-x)^2 - x + 1} = k$

$\Rightarrow \lim_{x\to-\infty} \dfrac{\left(3+\frac{2}{x^2}\right)\frac{\sin(1/x)}{(1/x)} - 1 + \frac{5}{x^3}}{-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}} = k$

$\Rightarrow \dfrac{3-1}{-1} = k \Rightarrow \dfrac{k}{2} = -1$

5. (A) put $x = 1/y$ $\quad \lim_{y\to 0}\left(\dfrac{1}{1+y}\right)^{1/y}$

$e^{\lim_{y\to 0}\frac{1-y-1}{y(1+y)}} = e^{-1}$ z $\Rightarrow$ (S)

(B) $\lim_{y\to 0}(\sin y + \cos y)^{1/y} = e^{\lim_{y\to 0}\frac{\sin y + \cos y - 1}{y}} = e$

$\Rightarrow$ (R)

(C) $e^{\lim_{x\to 0}\frac{\cos x - 1}{\frac{\tan^2 x}{x^2}\cdot x^2}} = e^{-\frac{1}{2}}$ $\Rightarrow$ (Q)

(D) $e^{\lim_{x\to 0}\frac{\tan((\pi/4)+x) - \tan(\pi/4)}{x}}$

$= e^{\lim_{x\to 0}\frac{\tan x[1+\tan((\pi/4)+x)\cdot 1]}{x}} = e^2$ $\Rightarrow$ (P)

6. (A) $l = \lim_{x\to 0} \dfrac{e^{x^2} - 1 + x - e^x + 1}{2\frac{\sin^2 x}{x^2}\cdot x^2}$

$= \dfrac{1}{2}\left[\lim_{x\to 0}\dfrac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\dfrac{x-e^x+1}{x^2}\right]$

$= \dfrac{1}{2}\left[1 - \lim_{x\to 0}\dfrac{e^x - x - 1}{x^2}\right]$

$= \dfrac{1}{2}\left[1-\dfrac{1}{2}\right] = \dfrac{1}{4}$ $\Rightarrow \dfrac{1}{l} = 4$

$= e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{3+x}{3-x}-1\right)} = e^{\lim_{x\to 0}\frac{2x}{x(3-x)}} = e^{2/3}$

$\Rightarrow 2+3 = 5$

(C) $\lim_{x\to 0}\dfrac{(\tan^3 x - x^3) - (\tan x^3 - x^3)}{x^5}$

$= \lim_{x\to 0}\dfrac{\tan^3 x - x^3}{x^5} - \underbrace{\lim_{x\to 0}\dfrac{\tan x^3 - x^3}{x^5}}_{\text{zero (by expansion)}}$

$= \lim_{x\to 0}\dfrac{(\tan x - x)}{x^3}\cdot\dfrac{(\tan^2 x + x\tan x + x^2)}{x^2}$

$= \dfrac{1}{3}\times 3 = 1$

(D) rationalising gives

$\lim_{x\to 0}\dfrac{(x+2\sin x)\left[\sqrt{(x^2+2\sin x+1)} + \sqrt{\sin^2 x - x + 1}\right]}{(x^2+2\sin x+1) - (\sin^2 x - x + 1)}$

$2\cdot\lim_{x\to 0}\dfrac{x+\sin 2x}{x^2 - \sin^2 x + 2\sin x + x}$

$= 2\cdot\lim_{x\to 0}\dfrac{1+\frac{\sin 2x}{x}}{x - \frac{\sin^2 x}{x} + 2 + 1} = 2\left(\dfrac{1+2}{3}\right) = 2$

Part # II : Comprehension

Comprehension– 3

1 to 3

$$f(x) = \lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}\right)^n = e^{\lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}-1\right).n}$$

Substituting, $n = \dfrac{1}{t^2}$

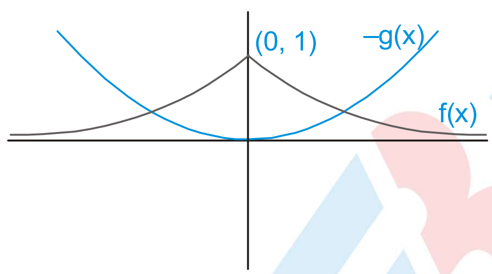
$$f(x) = e^{\lim_{t\to0}\left(\frac{\cos tx-1}{t^2}\right)} = e^{\lim_{t\to0}-x^2\left(\frac{1-\cos tx}{t^2x^2}\right)} = e^{-\frac{1}{2}x^2}$$

$$g(x) = -\,x^{4b}$$

$$b = \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}\right) = \frac{1}{2}$$

$$\therefore\quad g(x) = -\,x^{4.\frac{1}{2}} = -x^2$$



By observation, graphs of f(x) and – g(x) intersect each other at two points

$\therefore$ Number of solutions is 2.

