HYPERBOLA

EXERCISE # 1

Questions based on Various form of Hyperbola

Q.1 The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola, if its eccentricity is 2.

(A) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (B) $\frac{x^2}{4} - \frac{y^2}{12} = 2$

(C) $\frac{x^2}{9} - \frac{y^2}{12} = 1$ (D) None of these

Sol. **[A]**

Foci of ellipse (ae, 0) (4, 0)

$q = 25 (1 - e^2)$

$e = \frac{4}{5}$

foci (4, 0) of hyperbola

$e = 2$

$b^2 = a^2(e^2 - 1)$

$ae = 4$

$a = 2 \Rightarrow a^2 = 4$

$b^2 = 12$

equation of hyperbola

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

Q.2 The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13 is

(A) $9x^2 - 144 y^2 = 900$

(B) $25x^2 - 144 y^2 = 900$

(C) $25x^2 - 144 y^2 = 200$

(D) None of these

Sol. **[B]**

$2b = 5$

$b^2 = \frac{25}{4}$

$2ae = 13$

$(ae)^2 = \frac{169}{4}$

$b^2 = a^2 (e^2 - 1)$

$a^2 = \frac{144}{4}$

$$\frac{x^2}{144/4} - \frac{y^2}{25/4} = 1$$

$25x^2 - 144y^2 = 900.$

Q.3 If the hyperbolas, $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$ and $x^2 + 3xy + 2y^2 + 2x + 3y + c = 0$ are conjugate of each other, the value of 'c' is equal to -

(A) – 2 (B) 4 (C) 0 (D) 1

Sol. **[C]**

$H : x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

$x^2 + 3xy + 2y^2 + 2x + 3y + c = 0$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$A : x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$

$\Delta = 0$

$$2\lambda + \frac{9}{2} - \frac{9}{4} - 2 - \frac{9\lambda}{4} = 0$$

$\lambda = 1$

$C + H = 2A$

$C + 2 = 2$

$C = 0$

Q.4 $(x - 1)^2 + (y - 2)^2 = \frac{3(2x + 3y + 2)^2}{13}$ represents hyperbola whose eccentricity is -

(A) $\frac{\sqrt{13}}{\sqrt{3}}$ (B) $\frac{3}{\sqrt{13}}$ (C) $\sqrt{3}$ (D) 3

Sol. **[C]**

$$\frac{\text{distance of P from the fous}}{\text{distance of P from the directrix}}$$

Here P is (1, 2)

$\therefore (x - 1)^2 + (y - 2)^2 = \frac{3(2x+3y+2)^2}{13}$

$\Rightarrow$ taking square root both sides, we get

$$\sqrt{(x-1)^2 + (y-2)^2} = \frac{\sqrt{3}}{\sqrt{13}} (2x + 3y + 2)$$

$$\Rightarrow \frac{\sqrt{(x-1)^2 + (y-2)^2}}{\frac{(2x + 3y = 2)}{\sqrt{13}}} = \sqrt{3}$$

$\therefore e = \sqrt{3}$

Q.5 The equation of conjugate axis for the hyperbola $\frac{(x + y + 1)^2}{4} - \frac{(x - y + 2)^2}{9} = 1$ is -

(A) $x + y + 1 = 0$ (B) $x - y + 2 = 0$

(C) $x = - 3/2$ (D) None of these

Sol. **[A]**

$$\frac{(x+y+1)^2}{4} - \frac{(x-y+2)^2}{9} = 1$$

Let x + y + 1 = X and x – y + 2 = Y

$$\therefore \frac{X^2}{4} - \frac{y^2}{9} = 1$$

Put X = 0 for conjugate axis

$\therefore$ x + y + 1 = 0

Questions based on Position of point & Auxiliary circle

Q.6 Parametric form of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = -1$ is -

(A) (2tanθ, 3secθ) (B) (3secθ, 2tanθ)

(C) (9secθ, 4tanθ) (D) None of these

Sol. **[A]**

Parametric form of $\frac{x^2}{4} - \frac{y^2}{9} = -1$

$$\Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$$

on putting (3 secθ, 2 tanθ) the reqd form is given by

$\therefore$ x = 2 tanθ and y = 3secθ

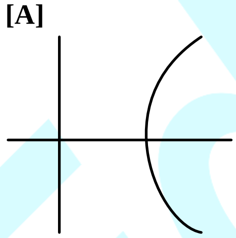
i.e. (2 tanθ, 3 secθ)

Q.7 The position of the point (2, 5) relative to the hyperbola $9x^2 - y^2 = 1$ is

(A) Inside (B) Outside

(C) lie on (D) None of these

Sol. **[A]**



Equation of hyperbola $\Rightarrow 9x^2 - y^2 = 1$

point (2, 5)

$S_1 = 9 \times 2^2 - 5^2 - 1$

= 36 – 25 – 1

= 36 – 25 = 11

11 > 0

$\therefore$ Point lies inside the hyperbola

Questions based on Line & Hyperbola, tangent & Normal

Q.8 The tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line 4y = 5x + 7. The points of contact are -

(A) $\left(\pm \frac{7}{2\sqrt{161}}, \pm \frac{8}{3\sqrt{161}}\right)$

(B) $\left(\pm \frac{5}{2\sqrt{163}}, \pm \frac{8}{3\sqrt{163}}\right)$

(C) $\left(\pm \frac{5}{2\sqrt{161}}, \pm \frac{8}{3\sqrt{161}}\right)$

(D) None of these

Sol. **[C]**

Q.9 The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines $x - 3\sqrt{5}\,y = 0$ and $\sqrt{5}\,x - 2y = 13$ and the length of its latus rectum is 4/3 units. The coordinates of its focus are -

(A) (± 2 $\sqrt{10}$, 1) (B) (± 3 $\sqrt{10}$, 0)

(C) (± 2 $\sqrt{10}$, 0) (D) None of these

Sol. **[C]**

Point of intersection, $P(3\sqrt{5}, 1)$ [Solving given lines]

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, passes through 'P'

$\therefore \frac{45}{a^2} - \frac{1}{b^2} = 1$ ….(i)

Also, $\frac{2b^2}{a} = 3$ [length of L.R.]

$b^2 = \frac{2a}{3}$ ….(ii)

And $b^2 = a^2 (e^2 - 1)$

$\frac{b^2 + a^2}{a^2} = e^2$ …..(iii)

From (i) & (ii)

$2a^2 + 3a - 90 = 0$

2a (a – 6) + 15 (a – 6) = 0

a = 6 or $a = \frac{-15}{2}$

$\therefore b^2 = \frac{2 \times 6}{3} = 4$

putting the value of a & b in (iii)

$e^2 = \frac{4 + 36}{36} = \frac{10}{9}$

$e = \pm \frac{\sqrt{10}}{3}$

Focus x = ± ae, y = 0

$(\pm 2\sqrt{10}, 0)$

Q.10 The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is -

(A) zero (B) 1 (C) 2 (D) 4

Sol. **[A]**

Given equation of curve :

$4x^2 - 9y^2 = 36$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$$

equation of straight line : $5x + 2y - 10 = 0$

slope of line perpendicular to it : $\frac{2}{5}$

for hyperbola condition of tangency is

$c^2 = m^2a^2 - b^2$

$$\Rightarrow c^2 = \frac{4}{25} \times 9 - 4$$

$$\Rightarrow c^2 = -\frac{64}{5}$$

which is not possible

Hence, no tangents are possible to the given curve

Questions based on — Chord of contacts, chord when mid point is given

Q.11 The locus of the mid-point of the chords of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ passing through a fixed point (α, β) is a hyperbola with centre at $(\alpha/2, \beta/2)$.

(A) $(x - \alpha/2)^2/a^2 - (y - \beta/2)^2/b^2 = \alpha^2/4a^2 - \beta^2/4b^2$

(B) $(x + \alpha/2)^2/a^2 - (y - \beta/2)^2/b^2 = \alpha^2/4a^2 + \beta^2/4b^2$

(C) $(x - \alpha/2)^2/a^2 - (y - \beta/2)^2/b^2 = \alpha^2/4a^2 + \beta^2/4b^2$

(D) None of these

Sol. **[A]**

Given equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let the coordinate of the middle point of the chord be (h, k)

Applying T = S, will give equation of chord with mid-point as (h, k)

$$\text{so } \left(\frac{xh}{a^2} - \frac{yk}{b^2} - 1\right) = \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right)$$

This line passes through (α, β) so we get

$$\frac{\alpha h}{a^2} - \frac{\beta k}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$$

$$\Rightarrow \frac{h^2}{a^2} - \frac{\alpha h}{a^2} - \frac{k^2}{b^2} + \frac{\beta k}{b^2} = 1$$

$$\Rightarrow \left[\frac{h^2}{a^2} - 2.\frac{h}{a}.\frac{\alpha}{2a} + \left(\frac{\alpha}{2a}\right)^2\right]$$

$$-\left[\left(\frac{k}{b}\right)^2 - 2.\frac{k}{b} + \frac{\beta}{2b} + \left(\frac{\beta}{2b}\right)^2\right] = \left(\frac{\alpha}{2a}\right)^2 - \left(\frac{\beta}{2b}\right)^2$$

$$\left(\text{Hint : Adding } \left[\left(\frac{\alpha}{2a}\right)^2 - \left(\frac{\beta}{2b}\right)^2\right] \text{ on L.H.S. \& R.H.S}\right)$$

$$\Rightarrow \left(\frac{h}{a} - \frac{\alpha}{2a}\right)^2 - \left(\frac{k}{b} - \frac{\beta}{2b}\right)^2 = \frac{\alpha^2}{4a^2} - \frac{\beta^2}{4b^2}$$

$$\Rightarrow \frac{\left(h - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(k - \frac{\beta}{2}\right)^2}{b^2} = \frac{\alpha^2}{4a^2} - \frac{\beta^2}{4b^2}$$

Hence the locus will be

$$\frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} = \frac{\alpha^2}{4a^2} - \frac{\beta^2}{4b^2}$$

with centre at $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Q.12 The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is -

(A) $3x - 4y = 4$ (B) $3y - 4x + 4 = 0$

(C) $4x - 4y = 3$ (D) $3x - 4y = 2$

Sol. **[A]**

Given equation of hyperbola is :

$3x^2 - 2y^2 + 4x - 6y = 0$

Let the coordinates of the middle points of the chords be (h, k).

It is given that the locus of chord is parallel to $y = 2x$

so let the equation of locus of chord be

$y = 2x + c$ …..(i)

pass (h, k)

we get $k = 2h + c$ …..(ii)

solving equation (i) with hyperbola we get

$3x^2 - 2(2x + c)^2 + 4x - 6(2x + c) = 0$

$\Rightarrow 3x^2 - 2(4x^2 + c^2 + 4cx) + 4x - 12x - 6c = 0$

$\Rightarrow 5x^2 + x\,8(c + 1) + 2c^2 + 6c = 0$ (roots x_1, x_2)

from the above quadratic equation

we get $x_1 + x_2 = -\frac{8(c+1)}{5} = 2h$

$\Rightarrow c = -\frac{10h}{8} - 1$ .....(iii)

putting value of 'c' in equation (ii)

$k = 2h - \frac{10h}{8} - 1$

$\Rightarrow 4k + 4 - 3h = 0$

$\Rightarrow 3h - 4k = 4$ is the required locus

Questions based on Diameter & Director circle & asymptotes & Rectangular Hyperbola

Q.13 The number of points from where a pair of perpendicular tangents can be drawn to the hyperbola, $x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = 1$, $\alpha \in (0, \pi/4)$ are -

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. **[D]**

Given, $x^2 \sec^2\alpha - y^2 \operatorname{cosec}^2 \alpha = 1$

$\Rightarrow \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

Here, $a^2 = \cos^2\alpha$, $b^2 = \sin^2\alpha$.

then, equation of its director circle, is

$x^2 + y^2 = a^2 - b^2$

$\Rightarrow x^2 + y^2 = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$

Director circle will exist as

$2\cos^2\alpha > 1$ (when, $\alpha \in (0, \pi/4)$)

$2\cos^2\alpha - 1 > 0$

so, there will exists infinite points.

Q.14 The asymptotes of the hyperbola $xy = hx + ky$ are-

(A) $x - k = 0$ & $y - h = 0$

(B) $x + h = 0$ & $y + k = 0$

(C) $x - k = 0$ & $y + h = 0$

(D) $x + k = 0$ & $y - h = 0$

Sol. **[A]**

Let, the equation of pair of asymptotes be

$hx + ky - xy + \lambda = 0$

then, $\Delta = 0$ $[\Delta = abc - 2fgh - af^2 - bg^2 - ch^2]$

$\Rightarrow \lambda = hk$

Now, $hx + ky - xy - hk = 0$

$x(-h + y) - k(y - h) = 0$

$\Rightarrow (y - h)(x - k) = 0$

then, asymptotes are $y - h = 0$

and $x - k = 0$

Q.15 For the rectangular hyperbola $xy = \frac{a^2}{2}$ the co-ordinates of its foci are -

(A) $(\sqrt{2}a, \sqrt{2}a), (-\sqrt{2}a, -\sqrt{2}a)$

(B) $(\sqrt{2}a, -\sqrt{2}a), (-\sqrt{2}a, \sqrt{2}a)$

(C) $(a, a), (-a, -a)$

(D) $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right) \left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$

Sol. **[C]**

Rectangular hyperbola $xy = \frac{a^2}{2}$ Referred to transverse and conjugate axes as co-ordinate axes, the equation of hyperbola is

$x^2 - y^2 = a^2$

It e be the eccentricity then

$a^2 = a^2(e^2 - 1)$

$\Rightarrow e = \sqrt{2}$

$\therefore$ co-ordinates of foci are $(\pm a\sqrt{2}, 0)$

Also its directories are $x = \pm \frac{a}{\sqrt{2}}$

Now rotating the co-ordinates axes though angle $-\frac{\pi}{4}$, the equation of hyperbola reduces to $xy = c^2$ where $c^2 = \frac{a^2}{2}$ by putting $\frac{x+y}{\sqrt{2}}$ for x and y

$\therefore$ for foci, we get $\frac{x+y}{\sqrt{2}} \pm a\sqrt{2}$ and $\frac{y-x}{\sqrt{2}} = 0$

