

HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

$$2. \int \frac{dx}{\sin x \cdot \sin(x+\alpha)} = \frac{1}{\sin \alpha} \int \frac{\sin(\alpha+x-x)}{\sin x \sin(x+\alpha)} dx$$

$$= \operatorname{cosec} \alpha \int \frac{\sin(x+\alpha) \cos x - \cos(x+\alpha) \sin x}{\sin x \sin(x+\alpha)} dx$$

$$= \operatorname{cosec} \alpha \left[\int \cot x dx - \int \cot(x+\alpha) \right] + C$$

$$= \operatorname{cosec} \alpha [\log |\sin x| - \log |\sin(x+\alpha)|] + C$$

$$= \operatorname{cosec} \alpha \log \left| \frac{\sin x}{\sin(x+\alpha)} \right| + C$$

$$3. x \cdot \int \frac{dx}{x} = x(\ln|x| + C) = x \ln|x| + Cx$$

$$5. \int \frac{2x+1}{(x^2+4x+1)^{3/2}} dx = \int \frac{2x+1}{x^3 \left(1 + \frac{4}{x} + \frac{1}{x^2}\right)^{3/2}} dx$$

$$= \int \frac{2x^{-2} + x^{-3}}{\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)^{3/2}} dx$$

now put $\frac{1}{x^2} + \frac{4}{x} + 1 = t^2$

$$7. I = \int \frac{dx}{x(1+x^7)} - \int \frac{x^6}{1+x^7} dx$$

$$8. \int \left(\frac{\cos^8 x - \sin^8 x}{1 - 2 \sin^2 x \cos^2 x} \right) dx$$

$$= \int \frac{(\cos^4 x + \sin^4 x)(\cos^2 x - \sin^2 x)}{(1 - 2 \sin^2 x \cos^2 x)} dx$$

$$= \int (\cos^2 x - \sin^2 x) dx = \int \cos 2x dx = \frac{\sin 2x}{2} + C$$

$$9. \int \left\{ \ln(1 + \sin x) + x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right\} dx$$

$$= \int \ln(1 + \sin x) dx + \int x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx$$

$$= x \cdot \ln(1 + \sin x) - \int \frac{1}{1 + \sin x} \cos x \cdot x dx + \int x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx$$

$$x \cdot \ln(1 + \sin x) - \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} \cdot x dx + \int x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx$$

$$= x \cdot \ln(1 + \sin x) + C$$

$$10. \left[\frac{1}{4} \tan^{-1} \frac{x+3}{4} + \frac{1}{2 \cdot 6} \ln \left| \frac{x-9}{x+3} \right| \right]$$

$$= 3 \tan^{-1} \left(\frac{x+3}{4} \right) + \ln \left| \frac{x-9}{x+3} \right| \Rightarrow \lambda = 3, \mu = 1 \Rightarrow 4$$

$$12. \int \frac{x^{1/3}}{(x^4-1)^{4/3}} dx = \frac{1}{4} \int \frac{4x^{-5}}{(1-x^{-4})^{4/3}} dx$$

$$= \frac{-3}{4} (1-x^{-4})^{-1/3} + C \quad (\text{Put } 1-x^{-4} = t)$$

$$14. \int \frac{x dx}{\sqrt{(1+x^2)(1+\sqrt{1+x^2})}} ; \text{ put } 1 + \sqrt{1+x^2} = t^2$$

$$15. f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx$$

$$f'(x) = \frac{2 \sin x - \sin 2x}{x^3} = \frac{2 \sin x}{x} \cdot \frac{1 - \cos x}{x^2}$$

$$= 2 \left(\frac{\sin x}{x} \right) \cdot \frac{2 \sin^2 \frac{x}{2}}{\frac{x^2}{4} \times 4} = \frac{2 \times 2}{4} \left(\frac{\sin x}{x} \right) \cdot \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 1$$

16. reduction formula of $(\sin x)^n$

$$\begin{aligned}
 18. I &= \int \sqrt{\sec x - 1} \, dx = \int \frac{\sqrt{1 - \cos x}}{\sqrt{\cos x + 1 - 1}} \, dx \\
 &= \int \frac{\sqrt{2} \sin \frac{x}{2} \, dx}{\sqrt{\left(\sqrt{2} \cos \frac{x}{2}\right)^2 - 1}} \quad \text{put } \sqrt{2} \cos \frac{x}{2} = t \\
 &\Rightarrow \sin \frac{x}{2} \, dx = -\sqrt{2} \, dt \\
 &\Rightarrow I = -2 \int \frac{dt}{\sqrt{t^2 - 1}} = -2 \bullet_n \left(t + \sqrt{t^2 - 1} \right) \\
 &= -2 \bullet_n \left(\sqrt{2} \cos \frac{x}{2} + \sqrt{2 \cos^2 \frac{x}{2} - 1} \right) + C_1 \\
 &= -2 \times \frac{1}{2} \bullet_n 2 - 2 \bullet_n \left(\cos \frac{x}{2} + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}} \right) + C_1 \\
 &= -2 \bullet_n \left(\cos \frac{x}{2} + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 19. I &= \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x} \, dx \\
 &= \ln(e^x + \sin x + x) - x + C \\
 \therefore f(x) &= e^x + \sin x + x \quad \text{and } g(x) = -x \\
 f(x) + g(x) &= e^x + \sin x
 \end{aligned}$$

22. use I.B.P. taking $\ln(x + \sqrt{1+x^2})$ as the first function

and $\frac{x}{\sqrt{1+x^2}}$ as the second function

$$\begin{aligned}
 23. \int \frac{(x^4 + 1)}{x(x^4 + 1 + 2x^2)} \, dx &= \int \left(\frac{-2x}{x^4 + 1 + 2x^2} + \frac{1}{x} \right) \, dx \\
 &= \bullet_n |x| - \int \left(\frac{2x}{(x^2 + 1)^2} \right) \, dx = \bullet_n |x| + \frac{1}{x^2 + 1} + c
 \end{aligned}$$

$$24. \int \left(\frac{x}{(1+x^5)} \right)^{3/2} \, dx = \int \frac{x^{-6}}{(1+x^{-5})^{3/2}} \, dx$$

$$\{\text{Let } t = 1 + x^{-5}; \, dt = -5x^{-6} dx\}$$

$$= -\frac{1}{5} \int \frac{dt}{(t)^{3/2}} = \frac{2}{5} (1 + x^{-5})^{-1/2} + C$$

$$25. \int \frac{1 - \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} + 3\right) \tan^{-1}\left(x + \frac{1}{x}\right)} \, dx ;$$

$$\int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(\left(x + \frac{1}{x}\right)^2 + 1\right) \tan^{-1}\left(x + \frac{1}{x}\right)} \, dx$$

$$27. I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cdot (\operatorname{cosec}^2 x - 1) \, dx$$

$$I_n = -\frac{u^{n-1}}{n-1} - I_{n-2}$$

$$\text{or } I_n + I_{n-2} = -\frac{u^{n-1}}{n-1} \quad (\text{Put } n = 2, 3, 4, \dots, 10)$$

$$I_2 + I_0 = -\frac{u}{1}$$

$$I_3 + I_1 = -\frac{u^2}{2}$$

$$I_4 + I_2 = -\frac{u^3}{3}$$

$$\bullet_n \bullet_n \bullet_n$$

$$I_{10} + I_9 = -\frac{u^9}{9}$$

$$\text{adding } I_0 + I_1 + 2(I_2 + I_3 + \dots + I_8) + I_9 + I_{10}$$

$$= -\left(u + \frac{u^2}{2} + \dots + \frac{u^9}{9}\right) \Rightarrow (B)$$

$$\begin{aligned}
 29. I &= \int \frac{dx}{x(x^{2007} + 1)} = \int \frac{x^{2007} + 1 - x^{2007}}{x(x^{2007} + 1)} \, dx \\
 &= \int \left(\frac{1}{x} - \frac{x^{2006}}{1 + x^{2007}} \right) \, dx
 \end{aligned}$$

$$= \ln x - \frac{1}{2007} \ln(1+x^{2007})$$

$$= \frac{\ln x^{2007} - \ln(1+x^{2007})}{2007}$$

$$= \frac{1}{2007} \ln \left(\frac{x^{2007}}{1+x^{2007}} \right) + C$$

$$p+q+r=6021$$

$$31. e^{\sqrt{\sin x}} = t \quad \frac{e^{\sqrt{\sin x}}}{2\sqrt{\sin x}} \cos x ;$$

$$e^{\sqrt{\sin x}} \sqrt{\cot x} \sqrt{\cos x} dx = 2 dt$$

$$\text{Hence } \int 2 dt = 2t + C \Rightarrow 2e^{\sqrt{\sin x}} + C$$

$$33. \int \frac{x^4 - 4}{x^3 \sqrt{\frac{4}{x^2} + 1 + x^2}} dx = \int \frac{x - 4x^{-3}}{\sqrt{4x^{-2} + 1 + x^2}} dx$$

$$= \frac{1}{2} \int t^{-1/2} dt \quad [\text{put } t = 4x^{-2} + 1 + x^2]$$

$$= \sqrt{\frac{4}{x^2} + 1 + x^2} + C = \frac{\sqrt{4 + x^2 + x^4}}{x} + C$$

$$34. I = \int \frac{x^2(1 - \ln x)}{x^4 \left(\left(\frac{\ln x}{x} \right)^4 - 1 \right)} dx = \int \frac{1 - \ln x}{x^2 \left(\left(\frac{\ln x}{x} \right)^4 - 1 \right)} dx$$