EXERCISE - 4

Subjective Type

1. Using L' Hospital rule

$$\lim_{x\to\frac{3\pi}{4}}\frac{1+(\tan x)^{\frac{1}{3}}}{1-2\cos^2 x} = \lim_{x\to\frac{3\pi}{4}}\frac{\frac{1}{3}\tan x^{\frac{-2}{3}}\sec^2 x}{4\cos x\sin x} = -\frac{1}{3}.$$

3. $\lim\limits_{x\to\infty} x^2\sin\left(\ln\sqrt{\cos\frac{\pi}{x}}\right)$

$$= \lim_{x\to\infty}\frac{x^2\sin\left(\ln\sqrt{\cos(\pi/x)}\right)}{\left(\ln\sqrt{\cos(\pi/x)}\right)}\times\left(\ln\sqrt{\cos\left(\frac{\pi}{x}\right)}\right)$$

$$= \lim_{x\to\infty}\frac{x^2}{2}\ln\left(1+\cos\left(\frac{\pi}{x}\right)-1\right)$$

$$= \lim_{x\to\infty}\frac{x^2}{2}\frac{\ln\left(1+\left(\cos\left(\frac{\pi}{x}\right)-1\right)\right)}{\left(\cos\left(\frac{\pi}{x}\right)-1\right)}\times\left(\cos\left(\frac{\pi}{x}\right)-1\right)$$

5. Let $f(x) = \left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2}}+\ldots+\dfrac{1}{\sqrt{n^2}}\right)$

$$= \frac{2n+1}{\sqrt{n^2}} \qquad \text{... (2n+1) terms}$$

$$\lim_{n\to\infty} f(x) = 2$$

$$g(x) = \left(\frac{1}{\sqrt{n^2+2n}}+\frac{1}{\sqrt{n^2+2n}}+\ldots.+\frac{1}{\sqrt{n^2+2n}}\right)$$

...(2n+1) terms

$$\lim_{n\to\infty} g(x) = 2$$

$$\therefore\quad \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots.+\frac{1}{\sqrt{n^2+2n}}\right) = 2$$

6. $\lim\limits_{x\to0}\dfrac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$

$$= \lim_{x\to0}\frac{ae^x+ce^{-x}-b\cos x}{x^2} = 2$$

$$=\lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}+\ldots\right)+c\left(1-x+\frac{x^2}{2!}+\ldots\right)-b+b-b\cos x}{x^2}$$

$= 2$

$\Rightarrow a + c - b = 0,\ a - c = 0\ \&\ \frac{a+c}{2}+\frac{b}{2} = 2$

$\Rightarrow a = c = 1,\ b = 2$

$$=\lim_{x\to\infty}\frac{x^2}{2}\left(\cos\left(\frac{\pi}{x}\right)-1\right)$$

Now put $x = 1/y$ $\Rightarrow$ $\lim_{y\to 0}\frac{\cos(\pi y)-1}{2y^2}=\frac{-\pi^2}{4}$

7. $\lim_{x\to\pi/4}\frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$

$$\text{LHL}=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{\sin\left(\frac{\pi}{2}-2h\right)}}}{\pi-\pi+4h}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{\cos 2h}}}{4h}\times\frac{\sqrt{1+\sqrt{\cos 2h}}}{\sqrt{1+\sqrt{\cos 2h}}}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-\cos 2h}}{4h}\times\frac{1}{\sqrt{1+\sqrt{\cos 2h}}}$$

$$=\frac{\sqrt{2}\,|\sin h|}{4h\times\sqrt{2}}=\frac{1}{4}$$

Similarly $\text{RHL}=\frac{-1}{4}$

Hence LHL $\neq$ RHL $\therefore$ limit does not exist.

10. $\lim_{x\to 1}\frac{\ln\left(\frac{1+x}{2}\right)\times 3.(4^{x-1}-x)}{\left[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right]\sin(x-1)}$

$$=\lim_{h\to 0}\frac{\ln\left(\frac{2+h}{2}\right)\times 3\,(4^h-(1+h))}{\left[(8+h)^{\frac{1}{3}}-(4+3h)^{1/2}\right].\sin h}$$

$$=\lim_{h\to 0}\frac{\frac{\ln\left(1+\frac{h}{2}\right)}{2(h/2)}.3.(4^h-(1+h))}{2\left\{\left[1+\frac{h}{8}\times\frac{1}{3}+\ldots\right]-\left[1+\frac{3h}{4}\times\frac{1}{2}+\ldots\right]\right\}\frac{\sin h}{h}}$$

$$=\lim_{h\to 0}\frac{\frac{3}{4}\left[\frac{4^h-1}{h}-1\right]}{\frac{1}{h}\left[\frac{h}{24}-\frac{3h}{8}\right]}=\frac{-9}{4}\ln\frac{4}{e}.$$

11. (i) $\lim_{x\to 0}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}$

$$=\lim_{x\to 0}\left(\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\ldots\right)}{e}\right)^{\frac{1}{x}}$$

$$=\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\ldots\right)^{\frac{1}{x}}$$

$$=e^{\lim_{x\to 0}\left(-\frac{x}{2}+\frac{11}{24}x^2-\ldots\ldots\right)\cdot\frac{1}{x}}=e^{-\frac{1}{2}}$$

(ii) $\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$ (1^∞ form)

$$=e^{\lim_{x\to 0}\left(\sin^2\left(\frac{\pi}{2-ax}\right)-1\right)\cdot\sec^2\left(\frac{\pi}{2-bx}\right)}$$

$$=e^{\lim_{x\to 0}\frac{-\cos^2\left(\frac{\pi}{2-ax}\right)}{\cos^2\left(\frac{\pi}{2-bx}\right)}}\qquad\left(\frac{0}{0}\text{ form}\right)$$

Applying L.Hospital rule

$$=e^{\lim_{x\to 0}\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{a\pi(2-bx)^2}{b\pi(2-ax)^2}}=e^{\lim_{x\to 0}\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{a\pi}{b\pi}}$$

$$=e^{\lim_{x\to 0}\frac{-\cos\left(\frac{2\pi}{2-ax}\right)}{\cos\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{\frac{2a^2\pi^2}{(2-ax)^2}}{\frac{2b^2\pi^2}{(2-bx)^2}}}=e^{-\frac{a^2}{b^2}}$$

12. $\lim\limits_{x\to1}\dfrac{\left[\sum\limits_{K=1}^{100}x^K-100\right]}{x-1}$

