SOLVED EXAMPLES

Ex. 1 The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Sol. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

Ex. 2 Evaluate the following :

(i) $\sin(\cos^{-1}3/5)$ **(ii)** $\cos(\tan^{-1} 3/4)$ **(iii)** $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Sol. (i) Let $\cos^{-1} 3/5 = \theta$. Then,

$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,

$\tan\theta = 3/4$

$\Rightarrow \cos\theta = \frac{4}{5}$ $\left\{\because \text{ as } \cos^2\theta = \frac{1}{1+\tan^2\theta}\right\}$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Ex. 3 The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6)$ is -

Sol. $\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$

Ex. 4 Evaluate the following :

(i) $\sin^{-1}(\sin\pi/4)$ **(ii)** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Sol. (i) $\sin^{-1}(\sin\pi/4) = \frac{\pi}{4}$

(ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$ $\left\{\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right\}$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

Ex. 5 Prove that $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1} 3) = 15$

Sol. We have,

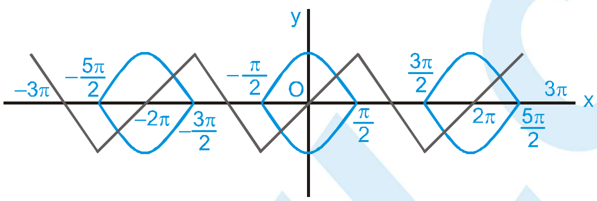
$\sec^2 (\tan^{-1}2) + \text{cosec}^2 (\cot^{-1}3)$

$$= \left\{\sec\left(\tan^{-1} 2\right)\right\}^2 + \left\{\text{cosec}\left(\cot^{-1} 3\right)\right\}^2 = \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2 + \left\{\text{cosec}\left(\text{cosec}^{-1}\sqrt{10}\right)\right\}^2 = \left(\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2 = 15$$

Ex. 6 Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \le 3\pi$.

Sol. Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



Ex. 7 Prove that

(i) $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$ **(ii)** $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Sol. (i) L.H.S. $= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left\{\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right);\ \text{if } xy < 1\right\}$$

$$= \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right) = \text{R.H.S.}$$

(ii) $\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad = \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Ex. 8 Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Sol. We have, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$ $\left[\because \cos^{-1}\frac{12}{13}=\sin^{-1}\frac{5}{13}\right]$

$$=\sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}=\sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right\}=\sin^{-1}\frac{56}{65}$$

Ex. 9 If $x=\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y=\sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1}a)))))$, where $a\in[0,1]$. Find the relationship between x and y in terms of 'a'

Sol. Here, $x=\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ $\left\{\text{Let } \sin\theta=a \Rightarrow \sec\theta=\frac{1}{\sqrt{1-a^2}}\right.$

$=\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec\theta))))$

$\Rightarrow$ $x=\operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$ $\left\{\text{Let } \cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)=\phi \Rightarrow \cot\phi=\frac{1}{\sqrt{1-a^2}}\right.$

$=\operatorname{cosec}(\tan^{-1}(\cos\phi))$ therefore $\cos\phi=\frac{1}{\sqrt{2-a^2}}$

$\Rightarrow$ $x=\operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$ $\left\{\text{Let, } \tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)=\psi \Rightarrow \tan\psi=\frac{1}{\sqrt{2-a^2}}\right.$

$=\operatorname{cosec}\psi$ therefore $\operatorname{cosec}\psi=\sqrt{3-a^2}$

$\Rightarrow$ $x=\sqrt{3-a^2}$ ...... (i)

and $y=\sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1}a)))))$ $\left\{\text{Let } \cos^{-1}a=\alpha \Rightarrow \cos\alpha=a \Rightarrow \operatorname{cosec}\alpha=\frac{1}{\sqrt{1-a^2}}\right.$

$=\sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}\alpha))))$

$\Rightarrow$ $y=\sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$ $\left\{\text{Let, } \tan^{-1}\frac{1}{\sqrt{1-a^2}}=\beta \Rightarrow \tan\beta=\frac{1}{\sqrt{1-a^2}}\right.$

$=\sec(\cot^{-1}(\sin(\beta)))$ $\Rightarrow \sin\beta=\frac{1}{\sqrt{2-a^2}}$

$\Rightarrow$ $y=\sec\left(\cot^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$

$$\left\{\text{Let } \cot^{-1}\frac{1}{\sqrt{2-a^2}} = \gamma \Rightarrow \cot\gamma = \frac{1}{\sqrt{2-a^2}} \Rightarrow \sec\gamma = \sqrt{3-a^2}\right.$$

$$= \sec\gamma$$

$\Rightarrow$ $y = \sqrt{3-a^2}$ ...... **(ii)**

from **(i)** and **(ii)**, $x = y = \sqrt{3-a^2}$.

Ex. 10 Evalulate : **(i)** $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ **(ii)** $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

Sol. (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\} = \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\}$ $\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right),\ if\ |x|<1\right]$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1}1\right\} = \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}$$

(ii) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$, $0 < \theta < \pi/2$

Now, $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

$$= \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sqrt{1-\frac{\sqrt{5}}{3}}}{\sqrt{1+\frac{\sqrt{5}}{3}}} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}} = \sqrt{\frac{(3-\sqrt{5})^2}{9-5}} = \frac{3-\sqrt{5}}{2}$$

Ex. 11 Prove that : $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Sol. We have, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\frac{1}{7}$$ $\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1\right]$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

Ex. 12 Prove that :

(i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$ **(ii)** $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Sol. (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left\{2\tan^{-1}\frac{1}{5}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}, if\ |x|<1\right]$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}=\tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\}-\tan^{-1}.\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$

$$=2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\}+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1} \qquad \left[\because \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\right]$$

$$=2\tan^{-1}\frac{13}{39}+\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2},\ \text{if } |x|<1\right]$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

Ex. 13 Find value of x for the equation $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$.