$\therefore x + y = \pm 2a$ and $y = x$

$\Rightarrow x = y = \pm a = \pm c\sqrt{2}$

$\therefore$ foci are $(\pm\sqrt{2}, \pm c\sqrt{2})$

put $c = \frac{a}{\sqrt{2}}$

foci are $(\pm a, \pm a)$

$\therefore$ foci are (a, a) and $(-a, -a)$

Q.16 Equation of asymptotes to the hyperbola $(x - 1)(y - 2) = 4$ is -

(A) $xy - 2x - y + 2 = 0$

(B) $xy - 2x - y = 0$

(C) $xy - 2x - y - 2 = 0$

(D) None of these

Sol. **[A]**

$(x - 1)(y - 2) = 4$ is the given hyperbola its equation of asymptotes is

$(x - 1)(y - 2) = 0$

$\Rightarrow xy - 2x - y + 2 = 0$

Hence $xy - 2x - y + 2 = 0$

Q.17 Equations of directrices for the hyperbola $xy = \frac{a^2}{2}$ are -

(A) $x + y = \pm \frac{a}{2}$ (B) $x + y = \pm a$

(C) $x + y = \pm \frac{a}{\sqrt{2}}$ (D) None of these

Sol. **[B]**

$xy = \frac{a^2}{2}$ is the eq^n of rectangular hyperbola by putting $\frac{x+y}{\sqrt{2}}$ and $\frac{y-x}{\sqrt{2}}$ for x and y the equation of the directrices reduces to

$\frac{x+y}{\sqrt{2}} = \pm \frac{a}{\sqrt{2}}$

or $x = y = \pm a$

or $x + y + a = 0$ and $x + y - a = 0$

Q.18 If a ray incident on rectangular hyperbola $xy = \frac{a^2}{2}$ along $y = x$ then equation of reflected ray will be -

(A) $y = -x$ (B) $y = x$

(C) $y = 2x$ (D) None of these

Sol. **[B]**

It any ray incident on rectangular hyperbola $xy = \frac{a^2}{2}$ dong $y = x$ then reflected ray will be the same $y = x$

Q.19 C is the centre of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the tangent at any point P meets the asymptotes in the points Q and R. The equation to the locus of the centre of the circle circumscribing the triangle CQR is

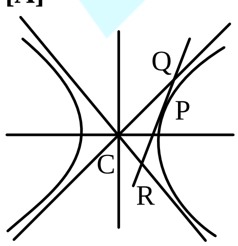
(A) $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$

(B) $4(a^2x^2 + b^2y^2) = (a^2 + b^2)^2$

(C) $4(a^2x^2 - b^2y^2) = (a^2 - b^2)^2$

(D) None of these

Sol. **[A]**



Let, tangent at P(a secθ, b tanθ), is

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ .....(i)

then,

Asymptotes ; $y = \frac{b}{a}x$ ......(ii)

$y = -\frac{b}{a}x$ ......(iii)

By solving (i) & (ii), we get

$Q \equiv (a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$

By solving (i) and (iii), we get

$R \equiv (a(\sec\theta - \tan\theta), b(\tan\theta - \sec\theta))$

Now, Let, circumcentre of ΔCQR be O(h, k). Then

$OC^2 = OQ^2$

$\Rightarrow h^2 + k^2 = \{h - a(\sec\theta + \tan\theta)\}^2 + \{k - b(\sec\theta + \tan\theta)\}^2$

$\Rightarrow 2(ha + kb) = (\sec\theta + \tan\theta)(a^2 + b^2)$ ....(iii)

and, $OC^2 = OR^2$, we get

$2(ha - kb) = (\sec\theta - \tan\theta)(a^2 + b^2)$ ....(iv)

Now, (iii) × (iv), we get

$4\{(ha)^2 - (kb)^2\} = (\sec^2\theta - \tan^2\theta)(a^2 + b^2)^2$

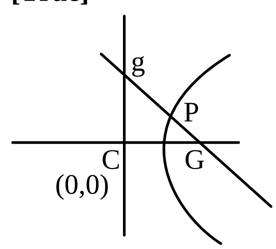
$\Rightarrow 4(h^2a^2 - k^2b^2) = (a^2 + b^2)^2$

$\Rightarrow 4(x^2a^2 - y^2b^2) = (a^2 + b^2)^2$

➤ True or false type questions

Q.20 If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola then 2PC = Gg.

Sol. **[True]**



Rectangular hyperbola : $x^2 - y^2 = 4$

Normal : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

$\frac{2x}{\sec\theta} + \frac{2y}{\tan\theta} = 8$

P (2 secθ, 2 tanθ)

G (4 secθ, 0) [Lies on x-axis]

C(0, 0), g (0, 4 tanθ) [Lies on y-axis]

PC : $\sqrt{4\sec^2\theta + 4\tan^2\theta}$

= $2\sqrt{\sec^2\theta+\tan^2\theta}$ .....(i)

Gg : $\sqrt{16\sec^2\theta+16\tan^2\theta}$

= $4\sqrt{\sec^2\theta+\tan^2\theta}$ .....(ii)

from (i) & (ii)

2 PC = Gg (True)

Q.21 If P($\sqrt{2}$, 1) are point on the rectangular hyperbola $x^2 - y^2 = 1$, C is centre and SS′ are the two foci then SP . S′ P is equal to 4.

Sol. Rectangular hyperbola : $x^2 - y^2 = 1$

$\therefore a^2 = b^2 = 1$

$b^2 = a^2 (e^2 - 1)$

$1 = e^2 - 1$

$e^2 = 2 \Rightarrow e = \sqrt{2}$

S(ae, 0) & S'(–ae, 0)

$S \equiv (\sqrt{2}, 0)$ & $S' (-\sqrt{2}, 0)$

$\therefore$ SP = $\sqrt{(\sqrt{2}-\sqrt{2})^2+1} = 1$

S'P = $\sqrt{(\sqrt{2}+\sqrt{2})^2+1} = \sqrt{8+1} = \sqrt{9} = 3$

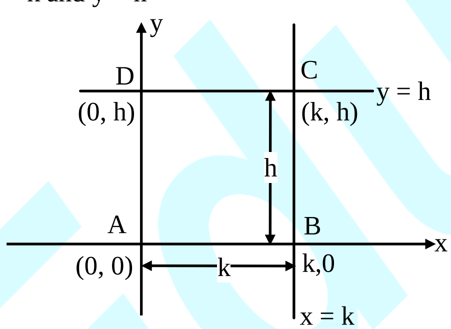
$\therefore$ SP . S'P = 3

($\therefore$ False)

Fill in the blanks type questions

Q.22 The area of a quadrilateral formed by asymptotes of the hyperbola xy = hx + ky with co-ordinates axes is ....

Sol. Asymptotes of the hyperbola xy = hx + ky are x = k and y = h



$\therefore$ Area of quadrilateral ABCD

i.e. rectangle ABCD is

= k × h

= hk sq. unit

Q.23 If the normals at P, Q, R on the rectangular hyperbola $xy = c^2$ intersect at a point S on the hyperbola, then co-ordinates of centroid of ΔPQR will be ....

Sol. Equation of the normal at point $\left(ct, \frac{c}{t}\right)$ on $xy = c^2$ is

$xt^3 - yt - ct^4 + c = 0$

it will pass through the point (h, k)

$\therefore ht^3 - kt - ct^4 + c = 0$ ....(i)

Also T(h, k) lies on $xy = c^2$

$\therefore hk = c^2$

therefore h = cq and k = $\frac{c}{q}$

from (i) $cqt^3 - \frac{c}{q}t - ct^4 + c = 0$

or qt3 – $\frac{t}{q}$ t – ct^4 + c = 0

or $q2t^3 - t - qt^4 + q = 0$

or $q(qt3 + 1) - t(qt^3 + 1) = 0$

or $(q - t)(qt^3 + 1) = 0$

$q \neq t$

$\therefore qt^3 + 1 = 0$

The three points other than T are given by

$qt^3 + 1 = 0$

It co-ordinates of $P\left(ct_1, \frac{c}{t_1}\right)$, $Q\left(ct_2, \frac{c}{t_2}\right)$ and $R\left(ct_3, \frac{c}{t_3}\right)$ then from (ii)

t1 + t2 + t3 = 0 ....(iii)

$t_1t_2 + t_2t_3 + t_3t_1 = 0$ ....(iv)

and $t_1t_2t_3 = -$ ....(v)

from (iii), $c(t_1 + t_2 + t_3) = 0$

$t_1t_2t_3\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right) = 0$

$\Rightarrow (-1)\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right) = 0$

$\therefore \frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3} = 0$

or $c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right) = 0$

Hence $c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right) = 0$

$\Rightarrow$ centroid of DPQR is the origin

i.e. (0, 0)

EXERCISE # 2

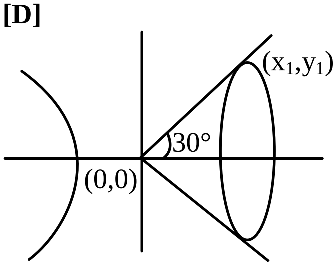
Part-A Only single correct answer type questions

Q.1 If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔOAB (O is the origin) is an equilateral triangle, then the eccentricity 'e' of the hyperbola satisfies -

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$

(C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

Sol. **[D]**



Hyperbola : $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ……(i)

$\tan 30° = \frac{y_1}{x_1} = \frac{1}{\sqrt{3}}$

$\therefore \frac{y_1^2}{x_1^2} = \frac{1}{3}$ ……(ii)

$\frac{3y_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$y_1^2 = \frac{1}{\left(\frac{3}{a^2} - \frac{1}{b^2}\right)} > 0$

$\frac{3}{a^2} - \frac{1}{b^2} > 0 \Rightarrow \frac{b^2}{a^2} > \frac{1}{3}$

$\because b^2 = a^2 (e^2 - 1)$

$\frac{b^2}{a^2} = e^2 - 1$

$\therefore (e^2 - 1) > \frac{1}{3}$

$e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

Q.2 The point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the product of whose slopes is c^2, lies on the curve -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$

(B) $y^2 + a^2 = c^2(x^2 - b^2)$

(C) $y^2 + b^2 = c^2(x^2 - a^2)$

(D) $y^2 - a^2 = c^2(x^2 + b^2)$

Sol. **[C]**

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$(y - mx)^2 = a^2m^2 - b^2$

$y^2 + m^2x^2 - 2mxy - a^2m^2 + b^2 = 0$

$m^2 (x^2 - a^2) - 2mxy + y^2 + b^2 = 0$ → m_1, m_2

$m_1 m_2 = c^2$

$\frac{y^2 + b^2}{x^2 - a^2} = c^2$

$(y^2 + b^2) = c^2 (x^2 - a^2)$

Q.3 If x cos α + y sin α = p, a variable chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$ subtends a right angle at the centre of the hyperbola, then the chords touch a fixed circle whose radius is equal to -

(A) $\sqrt{2}$ a (B) $\sqrt{3}$ a (C) 2 a (D) $\sqrt{5}$ a

Sol. **[A]**

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$

Chord : $\frac{x\cos\alpha + y\sin\alpha}{p} = 1$

$\frac{x^2}{a^2} - \frac{y^2}{2a^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$

coeff. of x^2 + coeff. of y^2 = a

$1 - \frac{a^2}{p^2}\cos^2\alpha - \frac{1}{2} - \frac{a^2}{p^2}\sin^2\alpha = 0$

$\frac{1}{2} - \frac{a^2}{p^2} = 0$

$\frac{1}{2} = \frac{a^2}{p^2}$

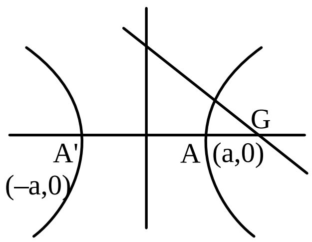
$p = \sqrt{2}a$

Q.4 If the normal at 'θ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at G then AG × A'G is (where A and A' are the vertices of the hyperbola)-

(A) $a^2\sec\theta$ (B) $a^2(e^4\sec^2\theta + 1)$

(C) $a^2(e^4\sec^2\theta - 1)$ (D) None of these

Sol. **[C]**



Normal : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

$G\left(\frac{\sec\theta(a^2+b^2)}{a}, 0\right)$

AG = $\sqrt{a - \frac{\sec\theta(a^2+b^2)}{a}}$

A'G = $\sqrt{a + \frac{\sec\theta(a^2+b^2)}{a}}$

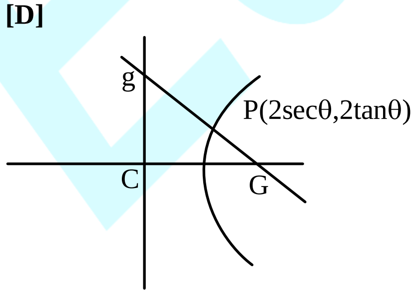
AG × A'G = $a^2(e^4\sec^2\theta - 1)$

Q.5 If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axis in G and g and C is centre of the hyperbola, then -

(A) 2PG = PC (B) $\frac{1}{2}$ Pg = PC

(C) $\sqrt{2}$ PG = Pg (D) Gg = 2PC

Sol. **[D]**



$\frac{x^2}{4} - \frac{y^2}{4} = 1$

Normal : $\frac{2x}{\sec\theta} + \frac{2y}{\tan\theta} = 8$

$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 4$

G(4secθ, 0), g(0, 4tanθ), C(0, 0)

Gg = $\sqrt{16\sec^2\theta + 16\tan^2\theta} = 4\sqrt{\sec^2\theta + \tan^2\theta}$

PC = $\sqrt{4\sec^2\theta + 4\tan^2\theta} = 2\sqrt{\sec^2\theta + \tan^2\theta}$

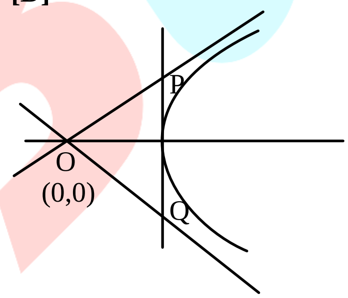
Gg = 2PC

Q.6 From any point to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$

The area cut off by the chord of contact on the region between the asymptotes is equal to -

(A) $\frac{ab}{2}$ (B) ab (C) 2ab (D) 4ab

Sol. **[D]**



P (a secθ, b tanθ)

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 2$

$y = \frac{b}{a}x$, $y = -\frac{b}{a}x$

P [2a(sec θ + tan θ), 2b(sec θ + tan θ)]

Q [2a(sec θ – tan θ), 2b(sec θ – tan θ)]