$=\lim\limits_{x\to1}\left[\dfrac{x+x^2+x^3+\dots\dots+x^{100}-100}{x-1}\right]$

$=\lim\limits_{x\to1}\left[\dfrac{1+2x+3x^2+\dots\dots+100x^{99}}{1}\right]$

(using L' Hospital)

$=5050.$

13. Case **(i)** $\lim\limits_{x\to0^+}f(x)$

$=\lim\limits_{x\to0^+}\dfrac{\sin^{-1}(1-\{x\}.\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,(1-\{x\})}$

$=\lim\limits_{h\to0}\dfrac{\sin^{-1}(1-h).\cos^{-1}(1-h)}{\sqrt{2h}\,(1-h)}$

$=\lim\limits_{h\to0}\dfrac{\sin^{-1}(1-h)\sin^{-1}\sqrt{h(2-h)}}{(1-h)\sqrt{2h}}$

$=\lim\limits_{h\to0}\dfrac{\frac{\pi}{2}}{1}\dfrac{\sin^{-1}\sqrt{h(2-h)}}{\sqrt{2h-h^2}\,\sqrt{2h}}\sqrt{2h-h^2}$

$=\lim\limits_{h\to0}\dfrac{\pi}{2}.1.\sqrt{1-\dfrac{h}{2}}=\dfrac{\pi}{2}$

Case **(ii)** $\lim\limits_{x\to0^-}f(x)$

$=\lim\limits_{x\to0^-}\dfrac{\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,(1-\{x\})}$

$=\lim\limits_{h\to0}\dfrac{\sin^{-1}h.\cos^{-1}h}{\sqrt{2(1-h)}\,h}=\dfrac{\frac{\pi}{2}}{\sqrt{2}}=\dfrac{\pi}{2\sqrt{2}}$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. $y=\lim\limits_{x\to1}\dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1}$

$\Rightarrow y=\lim\limits_{x\to1}\dfrac{\left(\sqrt{f(x)}-1\right)}{\left(\sqrt{x}-1\right)}.\dfrac{\left(\sqrt{f(x)}+1\right)}{\left(\sqrt{x}+1\right)}.\dfrac{\left(\sqrt{x}+1\right)}{\left(\sqrt{f(x)}+1\right)}$

$\Rightarrow y=\lim\limits_{x\to1}\dfrac{f(x)-1}{x-1}.\dfrac{\sqrt{x}+1}{\sqrt{f(x)}+1}$

$\Rightarrow y=\lim\limits_{x\to1}\dfrac{f(x)-f(1)}{x-1}.\lim\limits_{x\to1}\dfrac{\sqrt{x}+1}{\sqrt{f(x)}+1}$

$\Rightarrow y=f'(1).\dfrac{2}{\sqrt{f(1)}+1}\quad\Rightarrow\quad y=2.\dfrac{2}{2}=2$

Applying L-Hospital's rule.

3. We have,

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$

$=\lim\limits_{x\to\infty}\left(1+\dfrac{4x+1}{x^2+x+2}\right)^x=e^{\lim\limits_{x\to\infty}\frac{x(4x+1)}{x^2+x+2}}=e^4$

4. $\lim\limits_{x\to\infty}\dfrac{\log x^n-[x]}{[x]}=\lim\limits_{x\to\infty}\dfrac{\log x^n}{[x]}-\lim\limits_{x\to\infty}\dfrac{[x]}{[x]}=0-1=-1$

5. $\lim\limits_{x\to0}\dfrac{\log(3+x)-\log(3-x)}{x}=k$

By L-Hospital's rule,

$\lim\limits_{x\to0}\dfrac{\frac{1}{3+x}+\frac{1}{3-x}}{1}=k\Rightarrow\dfrac{2}{3}=k$

6. We have, $\lim\limits_{x\to a}\dfrac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$

$\Rightarrow\lim\limits_{x\to a}\dfrac{f(a)g'(x)-0-g(a)f'(x)+0}{g'(x)-f'(x)}=4$

$\Rightarrow\dfrac{f(a)g'(a)-g(a)f'(a)}{g'(a)-f'(a)}=4$

$\Rightarrow\dfrac{k\{g'(a)-f'(a)\}}{\{g'(a)-f'(a)\}}=4\qquad[\because f(a)=g(a)=k]$

$\Rightarrow k=4$

7. $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let $x=\frac{\pi}{2}+y : y\to 0$

$\Rightarrow \lim_{y\to 0} \dfrac{\tan\left(-\frac{y}{2}\right)(1-\cos y)}{(-2y)^3}$

$= \lim_{y\to 0} \dfrac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y\to 0} \dfrac{1}{32}\dfrac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\dfrac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \dfrac{1}{32}$

8. Since, $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$

$\therefore \lim_{x\to\infty}\left[\left(1+\frac{ax+b}{x^2}\right)^{\frac{x^2}{ax+b}}\right]^{\frac{2(ax+b)}{x}} = e^2$

$\Rightarrow \lim_{x\to\infty} e^{\frac{2(ax+b)}{x}} = e^2$

$\Rightarrow \lim_{x\to\infty}\dfrac{2(ax+b)}{x} = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$

Thus $a = 1$ and $b \in R$

12. $\lim_{x\to 5}\dfrac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$

$\therefore$ Question must be in $\frac{0}{0}$ form

$\therefore \ (f(5))^2-9=0$

$\Rightarrow \ f(5)=3$

13. $\lim_{x\to a}\dfrac{x^2 f(a)-a^2 f(x)}{x-a} \quad \left[\frac{0}{0} \text{ form}\right]$