Sol. Given equation is $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$

$\Rightarrow \quad \cos^{-1}x + (\cos^{-1}x + \sin^{-1}x) = \frac{11\pi}{6}$

$\Rightarrow \quad \cos^{-1}x + \frac{\pi}{2} = \frac{11\pi}{6} \Rightarrow \cos^{-1}x = 4\pi/3$

which is not possible as $\cos^{-1}x \in [0, \pi]$

Ex. 14 Solve the equation : $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

Sol. $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

taking tangent on both sides

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)\right) = 1$$

$$\Rightarrow \quad \frac{\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)} = 1$$

$$\Rightarrow \quad \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}} = 1$$

$$\Rightarrow \quad \frac{(x-1)(x+2) + (x-2)(x+1)}{x^2 - 4 - (x^2 - 1)} = 1 \quad \Rightarrow \quad 2x^2 - 4 = -3 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

Now verify $x = \frac{1}{\sqrt{2}}$

$$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} - 2}\right) + \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} + 2}\right) = \tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right) + \tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)$$

$$= \tan^{-1}\left\{\frac{(2\sqrt{2}+1)(\sqrt{2}-1)+(2\sqrt{2}-1)(\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)-(\sqrt{2}-1)(\sqrt{2}+1)}\right\} = \tan^{-1}\left(\frac{6}{6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$x = -\frac{1}{\sqrt{2}}$$

$$= \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}-1}{-\frac{1}{\sqrt{2}}-2}\right) + \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}+1}{-\frac{1}{\sqrt{2}}+2}\right) = \tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right) + \tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right) \text{ \{same as above\}}$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

$\therefore$ $x = \pm\frac{1}{\sqrt{2}}$ are solutions

Ex. 15 Solve the equation : $2\tan^{-1}(2x+1) = \cos^{-1}x$.

Sol. Here, $2\tan^{-1}(2x+1) = \cos^{-1}x$

or $\cos(2\tan^{-1}(2x+1)) = x$ $\left\{\text{We know } \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$

$\therefore$ $\frac{1-(2x+1)^2}{1+(2x+1)^2} = x$ $\Rightarrow$ $(1-2x-1)(1+2x+1) = x(4x^2+4x+2)$

$\Rightarrow$ $-2x\,.\,2(x+1) = 2x(2x^2+2x+1)$ $\Rightarrow$ $2x(2x^2+2x+1+2x+2) = 0$

$\Rightarrow$ $2x(2x^2+4x+3) = 0$

$\Rightarrow$ $x=0$ or $2x^2+4x+3=0$ {No solution}

Verify $x=0$

$2\tan^{-1}(1) = \cos^{-1}(1)$ $\Rightarrow$ $\frac{\pi}{2} = \frac{\pi}{2}$

$\therefore$ $x = 0$ is only the solution

Ex. 16 Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Sol. $\sin^{-1}(\sin 5) > x^2 - 4x$ $\Rightarrow$ $\sin^{-1}[\sin(5-2\pi)] > x^2 - 4x$

$\Rightarrow$ $x^2 - 4x < 5 - 2\pi$ $\Rightarrow$ $x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow$ $2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi}$ $\Rightarrow$ $x \in (2-\sqrt{9-2\pi},\ 2+\sqrt{9-2\pi})$

Ex. 17 Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Sol. $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow$ $([\cot^{-1}x] - 3)^2 \le 0$ $\Rightarrow$ $[\cot^{-1}x] = 3$

$\Rightarrow$ $3 \le \cot^{-1}x < 4$ $\Rightarrow$ $x \in (-\infty, \cot 3]$

Ex. 18 If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of n.

Sol. $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$

$\Rightarrow \quad \cot\left(\cot^{-1}\left(\frac{n}{\pi}\right)\right) < \cot\left(\frac{\pi}{6}\right) \Rightarrow \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \pi\sqrt{3} \qquad \Rightarrow \quad n < 5.5$ (approx)

$\Rightarrow \quad n = 5 \qquad$ ✈ $(n \in N)$

Ex. 19 Prove that :

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

Sol. L.H.S. $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$

$$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1}.\frac{1}{c_2}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2}.\frac{1}{c_3}}\right)+\ldots+\tan^{-1}\left(\frac{\frac{1}{c_{n-1}}-\frac{1}{c_n}}{1+\frac{1}{c_{n-1}}.\frac{1}{c_n}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)+\left(\tan^{-1}\frac{1}{c_2}\tan^{-1}\frac{1}{c_3}\right)+\ldots\ldots$$

$$+\left(\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$=\tan^{-1}\left(\frac{x}{y}\right)=$ R.H.S.

Ex. 20 If $\tan^{-1}y = 4\tan^{-1}x, \left(|\ x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Sol. We have $\tan^{-1}y = 4\tan^{-1}x$

$\Rightarrow \quad \tan^{-1}y = 2\tan^{-1}\frac{2x}{1-x^2} \qquad$ (as $|x| < 1$)

$= \tan^{-1}\frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\frac{4x(1-x^2)}{x^4-6x^2+1} \qquad \left(\text{as}\left|\frac{2x}{1-x^2}\right|<1\right)$

$\Rightarrow \quad y = \dfrac{4x(1-x^2)}{x^4 - 6x^2 + 1}$

If $\quad x = \tan\dfrac{\pi}{8}$

$\Rightarrow \quad \tan^{-1} y = 4\tan^{-1} x = \dfrac{\pi}{2}$

$\Rightarrow$ y is not defined $\Rightarrow x^4 - 6x^2 + 1 = 0$

Ex. 21 If $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show A > B.

Sol. We have, $A = 2\tan^{-1}\left(2\sqrt{2}-1\right) = 2\tan^{-1}(1.828)$

$\Rightarrow \quad A > 2\tan^{-1}\left(\sqrt{3}\right) \quad \Rightarrow \quad A > \dfrac{2\pi}{3}$ ...... **(i)**

also we have, $\sin^{-1}\left(\dfrac{1}{3}\right) < \sin^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \sin^{-1}\left(\dfrac{1}{3}\right) < \dfrac{\pi}{6}$

$\Rightarrow \quad 3\sin^{-1}\left(\dfrac{1}{3}\right) < \dfrac{\pi}{2}$

also, $\quad 3\sin^{-1}\left(\dfrac{1}{3}\right) = \sin^{-1}\left(3.\dfrac{1}{3} - 4\left(\dfrac{1}{3}\right)^3\right) = \sin^{-1}\left(\dfrac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow \quad 3\sin^{-1}(1/3) < \sin^{-1}\left(\sqrt{3}/2\right) \quad \Rightarrow \quad 3\sin^{-1}(1/3) < \pi/3$

also, $\quad \sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}\left(\sqrt{3}/2\right) \quad \Rightarrow \quad \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \dfrac{2\pi}{3}$ ...... **(ii)**

From **(i)** and **(ii)**, we have A > B.

Ex. 22 Solve for x : If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where [.] denotes the greatest integer function.

Sol. We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$\Rightarrow \quad 1 \le \sin^{-1}.\cos^{-1}.\sin^{-1}.\tan^{-1}x \le \dfrac{\pi}{2} \quad \Rightarrow \quad \sin 1 \le \cos^{-1}.\sin^{-1}.\tan^{-1}x \le 1$

$\Rightarrow \quad \cos\sin 1 \ge \sin^{-1}.\tan^{-1}x \ge \cos 1 \quad \Rightarrow \quad \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1$

$\Rightarrow \quad \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$

Ex. 23 If $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ then find the sum of all possible values of $\tan\theta$.