Area of ΔOPQ = 4ab

Q.7 The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is -

(A) $(x^2 - y^2) = 4c^2xy$ (B) $(x^2 + y^2)^2 = 2c^2xy$

(C) $(x^2 + y^2) = 4c^2xy$ (D) $(x^2 + y^2)^2 = 4c^2xy$

Sol. **[D]**

$\frac{xc}{t} + yct = 2c$ …..(i)

tangent : $y - k = -\frac{h}{k}(x - h)$

$xh + yk = h^2 + k^2$ ……(ii)

compare :

$$\frac{c}{ht} = \frac{ct}{k} = \frac{2c^2}{h^2 + k^2}$$

$h^2 + k^2 = 2hct$

$(h^2 + k^2)^2 = c^2 4h^2t^2$ .....(iii)

$k = ht^2$ .....(iv)

divide (iii) by (iv)

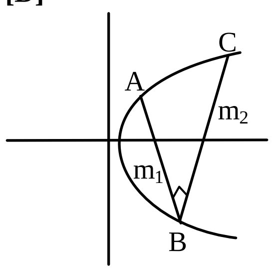
$$\frac{(x^2 + y^2)^2}{y} = \frac{4x^2c^2}{x}$$

$(x^2 + y^2)^2 = 4yxc^2$

Q.8 If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to -

(A) $-\frac{a^2}{b^2}$ (B) $-\frac{b^2}{a^2}$ (C) $-\frac{b^4}{a^4}$ (D) $-\frac{a^4}{b^4}$

Sol. **[D]**



Equation of AB

T = 0

$$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$$

equation of BC, T = 0

$$\frac{x x_2}{a^2} - \frac{y y_2}{b^2} = 1$$

AB ⊥ BC

$m_1m_2 = -1$ [product of slope = –1]

$$\left[\frac{x_1/a^2}{y_1/b^2}\right]\left[\frac{x_2/a^2}{y_2/b^2}\right] = -1$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2} \times \frac{b^4}{a^4} = -1$$

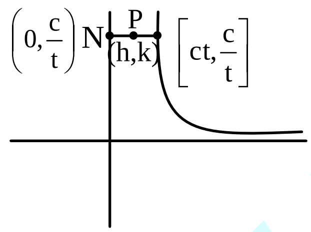
$$\Rightarrow \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

Q.9 If PN is the perpendicular from a point P on a rectangular hyperbola to its asymptotes, the locus of the mid-point of PN is -

(A) circle (B) parabola

(C) ellipse (D) hyperbola

Sol. **[D]**



Let equation of hyperbola

$xy = c^2$

$h = \frac{ct}{2}$ & $k = \frac{c}{t}$

$t = \frac{2h}{c}$ & $t = \frac{c}{k}$

$\frac{2h}{c} = \frac{c}{k} \Rightarrow 2hk = c^2$

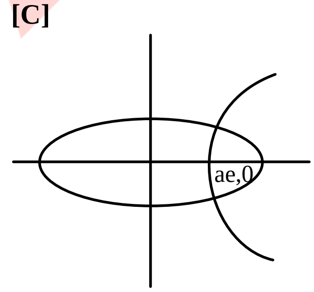
$\Rightarrow 2xy = c^2$

∴ equation of hyperbola

Q.10 If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then eccentricity of hyperbola is -

(A) $\sqrt{2}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) None

Sol. **[C]**



$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$$

passes through (ae, 0)

$$\frac{a^2e^2}{b^2} - 0 = 1$$

$$\frac{a^2}{b^2} = \frac{1}{e^2} \quad ....(i)$$

Let eccentricity of hyperbola be e'

$a^2 = b^2 (e'^2 - 1)$

$$\frac{a^2}{b^2} = e'^2 - 1 \quad ....(ii)$$

from (i)

$$\frac{b^2}{a^2} = e^2$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} \quad [b^2 = a^2(1 - e^2)]$$

$$\frac{2b^2}{a^2} = 1$$

$$\frac{a^2}{b^2} = 2 \qquad \text{......(iii)}$$

put it in (iii)

$2 + 1 = e'^2$

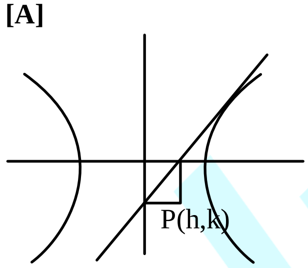
$e = \sqrt{3}$

Q.11 Tangent at any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the axes at A and B respectively. If the rectangle OAPB (where O is origin) is completed then locus of point P is given by

(A) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ (B) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

(C) $\frac{a^2}{y^2} - \frac{b^2}{x^2} = 1$ (D) None of these

Sol. **[A]**



Tangent :

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

$$A\left[\frac{a}{\sec\theta}, 0\right], \; B\left[0, \frac{-b}{\tan\theta}\right]$$

P(h, k)

$$k = \frac{-b}{\tan\theta} \Rightarrow \tan^2\theta = \frac{b^2}{k^2} \qquad \text{....(i)}$$

$$h = \frac{a}{\sec\theta} \Rightarrow \sec^2\theta = \frac{a^2}{h^2} \qquad \text{.....(ii)}$$

subtract equation (ii) – (i)

$$\frac{a^2}{h^2} - \frac{b^2}{k^2} = 1$$

$$\text{locus : } \frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$$

Q.12 The asymptotes of a hyperbola are parallel to $2x + 3y = 0$ and $3x + 2y = 0$. Its centre is (4, 2) and hyperbola pass through (5, 3). Then its equation is -

(A) $x^2 + xy + y^2 - 74x - 76y + 199 = 0$

(B) $6x^2 + 13xy + 6y^2 - 74x - 76y + 199 = 0$

(C) $6x^2 + 13xy + y^2 - 74x - 76y + 199 = 0$

(D) None of these

Sol. **[B]**

Asymptotes are

$2x + 3y = \lambda_1$

$3x + 2y = \lambda_2$

passes through (4, 2)

$2x + 3y = 14 \rightarrow L_1$

$3x + 2y = 16 \rightarrow L_2$

$L_1 . L_2 + \lambda = 0$

$(2x + 3y - 14)(3x + 2y - 16) + \lambda = 0$

passes through (5, 3) $\Rightarrow \lambda = -25$

so equation of hyperbola :

$6x^2 + 6y^2 + 13xy - 74x - 76y + 199 = 0$

Q.13 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{3} = 1$ then incorrect statement is -

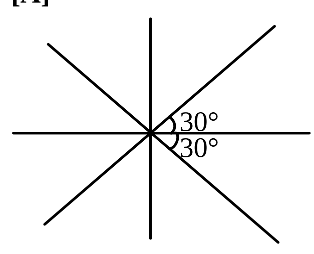
(A) the acute angle between its asymptotes is 60º

(B) its eccentricity is $\frac{2}{3}$

(C) length of the latus rectum is 2

(D) product of the perpendicular distances from any point on the hyperbola on its asymptotes is less than the length of its latus rectum.

Sol. **[A]**



$$\frac{x^2}{9} - \frac{y^2}{3} = 1$$

$$b^2 = a^2(e^2 - 1)$$

$$\frac{3}{9} + 1 = e^2 \Rightarrow e = \sqrt{\frac{12}{9}} = \frac{2}{\sqrt{3}}$$

Length of L.R. = $\frac{2b^2}{a} = \frac{2\times 3}{3} = 2$

Asymptotes

$y = \frac{1}{\sqrt{3}x} \Rightarrow y = -\frac{1}{\sqrt{3}x}$

angle = 60°

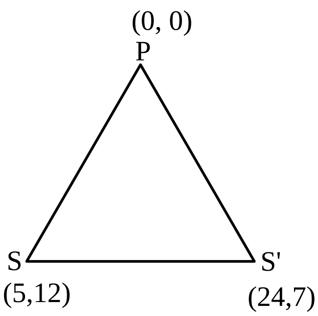
Part-B **One or more than one correct answer type questions**

Q.14 If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is -

(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$

(C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

Sol. **[A, D]**



Given foci of conic is : (5, 12) and (24, 7)

it is also given that the conic passes through (0,0). There are two possibilities that stand :

(i) |SP – S'P| = 2a using distance formula we can calculate :

$\Rightarrow$ |13 – 25| = 2a SP = 13

$\Rightarrow$ 2a = 12 S'P = 25

$\Rightarrow$ a = 6 SS' = $\sqrt{386}$

we know that

2ae = $\sqrt{386}$ ($\because$ 2ae = distance between foci)

$\Rightarrow e = \frac{\sqrt{386}}{12}$

(ii) Again |SP + S'P| = 2a

$\Rightarrow$ 13 + 25 = 2a

$\Rightarrow$ a = 19

S . 2ae = $\sqrt{386}$

$\Rightarrow e = \frac{\sqrt{386}}{38}$

Q.15 Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is -

(A) $y = x + \sqrt{a^2 - b^2}$

(B) $y = x - \sqrt{a^2 - b^2}$

(C) $y = -x + \sqrt{a^2 - b^2}$

(D) $y = -x - \sqrt{a^2 - b^2}$

Sol. **[A, B, C, D]**

Given equation of hyperbolas are :

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$

equation of tangent to hyperbola in slope form :

for first hyperbola : $y = mx \pm \sqrt{a^2m^2 - b^2}$

for second hyperbola : $y = mx \pm \sqrt{(-b^2)m^2 + a^2}$

since they are common to the hyperbolas so

$a^2m^2 - b^2 = (-b^2)m^2 + a^2$

$\Rightarrow a^2m^2 + b^2m^2 = a^2 + b^2$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

so the possible equation of tangents are

$y = x \pm \sqrt{a^2 - b^2}$

$y = -x \pm \sqrt{a^2 - b^2}$

Q.16 If (a secθ, b tan θ) and (a secϕ, b tan ϕ) are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equals to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$

(C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Sol. **[B, C]**

Given ends of focal chord are (a secθ, b tanθ) and (a sec ϕ, b tan ϕ)

Let the equation of focal chord be

$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$

it is given that it is a focal chord so it will pass through (± ae, 0)

so passing (± ae, 0) we get

$$\pm\frac{ae}{a}\cos\frac{\alpha-\beta}{2}-\frac{0}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow \pm e\ \cos\frac{\alpha-\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow \pm e=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

using compoundo and dividendo we can write it as

$$\frac{\pm e+1}{\pm e-1}=\frac{\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}}=\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \frac{e+1}{e-1}=-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\quad\text{and}\quad\frac{-e+1}{-e-1}=-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$\frac{1-e}{1+e}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\ \Rightarrow\ \frac{-e-1}{-e+1}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\Rightarrow \frac{1+e}{1-e}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

Q.17 The equation of a tangent to the hyperbola $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is -

(A) y = x + 2 (B) y = x – 4

(C) x = y + 3 (D) x = y + 2

Sol. **[A, B]**

$16x^2 - 96x - 25y^2 + 100y - 356 = 0$

$\Rightarrow 16(x^2 - 6x) - 25(y^2 - 4y) - 356 = 0$

$\Rightarrow 16(x-3)^2 - 25(y-2)^2 = 356 + 144 - 100$

$\Rightarrow 16(x-3)^2 - 25(y-2)^2 = 400$

$$\Rightarrow \frac{(x-3)^2}{25}-\frac{(y-2)^2}{16}=1$$

eq^n of hyperbola where sbpe is m is

$Y = mx \pm \sqrt{a^2m^2 - b^2}$

$y - 2 = m(x-3) \pm \sqrt{a^2m^2-b^2}$

$y - 2 = (x-3) \pm \sqrt{25-16}$

$y - 2 = (x-3) \pm 3 \Rightarrow y - 2 = x - 3 \pm 3$

$\Rightarrow y - 2 = x - 3 + 3$ & $y - 2 = x - 3 - 3$

$\Rightarrow y - 2 = x$ & $y - 2 = x - 6$

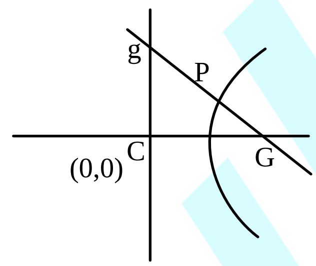
$\Rightarrow y = x + 2$ & $y = x - 4$

Q.18 If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then -

(A) PG = PC (B) Pg = PC

(C) PG = Pg (D) Gg = 2PC

Sol. **[D]**



Rectangular hyperbola

$$\frac{x^2}{4}-\frac{y^2}{4}=1$$

$$\text{Normal}:\ \frac{2x}{\sec\theta}+\frac{2y}{\tan\theta}=8$$

$$\frac{x}{\sec\theta}+\frac{y}{\tan\theta}=4$$

P (2 secθ, 4 tanθ)

g (0, 4 tanθ) [Θ lies on y-axis]

G (4 secθ, 0) [Θ lies on x-axis]

c (0, 0)

$PC = \sqrt{4\sec^2\theta + 4\tan^2\theta}$

$= 2\sqrt{\sec^2\theta + \tan^2\theta}$ ……(i)

$Gg = \sqrt{16\sec^2\theta + 16\tan^2\theta}$

$= 4\sqrt{\sec^2\theta + \tan^2\theta}$ ….(ii)

from (i) & (ii)

2PC = Gg

Q.19 If the tangent at the point (a sec α, b tan α) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at T, then the distance of T from a focus of the hyperbola is -

(A) a (e – cos α) (B) b (e + cos α)

(C) a (e + cos α) (D) $\sqrt{a^2c^2 + b^2\cot^2\alpha}$

Sol. **[C]**

eq^n of tangent at point (a secα, b tanα) to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x\sec\alpha}{a} - \frac{y\tan\alpha}{b} = 1$$

it meets its transverse axis i.e. x-axis at T.

∴ co-ordinates of T is $\left(\frac{a}{\sec},0\right) \equiv (a\cos\alpha, 0)$

focus of hyperbola is (ae, 0) or (–ae, 0)

∴ distance of T from the focus (ae, 0) is $\sqrt{(ae - a\cos\alpha)^2}$

= ae – a cosα

= a(e – cosα)

and distances of T from focus (–ae, 0) is

$\sqrt{(-ae - a\cos\alpha)^2}$

= ae + a cos α

= a(e + cos α)

Q.20 The locus of the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is -

(A) directors circle (B) $x^2 + y^2 = a^2$

(C) $x^2 + y^2 = a^2 - b^2$ (D) $x^2 + y^2 = a^2 + b^2$

Sol. **[B, C]**

since the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the director circle

∴ Equation of director circle is given by

$x^2 + y^2 = a^2 - b^2$

Q.21 If H(x, y) = 0 represent the equation of a hyperbola and A (x, y) = 0, C (x, y) = 0 the equations of its asymptotes and the conjugate hyperbola respectively then for any point (α, β) in the plane : H (α, β), A (α, β) and C (α, β) are in -

(A) A.P. (B) G.P.