Use L'Hospital rule

$= \lim_{x\to a}\dfrac{2x f(a)-a^2 f'(x)}{1}$

$= 2af(a)-a^2f'(a)$

14. $\lim_{x\to 0}\dfrac{1-\cos 2x}{4x^2}.(4x^2).\dfrac{(3+1)}{\frac{x\tan 4x}{4x}(4x)}$

$\frac{1}{2}.4 = 2$

18. $p = \lim_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}} = \lim_{x\to 0^+} e^{\frac{1}{2x}\left(\tan^2\sqrt{x}\right)} = e^{\frac{1}{2}}$

$\log_e p = \frac{1}{2}$

19. $l = \left(\frac{(n+1)}{n}.\frac{n+2}{n}.\frac{n+3}{n}.\ldots\ldots.\frac{n+2n}{n}\right)^{1/n}$

$\log l = \frac{1}{n}\left[\log\left(\frac{n+1}{n}\right)+\log\left(\frac{n+2}{n}\right)+\ldots\ldots\ldots+\log\left(\frac{n+2n}{n}\right)\right]$

$\log l = \int_0^2 \log(1+x)\,dx$

$\log l = \int_1^3 \log t \ \ dt$

$\log l = t\log t - \int \frac{1}{t}.t\ dt$

$\log l = t(\log t - 1)$

$\log l = 3(\log 3 - 1) - 1(\log 1 - 1)$

$= 3\log 3 - 2$

$= \log 27 - \log e^2$

$= \log l = \log\frac{27}{e^2}$

$l = \frac{27}{e^2}$

Part # II : IIT-JEE ADVANCED

6. $\dfrac{a-a\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}-\frac{x^2}{4}}{x^4} = \dfrac{a-a\left(1-\frac{x^2}{2a^2}-\frac{1}{8}\frac{x^4}{a^4}\right)-\frac{x^2}{4}}{x^4}$

$a=2, \quad (\text{coefficient of } x^2=0)$

$\therefore \ L=\frac{1}{64}.$

7. $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$

$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2+x(1-a-b)+1-b}{x+1}\right)=4$

$\Rightarrow \lim_{x\to\infty}\frac{\left((1-a).x+1-a-b+\left(\frac{1-b}{x}\right)\right)}{1+\frac{1}{x}}=4$

for limit to exist finitely

$1-a=0$ and $1-a-b=4$

$\Rightarrow a=1$ and $b=-4.$

8. $\left(\left(1+\frac{a}{3}\right)-1\right)x^2+\left(\left(1+\frac{a}{2}\right)-1\right)x+\left(1+\frac{a}{6}-1\right)=0$

$a\left(\frac{x^2}{3}+\frac{x}{2}+\frac{1}{6}\right)=0 \Rightarrow 2x^2+3x+1=0$

$\Rightarrow x=-\frac{1}{2},-1$

$\Rightarrow \lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are $-\frac{1}{2}$ and -1

10. $\lim_{x\to 0}\frac{x^2\left\{\beta x-\frac{(\beta x)^3}{3!}+...\right\}}{\alpha x-\left(x-\frac{x^3}{3!}+...\right)}=1$

$\lim_{x\to 0}\frac{x^3\left(\beta-\frac{\beta^3x^2}{3!}+...\right)}{(\alpha-1)x+x-\frac{x^3}{3!}-\frac{x^5}{5!}+...}=1$

As limit 1 $\Rightarrow \alpha=1$

$\lim_{x\to 0}\frac{\beta-\frac{\beta^3}{3!}x^2+...}{\frac{1}{3!}-\frac{x^2}{5!}+...}=1$

$\Rightarrow \beta=\frac{1}{3!}=\frac{1}{6}$

$\therefore 6(\alpha+\beta)=6\left(1+\frac{1}{6}\right)=7$

11. $\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{n}{r}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{n^2}{r^2}\right)}\cdot\frac{1}{\prod_{r=1}^{n}\left(\frac{r}{n}\right)}\right]$

$\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{\frac{r}{n}}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{\left(\frac{r}{n}\right)^2}\right)}\cdot\frac{1}{\prod_{r=1}^{n}\left(\frac{r}{n}\right)}\right]$

$=x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left[\frac{x\left(\frac{r}{n}\right)+1}{x\left(\frac{r}{n}\right)^2+1}\right]$

$=x\int_0^1\ln\left(\frac{1+tx}{1+t^2x^2}\right)dt$

Put, $tx=p$, we get

$\ln f(x)=\int_0^x\ln\left(\frac{1+p}{1+p^2}\right)dp$

$\Rightarrow \frac{f'(x)}{f(x)}=\ln\left(\frac{1+x}{1+x^2}\right)$

sign scheme of $f'(x)$

Also $f'(1)=0.$

$\Rightarrow f\left(\frac{1}{2}\right)<f(1), f\left(\frac{1}{3}\right)<f\left(\frac{2}{3}\right), f'(2)<0$

Also, $\frac{f'(3)}{f(3)}-\frac{f'(2)}{f(2)}=\ln\left(\frac{4}{10}\right)-\ln\left(\frac{3}{5}\right)$

$=\ln\left(\frac{4}{6}\right)<0 \Rightarrow \frac{f'(3)}{f(3)}<\frac{f'(2)}{f(2)}$

MOCK TEST

2. (D)

$$\lim_{x\to 1} \sec\frac{\pi}{2^x}\cdot \ell nx = \lim_{x\to 1}\frac{\ell n x}{\cos\frac{\pi}{2^x}}$$

$$= \lim_{x\to 1}\frac{\frac{1}{x}}{\pi\sin\frac{\pi}{2^x}\cdot\frac{1}{2^x}\cdot \ell n2} = \frac{1}{\pi \ell n2\cdot\frac{1}{2}} = \frac{2}{\pi \ell n2}$$