Sol. $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ $\Rightarrow$ $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$

$\Rightarrow$ $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}\right]$ $\Rightarrow$ $\theta = \tan^{-1}(2\tan^2\theta) - \frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)$

$\Rightarrow$ $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right)$ ...... **(i)**

taking tangent on both sides

$\Rightarrow$ $\tan\theta = \frac{6\tan^2\theta - \tan\theta}{3+2\tan^3\theta}$ $\Rightarrow$ $2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$

$\Rightarrow$ $2\tan\theta(\tan^3\theta - 3\tan\theta + 2) = 0$ $\Rightarrow$ $2\tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$

$\Rightarrow$ $\tan\theta = 0, 1, -2$

which satisfy equation **(i)**

$\therefore$ $\text{sum} = 0 + 1 - 2 = -1$

Ex. 24 Find the sum of the series $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right) + ... + \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right) + ...\text{to } \infty$

Sol. Let S_n denote the sum of n-terms of the given series.

Then,

$$S_n = \sum_{r=1}^{n}\sin^{-1}\left\{\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right\}$$

$$= \sum_{r=1}^{n}\sin^{-1}\left\{\sqrt{\frac{r}{r(r+1)}} - \sqrt{\frac{r-1}{r(r+1)}}\right\}$$

$$= \sum_{r=1}^{n}\sin^{-1}\left\{\frac{1}{\sqrt{r}}\sqrt{1-\frac{1}{r+1}} - \frac{1}{\sqrt{r+1}}\sqrt{1-\frac{1}{r}}\right\}$$

$$= \sum_{r=1}^{n}\left\{\sin^{-1}\frac{1}{\sqrt{r}} - \sin^{-1}\frac{1}{\sqrt{r+1}}\right\}$$

$$= \sin^{-1} 1 - \sin^{-1} \frac{1}{\sqrt{n+1}}$$

$$= \frac{\pi}{2} - \sin^{-1} \frac{1}{\sqrt{n+1}}$$

$$= \cos^{-1} \frac{1}{\sqrt{n+1}}$$

$\therefore$ Required sum

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \cos^{-1} \frac{1}{\sqrt{n+1}}$$

$$= \cos^{-1} 0 = \frac{\pi}{2}$$

Ex. 25 If $y = \cot^{-1}\sqrt{\cos x} - \tan^{-1}\sqrt{\cos x}$, prove that $\sin y = \tan^2 \frac{x}{2}$

Sol. We have

$$y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$$

$\Rightarrow$ $y = \frac{\pi}{2} - \tan^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow$ $y = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos x})$

$\Rightarrow$ $y = \frac{\pi}{2} - \cos^{-1}\left\{\frac{1-(\sqrt{\cos x})^2}{1+(\sqrt{\cos x})^2}\right\}$ $\quad \left[\because 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

$\Rightarrow$ $y = \frac{\pi}{2} - \cos^{-1}\left(\tan^2 \frac{x}{2}\right)$

$\Rightarrow$ $\cos^{-1}\left(\tan^2 \frac{x}{2}\right) = \frac{\pi}{2} - y$

$\Rightarrow$ $\tan^2 \frac{x}{2} = \cos\left(\frac{\pi}{2} - y\right)$

$\Rightarrow$ $\tan^2 \frac{x}{2} = \sin y$

Ex. 26 Solve for x : $\sin^{-1}\left\{\sin\left(\frac{2x^2+4}{1+x^2}\right)\right\} < \pi - 3$

Sol. We have,

$$\Rightarrow \quad \left(\frac{2x^2+4}{1+x^2}\right) \text{ radians} = \left(2+\frac{2}{1+x^2}\right) \text{ radians} > 90^0$$

$$\therefore \quad \sin^{-1}\left\{\sin\left(\frac{2x^2+4}{1+x^2}\right)\right\} < \pi - 3$$

$$\Rightarrow \quad \sin^{-1}\left[\left\{\pi - \left(2+\frac{2}{1+x^2}\right)\right\}\right] < \pi - 3$$

$$\Rightarrow \quad \pi - \left(2+\frac{2}{1+x^2}\right) < \pi - 3$$

$$\Rightarrow \quad -2 - \frac{2}{1+x^2} < -3$$

$$\Rightarrow \quad -\frac{2}{1+x^2} < -1$$

$$\Rightarrow \quad \frac{2}{1+x^2} > 1$$

$$\Rightarrow \quad 2 > 1 + x^2$$

$$\Rightarrow \quad x^2 - 1 < 0$$

Ex. 27 Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Sol. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}$.

Ex. 28 Find the value of cosec $\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$.

Sol. $\cot(\cot^{-1} x) = x, \ \forall\, x \in R$

$$\therefore \quad \cot\left(\cot^{-1}\frac{3\pi}{4}\right) = \frac{3\pi}{4}$$

$$\text{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\} = \text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}.$$

Ex. 29 Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol. Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ...... **(i)**

✈ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

(i) can be written as

$$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$$

$$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$$

✈ $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ if $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \Rightarrow y = -\frac{3}{2}$

Ex. 30 Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Sol. $\cos\left(2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right) = \cos\left(\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$

$$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right) \quad \text{...... (i)}$$

$$= -\sqrt{1 - \left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}.$$

Ex. 31 Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\frac{84}{85}$

Sol. ✈ $\frac{3}{5} > 0,\ \frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore$ $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1 - \frac{225}{289}} + \frac{15}{17}\sqrt{1 - \frac{9}{25}}\right)$

$$= \pi - \sin^{-1}\left(\frac{3}{5}\cdot\frac{8}{17} + \frac{15}{17}\cdot\frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right)$$

Ex. 32 Evaluate $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{4}{5}-\tan^{-1}\frac{63}{16}$

Sol. Let $z=\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{4}{5}-\tan^{-1}\frac{63}{16}$

✈ $\sin^{-1}\frac{4}{5}=\frac{\pi}{2}-\cos^{-1}\frac{4}{5}$

∴ $z=\cos^{-1}\frac{12}{13}+\left(\frac{\pi}{2}-\cos^{-1}\frac{4}{5}\right)-\tan^{-1}\frac{63}{16}$.