(C) H.P. (D) None

Sol. **[A]**

Let the equation of the hyperbola be

$$H(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

then A(x, y) º $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

is the equation of asymptotes and $C(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ is the equation of its conjugate hyperbola

since H(α, β) + C(α, β) = 2A(α, β)

therefore H(α, β), A(α, β) and C(α, β) are in A.P.

Q.22 If the normals at (x_i, y_i), i = 1, 2, 3, 4 to the rectangular hyperbola xy = 2 meet at the point (3, 4), then -

(A) $x_1 + x_2 + x_3 + x_4 = 3$

(B) $y_1 + y_2 + y_3 + y_4 = 4$

(C) $x_1 x_2 x_3 x_4 = -4$

(D) $y_1 y_2 y_3 y_4 = -4$

Sol. **[B]**

Equation of the normal to the hyperbola $xy = c^2$ at any point whose parameter is t is given by

$xt^3 - yt - ct^4 + c = 0$ ....(i)

It fasces through (3, 4) then

ct4 –3t3 + 0 t2 + 4t – c = 0 ....(ii)

Above equation is of fourth degree in t given us four values of t showing that there will be four points the normals at are of which will pass through (3, 4)

$\therefore t_1 + t_2 + t_3 + t_4 = \frac{3}{c}$

or $ct_1 + ct_2 + ct_3 + ct_4 = 3$

or $x_1 + x_2 + x_3 + x_4 = 4$

similarly $y_2 + y_2 + y_3 + y_4 = 4$

and also $x_1 x_2 x_3 x_4 = -4$

and $y_1 y_2 y_3 y_4 = -4$

Part-C Assertion-Reason type questions

The following questions consist of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.
(C) If Assertion is true but the Reason is false.
(D) If Assertion is false but Reason is true.

Q.23 **Assertion :** The point P $\left[\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right]$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for infinite values is of t.

Reason : Locus of point P is hyperbola if $t \in R$.

Sol. [A] Let $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$

$\Rightarrow \frac{2x}{a} = t + \frac{1}{t}$

$\Rightarrow \left(\frac{2x}{a}\right)^2 = t_2 + \frac{1}{t^2} + 2$ ....(i)

and let $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$

$\Rightarrow \frac{2y}{b} = t - \frac{1}{t}$

$\Rightarrow \left(\frac{2y}{b}\right)^2 = t2 + \frac{1}{t^2} - 2$ ....(iii)

subtracting (ii) from (i), me get

$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$

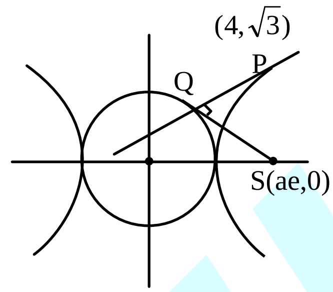
or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

which is a hyperbola

Q.24 **Assertion :** Tangent to the hyperbola $\frac{x^2}{4} - y^2 = 1$ at point P(4, $\sqrt{3}$) meets circle $x^2 + y^2 = 4$ at Q, then line passing through Q and parallel to normal at P to the hyperbola will pass through focus of hyperbola.

Reason : Foot of perpendicular on any tangent to the hyperbola drawn from focus of hyperbola will always lie on auxiliary circle of hyperbola.

Sol.[A]



Θ foot of $\perp^r$ on any tangent to hyperbola drawn from focus of hyperbola will always lie on auxiliary circle.

Q.25 **Assertion :** The locus of the mid-points of the chords of the hyperbola $\frac{x^2}{36} - \frac{y^2}{25} = 1$ passing through a fixed point (2, 4) is a hyperbola with centre at (1, 2).

Reason : The equation of the chord is $T = S_1$ whose mid point is (x_1, y_1) i.e.

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$ for standard hyperbola.

Sol.[A] Hyperbola : $\frac{x^2}{36} - \frac{y^2}{25} = 1$

Let mid point of chord be (h, k)

$T = S_1$

$\frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

passes through (2, 4)

$\frac{2h}{36} - \frac{4k}{25} = \frac{h^2}{36} - \frac{k^2}{25}$

$\frac{2x}{36} - \frac{4y}{25} = \frac{x^2}{36} - \frac{y^2}{25}$

$50x - 144y = 25x^2 - 36y^2$

$25(x^2 - 2x + 1) - 36(y^2 - 4y + 4) = 25 - 144$

$25(x - 1)^2 - 36(y - 2)^2 = -119$

$\therefore$ Above equation is a hyperbola whose centre is (1, 2)

Q.26 **Assertion :** If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ at 4 points then $t_1 + t_2 + t_3 + t_4 = -\frac{2g}{c}$.

Reason : The mean position of the point of intersections of a rectangular hyperbola and a circle lies at the mid point of the line joining their centres.

Sol.[A] Parametric equation of hyperbola $\left(ct, \frac{c}{t}\right)$

solving hyperbola with circle

$x^2 + y^2 + 2gx + 2fy + k = 0$

$(ct)^2 + (c/t)^2 + 2g(ct) + 2f(c/t) + k = 0$

$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$

$c^2t^4 + c^2 + 2gct^3 + 2fct + kt^2 = 0$

$\therefore t_1 + t_2 + t_3 + t_4 = -\frac{2g}{c}$ .....(i)

let, point of intersection of hyperbola and circle is (x_i, y_i) i = 1,2,3,4

so, from equation (i)

$x_1 + x_2 + x_3 + x_4 = -2g$

so, $\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{-g+0}{2}$

similarly,

$\frac{y_1 + y_2 + y_3 + y_4}{4} = \frac{-f+0}{2}$

Hence, the reason is true.

Part-D Column Matching type questions

Q.27

Column I	Column II
(A) The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ makes with its asymptotes	(P) 12
(B) If the line $y = 3x + \lambda$ touches the curve $9x^2 - 5y^2 = 45$ then $\lvert\lambda\rvert$ is	(Q) 6
(C) If the chord $x \cos \alpha + y \sin \alpha = P$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, then the radius of the circle concentric with the hyperbola to which the given chord is a tangent is	(R) 24
(D) If λ be the length of the latus rectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is equal to	(S) 32

Sol. **A → P; B → Q; C → P; D → S**

(A) Area of the triangle is constant

By the property of hyperbola and is equal to ab $= 4 \times 3 = 12$

(B) $\frac{x^2}{5} - \frac{y^2}{a} = 1$

tangent : $\frac{x \sec\theta}{\sqrt{5}} - \frac{y \tan\theta}{3} = 1$

$3x - y = -\lambda$

$\frac{\sec\theta}{3\sqrt{5}} = \frac{\tan\theta}{3} = -\frac{1}{\lambda}$

$\sec\theta = -\frac{3\sqrt{5}}{\lambda}$, $\tan\theta = -\frac{3}{\lambda}$

$\frac{45}{\lambda^2} - \frac{9}{\lambda^2} = 1$

$\lambda^2 = 36 \Rightarrow \lambda = 6$

(C) $\frac{x^2}{16} - \frac{y^2}{18} = 1$

$\left(\frac{x\cos\alpha + y\sin\alpha}{P}\right) = 1$

$\frac{x^2}{16} - \frac{y^2}{18} = \left(\frac{x\cos\alpha + y\sin\alpha}{P}\right)^2$

$\frac{x^2}{16} - \frac{y^2}{18} = \frac{x^2\cos^2\alpha}{P^2} + \frac{y^2\sin^2\alpha}{P^2} + \frac{2xy\sin\alpha\cos\alpha}{P^2}$

coeff. of x^2 + coeff. of y^2 = 0

$\left(\frac{1}{16} - \frac{\cos^2\alpha}{P^2}\right) + \left(-\frac{1}{18} - \frac{\sin^2\alpha}{P^2}\right) = 0$

$\frac{1}{16} - \frac{1}{18} - \frac{1}{P^2}(\cos^2\alpha + \sin^2\alpha) = 0$

$\frac{1}{P^2} = \frac{1}{16} - \frac{1}{18}$

$\frac{1}{P^2} = \frac{9-8}{144}$

$\frac{1}{P} = \frac{1}{12}$

P = 12

(D) $16(x^2 + 2x + 1) - 9(y^2 - 4y + 4) = 164 + 16 - 36$

$16(x + 1)^2 - 9(y - 2)^2 = 144$

$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

L.R. $= \frac{2b^2}{a} = \frac{2 \times 16}{3} = \frac{32}{3}$

$3\lambda = 3 \times \frac{32}{3} = 32$.

Q.28 **Column I** **Column II**

Column I	Column II
(A) The equation of the axis of the parabola $9y^2 - 16x - 12y - 57 = 0$ is	(P) 0
(B) The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. The cartesian equation of its directrix is	(Q) 7
(C) The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is	(R) 3y = 2
(D) If the tangent and the normal to $x^2 - y^2 = 4$ at a point cut off intercepts a_1, a_2 on the x-axis and b_1, b_2 on the y-axis, respectively, then the value of $a_1a_2 + b_1b_2$ is	(S) x = 0

Sol. **A → R; B → S; C → Q; D → P**

(A) $9y^2 - 16x - 12y - 57 = 0$

$$9\left(y^2 - \frac{4}{3}y + \frac{4}{9}\right) - 16x = 57 + 4$$

$$9\left(y - \frac{2}{3}\right)^2 - 16x = 61$$

$$\left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right)$$

Axis Y = 0

$$y - \frac{2}{3} = 0$$

$$y = \frac{2}{3} \Rightarrow 3y = 2$$

(B) $x = t^2 + 1$

$t^2 = x - 1$ ….(i)

$y = 2t + 1$

$$t = \frac{y-1}{2}$$

$$t^2 = \left(\frac{y-1}{2}\right)^2 \quad ….(ii)$$

Divide (i) & (ii)

$$\frac{(x-1)4}{(y-1)^2} = 1$$

$(y - 1)^2 = 4(x - 1)$

directrix : X + a = 0

x – 1 + 1 = 0

x = 0

(C) $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

$$\frac{x^2}{144/25} - \frac{y^2}{81/25} = 1$$

$$\frac{81}{25} = \frac{144}{25}(e^2 - 1)$$

$$\frac{81}{144} + 1 = e^2$$

$$e = \frac{15}{12}$$

foci (3, 0)

ae = 3

for ellipse

$b^2 = a^2(1 - e^2)$

$= a^2 - a^2 e^2$

$= 16 - 9$

$= 7$

(D) $\frac{x^2}{4} - \frac{y^2}{4} = 1$

tangent : $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{2} = 1$

Normal : $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 4$

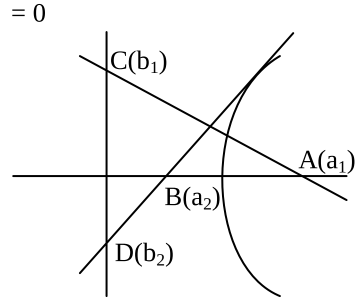
A(4 secθ, 0), $B\left(\frac{2}{\sec\theta}, 0\right)$

C(0, 4 tanθ), $D\left(0, \frac{-2}{\tan\theta}\right)$

$$a_1a_2 + b_1b_2 = \left(4\sec\theta \times \frac{2}{\sec\theta}\right) + \left(4\tan\theta \times \frac{-2}{\tan\theta}\right)$$

= 8 – 8

= 0



EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, whose focus is (a, 0), and whose directrix is 4x – 3y = a. Find also the coordinates of the centre and the equation to the other directrix.

Sol. $e = \frac{5}{4}$, fours (a, 0) directrix 4x –3y = a

Let P(x, y) be any point on the hyperbola, then by focus directrix property.

$$\frac{\text{distance of P from the focus}}{\text{distance of P from the directrix}} = e = \frac{5}{4}$$

$$\therefore \frac{\sqrt{(x-a)^2+(y-0)^2}}{\frac{4x-3y-a}{\sqrt{16+9}}} = \frac{5}{4}$$

$\Rightarrow 4\sqrt{(x-a)^2+y^2} = (4x - 3y - a)$

$\Rightarrow 16\{(x - a)^2 + y^2\} = (4x - 3y - a)^2$

$\Rightarrow 16\{x^2 + b^2 - 2ax + a^2\} = \{(4x - 3y) - a\}^2$

$\Rightarrow 16x^2 + 16y^2 - 32ax + 16a^2$

$= 16x2 + 9y2 - 24xy + a^2 - 8ax + 6ay$

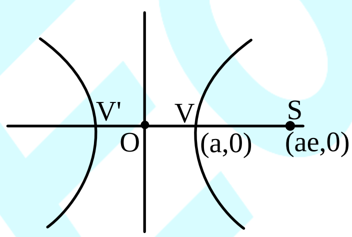
$\Rightarrow 7y^2 + 24xy - 24ax + 15a^2 - 6ay = 0$

$\therefore$ Reqd eq^n is

$7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

Q.2 Find the equation to the hyperbola of given transverse axis whose vertex bisects the distance between the centre and the focus.

Sol.



'V' bisects OS, then,

2a = ae

$\Rightarrow e = 2$

Now, $b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = 3a^2$

equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{3a^2} = 1$$

$\Rightarrow 3x^2 - y^2 = 3a^2$

Q.3 The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, 7x + 13y – 87 = 0 & 5x – 8y + 7 = 0, the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given hyperbola

point of intersection

35x + 65y – 435 = 0

35x – 56y + 49 = 0

– + –

121y = 484

y = 4

$\therefore$ 5x = 8y –7

$\therefore$ 5x = 32 –7 = 25

x = 5

$\therefore$ (5, 4) is the point of intention through which hyperbola passes $\frac{25}{a^2} - \frac{16}{b^2} = 1$

$\Rightarrow \frac{25}{a^2} = 1 + \frac{16}{b^2}$ ....(i)

and $\frac{2b^2}{a} = \frac{32\sqrt{2}}{5}$

$\Rightarrow b2 = \frac{16\sqrt{2}a}{5}$

$\Rightarrow b2 = \frac{16\sqrt{2}a}{5}$ ....(ii)

from (i) and (ii)

$\frac{25}{a^2} = 1 + \frac{16}{b^2}$

$\Rightarrow \frac{25}{a^2} = 1 + \frac{5}{a\sqrt{2}}$

$\Rightarrow \left(\frac{5}{a}\right)^2 - \frac{5}{a\sqrt{2}} - 1 = 0$

$\Rightarrow \left(\frac{5}{a} - \frac{1}{2\sqrt{2}}\right)^2 = 1 + \frac{1}{8} = \frac{9}{8}$

$\frac{5a}{a} - \frac{1}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}$

$\frac{5}{a} = \frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}$

$\frac{5}{a} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

$\frac{5}{a} = \sqrt{2}$

$a = \frac{5}{\sqrt{2}}$

$\therefore b2 = \frac{16\sqrt{2}}{5}.a$

$\therefore b2 = \frac{16\sqrt{2}}{5} . \frac{5}{\sqrt{2}}$

$\therefore b2 = 16$

$b = 4$

Q.4 If two points P and Q on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ whose centre is C be such that CP is perpendicular to CQ and a < b, then prove that $(1/CP^2) + (1/CQ^2) = (1/a^2) - (1/b^2)$.