3. (A)

$$RHL = \lim_{h\to 0^+} |h(h-1)|^{[\cos 2h]} = \lim_{h\to 0^+} (h(1-h))^0 = 1$$

$$LHL = \lim_{h\to 0^-} |-h(-h-1)|^{[\cos 2h]} = \lim_{h\to 0^-} (h(1+h))^0 = 1$$

$\therefore$ Required limit = 1

7. (C)

$$\lim_{x\to 0}\frac{\sin x^4 - x^4\cos x^4 + x^{20}}{x^4(e^{2x^4}-1-2x^4)}$$

$$= \lim_{t\to 0}\frac{\sin t - t\cos t + t^5}{t(e^{2t}-1-2t)} \qquad \{\text{Let } x^4 = t\}$$

$$= \lim_{t\to 0}\frac{\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-.....\right)-t\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-.....\right)+t^5}{t\left(1+2t+\frac{4t^2}{2!}+\frac{8t^3}{3!}+\frac{16t^4}{4!}+....-1-2t\right)}$$

$$= \lim_{t\to 0}\frac{-\frac{t^3}{6}+\frac{t^3}{2}+\frac{t^5}{5!}-\frac{t^5}{4!}+...+t^5}{2t^3+\frac{8t^4}{3!}+...}$$

$$= \frac{-\frac{1}{6}+\frac{1}{2}}{2} = \frac{-1+3}{12} = \frac{1}{6}$$

9. (C)

$$\lim_{x\to 0^+}(x)^{\frac{1}{\ell n \sin x}} = \lim_{x\to 0^+} e^{\frac{\ell n x}{\ell n x + \ell n\frac{\sin x}{x}}} = e$$

10. (A)

S_1 : true

$$S_2 : \lim_{x\to 0^+}\sqrt{x}(\log x)^2 = \lim_{x\to 0^+}\frac{(\log x)^2}{x^{-\frac{1}{2}}}$$

$$= \lim_{x\to 0^+}\frac{2\log x\cdot\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+}(-4)\frac{\log x}{x^{-1/2}}$$

$$= \lim_{x\to 0^+}(-4)\frac{1/x}{-\frac{1}{2}x^{-3/2}} = 8\lim_{x\to 0^+}\sqrt{x} = 0$$

S_3 : Since $0 < \frac{\sin x}{x} < 1$ for all x near to 0

$$\therefore \quad \left\{\frac{\sin x}{x}\right\} = \frac{\sin x}{x}$$

$$\therefore \quad \lim_{x\to 0}\left\{\frac{\sin x}{x}\right\} = 1$$

S_4 : When $x\to\infty$, through the values $n\pi$, $n\in N$

then $\frac{\ell n\cos^2 x}{x^2} = 0 \quad \therefore \quad S_4$ is false

11. (A, B)

(A) $\lim_{x\to 0^-}\left[x+\frac{1}{2}\right] = 0,\ \lim_{x\to 0^+}\left[2x+\frac{1}{3}\right] = 0$

$\therefore \lim_{x\to 0}[f(x)] = 0$

(B) $\lim_{x\to 0^-}\left(x+\frac{1}{2}\right) = \frac{1}{2},\ \lim_{x\to 0^+}\left(2x+\frac{1}{3}\right) = \frac{1}{3}$

$\therefore f(0^-)\neq f(0^+) \quad \therefore \lim_{x\to 0} f(x)$ does not exist

(C) $\lim_{x\to 0} f(x)$ does not exist so $\left[\lim_{x\to 0} f(x)\right]$ does not exist.

(D) $\lim_{x\to 0^-}\frac{\left[x+\frac{1}{2}\right]}{x} = 0,\ \lim_{x\to 0^+}\frac{\left[2x+\frac{1}{3}\right]}{x} = 0$

$\therefore \lim_{x\to 0}\frac{[f(x)]}{x} = 0$

12. (A, B)

$$\lim_{t\to 0}\frac{\sin(\tan t)}{\sin t}=\lim_{t\to 0}\frac{\tan t}{t}=1$$

$$\lim_{t\to \pi/2}\frac{\sin(\cos x)}{\cos x}=1$$

$$\lim_{t\to 0}\frac{|x|}{x}=\text{DNE}$$

$$\lim_{t\to \pi/2}\frac{1-\cos x}{x^2}=0$$

13. (B, C)

By standard results

14. (C)

$$f(x)=\left(\frac{x}{2+x}\right)^{2x}$$

$$\Rightarrow \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{2x}$$

$$=\lim_{x\to\infty}\left(1+\frac{x}{2+x}-1\right)^{2x}$$

$$=e^{\lim_{x\to\infty}2x\left(-\frac{2}{2+x}\right)}=e^{\lim_{x\to\infty}-4\left(\frac{x}{2+x}\right)}=e^{-4}$$

Also $\lim_{x\to 1} f(x)=\left(\frac{1}{3}\right)^2=\frac{1}{9}$

15. (A, D)

If $x\in Q$

$n!\pi x \to$ multiple of π

$\cos(n!\pi x)\to \pm 1$

$1+1=2$

if $x\notin Q$ $\cos((n!\pi x)$ be any nubmer between -1 & 1

$\lim_{m\to\infty}$ [1 + [any no between –1 & 1]^{2m}]

$[1]=1$

16. (B)

$$\lim_{x\to 0^-}\sin^{-1}\{x\}=\lim_{h\to 0}\sin^{-1}\{0-h\}$$

$$=\lim_{h\to 0}\sin^{-1}(1-\{h\})=\frac{\pi}{2}$$

$$\lim_{x\to 0^+}\sin^{-1}\{x\}=\lim_{h\to 0}\sin^{-1}\{0+h\}=\lim_{h\to 0}\sin^{-1}\{h\}=0$$

$f(0^-)\neq f(0^+)$ so limit does not exist

Statement -I is true, Statement-II is true but not explaning statement -I.