$z=\frac{\pi}{2}-\left(\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{12}{13}\right)-\tan^{-1}\frac{63}{16}$ ...... **(i)**

✈ $\frac{4}{5}>0,\ \frac{12}{13}>0$ and $\frac{4}{5}<\frac{12}{13}$

∴ $\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{12}{13}=\cos^{-1}\left[\frac{4}{5}\times\frac{12}{13}+\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right]=\cos^{-1}\left(\frac{63}{65}\right)$

∴ equation **(i)** can be written as

$z=\frac{\pi}{2}-\cos^{-1}\left(\frac{63}{65}\right)-\tan^{-1}\left(\frac{63}{16}\right)$

$z=\sin^{-1}\left(\frac{63}{65}\right)-\tan^{-1}\left(\frac{63}{16}\right)$ ...... **(ii)**

✈ $\sin^{-1}\left(\frac{63}{65}\right)=\tan^{-1}\left(\frac{63}{16}\right)$

∴ from equation **(ii)**, we get

∴ $z=\tan^{-1}\left(\frac{63}{16}\right)-\tan^{-1}\left(\frac{63}{16}\right)$ $\Rightarrow$ $z=0$

Exercise # 1 [Single Correct Choice Type Questions]

1. $\cos\left(2\tan^{-1}\left(\frac{1}{7}\right)\right)$ equals -

(A) $\sin(4\cot^{-1}3)$ (B) $\sin(3\cot^{-1}4)$ (C) $\cos(3\cot^{-1}4)$ (D) $\cos(4\cot^{-1}4)$

2. $\alpha = \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)$ and $\beta = \cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)$, then :

(A) $\tan\alpha = \cot\beta$ (B) $\tan\alpha = -\cot\beta$ (C) $\tan\alpha = \tan\beta$ (D) $\tan\alpha = -\tan\beta$

3. $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 + 2(\sin^{-1}x)(\sin^{-1}y) = \pi^2$, then x^2+y^2 is equal to -

(A) 1 (B) 3/2 (C) 2 (D) 1/2

4. If $x = \sin(2\tan^{-1}2)$, $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

(A) $x = 1 - y$ (B) $x^2 = 1 - y$ (C) $x^2 = 1 + y$ (D) $y^2 = 1 - x$

5. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}] = y$, then

(A) $y = \frac{4}{5}$ (B) $y = \frac{2}{\sqrt{5}}$ (C) $y = -\frac{2}{\sqrt{5}}$ (D) $y^2 = \frac{10}{11}$

6. The value of $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$ is

(A) $\frac{6}{17}$ (B) $\frac{22}{7}$ (C) $\frac{19}{9}$ (D) $\frac{17}{6}$

7. The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(A) $\frac{6}{17}$ (B) $\frac{7}{16}$ (C) $\frac{5}{7}$ (D) $\frac{17}{6}$

8. The equation $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has :

(A) no solution (B) unique solution
(C) infinite number of solutions (D) none of these

9. $\cos^{-1}\left\{\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{5} - \sin\frac{2\pi}{5}\right)\right\}$ is equal to

(A) $\frac{23\pi}{20}$ (B) $\frac{13\pi}{20}$ (C) $\frac{3\pi}{20}$ (D) $\frac{17\pi}{20}$

10. $\tan(\cos^{-1}x)$ is equal to

(A) $\frac{x}{1+x^2}$ (B) $\frac{\sqrt{1+x^2}}{x}$ (C) $\frac{\sqrt{1-x^2}}{x}$ (D) $\sqrt{1-2x}$

11. The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right)+\cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$ is equal to

(A) $\sec\frac{10\pi}{9}$ **(B)** $\sec\frac{\pi}{9}$ **(C)** 1 **(D)** –1

12. If $x = 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\sqrt{3}\right)$ and $y = \cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right)$ then which of the following statements holds good ?

(A) $y = \cos\frac{3\pi}{16}$ **(B)** $y = \cos\frac{5\pi}{16}$ **(C)** $x = 4\cos^{-1} y$ **(D)** none of these

13. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is -

(A) 0 **(B)** $\frac{1}{\sqrt{5}}$ **(C)** $\frac{2}{\sqrt{5}}$ **(D)** $\frac{\sqrt{3}}{2}$

14. $\tan^{-1}2 + \tan^{-1}3 = \text{cosec}^{-1}x$, then x is equal to -

(A) 4 **(B)** $\sqrt{2}$ **(C)** $-\sqrt{2}$ **(D)** none of these

15. Which of the following is the solution set of the equation $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x - 2)$?

(A) $\left\{\frac{1}{2}, 1\right\}$ **(B)** $\left[\frac{1}{2}, 1\right]$ **(C)** $\left[\frac{1}{3}, 1\right]$ **(D)** $\left\{\frac{1}{3}, 1\right\}$

16. The solution set of the equation $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) - \sin^{-1}x$

(A) $[-1, 1] - \{0\}$ **(B)** $(0, 1] \cup \{-1\}$ **(C)** $[-1, 0) \cup \{1\}$ **(D)** $[-1, 1]$

17. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then -

(A) $x^2+y^2+z^2+xyz=0$ **(B)** $x^2+y^2+z^2+xyz=1$ **(C)** $x^2+y^2+z^2+2xyz=0$ **(D)** $x^2+y^2+z^2+2xyz=1$

18. The solution of the inequality $(\tan^{-1} x)^2 - 3\tan^{-1} x + 2 \geq 0$ is -

(A) $(-\infty, \tan 1] \cup [\tan 2, \infty)$ **(B)** $(-\infty, \tan 1]$

(C) $(-\infty, -\tan 1] \cup [\tan 2, \infty)$ **(D)** $[\tan 2, \infty)$

19. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then

(A) $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$ **(B)** $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(C) $x^{50} + y^{25} + z^{5} = 0$ **(D)** $\frac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

20. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, where $\frac{\pi}{2}<x<\pi$, is:

(A) $\pi-\frac{x}{2}$ **(B)** $\frac{\pi}{2}+\frac{x}{2}$ **(C)** $\frac{x}{2}$ **(D)** $2\pi-\frac{x}{2}$

21. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then $\tan\theta$ is equal to

(A) 1/3 **(B)** 3 **(C)** 1 **(D)** – 1

22. The value of the angle $\tan^{-1}(\tan 65° - 2\tan 40°)$ in degrees is equal to

(A) – 20° **(B)** 20° **(C)** 25° **(D)** 40°

23. Let $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$. If x satisfies the cubic $ax^3+bx^2+cx-1=0$, then (a + b + c) has the value equal to

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27

24. If $x=\frac{1}{2}$ and $(x+1)(y+1)=2$ then the radian measure of $\cot^{-1}x+\cot^{-1}y$ is

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{3\pi}{4}$

25. If $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}\right)$ is expressed as a rational $\frac{a}{b}$ in lowest form then (a + b) has the value equal to

(A) 19 **(B)** 27 **(C)** 38 **(D)** 45

26. The number of real solutions of equation $\sqrt{1+\cos 2x}=\sqrt{2}\sin^{-1}(\sin x)$, $-\pi\le x\le\pi$, is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