Sol. Let the equation to cp be y = mx. its intersection with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$x^2(b^2 - a^2m^2) = a^2b^2$ ....(i)

$\therefore cp^2 = x^2 + y^2 = x2 + m^2x^2$

Replacing m by $-\frac{1}{m}$, are get

$cQ^2 = \frac{1+m^2}{m^2} . \frac{a^2b^2.m^2}{b^2m^2 - a^2}$

$= \frac{(1+m^2)a^2b^2}{b^2m^2 - a^2}$

$\therefore \frac{1}{cP^2} + \frac{1}{cQ^2}$

$= \frac{1}{(1+m^2)} \frac{b2(1+m^2) - a2(1+m^2)}{a^2b^2}$

$\therefore \frac{1}{cP^2} + \frac{1}{cQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$

Q.5 Chords of the hyperbola $x^2 - y^2 = a^2$ touch the parbola $y^2 = 4$ ax. Prove that the locus of their mid points is the curve $(x - a)\, y^2 = x^3$.

Sol. The equation of the chord of the hyperbola whose middle point is (h, k) is given by

$T = S_1$

$\Rightarrow xh - yk = h^2 - k^2$

$\Rightarrow yk = xh - h^2 + k^2$

$\Rightarrow y = \frac{h}{k}x - \frac{h^2 - k^2}{k}$

me know that the line y = mx + c will touch the parabola if $c = \frac{a}{m}$

Hence $-\frac{h^2 - k^2}{k} = \frac{ka}{h}$

$\Rightarrow -\frac{h}{k^2}(h^2 - k^2) = a$

$\Rightarrow h(h^2 - k^2) = -ak^2$

$\Rightarrow h^3 - hk^2 = -ak^2$

$\Rightarrow h^3 = hk^2 - ak^2$

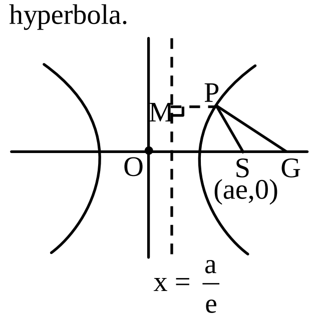
$\Rightarrow h^3 = (h - a)k^2$

$\therefore$ locus is

$x^3 = (x - a)$

Q.6 If the normal at a point P to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the x-axis at G, show that SG = e.SP, S being the focus of the hyperbola.

Sol.



SP = e(MP)

$\Rightarrow$ e(SP) = e^2(MP)

$= e^2\left\{a\sec\theta - \frac{a}{e}\right\}$

= ae {e secθ – 1} …..(i)

Now,

Normal at P : $\frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2 - b^2$

$\Rightarrow \frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2e^2$

putting y = 0, we get

G $\equiv$ (ae^2 secθ, 0)

then, SG = ae^2 secθ – ae

= ae(e secθ – 1)

= e(SP) [$\because$ from (i)]

Q.7 Find the coordinates of the points of contact of common tangents to the two hyperbolas $x^2 - y^2 = 3a^2$ and $xy = 2a^2$.

Sol. Let, the tangent of $xy = 2a^2$ be $\frac{x}{t} + yt = 2\sqrt{2}a$

$\Rightarrow y = \left(-\frac{1}{t^2}\right)x + \left(\frac{2\sqrt{2}a}{t}\right)$

But, it is also tangent to $x^2 - y^2 = 3a^2$

So, $c^2 = 3a^2(m^2 - 1)$

$\Rightarrow \left(\frac{2\sqrt{2}a}{t}\right)^2 = 3a^2\left(\frac{1}{t^4} - 1\right)$

$\Rightarrow \frac{8}{t^2} = \frac{3}{t^4} - 3$

$\Rightarrow \frac{8}{t^2} - \frac{3}{t^4} + 3 = 0$

$\Rightarrow 3t^4 + 8t^2 - 3 = 0$

$\Rightarrow 3t^4 + 9t^2 - t^2 - 3 = 0$

$\Rightarrow 3t^2(t^2 + 3) - 1(t^2 + 3) = 0$

$\Rightarrow (3t^2 - 1)(t^2 + 3) = 0$

$\Rightarrow t^2 = \frac{1}{3}$ or –3

$\Rightarrow t = \pm\frac{1}{\sqrt{3}}$ [$\because t^2 = -3$ not possible]

then, points are $\left(ct, \frac{c}{t}\right)$

$\equiv \left(\pm\sqrt{2}a \times \frac{1}{\sqrt{3}}, \pm\left(\sqrt{2}a\right)\left(\sqrt{3}\right)\right)$

$\equiv \left(\pm\frac{\sqrt{2}a}{\sqrt{3}}, \pm\sqrt{6}a\right)$

Now, the equation of tangent

$y = \left(-\frac{1}{t^2}\right)x + \left(\frac{2\sqrt{2}a}{t}\right)$

$\Rightarrow y = -3x \pm 2\sqrt{6}a$ …..(i)

Now, $x^2 - y^2 = 3a^2$ …..(ii)

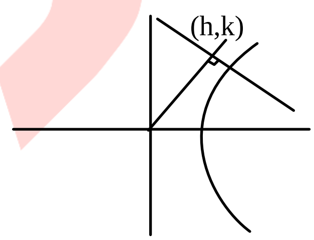
solving (i) and (ii), we get,

$x = \pm\frac{3}{4\sqrt{6}}a$, then $y = \frac{1}{4\sqrt{6}}a$

so, points $\left(\pm\frac{3}{4\sqrt{6}}a, \frac{1}{4\sqrt{6}}a\right)$

Q.8 The perpendicular from the centre upon the normal on any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets at R . Find the locus of R.

Sol.



Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

it is given that perpendicular is drawn from origin to the normal of hyperbola.

So we can write equation of normal in slope point form as

$y - k = -\frac{h}{k}(x - h)$

$\Rightarrow yk + xh + h^2 + k^2 = 0$ ….(i)

write parametric equation of normal

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ …..(ii)

comparing (i) and (ii) we get

$\frac{\frac{a}{\sec\theta}}{h} = \frac{\frac{b}{\tan\theta}}{k} = \frac{a^2 + b^2}{k^2 + h^2}$

from above relation we get

$$\frac{a}{h\sec\theta} = \frac{a^2+b^2}{k^2+h^2} \quad \text{and} \quad \frac{b}{k\tan\theta} = \frac{a^2+b^2}{k^2+h^2}$$

$$\Rightarrow \sec\theta = \frac{a(k^2+h^2)}{h(a^2+b^2)} \quad \text{....(iii)}$$

$$\Rightarrow \tan\theta = \frac{b(k^2+h^2)}{k(a^2+b^2)} \quad \text{.....(iv)}$$

$(iii)^2 – (iv)^2 = 1$

$$\Rightarrow \left(\frac{a(k^2+h^2)}{h(a^2+b^2)}\right)^2 - \left(\frac{b(k^2+h^2)}{k(a^2+b^2)}\right)^2 = 1$$

$$\Rightarrow \frac{a^2(k^2+h^2)^2}{b^2(a^2+b^2)^2} - \frac{b^2(k^2+h^2)^2}{k^2(a^2+b^2)^2} = 1$$

$$\Rightarrow \frac{(k^2+h^2)^2}{(a^2+b^2)^2}\left(\frac{a^2}{h^2} - \frac{b^2}{k^2}\right) = 1$$

$\Rightarrow (k^2 + h^2)^2 (a^2k^2 – b^2h^2) = (a^2 + b^2)^2 (h^2 k^2)$

Hence required locus is

$(x^2 + y^2)^2 (a^2y^2 – b^2x^2) = (a^2 + b^2)^2 x^2y^2$

Q.9 From points on the circle $x^2 + y^2 = a^2$ tangents are drawn to the hyperbola $x^2 – y^2 = a^2$; prove that the locus of middle points of chord of contact is the curve $(x^2 – y^2)^2 = a^2 (x^2 + y^2)$.

Sol. Any point on the circle $x^2 + y^2 = a^2$ is given by (a cosθ, a sinθ).

chord of contact of this point w.r.t. the hyperbola $x^2 – y^2 = a^2$ is

$x(a\cos\theta) – y(a\sin\theta) = a^2$

or $x\cos\theta – \sin\theta = a$ ....(i)

It its mid-point be (h, k), then it is same as

$T = S_1$

or $hx – ky = h^2 – k^2$ ....(ii)

comparing (i) and (ii), we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{h^2-k^2}$$

But $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{ah}{h^2-k^2}\right) + \left(\frac{ak}{h^2-k^2}\right) = 1$$

Hence the locus o the mid-point (h, k) is

$a^2(x^2 + y^2) = (x^2 – y^2)^2$

Q.10 A point P divides the focal length of the hyperbola $9x^2 – 16y^2 = 144$ in the ratio S'P : PS = 2 : 3 where S & S' are the foci of the hyperbola. Through P a straight line is drawn at an angle of 135° to the axis OX. Find the points of intersection of this line with the asymptotes of the hyperbola.

Sol. Equation of hyperbola

$9x^2 – 16y^2 = 144 \qquad a = 4$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad e = \sqrt{\frac{b^2}{a^2}+1} = \frac{5}{4}$$

coordinates of foci are (± ae, 0) ≡ (±5, 0)

Let P be (h, 0)

$$\text{then, } \frac{-5\times3+5\times2}{5} = h$$

h = –1

hence coordinates of ‘P’ (–1, 0)

using slope point we write equation of a line with m = –1 (as θ = 135°)

so equation of line is

y = – (x + 1)

⇒ y + x + 1 = 0

Now, equation of asymptotes are

$$\frac{x}{a} = \frac{y}{b} \quad \text{and} \quad \frac{x}{a} = -\frac{y}{b}$$

solving the line with the asymptotes separately

$$\frac{x}{a} = \frac{-(x+1)}{b} \qquad \frac{x}{a} = \frac{-(-(x+1))}{b}$$

$$\Rightarrow \frac{x}{4} = \frac{-(x+1)}{3} \qquad \Rightarrow \frac{x}{4} = \frac{x+1}{3}$$

$\Rightarrow 3x + 4x = – 4 \qquad \Rightarrow x = – 4$

$$\Rightarrow x = -\frac{4}{7} \;\&\; \frac{y}{3} = \frac{x}{4} \qquad \text{so } \frac{-4}{4} = -\frac{y}{3}$$

$$\text{so } y = \frac{3\times(-4)}{4\times7} \qquad \Rightarrow y = 3$$

$$\Rightarrow y = \frac{-3}{7}$$

So intersection points are $\left(\frac{-4}{7}, \frac{-3}{7}\right)$ & (–4, 3)

Q.11 Prove that the locus of the middle point of the chord of contact of tangents from any point of the circle $x^2 + y^2 = r^2$ to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by the equation $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = (x^2 + y^2)/r^2$

Sol. Equation of hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

it is given that chord of contact is drawn from contact points of tangents from circle $x^2 + y^2 = r^2$

so let the points be (rcosθ, rsinθ)

then equation of tangent from (rcosθ, rsinθ) to the hyperbola will be

$$\frac{x\, r\cos\theta}{a^2} - \frac{y\, r\sin\theta}{b^2} = 1 \quad \text{....(i)}$$

so, let the coordinate of the middle points of the chord be (h, k)

then applying T = S, we get

$$\left(\frac{xh}{a^2} - \frac{yk}{b^2} - 1\right) = \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right)$$

$$\Rightarrow \frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{.....(ii)}$$

comparing (i) & (ii) we get

$$\frac{\frac{r\cos\theta}{a^2}}{\frac{h}{a^2}} = \frac{\frac{r\sin\theta}{b^2}}{\frac{k}{b^2}} = \frac{1}{\frac{h^2}{a^2} - \frac{k^2}{b^2}}$$

$$\frac{r\cos\theta}{h} = \frac{1}{\frac{h^2}{a^2} - \frac{k^2}{b^2}} \quad \text{and} \quad \frac{r\sin\theta}{k} = \frac{1}{\frac{h^2}{a^2} - \frac{k^2}{b^2}}$$

$$\Rightarrow \cos\theta = \frac{h}{r\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)} \quad \text{....(i)}$$

$$\Rightarrow \sin\theta = \frac{k}{r\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)} \quad \text{....(ii)}$$

squaring (i) and (ii) and adding them we get

$\cos^2\theta +$

$$\sin^2\theta = \left(\frac{h}{r\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)}\right)^2 + \left(\frac{k}{r\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)}\right)^2$$

$$\Rightarrow \frac{h^2}{r^2} + \frac{k^2}{r^2} = \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2$$

Hence the required locus is

$$\frac{1}{r^2}(x^2 + y^2) = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

Q.12 Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$. Also find the equation to the conjugate hyperbola & the equation of the principal axes of the curve.

Sol. $H : 2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$

$A : 2x^2 - 3xy - 2y^2 + 3x - y + c = 0$

$\Delta = 0$

$abc + 2fgh - a^2f^2 - bg^2 - ch^2 = 0$

$c = 1$

$A : 2x^2 - 3xy - 2y^2 + 3x - y + 1 = 0$

$C + H = 2A$

$C + 8 = 2$

$C = -6$

$C : 2x^2 - 3xy - 2y^2 + 3x - y - 6 = 0$

Asymptotes are

$x - 2y + 1 = 0$

$2x + y + 1 = 0$

equations of bisectors

$$\frac{x - 2y + 1}{\sqrt{5}} = \pm\frac{2x + y + 1}{\sqrt{5}}$$

$x - 2y + 1 = 2x + y + 1$	$x - 2y + 1 = -(2x + y + 1)$
$x + 3y = 0$	$3x - y + 2 = 0$

Q.13 A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P & Q. Prove than the middle point of PQ lies on a parabola.