$$\text{put } \frac{\ln x}{x} = t \Rightarrow \frac{1 - \ln x}{x^2} = dt$$

$$I = \int \frac{dt}{t^4 - 1} = \int \frac{dt}{(t^2 + 1)(t^2 - 1)}$$

$$= \frac{1}{2} \int \frac{(t^2 + 1) - (t^2 - 1)}{(t^2 + 1)(t^2 - 1)} dt$$

$$I = \frac{1}{2} \left(\int \frac{dt}{t^2 - 1} - \int \frac{dt}{t^2 + 1} \right) = \frac{1}{2} \left(\frac{1}{2} \ln \frac{t-1}{t+1} - \tan^{-1} t \right)$$

$$= \frac{1}{4} \ln \left(\frac{\ln x - x}{\ln x + x} \right) - \frac{1}{2} \tan^{-1} \left(\frac{\ln x}{x} \right) + C$$

$$36. \int \frac{2x+3}{(x^2+3x)(x^2+3x+2)+1} dx$$

$$\text{put } x^2+3x=t \Rightarrow (2x+3) dx = dt$$

$$\int \frac{dt}{t(t+2)+1} ; \int \frac{dt}{(t+1)^2} = C - \frac{1}{t+1} = C - \frac{1}{x^2+3x+1}$$

$$\Rightarrow a=1, b=3, c=1$$

$$\Rightarrow a+b+c=5$$

$$38. I = \int (\sin(100x+x) \cdot (\sin x)^{99}) dx$$

$$= \int ((\sin(100x)\cos x + \cos 100x \cdot \sin x)(\sin x)^{99}) dx$$

$$= \int \sin(100x)\cos x \cdot (\sin x)^{99} dx + \int \cos(100x) \cdot (\sin x)^{100} dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100} \int \cos(100x)(\sin x)^{100} dx$$

$$+ \int \cos(100x)(\sin x)^{100} dx$$

$$\Rightarrow \frac{\sin(100x)(\sin x)^{100}}{100} + C$$

EXERCISE - 2

Part # I : Multiple Choice

1. $\sqrt{1+2 \tan x \sec x+2 \tan ^2 x}=|\sec x+\tan x|$

$\therefore \int |\sec x+\tan x| dx=I$

$I=\bullet n|\sec x+\tan x|+\bullet n|\sec x|+c \quad \dots \dots \text{(i)}$

$I=\bullet n|\sec ^2 x+\sec x \tan x|+c$

$I=-\bullet n|\sec x-\tan x|+\bullet n|\sec x|+c \quad \dots \dots \text{(ii)}$

$I=\bullet n|1+\tan x(\sec x+\tan x)|+c \quad \dots \dots \text{(iv)}$

2. $I=\int e^{3 x} \cos 4 x dx=e^{3 x}(A \sin 4 x+B \cos 4 x)+C \quad \dots \text{(i)}$

$I=\frac{1}{4} e^{3 x} \sin 4 x-\int \frac{3}{4} e^{3 x} \sin 4 x dx$

$=\frac{1}{4} e^{3 x} \sin 4 x+\frac{3}{16} e^{3 x} \cos 4 x-\int \frac{9}{16} e^{3 x} \cos 4 x dx$

$\frac{25}{16} I=\frac{1}{16}\left(4 e^{3 x} \sin 4 x+3 e^{3 x} \cos 4 x\right)$

comparing with equation (i)

$\Rightarrow A=\frac{4}{25}, B=\frac{3}{25} \Rightarrow \frac{A}{B}=\frac{4}{3}$

$\Rightarrow 3 A=4 B \Rightarrow 4 A+3 B=1$

3. $f(x)=3 x^2 \cdot \sin \frac{1}{x}-x \cdot \cos \frac{1}{x}$

$\Rightarrow f(x)=\int\left(3 x^2 \cdot \sin \frac{1}{x}-x \cos \frac{1}{x}\right) dx$

$=x^3 \sin \frac{1}{x}-\int \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) x^3 dx-\int x \cos \frac{1}{x} dx$

$=x^3 \sin \frac{1}{x}+C$

since $f\left(\frac{1}{\pi}\right)=0+C \Rightarrow C=0$

$\Rightarrow f(x)=\begin{cases} x^3 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$

$f(x)$ is clearly continuous and differentiable at $x=0$ zero with $f'(0)=0$.

$f''(0)=\lim _{h \rightarrow 0} \frac{3 h^2 \sin \frac{1}{h}-h \cos \frac{1}{h}}{h}$

$=3 h \sin \frac{1}{h}-\cos \frac{1}{h}$

This limit doesn't exist, hence $f'(x)$ is non-differentiable at $x=0$.

Also $\lim _{x \rightarrow 0} f(x)=0$.

Thus $f(x)$ is continuous at $x=0$.

5. put $\ln (\tan x)=t$

6. $I=\int 2^{m x} \cdot 3^{n x} dx=\int\left(2^m \cdot 3^n\right)^x dx=\frac{2^{m x} \cdot 3^{n x}}{\ln \left(2^m \cdot 3^n\right)}+C$

13. $I_n=\int \cot ^n x dx=\int \cot ^{n-2} x\left(\operatorname{cosec}^2 x-1\right) dx=\frac{-\cot ^{n-1} x}{n-1}-I_{n-2}+C_1$

$\Rightarrow I_n+I_{n-2}=\frac{-\cot ^{n-1} x}{n-1}+C_{10}$

Now, $I_0+I_1+2\left(I_2+\dots+I_8\right)+I_9+I_{10}$
 $=\left(I_2+I_0\right)+\left(I_3+I_1\right)+\left(I_4+I_2\right)+\left(I_5+I_3\right)+\left(I_6+I_4\right)+\left(I_7+I_5\right)$
 $+\left(I_8+I_6\right)+\left(I_9+I_7\right)+\left(I_{10}+I_8\right)$

$=-\left(\frac{\cot x}{1}+\frac{\cot ^2 x}{2}+\dots+\frac{\cot ^9 x}{9}\right)+C$

$\therefore A=-1$.

14. $J+K=\int \frac{1+\sin x+\cos x}{1+\sin x+\cos x} dx \quad J+K=x+C \quad \dots \text{(1)}$

$\Rightarrow \text{(C)}$

again $J-K=\int \frac{\left(\sin ^2 x-\cos ^2 x\right)+\sin x-\cos x}{1+\sin x+\cos x} dx$

$=\int \frac{(\sin x-\cos x)+(\sin x+\cos x+1)}{1+\sin x+\cos x} dx$

$J-K=-\cos x-\sin x+C \quad \dots \text{(2)}$

hence $J=K-(\sin x+\cos x)+C \Rightarrow \text{(B)}$

Also $(1)+(2)$

$2 J=x-(\cos x+\sin x)+C$



$$J = \frac{1}{2} [x - \sin x - \cos x] + C$$

and (1) - (2)

$$2K = x + (\sin x + \cos x) + C$$

$$K = \frac{1}{2} (x + \sin x + \cos x) + C$$

from (1) $J = x - K + C \Rightarrow$ (C)

Part # II : Assertion & Reason

1. Let $D(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x)$

where $\lambda_1 = (b_2 c_3 - b_3 c_2)$, $\lambda_2 = a_2 c_3 - a_3 c_2$,

$$\lambda_3 = a_2 b_3 - a_3 b_2$$

then $\int D(x) dx = \int \lambda_1 f_1(x) dx + \int \lambda_2 f_2(x) dx$
 $+ \int \lambda_3 f_3(x) dx + C \dots$ (i)

$$= \begin{vmatrix} \int f_1(x) dx & \int f_2(x) dx & \int f_3(x) dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + C$$

Thus statement-I is true and follows from statement-II which we have applied at Eq. (1)

4. Statement-1 is true, statement-2 is false

e.g. $F(x) = \int \frac{1}{\sqrt{x}} dx,$

although $\frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$

but $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$ is continuous at $x = 0$.

5. $I = \int ((\log_x e) - (\log_x e)^2) dx = \int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} \right) dx$

put $\bullet \ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$\Rightarrow I = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt = \frac{e^t}{t} + C = \frac{x}{\ln x} + C = x \log_x e + C$$

7. The statement-II is false since in

$$\int \frac{dx}{x-3y} = \log(x-3y) + C,$$

we are assuming that y is a constant.

We will now prove the statement-I.

From the given relation $(x-y)^2 = \frac{x}{y},$

and $2 \log(x-y) = \log x - \log y \dots$ (i)

Also, $\frac{dy}{dx} = \left(-\frac{y}{x} \right) \cdot \frac{x+y}{x-3y}.$

To prove the integral relation, it is sufficient to show

that $\frac{d}{dx} \text{ RHS} = \frac{1}{x-3y}$

Now, $\text{RHS} = \frac{1}{2} \log \left[\frac{x}{y} - 1 \right] \quad \left(Q \ (x-y)^2 = \frac{x}{y} \right)$

$$= \frac{1}{2} [\log(x-y) - \log y]$$

$$= \frac{1}{2} \left[\frac{\log x - \log y}{2} - \log y \right] \quad [\text{From Eq. (i)}]$$

$$= \frac{1}{4} [\log x - 3 \log y]$$

$$\Rightarrow \frac{d}{dx} \text{ RHS} = \frac{1}{4} \left[\frac{1}{x} - \frac{3}{y} \frac{dy}{dx} \right]$$

$$= \frac{1}{4} \left[\frac{1}{x} - \frac{3}{y} \left(-\frac{y}{x} \right) \frac{x+y}{x-3y} \right] = \frac{1}{x-3y}$$

Thus, statement-I is true.