17. (D)

Obvious

18. (A)

Statement-I :

$$\underset{x\to\infty}{\text{limit}}\left\{1+\left(\frac{a_1^{1/x}+a_2^{1/x}+a_3^{1/x}+\ldots..+a_n^{1/x}}{n}-1\right)\right\}^{nx}$$

$$=\underset{x\to\infty}{\text{limit}}\ e^{\frac{a_1^{1/x}-1}{1/x}}\ \underset{x\to\infty}{\text{limit}}\ e^{\frac{a_2^{1/x}-1}{1/x}}\ \ldots\ldots\ \underset{x\to\infty}{\text{limit}}\ e^{\frac{a_n^{1/x}-1}{1/x}}$$

$$=e^{\ln a_1}\,e^{\ln a_2}\ldots\ldots\ e^{\ln a_n}$$

$$=a_1.a_2\ldots\ldots..a_n$$

Statement-II : Similarly

19. (A)

Statement-II is true. Also $\lim_{x\to 1^-}\frac{3x^2+1}{(x-1)(x-2)}=+\infty$

and $\lim_{x\to 2^-}\frac{3x^2+1}{(x-1)(x-2)}=-\infty$

$\Rightarrow$ $f(1^-)=0$ & $f(2^-)=2$

20. (C)

$$\lim_{x\to\infty}\left(\frac{1^2}{x^3}+\frac{2^2}{x^3}+\frac{3^2}{x^3}+\ldots..+\frac{x^2}{x^3}\right)$$

$$\Rightarrow S_n=\frac{1^2+2^2+3^2+\ldots...+x^2}{x^3}$$

$$S_n=\lim_{x\to\infty}\frac{x(x+1)(2x+1)}{x^3}\ \Rightarrow\ S_n=\frac{\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)}{6}$$

$$S_n=\frac{1}{3}$$

Statement-II is correct only when R.H.S. does not take any indeterminate form.

21. (A) → (t), (B) → (r), (C) → (q), (D) → (p)

(A) $\lim_{x\to 1}\int_1^{x^2}\frac{f(t)-t}{(x-1)^2}dt = \lim_{x\to 1}\frac{\int_1^{x^2}(f(t)-t)dt}{(x-1)^2}$

$= \lim_{x\to 1}\frac{2x(f(x^2)-x^2)}{2(x-1)}$

$= \lim_{x\to 1}\frac{f(x^2)-x^2+2x^2f'(x^2)-2x^2}{1} = 4$

(B) $\lim_{n\to\infty}\left(\frac{1+\sqrt[n]{4}}{2}\right)^n = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{4}-1}{2}\right)^n$

$= e^{\lim_{n\to\infty}\frac{4^{1/n}-1}{2}\cdot n} = e^{\lim_{n\to\infty}\frac{1}{2}\frac{4^{1/n}-1}{1/n}} = e^{\frac{1}{2}\ln 4} = 2$

(C) $f(x) = \lim_{n\to\infty}\frac{2x}{\pi}\tan^{-1}(nx) = x,\ \ x>0$

$\therefore\ \lim_{x\to 0+}[f(x)-1] = \lim_{x\to 0+}[x-1] = -1$

(D) $\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right] = \lim_{n\to\infty}\left[1-\frac{1}{2^n}\right] = 0$

22. (A) → (r), (B) → (q), (C) → (p), (D) → (q)

(A) $\lim_{x\to 0}\left(\frac{f^2(a+x)}{f(a)}\right)^{1/x} = e^{\lim_{x\to 0}\left(\frac{f^2(a+x)-f(a)}{xf(a)}\right)}$

$= e^{\lim_{x\to 0}\frac{f^2(a+x)-(f(a))^2}{x}} = e^4$

$\therefore\ k = 4$

(B) $\lim_{x\to\frac{\pi}{2}^+}\frac{\cos(\tan^{-1}(\tan x))}{x-\frac{\pi}{2}}$

$\lim_{h\to 0^+}\frac{\cos\left(\tan^{-1}\left(\tan\left(\frac{\pi}{2}+h\right)\right)\right)}{h}$

$\lim_{h\to 0^+}\frac{\cos\left(h-\frac{\pi}{2}\right)}{h} = \lim_{h\to 0^+}\frac{\sin h}{h} = 1$

(C) $\lim_{x\to\pi}\frac{\sin(\cos x+1)}{\cos\left(\frac{x}{2}\right)} = \lim_{x\to\pi}\frac{\sin\left(2\cos^2\left(\frac{x}{2}\right)\right)}{\left(2\cos^2\frac{x}{2}\right)}\cdot$

$\left(2\cos\frac{x}{2}\right) = 1\times 0 = 0$

(D) $\lim_{x\to 0}\frac{xe^{\sin x}-e^x\sin^{-1}(\sin x)}{\sin^2 x - x\sin x}$

$= \lim_{x\to 0}\frac{xe^x}{\sin x}\cdot\frac{e^{\sin x - x}-1}{\sin x - x} = 1$

23. $\lim_{x\to 0^+} f(x) = \text{finite}$

$\Rightarrow\ \lim_{h\to 0} f(0+h) = \text{finite}$

$\Rightarrow\ \lim_{h\to 0}\frac{\sin h + ae^h + be^{-h} + c\ln(1+h)}{h^3} = \text{finite}$

at $h\to 0$, Numerator $\to a+b$ and Denominator $\to 0$

RHS is finite

$\therefore\ a+b=0$

$\Rightarrow\ b = -a$ ....(i)