Sol. $P : x^2 = 4ay$

$H : xy = k^2$

$y = mx - am^2 \Rightarrow mx - y - am^2 = 0 \quad \text{....(i)}$

Chord

kx + hy = 2hk ….(ii)

compare (i) & (ii) equation

$$\frac{k}{m} = -\frac{1}{h} = \frac{am^2}{2hk}$$

$2k = -am^2$

$-hk = m$

$2k = -a(-hk)^2$

$2k = -ah^2k^2$

$$h^2 = \frac{-2}{ak}$$

locus : $x^2 = -\frac{-2}{ay}$

it is a equation of parabola

Part-B Passage based objective questions

Passage I (Q. 14 to 16)

If Equation of hyperbola is S = 0 then the eq^n of its asymptotes is S + λ = 0, which will be a pair of lines for which using discriminant = 0 we get λ. Say we get eq^n of asymptote A = 0

Let eq^n of conjugate hyperbola is say S_1 = 0 then we have a formula $\frac{S+S_1}{2}$ = A. Also if eccentricities of hyperbola & its conjugate hyperbola is e_1 & e_2 then we have $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

Now, statement of question is as the equation of hyperbola is $16x^2 - 9y^2 = 144$.

On the basis of above information, answer the following questions :

Q.14 The equation of asymptotes is -

(A) $y^2 = \frac{16}{9}x^2$ (B) $x^2 = \frac{16}{9}y^2$

(C) $y^2 = \frac{4}{3}x^2$ (D) None of these

Sol. **[A]**

$16x^2 - 9y^2 = 144$

Asymptotes passes through origin

So, $16x^2 - 9y^2 = \lambda$

pass (0, 0)

$\lambda = 0$

$16x^2 - 9y^2 = 0$

$$y^2 = \frac{16}{9}x^2$$

Q.15 Eccentricity of conjugate hyperbola -

(A) $\frac{5}{3}$ (B) $\frac{5}{4}$ (C) $\frac{5}{2}$ (D) None

Sol. **[B]**

Conjugates hyperbola will be

$S_1 = 16x^2 - 9y^2 = -144$

eccentricity of conjugate hyperbola

$a^2 = b^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$$\frac{25}{16} = e^2$$

$$e = \frac{5}{4}$$

Q.16 For what value of λ does the line y = 2x + λ touches the given hyperbola -

(A) $2\sqrt{5}$ (B) $3\sqrt{5}$

(C) $4\sqrt{5}$ (D) None of these

Sol. **[A]**

$c^2 = a^2m^2 - b^2$ [Condition of tangency]

$\lambda^2 = 9 \times 4 - 16$

$\lambda = 2\sqrt{5}$

Passage II (Q. 17 to 19)

A rectangular hyperbola passing through (3, 2) having centre at (1, 2) and its asymptotes are parallel to lines x + y = 2 and x – y = 3.

On the basis of above information, answer the following questions :

Q.17 The equation of asymptotes are -

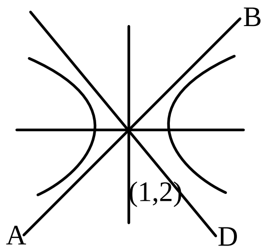
(A) (x + y – 3) (x + y –1) = 0

(B) (x + y – 3) (x – y +1) = 0

(C) (x + y + 3) (x – y +1) = 0

(D) None of these

Sol. **[B]**



AB : $x + y + \lambda_1 = 0$

pass (1, 2)

$\lambda_1 = -3$

AB : $(x + y - 3) = 0$

CD : $x - y + \lambda_2 = 0$

pass (1, 2)

$\lambda = 1$

$x - y + 1 = 0$

equation of asymptotes

$= (x + y - 3)(x - y + 1) = 0$

Q.18 The equation of rectangular hyperbola is -

(A) $(x + y - 3)(x - y + 1) = 4$

(B) $(x + y - 3)(x - y + 1) = 8$

(C) $(x + y + 3)(x - y + 1) = 4$

(D) None of these

Sol. **[A]**

Hyperbola : $(x + y - 3)(x - y + 1) - \lambda_3 = 3$

pass (3, 2)

$\lambda_3 = 4$

equation $(x + y - 3)(x - y + 1) = 4$

Q.19 Distance between foci is -

(A) 4 (B) $4\sqrt{2}$

(C) $2\sqrt{2}$ (D) None of these

Sol. **[B]**

The above hyperbola can be written in the form

$$\frac{(x-1)^2}{4} - \frac{(y-2)^2}{4} = 1$$

a = 2

2a = 4

$e = \sqrt{2}$

$\therefore$ Distance between foci = 2ae = $4\sqrt{2}$

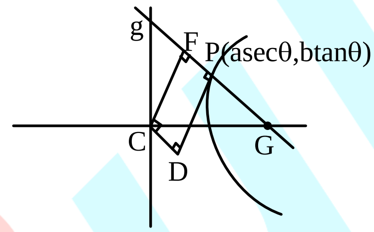
Passage III (Q. 20 to 22)

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normal at P meets the transverse axis AA′ in G and the conjugate axis BB′ in g and CF be perpendicular to the normal from the centre.

Q.20 PF . PG = k CB^2, then k =

(A) 2 (B) 1 (C) $\frac{1}{2}$ (D) 4

Sol. **[B]**



Equation of normal at P

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

solving this with x-axis & y-axis

$$G\left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right) \ \& \ g\left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$$

equation of tangent at P

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

length of ⊥ from origin

$$PF = CD = \left|\frac{-1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}}\right| = \frac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$

$$PG = \sqrt{\left(a\sec\theta - \frac{(a^2+b^2)\sec\theta}{a}\right)^2 + b^2\tan^2\theta}$$

$$PG = \sqrt{\left[\frac{a^2\sec\theta - a^2\sec\theta - b^2\tan\theta}{a}\right]^2 + b^2\tan^2\theta}$$

$$PG = \sqrt{\frac{b^4\sec^2\theta + a^2b^2\tan^2\theta}{a^2}}$$

$$PG = \frac{b}{a}\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}$$

Now PF . PG = kCB^2

$$\Rightarrow \frac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} \times \frac{b}{a}\sqrt{b^2\sec^2\theta + a^2\tan^2\theta} = kb^2$$

$b^2 \times 1 = kb^2$

$k = 1$

Q.21 PF . PG equal to –

(A) CA^2 (B) CF^2 (C) CB^2 (D) CA . CB

Sol. **[A]**

$$Pg = \sqrt{a^2\sec^2\theta + \left(b\tan\theta - \frac{(a^2+b^2)\tan\theta}{b}\right)^2}$$

$$= \sqrt{a^2\sec^2\theta + \frac{a^4\tan^2\theta}{b^2}}$$

$$= \frac{1}{b}\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}$$

Now, PF . Pg

$$= \frac{a}{b}\sqrt{b^2\sec^2\theta + a^2\tan^2\theta} \times \frac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$

PF . Pg = a^2

$a^2 = CA^2$

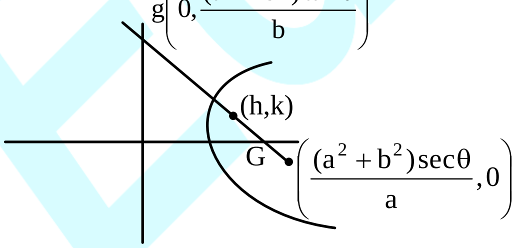
$\therefore$ PF . Pg = CA^2.

Q.22 Locus of middle point of G and g is a hyperbola of eccentricity –

(A) $\dfrac{1}{\sqrt{e^2-1}}$ (B) $\dfrac{e}{\sqrt{e^2-1}}$

(C) $2\sqrt{e^2-1}$ (D) $\dfrac{3}{2}$

Sol. **[B]**



$g\left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$, $G\left(\dfrac{(a^2+b^2)\sec\theta}{a}, 0\right)$

$$2h = \frac{(a^2+b^2)\sec\theta}{a}$$

$$\sec\theta = \frac{2ah}{a^2+b^2} \quad .....(i)$$

$$2k = \frac{(a^2+b^2)\tan\theta}{b}$$

$$\tan\theta = \frac{2kb}{a^2+b^2} \quad .....(ii)$$

$\sec^2\theta - \tan^2\theta = 1$

$$\frac{4a^2h^2}{(a^2+b^2)^2} - \frac{4k^2b^2}{(a^2+b^2)^2} = 1$$

$$a^2x^2 - b^2y^2 = \frac{(a^2+b^2)^2}{4}$$

$$\frac{x^2}{\frac{(a^2+b^2)^2}{4a^2}} - \frac{y^2}{\frac{(a^2+b^2)^2}{4b^2}} = 1$$

$$\frac{(a^2+b^2)^2}{4b^2} = \frac{(a^2+b^2)^2}{4a^2}\left(e_0^2 - 1\right)$$

$$\frac{a^2}{b^2} = e_0^2 - 1$$

$$\frac{1}{e^2-1} = e_0^2 - 1$$

$$e_0^2 = \frac{1}{e^2-1} + 1$$

$$e_0 = \sqrt{\frac{1+e^2-1}{e^2-1}} = \frac{e}{\sqrt{e^2-1}}$$

EXERCISE # 4

Old IIT-JEE questions

Q.1 The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is - **[IIT 2002]**

(A) $3y = 9x + 2$ (B) $y = 2x + 1$

(C) $2y = x + 8$ (D) $y = x + 2$

Sol.[D] Any point on $y^2 = 8x$ is $(2t^2, 4t)$, where the tangent is $yt = x + 2t^2$

solving it with $xy = -1$, and $y(yt - 2t^2) = -1$

or $ty^2 - 2t^2y + 1 = 0$

for common tangent, it should have equal roots

$\therefore 4t^4 - 4t = 0$

$\Rightarrow t = 0, 1$

$\therefore$ the common tangent is $y = x + 2$

(when t = 0, it is x = 0 which can torch xy = –1 at infinity only)

$\therefore$ Reqd common tangent is $y = x + 2$

Q.2 If $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas, where α varies then - **[IIT Scr.2003]**

(A) e remains constant

(B) abscissas of foci remain constant

(C) equation of directrices remain constant

(D) abscissas of vertices remains constant

Sol.[B] $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

Θ $ae = \sqrt{a^2 + b^2}$ where $a^2 = \cos^2\alpha$, $b^2 \sin^2\alpha$

$ae = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$\therefore$ abscissas of foci remain constant

Q.3 The point at which the line $2x + \sqrt{6}\,y = 2$ touches the curve $x^2 - 2y^2 = 4$, is - **[IIT Scr. 2004]**

(A) $(4, -\sqrt{6})$ (B) $(\sqrt{6}, 1)$

(C) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (D) $\left(\frac{\pi}{6}, \pi\right)$

Sol.[A] As we know, equation of tangent at (x_1, y_1) is

$xx_1 - 2yy_1 = 4$

which is same as

$2x + \sqrt{6}\,y = 2$

$\therefore \frac{x_1}{2} = -\frac{2y_1}{\sqrt{6}} = \frac{4}{2}$

$\Rightarrow$ x1 = 4 and y1 = $-\sqrt{6}$

$\therefore$ point of contact is

$(4, -\sqrt{6})$

Q.4 A tangent is drawn from a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to circle $x^2 + y^2 = 9$ find the locus of mid-point of chord of contact. **[IIT 2005]**

Sol. Let any point on the hyperbola is $(3\sec\theta, 2\tan\theta)$

$\therefore$ chord of contact of the circle x2 + y2 = 9 w.r.t. the point $(3\sec\theta, 2\tan\theta)y = 9$ ....(i)

let (x_1, y_1) be the mid-point of the chord of antact

$\Rightarrow$ equation of chord in mid-point form is

$xx_1 + yy_1 = x_1^2 + y_1^2$ ....(ii)

since (i) and (ii) are identically equal

$\therefore \frac{3\sec\theta}{x_1} = \frac{2\tan\theta}{y_1} = \frac{9}{x_1^2 + y_1^2}$

$\Rightarrow \sec\theta = \frac{9x_1}{3(x_1^2 + y_1^2)}$ and $\tan\theta = \frac{9y_1}{2(x_1^2 + y_1^2)}$

thus eliminating 'θ' from above equation, we get

$\frac{81x_1^2}{9(x_1^2 + y_1^2)^2} - \frac{81y_1^2}{4(x_1^2 + y_1^2)^2} = 1$

$\therefore$ Required locus $\frac{x^2}{9} - \frac{y^2}{4} = \frac{(x^2 + y^2)^2}{81}$

Q.5 The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents **[IIT 2006]**

(A) Pair of straight lines

(B) An ellipse

(C) A parabola

(D) A hyperbola

Sol.[C] Given $x = t^2 + t + 1$. $y = t^2 - t + 1$

Hence $x + y = 2(t^2 + 1)$

and $x - y = 2t$

$\frac{x + y}{2} = t^2 + 1 = \left(\frac{x - y}{2}\right)^2 + 1$

$\Rightarrow 2(x + y) = (x - y)^2 + 4$

$\Rightarrow x^2 - 2xy + y^2 - 2x - 2y + 4 = 0$

on comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 1, h = -1, b = 1, g = -1, f = -1, c = 4$

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1.1.4 + 2(-1)(-1)(-1) -1.1 -1.1 -1.4$

$= 4 -2 -2 - 4$

$\neq 0$

and $h^2 = ab$

$\therefore$ curve is a parabola

Q.6 If a hyperbola passes through the focus of the $\frac{x^2}{25}+\frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with the major and minor axis of ellipse, and product of eccentricities is 1, then **[IIT 2006]**

(A) Focus of hyperbola is (5, 0)

(B) Focus of hyperbola is $(5\sqrt{3}, 0)$

(C) The equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(D) The equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Sol. **[A,D]**

$\frac{x^2}{25}+\frac{y^2}{16} = 1$

Eccentricity of ellipse = $e = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\Rightarrow$ eccentricity of hyperbola = $\frac{5}{3}$

since transverse and conjugate axes of hyperbola coincide with the major and minor axes of ellipse in equation must be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

make it pars through foci (± 3, 0) of the ellipse to get $b^2 = 16$

$\Rightarrow$ equation of hyperbola is

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ whose foci are (± 5, 0)

Q.7 A hyperbola, having the transverse axis of length 2 $\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is - **[IIT-2007]**

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$

(B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$

(D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Sol.[A] The equation of ellipse 3x2 + 4y2 = 12 can be written as

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Here a = 2 and b = $\sqrt{3}$

It e is the eccentricity of the ellipse, then

$b^2 = a2(1 - e^2)$

$\Rightarrow 3 = 4(1 - e^2)$

$\Rightarrow e^2 = \frac{1}{4}$

$\Rightarrow e = \frac{1}{2}$

Foci of the ellipse are (ae, 0) and (–ae, 0)

i.e. (1, 0) and (–1, 0)

If 2a, is the length of transverse axis, and 2b, the length of the conjugate axis and e, is the eccentricity of the hyperbola, then a,e, = ae = 1 and $b_1 = \sin\theta$

Since $b_1^2 = a_1^2(e_1^2 - 1)$ me get

$\therefore$ An equation of the hyperbola is

$\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$

$\Rightarrow x^2\sec^2\theta - y^2\text{cosce}^2\theta = 1$

Q.8 Match the statements in Column I with the properties in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given the ORS **[IIT 2007]**

Column I

(A) Two intersecting circles

(B) Two mutually external circles

(C) Two circles, one strictly inside the other

(D) Two branches of a hyperbola

Column II

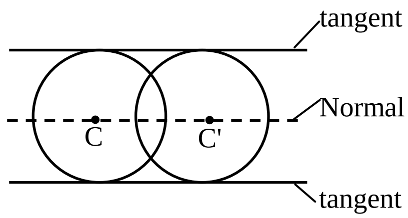
(P) have a common tangent

(Q) have a common normal

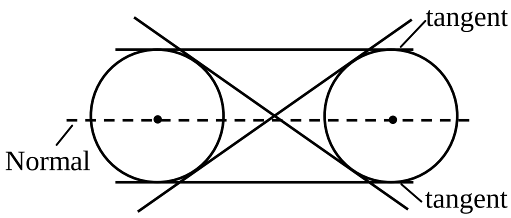
(R) do not have a common tangent

(S) do not have a common Normal

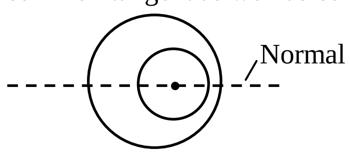
Sol. **A → P,Q; B → P,Q; C → Q,R ; D → Q,R**

(A)


i.e. two intersecting circles have a common tangent as well as a common normal

(B)


i.e. two mutually external circles have a common tangent as well as common normal

(C)


i.e. two circles, are strictly inside the other do not have a common tangent but have a common normal

(D) Two branches of a hyperbola do not have a common tangent but can have a common normal which is transverse axis of hyperbola.