27. There exists a positive real number x satisfying $\cos(\tan^{-1}x)=x$. The value of $\cos^{-1}\left(\frac{x^2}{2}\right)$ is

(A) $\frac{\pi}{10}$ **(B)** $\frac{\pi}{5}$ **(C)** $\frac{2\pi}{5}$ **(D)** $\frac{4\pi}{5}$

28. The solution of the equation $2\cos^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(A) [–1, 0] **(B)** [0, 1] **(C)** [–1, 1] **(D)** $\left[\frac{1}{\sqrt{2}},1\right]$

29. Number of real value of x satisfying the equation, $\arctan\sqrt{x(x+1)}+\arcsin\sqrt{x(x+1)+1}=\frac{\pi}{2}$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** more than 2

Exercise # 2 Part # I [Multiple Correct Choice Type Questions]

1. $2\tan\left(\tan^{-1}(x)+\tan^{-1}(x^3)\right)$ where $x \in R - \{-1, 1\}$ is equal to

(A) $\frac{2x}{1-x^2}$ **(B)** $\tan(2\tan^{-1}x)$

(C) $\tan\left(\cot^{-1}(-x)-\cot^{-1}(x)\right)$ **(D)** $\tan(2\cot^{-1}x)$

2. Let x_1, x_2, x_3, x_4 be four non zero numbers satisfying the equation $\tan^{-1}\frac{a}{x}+\tan^{-1}\frac{b}{x}+\tan^{-1}\frac{c}{x}+\tan^{-1}\frac{d}{x}=\frac{\pi}{2}$ then which of the following relation(s) hold good?

(A) $\sum_{i=1}^{4} x_i = a+b+c+d$ **(B)** $\sum_{i=1}^{4}\frac{1}{x_i}=0$

(C) $\prod_{i=1}^{4} x_i = abcd$ **(D)** $(x_1+x_2+x_3)(x_2+x_3+x_4)(x_3+x_4+x_1)(x_4+x_1+x_2)=abcd$

3. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]$ is :

(A) $\cos\left(-\frac{7\pi}{5}\right)$ **(B)** $\sin\left(\frac{\pi}{10}\right)$ **(C)** $\cos\left(\frac{2\pi}{5}\right)$ **(D)** $-\cos\left(\frac{3\pi}{5}\right)$

4. If numerical value of $\tan\left\{\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right\}$ is $\frac{a}{b}$, then -

(A) $a+b=23$ **(B)** $a-b=11$ **(C)** $3b=a+1$ **(D)** $2a=3b$

5. $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$, then roots of the equation are -

(A) 0 **(B)** 1 **(C)** –1 **(D)** –2

6. Which of the following is/are correct?

(A) $\cos\left(\cos(\cos^{-1}1)\right)<\sin\left(\sin^{-1}\left(\sin(\pi-1)\right)\right)<\sin\left(\cos^{-1}\left(\cos(2\pi-2)\right)\right)$

(B) $\cos\left(\cos(\cos^{-1}1)\right)<\sin\left(\cos^{-1}\left(\cos(2\pi-2)\right)\right)<\sin\left(\sin^{-1}\left(\sin(\pi-1)\right)\right)<\tan\left(\cot^{-1}(\cot 1)\right)$

(C) $\sum_{t=1}^{5000}\cos^{-1}\left(\cos(2t\pi-1)\right)=\sum_{t=1}^{2500}\cot^{-1}\left(\cot(t\pi+2)\right)$ where $t \in I$

(D) $\cot^{-1}\cot\,\text{cosec}^{-1}\text{cosec}\,\sec^{-1}\sec\,\tan\,\tan^{-1}\cos\,\cos^{-1}\sin^{-1}\sin 4 = 4-\pi$

7. For the equation $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$, which of the following is invalid?

(A) $a^2x+2a=x$ **(B)** $a^2+2ax+1=0$ **(C)** $a \neq 0$ **(D)** $a \neq -1, 1$

8. If $[\sin^{-1}x] + [\cos^{-1}x] = 0$, where 'x' is a non negative real number and [.] denotes the greatest integer function, then complete set of values of x is -

(A) $(\cos 1, 1)$ **(B)** $(-1, \cos 1)$ **(C)** $(\sin 1, 1)$ **(D)** $(\cos 1, \sin 1)$

9. $\sin^{-1} x > \cos^{-1} x$ holds for

(A) all values of x **(B)** $x \in \left(0, \frac{1}{\sqrt{2}}\right)$ **(C)** $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ **(D)** $x = 0.75$

10. $\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4 - 2n^2 + 2}$ is equal to:

(A) $\tan^{-1} 2 + \tan^{-1} 3$ **(B)** $4\tan^{-1} 1$ **(C)** $\pi/2$ **(D)** $\sec^{-1}\left(-\sqrt{2}\right)$

11. Identify the pair(s) of functions which are identical -

(A) $y = \tan(\cos^{-1}x)$; $y = \frac{\sqrt{1-x^2}}{x}$ **(B)** $y = \tan(\cot^{-1}x)$; $y = \frac{1}{x}$

(C) $y = \sin(\text{arc}\tan x)$; $y = \frac{x}{\sqrt{1+x^2}}$ **(D)** $y = \cos(\text{arc}\tan x)$; $y = \sin(\text{arc}\cot x)$

12. The sum of the infinite terms of the series -

$\cot^{-1}\left(1^2 + \frac{3}{4}\right) + \cot^{-1}\left(2^2 + \frac{3}{4}\right) + \cot^{-1}\left(3^2 + \frac{3}{4}\right) + \ldots\ldots\ldots$ is equal to -

(A) $\tan^{-1}(1)$ **(B)** $\tan^{-1}(2)$ **(C)** $\tan^{-1}(3)$ **(D)** $\frac{3\pi}{4} - \tan^{-1} 3$

13. Solution set of the inequality $\sin^{-1}\left(\sin\frac{2x^2+3}{x^2+1}\right) \le \pi - \frac{5}{2}$ is -

(A) $(-\infty, 1) \cup (1, \infty)$ **(B)** $[-1, 1]$ **(C)** $(-1, 1)$ **(D)** $(-\infty, -1] \cup [1, \infty)$

14. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to -

(A) $\frac{1}{2}\cos^{-1} x$ **(B)** $\cos^{-1}\sqrt{\frac{1+x}{2}}$ **(C)** $\sin^{-1}\sqrt{\frac{1-x}{2}}$ **(D)** $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

15. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is -

(A) zero **(B)** one **(C)** two **(D)** infinite

Part # II [Assertion & Reason Type Questions]

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.
(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.
(C) Statement-I is true, Statement-II is false.
(D) Statement-I is false, Statement-II is true.