EXERCISE - 3

Part # I : Matrix Match Type

2. (A) $v = 0 \Rightarrow a = b$,

Also $a + b = u \Rightarrow a = \frac{u}{2}$

Now,
$$I = \frac{1}{a} \int \frac{dx}{1 + \cos x} = \frac{2}{u} \int \frac{dx}{2 \cos^2 \frac{x}{2}}$$
$$= \frac{2}{u} \int \frac{1}{2} \sec^2 \frac{x}{2} dx = \frac{2}{u} \tan \frac{x}{2} + C$$

(B) $v > 0 \Rightarrow a > b$

Now,
$$I = \int \frac{dx}{a + b \cos x} = \int \frac{dx}{(a-b) + 2b \cos^2 \frac{x}{2}}$$
$$= \int \frac{\sec^2 \frac{x}{2} dx}{(a+b) + (a-b) \tan^2 \frac{x}{2}}$$

put $\tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

$$\Rightarrow I = \int \frac{2dt}{(a+b) + (a-b)t^2}$$

$$= \frac{2}{a-b} \int \frac{dt}{t^2 + \left(\frac{a+b}{a-b}\right)} \dots\dots(i)$$

$$= \frac{2}{a-b} \sqrt{\frac{a-b}{a+b}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} t \right) + C$$

$$= \frac{2}{\sqrt{uv}} \tan^{-1} \left(\sqrt{\frac{v}{u}} \tan \frac{x}{2} \right) + C$$

(C) $v < 0 \Rightarrow a - b < 0 \Rightarrow b - a > 0$

Now
$$I = \frac{2}{b-a} \int \frac{dt}{\frac{a+b}{b-a} - t^2} \text{ (using equation (1) of part (B))}$$

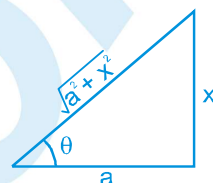
$$= \frac{2}{b-a} \cdot \frac{1}{2} \sqrt{\frac{b-a}{b+a}} \ln \left| \frac{\sqrt{\frac{b+a}{b-a}} + t}{\sqrt{\frac{b+a}{b-a}} - t} \right| + C$$

$$= \frac{1}{\sqrt{-uv}} \ln \left| \frac{\sqrt{u} + \sqrt{-v} \tan \frac{x}{2}}{\sqrt{u} - \sqrt{-v} \tan \frac{x}{2}} \right| + C$$

5. (A) $\int \frac{dx}{(a^2 + x^2)^{3/2}}$, put $x = a \tan \theta$

$$= \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \sin \theta$$

$$= \frac{x}{a^2 \sqrt{a^2 + x^2}} + C$$



(B) $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = -\int \left(\frac{-x^2 + a^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}} \right) dx$

$$= -\int \sqrt{a^2 - x^2} dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= -\frac{x}{2} \sqrt{a^2 - x^2} - \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + a^2 \sin^{-1} \frac{x}{a}$$

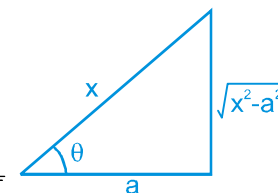
$$= -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

(C) $\int \frac{dx}{(x^2 - a^2)^{3/2}}$ Put $x = a \sec \theta$

$$= \int \frac{a \tan \theta \sec \theta}{a^3 (\tan^3 \theta)} d\theta = \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= \frac{1}{a^2} \left(\frac{-1}{\sin \theta} \right) + C$$

$$= c - \frac{x}{a^2 \sqrt{x^2 - a^2}}$$



(D) $\int \frac{1}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\frac{|x|}{a} \right) = \frac{1}{a} \cos^{-1} \left(\frac{a}{|x|} \right)$

$$= c + \frac{\pi}{2} - \frac{1}{a} \sin^{-1} \left(\frac{a}{|x|} \right)$$

$$= C - \frac{1}{a} \sin^{-1} \frac{a}{|x|}$$

Part # II : Comprehension

Comprehension-2

- $$I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= 2 \int \frac{(\sqrt{\tan x} + \sqrt{\cot x})(\tan x + \cot x)}{2(\tan x + \cot x)} dx$$

$$I = 2 \int \frac{d(\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2}$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + C$$
- If $x \in \left(0, \frac{\pi}{2}\right)$, then $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

$$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cdot \cos x}} dx = \int \frac{\sqrt{2} (\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$= \sqrt{2} \sin^{-1} (\sin x - \cos x) + C$$
- If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

$$= \int \left(\frac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \frac{\sqrt{-\cos x}}{\sqrt{-\sin x}} \right) dx$$

$$= - \int \frac{\sin x + \cos x}{\sqrt{\sin x \cdot \cos x}} dx = - \int \frac{\sqrt{2} (\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$= \sqrt{2} \sin^{-1} (\cos x - \sin x) + C$$

Comprehension # 4

- $$I_n = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \left[\int \left(\frac{x^2 + a^2}{(x^2 + a^2)^{n+1}} - \frac{a^2}{(x^2 + a^2)^{n+1}} \right) dx \right]$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{1}{(x^2 + a^2)^n} dx$$

$$-2na^2 \int \frac{1}{(x^2 + a^2)^{n+1}} dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n(I_n - a^2 I_{n+1})$$

Whence $I_{n+1} + \frac{1-2n}{2n} \frac{1}{a^2} I_n = \frac{1}{2na^2} \cdot \frac{x}{(x^2 + a^2)^n}$

$$2. I_{n, -m} = \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1} \int \frac{\sin^{n-2} x}{\cos^{m-2} x} dx$$

$$= \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1} I_{n-2, 2-m}$$

$$3. u_{n+1} = \int \frac{x^{n+1}}{\sqrt{ax^2 + 2bx + c}} dx$$

$$= \frac{1}{2a} \int \frac{x^n (2ax + 2b) - 2bx^n}{\sqrt{ax^2 + 2bx + c}} dx$$

$$= \frac{1}{2a} \int \frac{x^n (2ax + 2b)}{\sqrt{ax^2 + 2bx + c}} dx - \frac{b}{a} u_n$$

$$= I_n - \frac{b}{a} u_n, \text{ where } \dots \text{ (i)}$$

$$I_n = \frac{1}{2a} \int \frac{x^n (2ax + 2b)}{\sqrt{ax^2 + 2bx + c}} dx$$

$$= \frac{1}{2a} x^n 2\sqrt{ax^2 + bx + c}$$

$$- \int nx^{n-1} 2\sqrt{ax^2 + 2bx + c} dx$$

$$= \frac{x^n}{a} \sqrt{ax^2 + 2bx + c} -$$

$$\frac{n}{a} \int \frac{x^{n-1} (ax^2 + 2bx + c)}{\sqrt{ax^2 + 2bx + c}} dx \dots \text{ (ii)}$$

from (i) and (ii) be get

$$(n+1)au_{n+1} + (2n+1)bu_n + ncu_{n-1}$$

$$= x^n \sqrt{ax^2 + 2bx + c}$$

EXERCISE - 4

Subjective Type

1. $\operatorname{cosec}(b-a) \cdot \ln \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + c$

2. $\frac{1}{3} \left(x + \sqrt{x^2 + 2} \right)^{3/2} - \frac{2}{\left(x + \sqrt{x^2 + 2} \right)^{1/2}} + c$

3. $\sqrt{x} \sqrt{1-x} - 2\sqrt{1-x} + \arccos \sqrt{x} + c$

4. $\arccos x - \frac{\ln x}{\sqrt{x^2 - 1}} + c$

5. $-\frac{x+1}{x^5 + x + 1} + c$

6.

$\cos a \cdot \arccos \left(\frac{\cos x}{\cos a} \right) - \sin a \cdot \ln \left(\sin x + \sqrt{\sin^2 x - \sin^2 a} \right) + c$

7. Let $I = \int \frac{d(x^2 + 1)}{\sqrt{x^2 + 2}} = \int \frac{2x}{\sqrt{x^2 + 2}} dx$

Put $x^2 + 2 = t$

$\Rightarrow 2x dx = dt$

$\Rightarrow I = \int \frac{dt}{\sqrt{t}} = 2t^{1/2} + C = 2\sqrt{x^2 + 2} + C$

8. $\int \frac{\cot x dx}{(1 - \sin x)(\sec x + 1)} = \int \frac{\cos^2 x dx}{\sin x(1 - \sin x)(1 + \cos x)}$

$= \int \frac{(1 + \sin x)}{\sin x(1 + \cos x)} dx$

$= \int \frac{dx}{\sin x(1 + \cos x)} + \int \frac{1}{1 + \cos x} dx$

$= \int \frac{\sin x dx}{(1 - \cos^2 x)(1 + \cos x)} + \tan \frac{x}{2} + C$

$= \int \frac{-dt}{(1+t)(1-t^2)} + \tan \frac{x}{2} + C$

$= \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{4} \sec^2 \frac{x}{2} + \tan \frac{x}{2} + C$

9. $\frac{3}{2} \tan^{-1} x - \frac{1}{2} \ln(1+x) + \frac{1}{4} \ln(1+x^2) + \frac{x}{1+x^2} + c$

10. $\tan^{-1} \left(\frac{\sqrt{2} \sin 2x}{\sin x + \cos x} \right) + c$

11. $\frac{(x^2 + 1) \sqrt{x^2 + 1}}{9x^3} \cdot \left[2 - 3 \ln \left(1 + \frac{1}{x^2} \right) \right]$

12.