Q.9 Consider a branch of the hyperbola

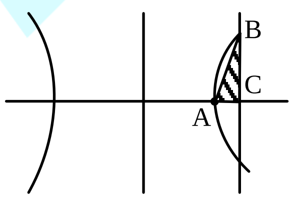
$x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$

with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is- **[IIT-2008]**

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$

(C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

Sol. **[B]**



Hight = $\frac{b^2}{a}$, Base = a e – a

$\Rightarrow \Delta = \frac{1}{2} \times \frac{b^2}{a} \times (ae - a)$

Hyperbola is

$(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$

$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$

$2 = 4(e^2 - 1)$

$\frac{1}{2} = e^2 - 1 \Rightarrow e^2 = \frac{3}{2} \Rightarrow e = \frac{\sqrt{3}}{\sqrt{2}}$

$\Delta = \frac{1}{2} \times \frac{2}{2} \times 2\left(\frac{\sqrt{3}}{\sqrt{2}} - 1\right) = \left(\sqrt{\frac{3}{2}} - 1\right)$

Q.10 The locus of the orthocentre of the triangle formed by the lines **[IIT-2009]**

$(1 + p)x - py + p(1 + p) = 0$

$(1 + q)x - qy + q(1 + q) = 0$,

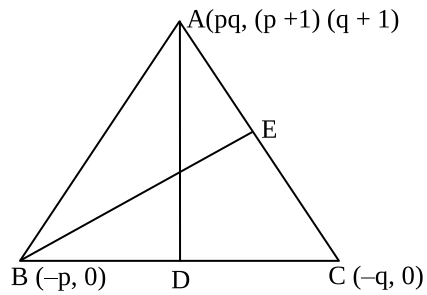
and $y = 0$, where $p \neq q$, is :

(A) a hyperbola (B) a parabola

(C) an ellipse (D) a straight line

Sol. **[D]**

Intersection points of given lines are (–p, 0), (–q, 0), [pq, (p +1) (q + 1)] respectively.



Now equation of altitudes AD and BE are x = pq, and qx + (q + 1)y + pq = 0

their point of intersection is (pq, –pq)

so, h = pq, k = –pq

so locus is h = –k

h + k = 0

$\Rightarrow x + y = 0$ which is a straight line.

Q.11 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then : **[IIT-2009]**

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Sol. **[A, B]**

Since both conic cuts orthogonally so foci coincides

Now foci of hyperbola is $(\pm1, 0)$

$\therefore$ foci of ellipse $(\pm1, 0)$

$\therefore$ eccentricity of rectangular hyperbola is $\sqrt{2}$

$\therefore$ eccentricity of ellipse $= \frac{1}{\sqrt{2}}$

$\therefore$ equation of ellipse is $\frac{x^2}{2} + y^2 = 1$

Q.12 The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[IIT-2010]**

Sol. [2] $1 = 4a^2 - b^2$ ... (1)

$\frac{2a}{e} = 1$

$a = \frac{e}{2}$ ... (2)

also $b^2 = a^2(e^2 - 1)$ ... (3)

(1) and (3)

$1 = 4a^2 - a^2e^2 + a^2 \Rightarrow 1 = 5a^2 - a^2e^2$

$\Rightarrow 1 = \frac{5e^2}{4} - \frac{e^4}{4}$

$\Rightarrow e^4 - 5e^2 + 4 = 0 \Rightarrow (e^2 - 4)(e^2 - 1) = 0$

$\therefore$ e = 2

Passage (Q. 13 & 14)

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B

[IIT-2010]

Q.13 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is –

(A) $2x - \sqrt{5}\,y - 20 = 0$

(B) $2x - \sqrt{5}\,y + 4 = 0$

(C) $3x - 4y + 8 = 0$

(D) $4x - 3y + 4 = 0$

Sol. **[B]**

$y = m(x - 4) \pm 4\sqrt{1+m^2}$

$y = mx \pm \sqrt{9m^2 - 4}$

$-4m \pm 4\sqrt{1+m^2} = \pm\sqrt{9m^2 - 4}$

$16m^2 + 16 + 16m^2 \mu\ 32m\sqrt{1+m^2} = 9m^2 - 4$

$\mu\ 32m\sqrt{1+m^2} = -23m^2 - 20$

$1024m^2 + 1024m^4 = 529m^4 + 400 + 920m^2$

$495m^4 + 104m^2 - 400 = 0$

$(5m^2 - 4)(99m^2 + 100) = 0$

$\therefore m^2 = \frac{4}{5}$ $\qquad \therefore m = \pm\frac{2}{\sqrt{5}}$

So tangent with positive slope

$y = \frac{2}{\sqrt{5}}x \pm \frac{4}{\sqrt{5}}$

$2x - \sqrt{5}\,y \pm 4 = 0$

Q.14 Equation of the circle with AB as its diameter is

(A) $x^2 + y^2 - 12x + 24 = 0$

(B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$

(D) $x^2 + y^2 - 24x - 12 = 0$

Sol. **[A]**

$x^2 + y^2 - 8x = 0$

$4x^2 - 9y^2 = 36$

$x^2 + \left(\frac{4x^2 - 36}{9}\right) - 8x = 0$

$13x^2 - 72x - 36 = 0$

$(x - 6)(13x + 6) = 0$

$x = 6, \frac{-6}{13}$

$x = 6, y = \pm \sqrt{12}$

$\therefore$ Equation of required circle

$(x-6)^2 + (y - \sqrt{12})(y + \sqrt{12}) = 0$

$x^2 + y^2 - 12x + 24 = 0$

Q.15 Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then **[IIT-2011]**

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

Sol. [B,D]

Let e_1 = eccentricity of hyperbola

e_2 = eccentricity of ellipse

$\therefore \quad e_1 = \frac{1}{e_2}$

so eccentricity of ellipse = $\frac{\sqrt{3}}{2} = e_2$

eccentricity of ellipse = $\frac{2}{\sqrt{3}} = e_1$

Now focus of ellipse is $(\pm ae_2, 0) \equiv (\pm\sqrt{3}, 0)$

Hyperbola passes through it

So, $\frac{(\sqrt{3})^2}{a^2} - 0 = 1 \Rightarrow a^2 = 3$

also $b^2 = a^2 (e_1^2 - 1)$

$b^2 = 3\left(\frac{4}{3} - 1\right) = 1$

and hyperbola

$\frac{x^2}{3} - \frac{y^2}{1} = 1$

also focus $(\pm ae_1, 0) \equiv (\pm 2, 0)$

Q.16 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is **[IIT-2011]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Sol. **[B]**

Equation of the normal at (6, 3) is

$\frac{a^2x}{6} + \frac{b^2y}{3} = a^2 + b^2$

it passes through (9, 0)

so $\frac{9a^2}{6} = a^2 + b^2$

$\Rightarrow b^2 = \frac{a^2}{2}$

Now $b^2 = a^2 (e^2 - 1)$

$\therefore e^2 - 1 = \frac{1}{2}$

$e^2 = \frac{3}{2}$

$\Rightarrow e = \sqrt{\frac{3}{2}}$

Q.17 Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line 2x – y = 1. The points of contact of the tangents on the hyperbola are **[IIT-2012]**

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

Sol. [A,B] Equation of tangent is

$2x - y + c = 0$

$y = 2x + c$

slope m = 2

$\therefore \quad a^2 = 9, b^2 = 4$

$\therefore \quad c^2 = a^2m^2 - b^2 = 9 \times 4 - 4$

$c = \pm 4\sqrt{2}$

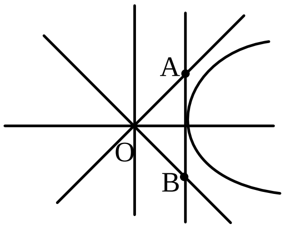
$\therefore$ point of contact is $\left(\pm \frac{a^2m}{c}, \pm \frac{b^2}{c}\right)$

$\left(\pm \frac{9}{2\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$

EXERCISE # 5

Q.1 Show that the area of the triangle formed by the lines x – y = 0 , x + y = 0 & any tangent to the hyperbola $x^2 - y^2 = a^2$ is a^2.

Sol.



Tangent

$\frac{x}{a}\sec\theta - \frac{y}{a}\tan\theta = 1$ .....(i)

$x + y = 0$ .....(ii)

$x - y = 0$ .....(iii)

solve (i) & (ii)

$A\left[\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right]$

solve (i) & (iii)

$B\left[\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right]$

Area of Δ

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{a}{\sec\theta - \tan\theta} & \frac{a}{\sec\theta - \tan\theta} & 1 \\ \frac{a}{\sec\theta + \tan\theta} & \frac{-a}{\sec\theta + \tan\theta} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\frac{-a^2}{\sec\theta^2 - \tan\theta^2} - \frac{a^2}{\sec^2\theta - \tan^2\theta}\right|$$

$$= \frac{1}{2}\left|\left[-2a^2\right]\right|$$

area of $\Delta = a^2$

Q.2 Show that the locus of mid points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Sol. If (h, k) be the middle point of the chord, then its equation is given by

$T = S_1$

$\Rightarrow hx - ky - a^2 = h^2 - k^2 - a^2$ ....(i)

$\Rightarrow hx - ky = h^2 - k^2$

since (i) is a normal, it should be tame as normal at ϕ, i.e. $x\cos\phi + y\cot\phi = 2a$ ....(ii)

comparing (i) and (ii) we get

$$\frac{h}{\cos\phi} = -\frac{k}{\cot\phi} = \frac{h^2 - k^2}{2a}$$

$\therefore \sec\phi = \frac{h^2 - k^2}{2ah}$ and $\tan\phi = -\frac{h^2 - k^2}{2ah}$

But $\sec^2\phi - \tan^2\phi = 1$

$$\therefore \frac{(h^2 - k^2)^2}{4a^2h^2} - \frac{(h^2 - k^2)^2}{4a^2h^2} = 1$$

or $(h^2 - k^2)^2(k^2 - h^2) = 4a^2h^2k^2$

or $(k^2 - h^2)^3 = 4a^2x^2y^2$

$\therefore$ required locus is

$(y^2 - x^2)^3 = 4a^2x^2y^2$

Q.3 Find the equation of the standard hyperbola passing through the point $(-\sqrt{3}, 3)$ and having the asymptotes as the straight lines $x\sqrt{5} \pm y = 0$.

Sol. Asymptotes

$\sqrt{5}x - y = 0$

$\sqrt{5}x + y = 0$

H : $\left(\sqrt{5}x - y\right)\left(\sqrt{5}x + y\right) + \lambda = 0$

pass through $\left(-\sqrt{3}, 3\right)$

$\lambda = -6$

H : $5x^2 - y^2 - 6 = 0$

$5x^2 - y^2 = 6$

Q.4 A circle of variable radius & centre (h, k) cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of the triangle PQR.

Sol. Let the circle be $(x - h)^2 + (y - k)^2 = r^2$ where r is a variable its intersection with $x^2 - y^2 = 9a^2$ is obtained by putting $y^2 = x^2 - 9a^2$

$x^2 + x^2 - 9a^2 - 2hx + h^2 + k^2 - r^2 = 2k\sqrt{x^2 - 9a^2}$

or $[2x^2 - 2hx + (h^2 + k^2 - r^2 - 9a^2)]^2 = 4k(x^2 - 9a^2)$

or $4x^4 - 8hx^3 + ... = 0$

$\therefore$ Above given the abscissas of the four points of intersection

$\therefore \Sigma x_1 = \frac{8h}{4} = 2h$

$\therefore x_1 + x_2 + x_3 + x_4 = 2h$

similarly $y_1 + y_2 + y_3 + y_4 = 2k$

Now if (α, β) be the centroid of ΔPQR, the

$3\alpha = x_1 + x_2 + x_3$ & $3\beta = y_1 + y_2 + y_3$

$\therefore x_4 = 2h - 3\alpha \leftarrow y_4 = 2k - 3\beta$

But (x_4, y_4) lies on $x^2 - y^2 = 9a^2$

$\therefore (2h - 3x)^2 + (2k - 3\beta)^2 = 9a^2$

Hence the locus of centroid (α, β) is

$(2h - 3x)^2 + (2k - 3y)^2 = 9a^2$

or $\left(x - \frac{2h}{3}\right)^2 + \left(y - \frac{2k}{3}\right)^2 = a^2$

Q.5 Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any points (x_1, y_1) and (x_2, y_2) in the region, the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is also in the region. The inequality defining this region is **[IIT-1981]**

(A) $x^2 + 2y^2 \le 1$ (B) Max. $\{|x|, |y|\} \le 1$

(C) $x^2 - y^2 \le 1$ (D) $y^2 - x \le 0$

Sol. **[B]**

(a) $x^2 + 2y^2 \le 1$ represents interior region of circle, where on taking any two points the mid-point of that segment will also lie inside that circle.