1. **Statement - I** Range of $\cos\left(\sec^{-1}\frac{1}{x}+\operatorname{cosec}^{-1}\frac{1}{x}+\tan^{-1}x\right)$ is $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

Statement - II Range of $\sin^{-1}x+\tan^{-1}x+\cos^{-1}x$ is $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$.

2. **Statement - I** If r, s & t be the roots of the equation : $x(x-2)(3x-7)=2$, then $\tan^{-1}r+\tan^{-1}s+\tan^{-1}t=3\pi/4$.

Statement - II The roots of the equation $x(x-2)(3x-7)=2$ are real & negative.

3. **Statement - I** If $\sum_{i=1}^{2n}\sin^{-1}x_i=n\pi,\ n\in N$. Then $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}x_i^3$

Statement - II $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2},\forall\ x\in[-1,1]$.

4. Let $f:R\to[0,\pi/2)$ defined by $f(x)=\tan^{-1}(x^2+x+a)$, then

Statement - I The set of values of a for which $f(x)$ is onto is $\left[\frac{1}{4},\infty\right)$.

Statement - II Minimum value of x^2+x+a is $a-\frac{1}{4}$.

5. **Statement - I** $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{9}{5}\right)=\pi-\frac{9}{5}$.

Statement - II $\operatorname{cosec}^{-1}(\operatorname{cosec}x)=\pi-x;\quad\forall\ x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-\{\pi\}$

6. **Statement - I** If α,β are roots of $6x^2+11x+3=0$ then $\cos^{-1}\alpha$ exist but not $\cos^{-1}\beta$, $(\alpha>\beta)$.

Statement - II Domain of $\cos^{-1}x$ is $[-1,1]$.

7. **Statement - I** $\tan^2(\sec^{-1}2)+\cot^2(\operatorname{cosec}^{-1}3)=11$.

Statement - II $\tan^2\theta+\sec^2\theta=1=\cot^2\theta+\operatorname{cosec}^2\theta$.

8. **Statement - I** If $a>0,b>0$, $\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)=\frac{\pi}{2}.\Rightarrow\ x=\sqrt{ab}$.

Statement - II If $m,n\in N$, $n\ge m$, then $\tan^{-1}\left(\frac{m}{n}\right)+\tan^{-1}\left(\frac{n-m}{n+m}\right)=\frac{\pi}{4}$.

Exercise # 3 Part # I [Matrix Match Type Questions]

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

1.

Column-I		Column-II	
(A)	$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)$	(p)	$-2\pi/7$
(B)	$\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$	(q)	$2\pi/7$
(C)	$\tan^{-1}\left(\tan\left(\frac{-33\pi}{7}\right)\right)$	(r)	$3\pi/7$
(D)	$\cot^{-1}\left(\cot\left(\frac{-46\pi}{7}\right)\right)$	(s)	$4\pi/7$

2.

Column-I		Column-II	
(A)	$\sin(\tan^{-1}x)$	(p)	x
(B)	$\cos(\tan^{-1}x)$	(q)	$\frac{x}{\sqrt{x^2+1}}$
(C)	$\sin(\cot^{-1}(\tan(\cos^{-1}x))).x \in (0,1]$	(r)	$\frac{1}{\sqrt{x^2+1}}$
(D)	$\sin(\text{cosec}^{-1}(\cot(\tan^{-1}x)))\ ;\ x \in (0,1]$	(s)	$\sqrt{1-x^2}$

3. $x \geq 0, y \geq 0, z \geq 0$ and $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = k$, the possible value(s) of k, if

Column-I		Column-II	
(A)	$xy + yz + zx = 1$, then	(p)	$k = \frac{\pi}{2}$
(B)	$x + y + z = xyz$, then	(q)	$k = \pi$
(C)	$x^2 + y^2 + z^2 = 1$ and $x + y + z = \sqrt{3}$, then	(r)	$k = 0$
(D)	$x = y = z$ and $xyz \geq 3\sqrt{3}$, then	(s)	$k = \frac{7\pi}{6}$

4. **Column - I** | **Column - II**

(A) Let a, b, c be three positive real numbers

$$\theta = \tan^{-1}\sqrt{\frac{a\,(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b\,(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c\,(a+b+c)}{ab}}$$

then θ equal — **(p)** π

(B) The value of the expression — **(q)** $-\frac{\pi}{2}$

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) \text{ for } 0 < A < (\pi/4)$$

(C) If $x < -1$, then $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1} x$ — **(r)** $-\pi$

(D) The value of $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$ — **(s)** $\frac{\pi}{2}$

Part # II [Comprehension Type Questions]

Comprehension # 1

Consider the two equations in x ; **(i)** $\sin\left(\frac{\cos^{-1} x}{y}\right) = 1$ **(ii)** $\cos\left(\frac{\sin^{-1} x}{y}\right) = 0$

The sets $X_1, X_2 \subseteq [-1, 1]$; $Y_1, Y_2 \subseteq I - \{0\}$ are such that

X_1 : the solution set of equation **(i)**

X_2 : the solution set of equation **(ii)**

Y_1 : the set of all integral values of y for which equation **(i)** possess a solution

Y_2 : the set of all intergral values of y for which equation **(ii)** possess a solution

Let : C_1 be the correspondence : $X_1 \to Y_1$ such that x C_1y for $x \in X_1$, $y \in Y_1$ & (x, y) satisfy **(i)**.

C_2 be the correspondence : $X_2 \to Y_2$ such that x C_2y for $x \in X_2$, $y \in Y_2$ & (x, y) satisfy **(ii)**.

On the basis of above information, answer the following questions :

1. The number of ordered pair (x, y) satisfying correspondence C_1 is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

2. The number of ordered pair (x, y) satisfying correspondence C_2 is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

3. $C_1 : X_1 \to Y_1$ is a function which is -

(A) one-one **(B)** many-one **(C)** onto **(D)** into

Comprehension # 2

Let $h_1(x) = \sin^{-1}(3x - 4x^3)$; $h_2(x) = \cos^{-1}(4x^3 - 3x)$ & $f(x) = h_1(x) + h_2(x)$

when $x \in [-1, \frac{-1}{2}]$; let $f(x) = a\cos^{-1}x + b\pi$; a, b $\in$ Q

$h_1(x) = p\sin^{-1}x + q\pi$; p, q $\in$ Q

$h_2(x) = r\cos^{-1}x + s\pi$; r, s $\in$ Q

Let C_1 be the circle with centre (p, q) & radius 1 & C_2 be the circle with centre (r, s) & radius 1.