$-\frac{1}{\sqrt{\sin \alpha}} \ln \left[\cot x + \cot \alpha + \sqrt{\cot^2 x + 2 \cot \alpha \cot x - 1} \right] + c$

13. Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$I = \int \frac{\tan^2 \theta \sec^2 \theta}{\left(\frac{\sin \theta}{\cos \theta} \sin(\tan \theta) + \cos(\tan \theta) \right)^2} d\theta$

$= \int \frac{\tan^2 \theta}{\cos^2(\tan \theta - \theta)} d\theta$

Put $\tan \theta - \theta = u \Rightarrow \tan^2 \theta d\theta = du$

$I = \int \frac{du}{\cos^2 u} = \int \sec^2 u du = \tan(\tan \theta - \theta)$

$= \tan(x - \tan^{-1} x) = \frac{\tan x - x}{1 + x \tan x}$

$= \frac{\sin x - x \cos x}{\cos x + x \sin x} + C$

14. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx = \int \frac{e^x(1+1-x^2)}{(1-x)\sqrt{1-x^2}} dx$

$= \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{1-x^2} \right) dx$

$= e^x \sqrt{\frac{1+x}{1-x}} + C$



15. $I = \int x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} dx$ put $x^2 = a^2 \cos 2\theta$

$$\Rightarrow 2x dx = -2a^2 \sin 2\theta d\theta$$

$$\Rightarrow I = \int \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} (-a^2 \sin 2\theta) d\theta$$

$$= -a^2 \int \frac{\sin \theta}{\cos \theta} \times 2 \sin \theta \cos \theta d\theta = -a^2 \int (1 - \cos 2\theta) d\theta$$

$$= -a^2 \theta + \frac{a^2 \sin 2\theta}{2} + C_1$$

$$= \frac{a^2}{2} \sqrt{1 - \frac{x^4}{a^4}} - \frac{a^2}{2} \cos^{-1} \left(\frac{x^2}{a^2} \right) + C_1$$

$$= \frac{1}{2} \sqrt{a^4 - x^4} + \frac{a^2}{2} \sin^{-1} \left(\frac{x^2}{a^2} \right) + C$$

16. $f'(x) = \frac{1}{1+x^2} + \frac{1}{2} \left[\frac{1}{1+x} + \frac{1}{1-x} \right]$

$$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{1-x^4}$$

Now $\int \frac{1}{2} \left(\frac{2}{1-x^4} \right) d(x^4) = \int \frac{4x^3}{1-x^4} dx$

$$= -\ln(1-x^4) + C$$

17. $I = \int \frac{\ln(\cos x + \sqrt{\cos 2x})}{\sin^2 x} dx$

$$= \int \operatorname{cosec}^2 x \ln \left[(\sin x)(\cot x + \sqrt{\cot^2 x - 1}) \right] dx$$

$$= \int \ln(\sin x) \operatorname{cosec}^2 x dx$$

$$+ \int \ln(\cot x + \sqrt{\cot^2 x - 1}) \operatorname{cosec}^2 x dx$$

$$= -\ln(\sin x) \cot x + \int \cot^2 x dx - \int \ln(t + \sqrt{t^2 - 1}) dt$$

[put $\cot x = t$]

$$= -\ln(\sin x) \cot x - \cot x - x - \left[\ln(t + \sqrt{t^2 - 1}) \right] t$$

$$- \int t \cdot \frac{1 + \frac{2t}{2\sqrt{t^2 - 1}}}{t + \sqrt{t^2 - 1}} dt$$

$$= -\ln(\sin x) \cot x - \cot x - x - t \ln(t + \sqrt{t^2 - 1}) + \int \frac{t}{\sqrt{t^2 - 1}} dt$$

$$= -\ln(\sin x) \cot x - \cot x - x - t \ln(t + \sqrt{t^2 - 1}) + \sqrt{t^2 - 1}$$

$$= -\ln(\sin x) \cot x - \cot x - x - \cot x \ln \left(\cot x + \frac{\sqrt{\cos 2x}}{\sin x} \right) + \frac{\sqrt{\cos 2x}}{\sin x}$$

$$= \frac{\sqrt{\cos 2x}}{\sin x} - x - \cot x - \cot x \ln(e(\cos x + \sqrt{\cos 2x})) + C$$

18. $-\left(1 + \frac{1}{x^4}\right)^{1/4} + c$

19. $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x} dx = \int \sqrt{\cot^2 x - 1} dx$$

Put $\cot x = \sec \theta$

$$\Rightarrow -\operatorname{cosec}^2 x dx = \sec \theta \tan \theta d\theta$$

$$\therefore I = \int \sqrt{\sec^2 \theta - 1} \cdot \frac{\sec \theta \tan \theta}{-(1 + \sec^2 \theta)} d\theta$$

$$= -\int \frac{\sec \theta \tan^2 \theta}{1 + \sec^2 \theta} d\theta = -\int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= -\int \frac{1 - \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= -\int \frac{(1 + \cos^2 \theta) - 2 \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= -\int \sec \theta d\theta + 2 \int \frac{\cos \theta}{1 + \cos^2 \theta} d\theta$$

$$= -\log |\sec \theta + \tan \theta| + 2 \int \frac{\cos \theta}{2 - \sin^2 \theta} d\theta$$

$$= -\log |\sec \theta + \tan \theta| + 2 \int \frac{dt}{2 - t^2} \quad (\text{put } \sin \theta = t)$$

$$= -\log |\sec \theta + \tan \theta| + 2 \cdot \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin \theta}{\sqrt{2} - \sin \theta} \right| + C$$

$$= -\log |\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \log \left[\frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right] - \log(\cot x + \sqrt{\cot^2 x - 1}) + C$$

20. $\int f(x) dx = \int \left(\ln(\ln x) + \frac{1}{(\ln x)^2} \right) dx$

Put $\bullet \ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$= \int e^t \left(\ln t + \frac{1}{t^2} \right) dt = \int e^t \left(\ln t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$= e^t \left(\ln t - \frac{1}{t} \right) + C \Rightarrow y = x \left(\ln(\ln x) - \frac{1}{\ln x} \right) + C$$

→ it passes through (e, e) $\Rightarrow C = 2$

Hence, $y = x \left[\ln(\ln x) - \frac{1}{\ln x} \right] + 2$

21. $\sin^{-1} \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) + c$

22. $3x - \ln \sqrt{x^2 + x + 1} + \sqrt{3} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$

23. $\int \frac{\sec^2 x dx}{\tan^2 x + 2 \tan x} = \int \frac{dt}{t^2 + 2t}$

[Put $\tan x = t$]

$$= \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t+2} \right) dt = \frac{1}{2} \ln \left| \frac{\tan x}{\tan x + 2} \right| + C$$

24. $\sin^{-1} \left(\frac{ax^2 + b}{cx} \right) + k$

25. $\int (\sin x)^{-11/3} \cos x^{-1/3} dx = \int \frac{1}{(\sin^{11} x \cos x)^{1/3}} dx$

$$= \int \frac{\cos \sec^4 x}{(\cot x)^{1/3}} dx = \int \frac{(1 + \cot^2 x) \cos \sec^2 x}{(\cot x)^{1/3}} dx$$

$$= -\int (t^{-1/3} + t^{5/3}) dt \quad [\text{Put } \cot x = t]$$

$$= -\left[\frac{t^{2/3}}{2/3} + \frac{t^{8/3}}{8/3} \right] + C = \frac{-3}{8} [4 \cot^{2/3} x + \cot^{8/3} x] + C$$

26. $(a+x) \arctan \sqrt{\frac{x}{a}} - \sqrt{ax} + c$

27. $I = \int \frac{2 \sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4 \sin \phi} d\phi$

$$= \int \frac{(4 \sin \phi - 1) \cos \phi}{6 - (1 - \sin^2 \phi) - 4 \sin \phi} d\phi$$

put $\sin \phi = t \Rightarrow \cos \phi d\phi = dt$

$$\Rightarrow I = \int \frac{(4t-1) dt}{5+t^2-4t} = 2 \int \frac{(2t-4)+7/2}{t^2-4t+5} dt$$

$$= 2 [\bullet \ln |t^2 - 4t + 5|] + 7 \int \frac{1}{(t-2)^2 + (1)^2} dt$$

$$= 2 [\bullet \ln |t^2 - 4t + 5|] + 7 \tan^{-1} \left(\frac{t-2}{1} \right) + C$$

$$= 2 \bullet \ln |\sin^2 \phi - 4 \sin \phi + 5| + 7 \tan^{-1} (\sin \phi - 2) + C$$