Now, $\lim_{h\to 0}\frac{\sin h + ae^h - ae^{-h} + c\ln(1+h)}{h^3} = \text{finite}$

$\Rightarrow\ \lim_{h\to 0}\frac{\frac{\sin h}{h}+\frac{a(e^h-1)}{h}-a\frac{(e^{-h}-1)}{h}+\frac{c\ln(1+h)}{h}}{h^2}$

$= \text{finite}$

at $h\to 0$, Numerator $\to 1+a+a+c$ and Denominator $\to 0$

RHS is finite

$\therefore\ 1+2a+c=0$

$\Rightarrow\ c = -1-2a$ ....(ii)

$\therefore\ \lim_{h\to 0}\frac{\sin h + ae^h - ae^{-h} - (1+2a)\ln(1+h)}{h^3} = \text{finite}$

(by L'Hospital'r sule) $\left(\text{from}\left(\frac{0}{0}\right)\right)$

$\lim_{h\to 0}\frac{\cos h + ae^h + ae^{-h} - \frac{(1+2a)}{(1+h)}}{3h^2}$

$$\lim_{h\to 0}\frac{-\sin h + ae^{h} - ae^{-h} + \dfrac{(1+2a)}{(1+h^2)}}{6h}$$

Numerator must be = 0

$$0 + a - a + \frac{1+2a}{(1)^2} = 0$$

$$\therefore \quad a = -\frac{1}{2}$$

23.

1. (A)

From eq, **(i)**, $b = \dfrac{1}{2}$

2. (B)

and from eq. **(ii)**, $\quad c = -1 + 1 = 0$

3. (C)

24. 1. (B) 2. (C) 3. (D)

$$f(x) = \lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n = \lim_{n\to\infty}\left(1+\left(\cos\sqrt{\frac{x}{n}}-1\right)\right)^n$$

$$= e^{\lim\limits_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}-1\right)n} = e^{-\lim\limits_{n\to\infty}2\sin^2\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)n}$$

$$= e^{-2\lim\limits_{n\to\infty}\frac{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}{\frac{1}{n}}} = e^{-2\lim\limits_{n\to\infty}\frac{\frac{1}{4}\frac{x}{n}}{\frac{1}{n}}} = e^{-\frac{x}{2}}$$

$y = f(x) = e^{-x/2}, x \ge 0 \qquad$ range = (0, 1]

$$g(x) = \lim_{n\to\infty}(1 - x + x\sqrt[n]{e})^n$$

$$= e^{\lim\limits_{n\to\infty} x\frac{(e^{\frac{1}{n}}-1)}{1/n}} = e^x \quad \forall\ x \in R$$

$h(x) = \tan^{-1}(g^{-1}(f^{-1}(x)))$

$$-\frac{x}{2} = \ell ny \quad \Rightarrow \quad x = 2\ell n\frac{1}{y}$$

$$\Rightarrow\ f^{-1}(x) = 2\ell n\frac{1}{x}\ ;\ 0 < x \le 1$$

$y = g(x) = e^x$

$x = \ell ny \quad \Rightarrow \quad g^{-1}(x) = \ell nx$

$$\therefore\ g^{-1}\left(2\ell n\frac{1}{x}\right) = \ell n\left(2\ell n\left(\frac{1}{x}\right)\right) \quad \text{for } 0 < x < 1$$

$$\therefore\ h(x) = \tan^{-1}\left(\ell n\left(\ell n\frac{1}{x^2}\right)\right) \quad \text{for } 0 < x < 1$$

1. $\displaystyle \lim_{x\to 0^+}\frac{\ell n f(x)}{\ell n g(x)} = \lim_{x\to 0^+}\frac{-x/2}{x} = -\frac{1}{2}$

2. domain of h(x) is (0, 1)

3. $h(x) = \tan^{-1}(\ell n(\ell n\, 1/x^2)) \qquad 0 < x < 1$

$$1 < \frac{1}{x^2} < \infty \quad \Rightarrow \quad 0 < \ell n\frac{1}{x^2} < \infty$$

$\therefore\ -\infty < \ell n(\ell n(1/x^2)) < \infty$

$\therefore$ range of h(x) is $(-\pi/2, \pi/2)$

25. $\displaystyle a = \lim_{n\to\infty}\lim_{\alpha\to 1^+}\frac{\alpha^n|\sin x| + \alpha^{-n}|\cos x|}{\alpha^n + \alpha^{-n}}$

$$= \lim_{n\to\infty}\lim_{\alpha\to 1^+}\frac{|\sin x| + \alpha^{-2n}|\cos x|}{1+\alpha^{-2n}} = |\sin x|$$

$$b = \lim_{n\to\infty}\lim_{\alpha\to 1^-}\frac{\alpha^n|\sin x| + \alpha^{-n}|\cos x|}{\alpha^n + \alpha^{-n}}$$

$$= \lim_{n\to\infty}\lim_{\alpha\to 1^-}\frac{\alpha^{2n}|\sin x| + |\cos x|}{\alpha^{2n}+1} = |\cos x|$$

$$c = \lim_{n\to\infty}\frac{\pi}{4n}\left[1 + \cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \dots\dots\dots + \cos\frac{(n-1)\pi}{2n}\right]$$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left(\frac{\sin\frac{\pi}{4}\cos\frac{(n-1)\pi}{4n}}{\sin\frac{\pi}{4n}}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{2}$$

$$\therefore\ b + c - \frac{1}{2} = |\cos x|$$

$$f(x) = \max\left\{|\sin x|, |\cos x|, \frac{1}{2}\right\}$$

$$\therefore\ \text{range of } f(x) \text{ is } \left[\frac{1}{\sqrt{2}}, 1\right]$$