On the basis of above information, answer the following questions :

1. $p + r + 2q - s =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4

2. If $b.\log_{|s|}|p + q| = k.a$, then value of k is -

(A) $\frac{9}{2}$ **(B)** 6 **(C)** $\frac{-3}{2}$ **(D)** none of these

3. Radical axis of circle C_1 & C_2 is -

(A) $12x - 2y - 3 = 0$ **(B)** $12x + 2y - 3 = 0$ **(C)** $-12x + 2y - 3 = 0$ **(D)** none of these

Comprehension # 3

Let the domain and range of inverse circular functions are defined as follows

Function	**Domain**	**Range**
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

1. $\sin^{-1}x < \frac{3\pi}{4}$ then solution set of x is

(A) $\left(\frac{1}{\sqrt{2}}, 1\right]$ **(B)** $\left(-\frac{1}{\sqrt{2}}, -1\right]$ **(C)** $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ **(D)** none of these

2. $\sin^{-1}x + \text{cosec}^{-1}x$ at $x = -1$ is

(A) π **(B)** 2π **(C)** 3π **(D)** $-\pi$

3. If $x \in [-1, 1]$, then range of $\tan^{-1}(-x)$ is

(A) $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$ **(B)** $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ **(C)** $[-\pi, 0]$ **(D)** $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Exercise # 4 [Subjective Type Questions]

1. Prove each of the following relations :

(i) $\tan^{-1} x = -\pi + \cot^{-1}\frac{1}{x} = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = -\cos^{-1}\frac{1}{\sqrt{1+x^2}}$ when $x < 0$.

(ii) $\cos^{-1}x = \sec^{-1}\frac{1}{x} = \pi - \sin^{-1}\sqrt{1-x^2} = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$ when $-1<x<0$

2. If X = cosec $\tan^{-1}$ cos $\cot^{-1}$ sec $\sin^{-1}$ a & Y = sec $\cot^{-1}$ sin $\tan^{-1}$ cosec $\cos^{-1}$ a; where $0 \le a < 1$. Find the relation between X & Y. Express them in terms of 'a'.

3. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y, z \le 1$, then find the value of $x^2 + y^2 + z^2 + 2xyz$

4. Express $f(x) = \text{arc}\cos x + \text{arc}\cos\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)$ in simplest form and hence find the values of

(A) $f\left(\frac{2}{3}\right)$ **(B)** $f\left(\frac{1}{3}\right)$

5. Let $\cos^{-1}x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then find the value of $a + b + c$.

6. If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$ then prove that $\alpha + \beta = \pi$. What is the value of $\alpha + \beta$ will be if $x > 1$?

7. Find the sum of the series :

(A) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \ldots\ldots + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots\infty$

(B) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots\ldots$ to n terms.

(C) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + \ldots\ldots$ to n terms.

8. Determine the integral values of 'k' for which the system, $(\tan^{-1} x)^2 + (\cos^{-1} y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \frac{\pi}{2}$ possess solution and find all the solutions.

9. Solve the following equation :

$\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a$ $a \ge 1$; $b \ge 1$, $a \ne b$.

10. Find the number of values of x satisfying the equation $\sin^2(2\cos^{-1}(\tan x)) = 1$.

Exercise # 5 Part # I [Previous Year Questions] [AIEEE/JEE-MAIN]

1. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to - **[AIEEE-2002]**

(A) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (C) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\frac{1}{2}$

2. $\cot^{-1}\left(\sqrt{\cos\alpha}\right)-\tan^{-1}\left(\sqrt{\cos\alpha}\right)=x$. then sin x is equal to - **[AIEEE-2002]**

(A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$ (C) $\tan\alpha$ (D) $\cot\left(\frac{\alpha}{2}\right)$

3. The Inverse trigonometric equation $\sin^{-1}x = 2\sin^{-1}\alpha$, has a solution for **[AIEEE-2003]**

(A) $-\frac{1}{2}<\alpha<\frac{1}{2}$ (B) all real values of α (C) $|\alpha|\le\frac{1}{\sqrt{2}}$ (D) $|\alpha|\ge\frac{1}{\sqrt{2}}$

4. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to- **[AIEEE-2005]**

(A) $2\sin 2\alpha$ (B) 4 (C) $4\sin^2\alpha$ (D) $-4\sin^2\alpha$

5. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$ then a value of x is- **[AIEEE-2007]**

(A) 1 (B) 3 (C) 4 (D) 5

6. The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$ is **[AIEEE-2008]**

(A) $\frac{3}{17}$ (B) $\frac{2}{17}$ (C) $\frac{5}{17}$ (D) $\frac{6}{17}$

7. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[AIEEE - 2013]**

(A) $x=y=z$ (B) $2x=3y=6z$ (C) $6x=3y=2z$ (D) $6x=4y=3z$

Part # II [Previous Year Questions][IIT-JEE ADVANCED]

1. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is: **[IIT-JEE-1999]**

(A) zero `(B) one (C) two (D) infinite

2. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-.......\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-.......\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals **[IIT-JEE-2001]**

(A) 1/2 (B) 1 (C) – 1/2 (D) –1

3. Prove that, $\cos\tan^{-1}\sin\cot^{-1}x=\sqrt{\frac{x^2+1}{x^2+2}}$. **[IIT-JEE-2002]**

4. The value of x for which $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$ is **[IIT-JEE-2005]**

(A) 1/2 **(B)** 1 **(C)** 0 **(D)** – 1/2

5. Match the column **[IIT-JEE-2007]**

Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$

	Column – I		**Column – II**
(A)	If a = 1 and b = 0, then (x, y)	**(p)**	lies on the circle $x^2 + y^2 = 1$
(B)	If a = 1 and b = 1, then (x, y)	**(q)**	lies on $(x^2 - 1)(y^2 - 1) = 0$
(C)	If a = 1 and b = 2, then (x, y)	**(r)**	lies on y = x
(D)	If a = 2 and b = 2, then (x, y)	**(s)**	lies on $(4x^2 - 1)(y^2 - 1) = 0$

6. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[IIT-JEE 2008]**

(A) $\frac{x}{\sqrt{1+x^2}}$ **(B)** x **(C)** $x\sqrt{1+x^2}$ **(D)** $\sqrt{1+x^2}$

7. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[IIT-JEE 2011]

8. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE Ad. 2013]**

(A) $\frac{23}{25}$ **(B)** $\frac{25}{23}$ **(C)** $\frac{23}{24}$ **(D)** $\frac{24}{23}$