28. $\frac{3}{2} x^{2/3} - \frac{12}{7} x^{7/12} + 2x^{1/2} - \frac{12}{5} x^{5/12} + 3x^{1/3} + 6x^{1/6} - 12x^{1/12} + 12 \log |x^{1/12} + 1| - 4x^{1/4} + c$

29. $\int \frac{f(x) dx}{x^2(x+1)^3} = \int \left(\frac{1}{x^2} - \frac{x}{(x+1)^3} \right) dx$

$$\int \frac{f(x) dx}{x^2(x+1)^3} = \int \frac{3x^2 + 3x + 1}{x^2(x+1)^3} dx$$

$$\Rightarrow f(x) = 3x^2 + 3x + 1$$

$$f'(0) = 3$$

30. $\int \cos 2\theta \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta$

$$= \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) \frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2} \cdot \frac{2}{\cos 2\theta} d\theta$$

$$= \frac{1}{2} \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) \sin 2\theta - \frac{1}{2} \ln(\sec 2\theta) + C$$



$$31. I = \int \left[\left(\frac{x}{e} \right)^x + \left(\frac{e}{x} \right)^x \right] \ln x \, dx = \int \left(1 + \frac{1}{\left(\frac{x}{e} \right)^{2x}} \right) \ln x \left(\frac{x}{e} \right)^x dx$$

$$\left\{ \text{Put } \left(\frac{x}{e} \right)^x = t \Rightarrow \ln x \left(\frac{x}{e} \right)^x dx = dt \right\}$$

$$I = \int \left(1 + \frac{1}{t^2} \right) dt = t - \frac{1}{t} + C = \left(\frac{x}{e} \right)^x - \left(\frac{e}{x} \right)^x + C$$

$$32. \int \frac{x}{(7x-10-x^2)^{3/2}} dx = \int \frac{x}{[(x-5)(2-x)]^{3/2}} dx$$

(put $x = 5 \cos^2 \alpha + 2 \sin^2 \alpha$)

$$= \int \frac{(5 \cos^2 \alpha + 2 \sin^2 \alpha)(-3 \sin 2\alpha)}{[-3 \sin^2 \alpha(-3 \cos^2 \alpha)]^{3/2}} d\alpha$$

$$= \int \frac{(5 \cos^2 \alpha + 2 \sin^2 \alpha)(-3 \sin 2\alpha)}{27(\sin \alpha \cos \alpha)^3} d\alpha$$

$$= \frac{-6}{27} \int \frac{(5 \cos^2 \alpha + 2 \sin^2 \alpha)}{(\sin \alpha \cos \alpha)^2} d\alpha$$

$$= \frac{-2}{9} \int (5 \operatorname{cosec}^2 \alpha + 2 \sec^2 \alpha) d\alpha$$

$$= \frac{-2}{9} [-5 \cot \alpha + 2 \tan \alpha]$$

$$= \frac{-2}{9} \int (-5 \cot \alpha + 2 \tan \alpha) d\alpha$$

$$= \frac{10}{9} \sqrt{\frac{2-x}{x-5}} - \frac{4}{9} \sqrt{\frac{x-5}{2-x}} + C \frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + c$$

$$33. \int \frac{1+x \cos x}{x(1-x^2 e^{2 \sin x})} dx$$

$$\text{Put } x e^{\sin x} = t \Rightarrow (x e^{\sin x} \cdot \cos x + e^{\sin x}) dx = dt$$

$$\Rightarrow e^{\sin x} (x \cos x + 1) dx = dt$$

$$\Rightarrow I = \int \frac{dt}{t(1-t^2)} = \int \frac{1}{t(1-t)(1+t)} dt$$

$$\text{Let } \frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{(1-t)} + \frac{C}{1+t}$$

$$1 = A(1-t)(1+t) + B(t)(1+t) + C(t)(1-t)$$

$$\text{Put } t=0 \Rightarrow A=1$$

$$\text{Put } t=1 \Rightarrow B=1/2$$

$$\text{Put } t=-1 \Rightarrow C=-1/2$$

$$\begin{aligned} \Rightarrow I &= \int \left\{ \frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)} \right\} dt \\ &= \bullet \ln|t| - \frac{1}{2} \bullet \ln|1-t| - \frac{1}{2} \bullet \ln|1+t| + K \\ &= \bullet \ln|x e^{\sin x}| - \frac{1}{2} \log|1-x^2 e^{2 \sin x}| + K \end{aligned}$$

$$34. \int \frac{e^{\cos x} (x \sin^3 x + \cos x)}{\sin^2 x} dx$$

$$\text{put } \cos x = t, \text{ we get } -\sin x dx = dt$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\int e^t \left(\cos^{-1} t - \frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} + \frac{t}{(1-t^2)^{3/2}} \right) dt$$

$$= e^t \left(\cos^{-1} t + \frac{1}{\sqrt{1-t^2}} \right) + C = e^{\cos x} (x + \operatorname{cosec} x) + C$$

$$35. I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$$

$$\text{put } \tan x = t^2 \Rightarrow (\sec^2 x) dx = 2t dt \Rightarrow dx = \frac{2t}{1+t^4} dt$$

$$I = \int \frac{t^2+1}{\sqrt{t^2}} \cdot \frac{2t}{t^4+1} dt = 2 \int \frac{t^2+1}{t^4+1} dt$$

$$= 2 \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 2 + 2} dt = 2 \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2} dt$$

$$= 2 \int \frac{du}{u^2 + (\sqrt{2})^2}, \text{ where } u = t - \frac{1}{t}$$

$$\Rightarrow I = \frac{2}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + C$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + C$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

$$8. \int \frac{5 \tan x}{\tan x - 2} dx = \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$5 \sin x = A(\sin x - 2 \cos x) + B \frac{d}{dx} (\sin x - 2 \cos x)$$

$$= A(\sin x - 2 \cos x) + B (\cos x + 2 \sin x)$$

$$\left. \begin{aligned} A + 2B &= 5 \\ -2A + B &= 0 \end{aligned} \right\} A = 1, B = 2$$

$$\text{Now } \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$= \int \left(\frac{(\sin x - 2 \cos x)}{\sin x - 2 \cos x} + \frac{2(\cos x + 2 \sin x)}{\sin x - 2 \cos x} \right) dx$$

$$= x + 2 \log |\sin x - 2 \cos x| + K$$

$$a = 2$$

$$9. \text{ put } x^3 = t \Rightarrow 3x^2 dx = dt \quad I = \frac{1}{3} \int f(t) dt$$

$$\frac{1}{3} \left[t \int f(t) dt - \int (1 \cdot \int f(t) dt) dt \right]$$

$$= \frac{1}{3} x^3 \Psi(x^3) - \int \Psi(x^3) x^2 dx$$

$$12. \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$$

$$\int \frac{2x^{12} + 5x^9}{x^{15} (1 + x^{-2} + x^{-5})^3} dx$$

$$\int \frac{2x^{-3} + 5x^{-6}}{(x^{-5} + x^{-2} + 1)^3} dx$$

$$x^{-5} + x^{-2} + 1 = t$$

$$(-5x^{-6} + 2x^{-3}) dx = -dt$$

$$\int \frac{dt}{t^3} = -\left(\frac{t^{-2}}{-2} \right) + C = \frac{1}{2t^2} + C = \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$$

Part # II : IIT-JEE ADVANCED

$$1. \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2+8x+13}} \right) dx$$

$$= \int \sin^{-1} \left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}} \right] dx$$

$$= \int \sin^{-1} \left[\frac{x+1}{\sqrt{(x+1)^2 + (3/2)^2}} \right] dx$$

$$\text{Put } x+1 = \frac{3}{2} \tan \theta, \quad dx = \frac{3}{2} \sec^2 \theta d\theta$$

$$= \int \sin^{-1} \left[\frac{\left(\frac{3}{2} \tan \theta \right)}{\sqrt{\frac{9}{4} \tan^2 \theta + \frac{9}{4}}} \right] \frac{3}{2} \sec^2 \theta d\theta$$

$$= \frac{3}{2} \int \sin^{-1} \left[\frac{\sin \theta \cos \theta}{\cos \theta \cdot 1} \right] \sec^2 \theta d\theta$$

$$= \frac{3}{2} \int \theta \sec^2 \theta d\theta = \frac{3}{2} \left[\theta \tan \theta - \int \tan \theta d\theta \right]$$

$$= \frac{3}{2} [\theta \tan \theta - \log |\sec \theta|] + C$$

$$I = \frac{3}{2} \left[\frac{2}{3} (x+1) \tan^{-1} \left[\frac{2}{3} (x+1) \right] - \log \sqrt{1 + \frac{4}{9} (x+1)^2} \right] + C$$

$$= (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log (9 + 4x^2 + 8x + 4) + \frac{3}{4} \log 9 + C$$

$$= (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log (4x^2 + 8x + 13) + C$$

$$2. I = \int (x^{3m} + x^{2m} + x^m) (2x^{2m} + 3x^m + 6)^{1/m} dx$$

$$= \int (x^{3m} + x^{2m} + x^m) \left[\frac{2x^{3m} + 3x^{2m} + 6x^m}{x^m} \right]^{1/m} dx$$

$$= \int \left(\frac{x^{3m} + x^{2m} + x^m}{x} \right) (2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$