(b) Max. $\{|x|, |y|\} \le 1$

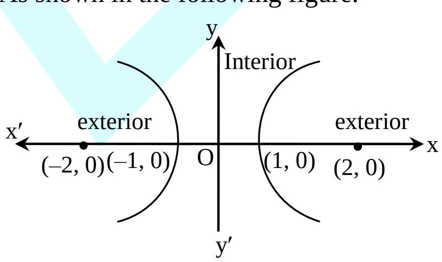
$\Rightarrow |x| \le 1, |y| \le 1$

$\Rightarrow -1 \le x \le 1$ and $-1 \le y \le 1$

which represents the interior region of a square with its sides $x = \pm 1$ and $y = \pm 1$ in which for any two points, their mid-point also lies inside the region.

(c) $x^2 - y^2 \ge 1$ represents the exterior region of hyperbola in which if we take two points (2, 0) and (–2, 0) then their mid-point (0, 0) does not lie in the same region.

As shown in the following figure.



(d) $y^2 \le x$ represents interior region of parabola in which for any two points, their mid-point also lie inside the region.

Q.6 If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$, $i = 1, 2, 3, 4$ are four distinct points on a circle, then show that $m_1m_2m_3m_4 = 1$. **[IIT-1989]**

Sol. Let the points $\left(m_i, \frac{1}{m_i}\right)$; $i = 1, 2, 3, 4$

Lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Then $m_i^2 + \frac{1}{m_i^2} + 2gm_i + \frac{2f}{m_i} + c = 0$; $i=1,2,3,4$

$\Rightarrow m_1, m_2, m_3$ and m_4 are the roots of the equation

$m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

$\Rightarrow m_1 m_2 m_3 m_4 = \frac{1}{1} = 1$

Q.7 The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents **[IIT-1994]**

(A) no locus if $k > 0$ (B) an ellipse if $k > 0$

(C) a point if $k = 0$ (D) a hyperbola if $k > 0$

Sol.[C] $2x^2 + 3y^2 - 8x - 18y + 35 - k = 0$

$\Rightarrow (2x^2 - 8x) + (3y^2 - 18y) + (35 - k) = 0$

$\Rightarrow 2(x^2 - 4x) + 3(y^2 - 6y) + (35 - k) = 0$

$\Rightarrow 2(x - 2)^2 + 3(y - 3)^2 = k - 35 + 8 + 27$

$\Rightarrow 2(x - 2)^2 + 3(y - 3)^2 = k$

$\Rightarrow \frac{(x-2)^2}{k/2} + \frac{(y-3)^2}{k/3} = 1$

if $k > 0$, it represents an ellipse therefore an ellipse, if $k > 0$

Also $\Delta = abc + 2fsh - af^2 - bg^2 - ch^2$

$= b(35 - k) + 0 - 162 - 48 - 0$

$= 210 - 6k - 210$

$= -6$

$\therefore \Delta = -6k$, $D = 0$, ib $k = 0$

so, given equation is a point if $k = 0$

Q.8 A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point which divides the line segment

between these two points in the ratio 1 : 2.
[IIT-97]

Sol. Let the live be y = 4x + c (where c is variable) its intersection with hyperbola xy = 1 is given by x (4x + c) = 1

$\Rightarrow 4x^2 + cx - 1 = 0$ ....(i)

$\Rightarrow y(y - c) = 4$

$\Rightarrow$ y2 – cy – 4 = 0 ....(ii)

Above given the abscissas of points of intersection P and Q say x_1, x_2. If (h, k) divides PQ in the ratio 1 : 2 then

$h = \frac{x_2 + 2x_1}{3}$, $k = \frac{y_2 + 2y_1}{3}$

Now from (i) and (ii)

$x_1 = -\frac{c + \sqrt{c^2 + 16}}{8}$, $x_2 = -\frac{c - \sqrt{c^2 + 16}}{8}$

and y1 = $\frac{c + \sqrt{c^2 + 16}}{2}$, $y_2 = \frac{c - \sqrt{c^2 + 16}}{2}$

$\therefore 3h = x_2 + 2x_1 = x_2 + x_1 + x_1$

$3k = y_2 + 2y_1 = y_2 + y_1 + y_1$

$3h = -\frac{2c}{8} + -\frac{c + \sqrt{c^2 + 16}}{8}$

or $24h = -3c + \frac{c + \sqrt{c^2 + 16}}{2}$ ....(iii)

$3k = \frac{2c}{2} + \frac{c + \sqrt{c^2 + 16}}{2}$

or $6k = 3c + \sqrt{c^2 + 16}$ ....(iv)

we have to eliminate the variable c

24h – 6k = – 6c

or c = k – 4h

putting the value of c in (iv), we get

$6k - 3(k - 4h) = \sqrt{c^2 + 16}$

or $(3k + 12h)^2$ = c2 + 16 = $(k - 4h)^2 + 16$

or $16h^2 + k^2 + 10hk - 2 = 0$

$\therefore$ Locus of (h, k) is

$16x^2 + y^2 + 10xy - 2 = 0$

Q.9 The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45º. Show that the locus of the point P is a hyperbola. **[IIT-98]**

Sol. Let P(α, β) be any point on the locus. Equation of pair of tangents from P(α, β) to the parabola y^2 = 4ax is

$[\beta y - 4a(x + \alpha)]^2 = (\beta^2 - 4a\alpha)(y^2 - 4ax)$

$\Rightarrow \beta^2y^2 + 4a^2x^2 + 4a^2\alpha^2 = \beta^2y^2 - 4\beta^2ax - 4a\alpha y^2 + 16a^2\alpha x - 2a\beta y\,(x + \alpha)$

$\Rightarrow \beta^2y^2 + 4a^2x^2 + 4a^2\alpha^2 = \beta^2y^2 - 4\beta^2ax - 4a\alpha y^2 + 16a^2\alpha x - 4a\beta xy - 4a\beta\alpha y$.........(i)

Now, coefficient of $x^2 = 4a^2$

coefficient of xy = – 4aβ

coefficient of y^2 = 4aα

again angle between the two of (i) is given as 45º

$\Rightarrow \tan 45º = \frac{2\sqrt{h^2 - ab}}{a + b}$

$\Rightarrow 1 = \frac{2\sqrt{h^2 - ab}}{a + b}$

$\Rightarrow a + b = 2\sqrt{h^2 - ab}$

$\Rightarrow (a + b)^2 = 4\,(h^2 - ab)$

$\Rightarrow (4a^2 + 4a\alpha)^2 = 4[4\alpha^2\beta^2 - (4a^2)(4a\alpha)]$

$\Rightarrow 16a^2\,(a + \alpha)^2 = 4.4a^2[b^2 - 4a\alpha]$

$\Rightarrow \alpha^2 + 6a\alpha + a^2 - \beta^2 = 0$

$\Rightarrow (\alpha + 3a)^2 - \beta^2 = 8a^2$

$\therefore$ locus is $(x + 3a)^2 - y^2 = 8a^2$

which is a hyperbola (proved)

Q.10 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then - **[IIT-98]**

(A) $x_1 + x_2 + x_3 + x_4 = 0$

(B) $y_1 + y_2 + y_3 + y_4 = 2$

(C) $x_1x_2x_3x_4 = 2c^4$

(D) $y_1y_2y_3y_4 = 2c^4$

Sol.[A] $x^2 + y^2 = a^2$ ....(i)

$xy = c^2$ ....(ii)

from (i) and (ii), we get

$$x^2 + \frac{c^4}{x^2} = a^2$$

$\Rightarrow x^4 - a^2x^2 + c^4 = 0$

$\therefore x_1 + x_2 + x_3 + x_4 = 0$

and $x_1 x_2 x_3 x_4 = c^4$

since both the curves are symmetrical in x and y

$\therefore y^4 - a^2y^2 + c^4 = 0$

$\therefore y_1 + y_2 + y_2 + y_3 = 0$

and $y_1 y_2 y_3 y_4 = c^4$

Q.11 If a circle cuts the rectangular hyperbola xy = 1 in the points (x_r, y_r) where r = 1, 2, 3, 4, then - **[IIT-98]**

(A) $x_1x_2x_3x_4 = 2$

(B) $x_1x_2x_3x_4 = 1$

(C) $x_1 + x_2 + x_3 + x_4 = 0$

(D) $y_1 + y_2 + y_3 + y_4 = 0$

Sol.[B] Let circle is $x^2 + y^2 = a^2$ ....(i)

$xy = 1$ ....(ii)

from (i) and (ii), we get

$$x^2 + \frac{1}{x^2} = a^2$$

$\Rightarrow x^4 - a^2x^2 + 1$

$\therefore x_1 + x_2 + x_3 + x_4 = 0$

and $x_1 x_2 x_3 x_4 = 1$

similarly $y^4 - a^2y^2 + 1 = 0$

$\therefore y_1 + y_2 + y_3 + y_4 = 0$ and $y_1 y_2 y_3 y_4 = 1$

Q.12 If x = 9 is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is - **[IIT-99]**

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x - 9 = 0$

(D) $9x^2 - 8y^2 + 18x + 9 = 0$

Sol.[B] The equation of chord of contact at point (h, k) is

$xy - yk = 9$

comparing with x = 9, we have h = 1 , k = 0

Hence equation of pair of tangents at point (1, 0) is

$SS_1 + T^2$

$\Rightarrow (x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x - 9)^2$

$\Rightarrow -8x^2 + 5y^2 + 72 = x^2 - 18x + 81$

$\Rightarrow 9x^2 - 8y^2 - 18x + 9 = 0$

Q.13 Let P (a sec θ, b tan θ) and Q (a sec ϕ, b tan ϕ) where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to - **[IIT-99]**

(A) $\frac{a^2+b^2}{a}$ (B) $-\frac{a^2+b^2}{a}$

(C) $\frac{a^2+b^2}{b}$ (D) $-\frac{a^2+b^2}{b}$

Sol.[D] Given P(a secθ, b tanθ) and Q(a secϕ, b tanϕ)

The equation of tangent at point P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

slope m of tangent = $\frac{b}{\tan\theta} \times \frac{\sec\theta}{a} = \frac{b}{a} . \frac{1}{\sin\theta}$

Hence the equation of perpendicular at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - \sec\theta)$$

or $by - b^2\tan\theta = -a\sin\theta . x + a^2\tan\theta$

or $a\sin\theta . x + by = (a^2 + b^2)\tan\theta$ ....(i)

similarly the equation of perpendicular at Q is

$a\sin\phi . x + by = (a^2 + b^2)\tan\phi$ ....(ii)

on multiplying (i) by sinϕ and (ii) by sinθ

$a\sin\phi x + b\sin\phi y = (a^2 + b^2)\tan\theta\sin\phi$

$a\sin\phi\sin\theta x + b\sin\theta y = (a^2 + b^2)\tan\phi\sin\theta$

on subtraction by

$(\sin\phi - \sin\theta) = (a^2 + b^2)(\tan\theta\sin\phi - \tan\phi\sin\theta)$

$$\therefore y = k = \frac{a^2+b^2}{b}, \frac{\tan\theta\sin\phi - \tan\phi\sin\theta}{\sin\phi - \sin\theta}$$

$$\Theta\ \theta + \phi + \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{2} - \theta$$

$\Rightarrow \sin\phi = \cos\theta$ and $\tan\phi = \cot\theta$

$$\therefore y = k = \frac{a^2+b^2}{b}, \frac{\tan\theta\cos\theta - \cot\theta\sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{a^2+b^2}{b}\left(\frac{\sin\theta - \cos\theta}{\cos\theta - \sin\theta}\right) = -\frac{a^2+b^2}{b}$$

$$\therefore k = -\frac{a^2+b^2}{b}$$

ANSWER KEY

EXERCISE # 1

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Ans.	A	B	C	C	A	A	A	C	C	A	A	A	D	A	C	A	B	B	A

20. True **21.** False **22.** |hk| **23.** (0, 0)

EXERCISE # 2

PART - A

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	D	C	A	C	D	D	D	D	D	C	A	B	D

PART - B

Qus.	14	15	16	17	18	19	20	21	22
Ans.	A,D	A,B,C,D	B,C	A,B	A,B,C,D	A,C	A,C	A	A,B,C,D

PART - C

Qus.	23	24	25	26
Ans.	A	A	A	A

PART - D

27. $A \to P$; $B \to Q$; $C \to P$; $D \to S$ **28.** $A \to R$; $B \to S$; $C \to Q$; $D \to P$

EXERCISE # 3

1. $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$; $\left(-\frac{a}{3}, a\right)$; $12x - 9y + 29a = 0$. **2.** $3x^2 - y^2 = 3a^2$

4. $a = 5/\sqrt{2}$; $b = 4$ **6.** $3x \pm 2y$ μ $5 = 0$ **7.** $(\pm 3/4\sqrt{6}\,a,\ 1/4\sqrt{6}\,a)$; $(\pm 1/3\sqrt{6}\,a, \pm\sqrt{6}\,a)$.

8. $(a^2y^2 - b^2x^2)(x^2 + y^2)^2 = (a^2 + b^2)^2 x^2y^2$ **10.** $(-4, 3)$ & $(-4/7, -3/7)$

12. $2x^2 - 3xy - 2y^2 + 3x - y + 1 = 0$; $2x^2 - 3xy - 2y^2 + 3x - y - 6 = 0$; $x + 3y = 0$; $3x - y + 2 = 0$

14. (A) **15.** (B) **16.** (A) **17.** (B) **18.** (A) **19.** (B) **20.** (B)

21. (A) **22.** (B)

EXERCISE # 4

1. (D) **2.** (B) **3.** (A) **4.** $(x^2 + y^2)^2 = 9x^2 - \frac{81}{4}y^2$ **5.** (C) **6.** (A,D)

7. (A) **8.** $A \to P,Q$; $B \to P,Q$; $C \to Q,R$; $D \to Q,R$ **9.** (B) **10.** (D)

11. (A,B) **12.** 2 **13.** (B) **14.** (A) **15.** (B,D) **16.** (B) **17.** (A, B)

EXERCISE # 5

3. $5x^2 - y^2 = 6$ **4.** $\left(x - \frac{2h}{3}\right)^2 - \left(y - \frac{2k}{3}\right)^2 = a^2$ **5.** (B) **7.** (B,C) **8.** $16x^2 + y^2 + 10xy - 2 = 0$

10. (A) **11.** (B,C,D) **12.** (B) **13.** (D)