26. (7)

$$\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$$

rationalizing

$$\Rightarrow \lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+c}+\sqrt{x^4+2x^3-cx^2+3x-d}}=4$$

Limit is finite so degree of N^r must be 2

So, $a-2=0 \Rightarrow a=2$

then divide N_r & D_r both by x^2

$$\lim_{x\to\infty}\frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\frac{a}{x}+\frac{3}{x^2}+\frac{b}{x^3}+\frac{2}{x^4}}+\sqrt{1+\frac{2}{x}-\frac{c}{x^2}+\frac{3}{x^3}-\frac{d}{x^4}}}=4$$

$$\Rightarrow \frac{3+c}{2}=4 \Rightarrow c=5,\ a=2 \text{ and } b, d\in R$$

27. $$\lim_{x\to0}\frac{\sin 3x}{x^3}+\frac{a}{x^2}+b=\lim_{x\to0}\frac{\sin 3x+ax+bx^3}{x^3}$$

$$=\lim_{x\to0}\frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$$

for existence of limit $3+a=0 \Rightarrow a=-3$

$$\therefore\ l=\lim_{x\to0}\frac{\sin 3x-3x+bx^3}{x^3}$$

$$=27.\frac{\sin t-t}{t^3}+b=0 \quad (3x=t)$$

$$=-\frac{27}{6}+b=0 \Rightarrow b=\frac{9}{2}$$

$a+2b=6$

28. (3)

$$\lim_{x\to0}\left([f(x)]+x^2\right)^{\frac{1}{\{f(x)\}}}=\lim_{x\to0}(1+x^2)^{\frac{1}{\frac{\tan x}{x}-1}}$$

$$=\lim_{x\to0}(1+x^2)^{\frac{x}{\tan x-x}}$$

$$=e^{\lim_{x\to0}\frac{x^3}{\tan x-x}}=e^{\lim_{x\to0}\frac{3x^2}{\sec^2 x-1}}$$

$$=e^{\lim_{x\to0}\frac{6x}{2\sec^2 x\tan x}}=e^3$$

29. $$\operatorname{Limit}_{x\to\infty}\frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)};\ \text{as}_{x\to\infty}\left(\frac{\log_a x}{x^a}\right)\to0$$

and $\left(\frac{a^x}{\log_a x}\right)\to\infty$ (using L'opital rule)

$$\therefore\ l=\frac{\pi/2}{\pi/2}=1$$

30. (2)

$$P=\frac{\prod_{r=1}^{n}\left((2r-1)^4+\frac{1}{4}\right)}{\prod_{r=1}^{n}\left((2r)^4+\frac{1}{4}\right)}$$

$$(2r-1)^4+\frac{1}{4}=(2r-1)^4+\frac{1}{4}+(2r-1)^2-(2r-1)^2$$

$$=\left((2r-1)^2+\frac{1}{2}\right)^2-(2r-1)^2$$

$$=\left((2r-1)^2+(2r-1)+\frac{1}{2}\right)\left((2r-1)^2-(2r-1)+\frac{1}{2}\right)$$

$$\Rightarrow (2r-1)^4+\frac{1}{4}$$

$$=\left((2r-1)2r+\frac{1}{2}\right)\left((2r-1)(2r-2)+\frac{1}{2}\right)\quad ....\text{(i)}$$

Now $$(2r)^4+\frac{1}{4}=(2r)^4+\frac{1}{4}+(2r)^2-(2r)^2$$

$$=\left((2r)^2+\frac{1}{2}\right)^2-(2r)^2$$

$$=\left((2r)^2+2r+\frac{1}{2}\right)\left((2r)^2-2r+\frac{1}{2}\right)$$

$\Rightarrow\ (2r)^4 + \frac{1}{4} = \left(2r(2r+1)+\frac{1}{2}\right)\left(2r(2r-1)+\frac{1}{2}\right)$

........(ii)

Now by equation (i) and (ii)

$\therefore\ P = \dfrac{\prod_{r=1}^{n}\left((2r-1)(2r-2)+\frac{1}{2}\right)}{\prod_{r=1}^{n}\left(2r(2r+1)+\frac{1}{2}\right)} =$

$$\frac{\frac{1}{2}\cdot\left(3.2+\frac{1}{2}\right)\left(5.4+\frac{1}{2}\right)\ldots\ldots\left((2n-1)(2n-2)+\frac{1}{2}\right)}{\left(2.3+\frac{1}{2}\right)\left(4.5+\frac{1}{2}\right)\ldots\ldots\left((2n-2)(2n-1)+\frac{1}{2}\right)\left(2n(2n+1)+\frac{1}{2}\right)}$$

$\Rightarrow\ P = \dfrac{1}{2.\left(2n(2n+1)+\frac{1}{2}\right)}$ ....(iii)

as $\lim\limits_{n\to\infty} n^{\alpha} P$ exists

$\Rightarrow\ \lim\limits_{n\to\infty} n^{\alpha}\dfrac{1}{2.\left(2n(2n+1)+\frac{1}{2}\right)}$ exists

$\Rightarrow\ \alpha = 2$ and $\lim\limits_{n\to\infty} n^{\alpha} P = \frac{1}{8}$