$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

Put $2x^{3m} + 3x^{2m} + 6x^m = y$

$$\therefore I = \frac{1}{6m} \int y^{1/m} dy = \frac{1}{6m} \frac{y^{(1/m)+1}}{\left(\frac{m+1}{m}\right)} + C$$

$$= \frac{1}{6} \frac{y^{\frac{m+1}{m}}}{(m+1)} + C$$

$$= \frac{1}{6} \frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{m+1} + C$$

4. Here $f f(x) = \frac{f(x)}{[1 + f(x)^n]^{1/n}} = \frac{x}{(1 + 2x^n)^{1/n}}$

and $f f f(x) = \frac{x}{(1 + 3x^n)^{1/n}}$

$$\therefore g(x) = (\text{fofo} \dots \text{of})(x) = \frac{x}{(1 + nx^n)^{1/n}}$$

Let $I = \int x^{n-2} g(x) dx = \int \frac{x^{n-1}}{(1 + nx^n)^{1/n}} dx$

$$= \frac{1}{n^2} \int \frac{n^2 x^{n-1}}{(1 + nx^n)^{1/n}} dx = \frac{1}{n^2} \int \frac{\frac{d}{dx}(1 + nx^n)}{(1 + nx^n)^{1/n}} dx$$

$$= \frac{1}{n(n-1)} (1 + nx^n)^{\frac{1}{n}-1} + K$$

6. $J - I = \int \left(\frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx$

$$= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1} dx = \int \frac{e^x(e^{2x} - 1)}{e^{4x} + e^{2x} + 1} dx$$

Let $e^x = t$ $e^x dx = dt$

$$= \int \frac{t^2 - 1}{t^4 + t^2 + 1} dt = \int \frac{1 - 1/t^2}{(t + 1/t)^2 - 1} dt$$

$$= \frac{1}{2} \ln \left| \frac{t + \frac{1}{t} - 1}{t + \frac{1}{t} + 1} \right| + C = \frac{1}{2} \ln \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + C$$

7. Let $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$

$$= \int \frac{\sec x (\sec x + \tan x) \sec x}{(\sec x + \tan x)^{11/2}} dx$$

Put $\sec x + \tan x = t$

$$\Rightarrow (\sec x \tan x + \sec^2 x) dx = dt$$

Also $\rightarrow \sec^2 x - \tan^2 x = 1$

$$\Rightarrow \sec x - \tan x = \frac{1}{t}$$

$$\therefore \sec x = \frac{1}{2} \left(t + \frac{1}{t} \right)$$

$$\therefore I = \frac{1}{2} \int \frac{\left(t + \frac{1}{t} \right) dt}{t^{11/2}} = \frac{1}{2} \int (t^{-9/2} + t^{-13/2}) dt$$

$$= \frac{1}{2} \left(-\frac{2t^{-7/2}}{7} - \frac{2t^{-11/2}}{11} \right) + K$$

$$= -\frac{1}{t^{11/2}} \left(\frac{1}{11} + \frac{t^2}{7} \right) + K$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$$

MOCK TEST

1. (B)

$$\begin{aligned} \frac{d}{dx} (A \bullet n | \cos x + \sin x - 2 | + Bx + C) \\ = A \frac{\cos x - \sin x}{\cos x + \sin x - 2} + B \\ = \frac{A \cos x - A \sin x + B \cos x + B \sin x - 2B}{\cos x + \sin x - 2} \end{aligned}$$

$$\therefore 2 = A + B, -1 = -A + B, \lambda = -2B$$

$$\therefore A = 3/2, B = 1/2, \lambda = -1$$

2. $\int x \cdot \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$

$$\begin{aligned} I &= \int \ln(x + \sqrt{1+x^2}) \cdot \frac{x}{\sqrt{1+x^2}} dx \\ &= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - \int \frac{1}{\sqrt{1+x^2}} \cdot \sqrt{1+x^2} dx \\ &\quad \text{(using integration by parts)} \\ &= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - x + C \end{aligned}$$

Alternate:

$$\begin{aligned} \text{Let } x + \sqrt{1+x^2} = e^t \text{ then } \sqrt{1+x^2} = x \\ = e^{-t} \frac{dx}{\sqrt{1+x^2}} = dt \\ I = \int x t dt = \int \frac{(e^t - e^{-t})t dt}{2} = \frac{1}{2} t(e^t + e^{-t}) \\ - \frac{1}{2} (e^t - e^{-t}) + C = \sqrt{1+x^2} \bullet n(x + \sqrt{1+x^2}) - x + C. \end{aligned}$$

3. (B)

$$\begin{aligned} \frac{2x}{(x-1)(x-4)} &= \frac{D}{x-1} + \frac{E}{x-4} \\ 2x &= D(x-4) + E(x-1) \\ \therefore D &= -2/3, E = 8/3 \\ \therefore \int \frac{e^{x-1}}{(x-1)(x-4)} 2x dx &= \int e^{x-1} \left(\frac{-2/3}{x-1} + \frac{8/3}{x-4} \right) dx \\ &= -\frac{2}{3} F(x-1) + \frac{8}{3} e^3 F(x-4) + C \\ \therefore A &= -2/3, B = 8/3 e^3 \end{aligned}$$

4. (A)

$$\begin{aligned} I_n &= \int (\log x)^n dx \\ \therefore I_{n-1} &= \int (\log x)^{n-1} dx \\ \text{Now } I_n &= \int (\log x)^n \cdot 1 \cdot dx \\ &= (\log x)^n \cdot x - n \int (\log x)^{n-1} \cdot \frac{1}{x} \cdot x dx \end{aligned}$$

$$\begin{aligned} I_n &= n(\log x)^n - n I_{n-1} \\ I_n + n I_{n-1} &= n(\log x)^n \end{aligned}$$

5. $\int \frac{\sin^8 x - \cos^8 x}{\sin^4 x + \cos^4 x} dx = \int (\sin^4 x - \cos^4 x) dx$

$$= \int (\sin^2 x - \cos^2 x) dx = -\frac{1}{2} \sin 2x + C$$

6. (A)

$$\begin{aligned} A &= \begin{bmatrix} x & x \\ x & x \end{bmatrix} \\ \Rightarrow A^2 &= \begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix}, \\ A^3 &= \begin{bmatrix} 2^2 x^2 & 2^2 x^2 \\ 2^2 x^2 & 2^2 x^2 \end{bmatrix} \dots \dots \text{so on} \end{aligned}$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} \dots \dots$$

$$e^A = \begin{bmatrix} 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\dots & x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\dots \\ x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\dots & 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\dots \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{2x} + 1 & e^{2x} - 1 \\ e^{2x} - 1 & e^{2x} + 1 \end{bmatrix}$$

$$\Rightarrow f(x) = e^{2x} + 1 \text{ \& } g(x) = e^{2x} - 1$$

Now solve all the options

7. $\int \{\sec^2 x + \tan^2 x + 2 \tan x \sec x\}^{1/2} dx$

$$\begin{aligned} &= \int (\sec x + \tan x) dx \\ &= \bullet n |(\sec x + \tan x)| + \bullet n |\sec x| + C \\ &= \bullet n |\sec x (\sec x + \tan x)| + C \end{aligned}$$



8. (C)

$$\int e^{(x \sin x + \cos x)} \left(\frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right) dx$$

$$= \int (e^{(x \sin x + \cos x)} x^2 \cos x) dx$$

$$- \int e^{x \sin x + \cos x} \left(\frac{x \sin x - \cos x}{(x \cos x)^2} \right) dx$$

Applying Integration by parts

$$= \left\{ x \cdot e^{x \sin x + \cos x} - \int e^{x \sin x + \cos x} \cdot dx \right\}$$

$$- \left\{ e^{x \sin x + \cos x} \cdot \frac{1}{x \cos x} - \int e^{x \sin x + \cos x} \cdot dx \right\}$$

$$= e^{x \sin x + \cos x} \left\{ x - \frac{1}{x \cos x} \right\} + C$$

9. $I = 2 \int \sin x \cdot \cos \sec 4x \, dx = 2 \int \frac{\sin x \, dx}{4 \sin x \cos x \cos 2x}$

$$= \frac{1}{2} \int \frac{\cos x \, dx}{(1 - \sin^2 x)(1 - 2 \sin^2 x)}$$

Put $\sin x = t$, then $\cos x \, dx = dt$

$$\Rightarrow I = \frac{1}{2} \int \frac{dt}{(1 - t^2)(1 - 2t^2)}$$

$$= \frac{1}{2} \left[-\int \frac{1}{1 - t^2} dt + \int \frac{2}{1 - 2t^2} dt \right] \text{ [By partial fraction]}$$

$$= \frac{1}{2} \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| + \frac{2}{2\sqrt{2}} \ln \left| \frac{1+\sqrt{2}t}{1-\sqrt{2}t} \right| \right]$$

$$= \frac{1}{2\sqrt{2}} \bullet \ln \left| \frac{(1+\sqrt{2} \sin x)}{(1-\sqrt{2} \sin x)} \right| - \frac{1}{4} \bullet \ln \left| \frac{(1+\sin x)}{(1-\sin x)} \right| + C$$

10. (B)

$S_1 : a = 0 \Rightarrow \int \sec x \, dx = \log \left(\tan \frac{x}{2} \right) + C$

$\Rightarrow f$ is a log function for $a = 0$

$S_2 : |a| < 1$

$$\Rightarrow \frac{2}{1-a} \int \frac{\frac{1}{2} \sec^2 \frac{x}{2} dx}{\left(\frac{a+1}{1-a} \right) - \tan^2 \frac{x}{2}} = \frac{2}{1-a} \int \frac{dt}{\left(\frac{a+1}{1-a} \right) - t^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} \ln \left(\frac{\sqrt{a+1} + \sqrt{1-a} \times \tan \frac{x}{2}}{\sqrt{a+1} - \sqrt{1-a} \times \tan \frac{x}{2}} \right) + c, f \text{ is a log}$$

function for $|a| < 1$

$S_3 : a = 1$

$$\Rightarrow \int \frac{dx}{1 + \cos x} = \int \frac{1}{2} \sec^2 \frac{x}{2} dx = \tan \frac{x}{2} + C$$

$\Rightarrow f$ is a polynomial for $a = 1$

$S_4 : \text{False}$

11. $I = \int \frac{3 \cot 3x - \cot x}{\tan x - 3 \tan 3x} dx = pf(x) + g(x) + C \dots\dots(i)$

Using

$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} \text{ and } \cot 3x = \frac{\cot^3 x - 3 \cot x}{3 \cot^2 x - 1}$$

$$I = \int \frac{1 - 3 \tan^2 x + 2 - 2}{3 - \tan^2 x} dx$$

$$I = \int 1 \cdot dx - 2 \int \frac{1 + \tan^2 x}{3 - \tan^2 x} dx = x - 2 \int \frac{\sec^2 x}{3 - \tan^2 x} dx + C$$

$$= x - \frac{2}{2\sqrt{3}} \bullet \ln \left| \frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x} \right| + C$$

$$= 1 \cdot x + \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x} \right| + C$$

comparing with R.H.S. of equation (i), we get result.

12. $I = \int \frac{(x-1)dx}{x^2 \sqrt{2x^2 - 2x + 1}} = \int \frac{(x-1)dx}{x^3 \sqrt{2 - \frac{2}{x} + \frac{1}{x^2}}}$

Put $2 - \frac{2}{x} + \frac{1}{x^2} = t^2$, then $\left(\frac{x-1}{x^3} \right) dx = t \, dt$

$$\Rightarrow I = \int \frac{t \, dt}{t} = t + C = \sqrt{2 - \frac{2}{x} + \frac{1}{x^2}} + C$$

$$= \frac{\sqrt{2x^2 - 2x + 1}}{x} + C$$

So $f(x) = 2x^2 - 2x + 1$ and $g(x) = x$



$$13. I_n = \int \cot^n x \, dx = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= \frac{-\cot^{n-1} x}{n-1} - I_{n-2} + C_1$$

$$\Rightarrow I_n + I_{n-2} = \frac{-\cot^{n-1} x}{n-1} + C_1$$

$$\begin{aligned} \text{Now, } I_0 + I_1 + 2(I_2 + \dots + I_8) + I_9 + I_{10} \\ = (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4) + \\ (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8) \\ = -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + \dots + \frac{\cot^9 x}{9}\right) + C \end{aligned}$$

$$\therefore A = -1.$$

$$14. I = \int e^{3x} \cos 4x \, dx = e^{3x} (A \sin 4x + B \cos 4x) + C$$

.....(i)

$$I = \frac{1}{4} e^{3x} \sin 4x - \int \frac{3}{4} e^{3x} \sin 4x \, dx = \frac{1}{4} e^{3x} \sin 4x +$$

$$\frac{3}{16} e^{3x} \cos 4x - \int \frac{9}{16} e^{3x} \cos 4x \, dx$$

$$\frac{25}{16} I = \frac{1}{16} (4e^{3x} \sin 4x + 3e^{3x} \cos 4x)$$

comparing with equation (i)

$$\Rightarrow A = \frac{4}{25}, B = \frac{3}{25}$$

$$\Rightarrow \frac{A}{B} = \frac{4}{3}$$

$$\Rightarrow 3A = 4B$$

$$\Rightarrow 4A + 3B = 1$$

$$15. I = \int 2^{mx} \cdot 3^{nx} \, dx = \int (2^m \cdot 3^n)^x \, dx = \frac{2^{mx} \cdot 3^{nx}}{\ln(2^m \cdot 3^n)} + C$$

16. (A)

$$\int ((\log_x e) - (\log_x e)^2) \, dx$$

$$= \int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} \right) dx = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

(where $t = \bullet \ln x$)

$$= \frac{e^t}{t} + C = \frac{x}{\ln x} + C = x \log_x e + C$$

17. (A)

Since $y = f^{-1}(x)$, therefore, $x = f(y)$.

$$\therefore dx = f'(y) dy. \text{ Thus } \int f^{-1}(x) dx = \int y f'(y) dy + c$$

$\therefore$ statement-2 is true

statement-1 is true for $f^{-1}(x) = \sin^{-1} x$ in statement-2

19. (D)

$$\text{Since } \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x,$$

$$\therefore d(\cot^{-1} x) = -d(\tan^{-1} x).$$

$$\text{Thus } \int 2^{\tan^{-1} x} d(\cot^{-1} x) = - \int 2^{\tan^{-1} x} d(\tan^{-1} x)$$

$$= - \frac{2^{\tan^{-1} x}}{\ln 2} + c$$

$\therefore$ Statement-I is false

Statement-II is true

20. (A)

Statement-II

$$\int e^{g(x)} (f(x) g'(x) + f'(x)) dx$$

$$= \int f(x) \cdot e^{g(x)} \cdot g'(x) dx + \int e^{g(x)} \cdot f'(x) dx$$

$$= f(x) \cdot e^{g(x)} - \int f'(x) e^{g(x)} dx + \int e^{g(x)} \cdot f'(x) dx$$

$$= e^{g(x)} f(x) + c$$

Statement-I

$$\text{Since } g(x) = \sin^{-1} x, f(x) = \sqrt{1-x^2}$$

$$\therefore \int e^{\sin^{-1} x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$$

$$= \int e^{\sin^{-1} x} \left(\frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} + \frac{-x}{\sqrt{1-x^2}} \right)$$

$$= e^{\sin^{-1} x} \cdot \sqrt{1-x^2} + c \text{ by statement-2}$$

21. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (r)

(A) If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$$\therefore \int \frac{\sin x - \cos x}{|\sin x - \cos x|} dx = \int 1 \cdot dx = x + c$$



$$(B) \int \frac{x^2 dx}{(x^3+1)(x^3+2)} = \frac{1}{3} \int 3x^2 \left(\frac{1}{x^3+1} - \frac{1}{x^3+2} \right) dx$$

$$= \frac{1}{3} \bullet \ln \left| \frac{x^3+1}{x^3+2} \right| + c$$

$$\therefore f(x) = \bullet \ln |x|$$

$$(C) \int \sin^{-1} x \cos^{-1} x dx = \int \left[\frac{\pi}{2} \sin^{-1} x - (\sin^{-1} x)^2 \right] dx$$

$$\Rightarrow \frac{\pi}{2} (x \sin^{-1} x + \sqrt{1-x^2}) - (x(\sin^{-1} x)^2 + \sin^{-1} x)$$

$$\sqrt{1-x^2} - x) + c \text{ By parts}$$

$$\Rightarrow \sin^{-1} x \left[\frac{\pi}{2} x - x \sin^{-1} x - 2\sqrt{1-x^2} \right] + \frac{\pi}{2}$$

$$\sqrt{1-x^2} + 2x + c$$

$$\therefore f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$$

$$(D) \int \frac{dx}{x \ln |x|} = \bullet \ln |\bullet \ln |x|| + c$$

$$\therefore f(x) = \bullet \ln |x|$$

22. (A) $\rightarrow$ (t), (B) $\rightarrow$ (t), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)

$$(A) F(x) = \int \frac{x + \sin x}{1 + \cos x} dx = \int \left(x \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx$$

$$= x \tan \frac{x}{2} + c$$

$$\text{Since } 0 = F(0)$$

$$\therefore c = 0 \text{ and } F(\pi/2) = \pi/2$$

$$(B) F(x) = \int e^{\sin^{-1} x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$$

$$= \int e^{\sin^{-1} x} \left(\frac{1}{\sqrt{1-x^2}} \sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}} \right) dx$$

$$F(x) = e^{\sin^{-1} x} \sqrt{1-x^2} + c$$

$$F(0) = 1 + c \Rightarrow c = 0 \quad (\rightarrow F(0) = 1)$$

$$F(1/2) = e^{\pi/6} \cdot \frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi} e^{\pi/6}$$

$$\therefore k = \frac{\pi}{2}$$

$$(C) F(x) = \int \frac{dx}{(x^2+1)(x^2+9)} = \frac{1}{8} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+9} \right) dx$$

$$= \frac{1}{8} \left[\tan^{-1} x - \frac{1}{3} \tan^{-1} \frac{x}{3} \right] + c$$

$$F(0) = c \Rightarrow c = 0$$

$$\therefore f(\sqrt{3}) = \frac{1}{8} \left(\frac{\pi}{3} - \frac{1}{3} \cdot \frac{\pi}{6} \right) = \frac{5\pi}{144} = \frac{5k}{36} \therefore k = \frac{\pi}{4}$$

$$(D) F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int (\tan x)^{-\frac{1}{2}} \sec^2 x dx$$

$$= 2 \sqrt{\tan x} + c$$

$$F(0) = c$$

$$\therefore c = 0$$

$$\therefore F(\pi/4) = 2 = \frac{2k}{\pi} \therefore k = \pi$$

23. 1. (A) 2. (D) 3. (B)

Differentiating both sides

$$\frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} = (Ax^2 + Bx + C) \frac{(x+2)}{\sqrt{x^2 + 4x + 3}}$$

$$+ (2Ax + B) \sqrt{x^2 + 4x + 3} + \frac{\lambda}{\sqrt{x^2 + 4x + 3}}$$

$$x^3 - 6x^2 + 11x - 6 = (Ax^2 + Bx + C)(x+2) + (2Ax + B)(x^2 + 4x + 3) + \lambda$$

comparing coefficients of like powers of x

$$x^3 : 1 = A + 2A$$

$$\Rightarrow A = 1/3$$

$$x^2 : -6 = 2A + B + 8A + B$$

$$2B = -6 - 10 \cdot \frac{1}{3}$$

$$\Rightarrow B = -\frac{14}{3}$$

$$x : 11 = 2B + C + 6A + 4B$$

$$C = 11 - 6 \left(-\frac{14}{3} \right) - 6 \cdot \frac{1}{3} = 11 + 28 - 2 = 37$$

$$\text{Constant terms : } -6 = 2C + 3B + \lambda$$

$$\lambda = -6 - 2 \cdot 37 - 3 \cdot \left(-\frac{14}{3} \right) = -6 - 74 + 14 = -66.$$



24.

$$1. \quad I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= 2 \int \frac{(\sqrt{\tan x} + \sqrt{\cot x})(\tan x + \cot x)}{2(\tan x + \cot x)} dx$$

$$I = 2 \int \frac{d(\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2}$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + C$$

2. If $x \in \left(0, \frac{\pi}{2}\right)$, then I

$$= \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cdot \cos x}} dx = \int \frac{\sqrt{2} d(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C$$

3. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then

$$I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= \int \left(\frac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \frac{\sqrt{-\cos x}}{\sqrt{-\sin x}} \right) dx$$

$$= - \int \frac{\sin x + \cos x}{\sqrt{\sin x \cdot \cos x}} dx$$

$$= - \int \frac{\sqrt{2} d(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$$= \sqrt{2} \sin^{-1}(\cos x - \sin x) + C$$

25.

1. (A)

$$\text{Let } P = \sin^3 x \cos^3 x$$

$$\frac{dP}{dx} = 3 \sin^2 x \cos^4 x - 3 \sin^4 x \cos^2 x = 3 \sin^2 x (1 - \sin^2 x)$$

$$\cos^2 x - 3 \sin^4 x \cos^2 x$$

$$= 3 \sin^2 x \cos^2 x - 6 \sin^4 x \cos^2 x$$

$$\therefore P = 3 I_{2,2} - 6 I_{4,2} \quad \therefore I_{4,2} = \frac{1}{6} (-P + 3 I_{2,2})$$

2. (A)

$$\text{Let } P = \sin^5 x \cos^3 x$$

$$\therefore \frac{dP}{dx} = 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x$$

$$= 5 \sin^4 x (1 - \sin^2 x) \cos^2 x - 3 \sin^6 x \cos^2 x$$

$$= 5 \sin^4 x \cos^2 x - 8 \sin^6 x \cos^2 x$$

$$\therefore P = 5 I_{4,2} - 8 I_{6,2}$$

$$\therefore I_{4,2} = \frac{1}{5} (P + 8 I_{6,2})$$

3. (B)

$$\text{Let } P = \sin^5 x \cos^3 x$$

$$\therefore \frac{dP}{dx} = 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x$$

$$= 5 \sin^4 x \cos^4 x - 3 \sin^4 x (1 - \cos^2 x) \cos^2 x$$

$$= 8 \sin^4 x \cos^4 x - 3 \sin^4 x \cos^2 x$$

$$\therefore P = 8 I_{4,4} - 3 I_{4,2}$$

$$\therefore I_{4,2} = \frac{1}{3} (-P + 8 I_{4,4})$$

$$26. \quad \int \cos 2x \bullet_n (1 + \tan x) dx = \frac{1}{2} \sin 2x \bullet_n (1 + \tan x) -$$

$$\frac{1}{2} \int \frac{\sec^2 x}{(1 + \tan x)} \sin 2x dx$$

$$= \frac{1}{2} \sin 2x \bullet_n (1 + \tan x) - \int \frac{\tan x + 1 - 1}{1 + \tan x} dx$$

$$= \frac{1}{2} \sin 2x \bullet_n (1 + \tan x) - x + \frac{1}{2}$$

$$\int \frac{2 \cos x + \sin x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \sin 2x \bullet_n (1 + \tan x) - x + \frac{x}{2} + \frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} [\sin 2x \bullet_n (1 + \tan x) - x + \bullet_n |\sin x + \cos x|] + C$$



27. (1)

$$\int \frac{x \cos \alpha + 1}{(x^2 + 2x \cos \alpha + 1)^{3/2}} dx = \int \frac{x \cos \alpha + 1}{((x + \cos \alpha)^2 + \sin^2 \alpha)^{3/2}} dx$$

Put $x + \cos \alpha = \sin \alpha \tan \theta$

$$\begin{aligned} &= \int \frac{\cos \alpha (\sin \alpha \tan \theta - \cos \alpha) + 1}{(\sin^2 \alpha \tan^2 \theta + \sin^2 \alpha)^{3/2}} \sin \alpha \cdot \sec^2 \theta d\theta \\ &= \int \frac{\cos \alpha \sin \alpha \tan \theta + \sin^2 \alpha}{\sin^3 \alpha \sec^3 \theta} \sin \alpha \sec^2 \theta d\theta \\ &= \int \frac{\cos \alpha \tan \theta + \sin \alpha}{\sin \alpha} \cos \theta d\theta \\ &= \int (\cot \alpha \cdot \sin \theta + \cos \theta) d\theta = -\cot \alpha \cos \theta + \sin \theta + c \\ &= -\cot \alpha \frac{\sin \alpha}{\sqrt{x^2 + 2x \cos \alpha + 1}} + \frac{x + \cos \alpha}{\sqrt{x^2 + 2x \cos \alpha + 1}} + c \\ &= \frac{x}{\sqrt{x^2 + 2x \cos \alpha + 1}} + c \end{aligned}$$

28. $I = \int \sqrt{\frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x + \cot x}} \cdot \frac{\sec x dx}{\sqrt{1 + 2 \sec x}}$

$$= \int \sqrt{\frac{1 - \cos x}{1 + \cos x}} \cdot \frac{\sec x dx}{\sqrt{1 + 2 \sec x}} = \int \tan \frac{x}{2} \cdot \frac{dx}{\sqrt{\cos x} \sqrt{\cos x + 2}}$$

$$= \int \tan \frac{x}{2} \cdot \frac{\sec^2 \frac{x}{2} dx}{\sqrt{1 - \tan^2 \frac{x}{2}} \sqrt{3 + \tan^2 \frac{x}{2}}}$$

$$= \int \frac{\tan \frac{x}{2} \sec^2 \frac{x}{2} dx}{\sqrt{2 - \sec^2 \frac{x}{2}} \sqrt{2 + \sec^2 \frac{x}{2}}} = \int \frac{\tan \frac{x}{2} \sec^2 \frac{x}{2} dx}{\sqrt{2^2 - \left(\sec^2 \frac{x}{2}\right)^2}}$$

Let $\sec^2 \frac{x}{2} = t$, then $\sec^2 \frac{x}{2} \tan \frac{x}{2} dx = dt$

$$\Rightarrow I = \int \frac{dt}{\sqrt{2^2 - t^2}} = \sin^{-1} \left(\frac{t}{2} \right) + C$$

$$= \sin^{-1} \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) + C$$

29. (2)

$$I = \int \left(\frac{x-1}{x+1} \right) \frac{dx}{x \sqrt{x+1 + \frac{1}{x}}}$$

So, $I = \int \frac{(x-1)dx}{(x+1)x \sqrt{x+1 + \frac{1}{x}}}$

$$= \int \frac{\left(1 - \frac{1}{x}\right) \left(1 + \frac{1}{x}\right) dx}{(x+1) \left(1 + \frac{1}{x}\right) \sqrt{x+1 + \frac{1}{x}}}$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x} + 2\right) \sqrt{x + \frac{1}{x} + 1}}$$

Put $x + 1 + \frac{1}{x} = t^2$

$$\left(1 - \frac{1}{x^2}\right) dx = 2t dt$$

$$= \int \frac{2t dt}{(t^2 + 1)t} = 2 \tan^{-1} t + c$$

$$= 2 \tan^{-1} \left(\sqrt{x + \frac{1}{x} + 1} \right) + c$$

30. $I = \int e^x \left(\frac{1 + nx^{n-1} - x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}} \right) dx$

$$= \int e^x \left(\sqrt{\frac{1+x^n}{1-x^n}} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}} \right) dx$$

Let $f(x) = \sqrt{\frac{1+x^n}{1-x^n}}$

$$\Rightarrow f'(x) = \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$$

$$\therefore I = e^x \cdot \sqrt{\frac{1+x^n}{1-x^n}} + C$$