9. Match List I with List II and select the correct answer using the code given below the lists :

	List - I		**List - II**
P	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4\right)^{1/2}$ takes value	1.	$\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If cos x + cos y + cox z = 0 = sin x + sin y + sin z then possible value of $\cos\frac{x-y}{2}$ is	2.	$\sqrt{2}$
R.	If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of sec x is	3.	$\frac{1}{2}$
S.	If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then possible value of x is	4.	1

[JEE Ad. 2013]

Codes

	p	q	r	s
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

MOCK TEST

SECTION - I : STRAIGHT OBJECTIVE TYPE

1. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(A) $\frac{6}{17}$ (B) $\frac{7}{16}$ (C) $\frac{16}{7}$ (D) none of these

2. $\sin^{-1}\left(\frac{x^2}{4}+\frac{y^2}{9}\right)+\cos^{-1}\left(\frac{x}{2\sqrt{2}}+\frac{y}{3\sqrt{2}}-2\right)$ equals to :

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi}{\sqrt{2}}$ (D) $\frac{3\pi}{2}$

3. If $x \in [-1, 0)$, then $\cos^{-1}(2x^2-1) - 2\sin^{-1} x =$

(A) $-\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) -2π

4. If $\sin^{-1}(x-1)+\cos^{-1}(x-3)+\tan^{-1}\left(\frac{x}{2-x^2}\right)=\cos^{-1}k+\pi$, then the value of k is equal to

(A) 1 (B) $-\frac{1}{\sqrt{2}}$ (C) $\frac{1}{\sqrt{2}}$ (D) none of these

5. If $\sin^{-1} a+\sin^{-1} b+\sin^{-1} c=\pi$, then the value of $a\sqrt{(1-a^2)}+b\sqrt{(1-b^2)}+c\sqrt{(1-c^2)}$ will be

(A) 2abc (B) abc (C) $\frac{1}{2}$abc (D) $\frac{1}{3}$abc

6. Range of the function $f(x)=\cos^{-1}(-\{x\})$, where $\{.\}$ is fractional part function, is

(A) $\left(\frac{\pi}{2}, \pi\right)$ (B) $\left(\frac{\pi}{2}, \pi\right]$ (C) $\left[\frac{\pi}{2}, \pi\right)$ (D) $\left(0, \frac{\pi}{2}\right]$

7. If $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$, then x =

(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) None of these

8. The solution of the inequality $\log_{1/2}\sin^{-1} x > \log_{1/2}\cos^{-1} x$ is

(A) $x \in \left[0, \frac{1}{\sqrt{2}}\right]$ (B) $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$ (C) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$ (D) None of these

9. S_1 : No. of solutions of the equation $\sin^{-1}x - \cos^{-1}(-x) = \frac{\pi}{2}$ is one

S_2 : Solution set of the equation $\sin^{-1}(x^2+4x+3) + \cos^{-1}(x^2+6x+8) = \frac{\pi}{2}$ is $\left\{-\frac{5}{2}\right\}$

S_3 : $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$ is equal to π

S_4 : $2[\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3]$ is equal to 2π

(A) FTFT (B) FTTF (C) FTTT (D) FFTT

10. If $\sin^{-1}\sin(5) > x^2 - 4x$, then the number of possible integral values of x is

(A) 1 (B) 2 (C) 3 (D) none of these

SECTION - II : MULTIPLE CORRECT ANSWER TYPE

11. $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2}$ equals to :

(A) $\pi + 3\tan^{-1}x$ if $x < -1$ (B) $\pi - 3\tan^{-1}x$ if $x > 1$

(C) $3\tan^{-1}x$ if $-1 < x < 0$ (D) $-\pi + 3\tan^{-1}x$ if $0 < x < 1$

12. If α is a real number for which f(x) = ●n $\cos^{-1}x$ is defined, then a possible value of $[\alpha]$ is (where [.] denotes greatest integer function).

(A) 0 (B) 1 (C) –1 (D) –2

13. If $\sin^{-1}x + 2\cot^{-1}(y^2-2y) = 2\pi$, then

(A) $x + y = y^2$ (B) $x^2 = x + y$ (C) $y = y^2$ (D) $x^2 - x + y = y^2$

14. Which of the following is a rational number :

(A) $\sin\left(\tan^{-1}3 + \tan^{-1}\frac{1}{3}\right)$ (B) $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right)$

(C) $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right)$ (D) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

15. The values of x satisfying $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ are

(A) 0 (B) 1/2 (C) 1 (D) 2

SECTION - III : ASSERTION AND REASON TYPE

16. **Statement-I :** $\cot^{-1}(x) - \tan^{-1}(x) > 0$ for all $x < 1$

Statement-II : Graph of $\cot^{-1}(x)$ is always above the graph of $\tan^{-1}(x)$ for all $x < 1$.

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

17. Consider $f(x) = \sin^{-1}\left(\sec(\tan^{-1} x)\right) + \cos^{-1}\left(\text{cosec}(\cot^{-1} x)\right)$

Statement-I : Domain of $f(x)$ is a singleton.

Statement-II : Range of the function $f(x)$ is a singleton

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

18. **Statement-I :** $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement-II : $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall\, x, y \in (0, 1)$

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

19. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Statement-I : $f'(2) = -\frac{2}{5}$

Statement-II : $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, \quad \forall\, x > 1.$

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

20. **Statement-I :** $\text{cosec}^{-1}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)$

Statement-II : $\text{cosec}^{-1}x > \sec^{-1}x$ if $1 \le x < \sqrt{2}$

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

SECTION - IV : MATRIX - MATCH TYPE

21. **Column – I** | **Column – II**

(A) Difference of greatest and least value of $\sqrt{2}\ (\sin 2x - \cos 2x)$ **(p)** $\frac{\pi}{4}$

(B) Difference of greatest and least value of $x^2 - 4x + 3$, $x \in [1, 3]$, is **(q)** $\frac{\pi}{6}$

(C) Greatest value of $\tan^{-1}\frac{1-x}{1+x}$, $x \in [0, 1]$, is **(r)** 4

(D) Difference of greatest and least value of **(s)** 1

$\cos^{-1} x^2$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, is **(t)** $\frac{\pi}{3}$

22. **Column – I** | **Column – II**

(A) If $f(x) = \sin^{-1} x$ and $\lim_{x\to\frac{1}{2}^+} f(3x - 4x^3) = a - 3\lim_{x\to\frac{1}{2}^+} f(x)$, then $[a] =$ **(p)** 2

(B) If $f(x) = \tan^{-1} g(x)$ where $g(x) = \frac{3x - x^3}{1-3x^2}$ and **(q)** 3

$\lim_{h\to 0}\frac{f(a+3h) - f(a)}{3h} = \frac{3}{1+a^2}$, when $\frac{-1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$,

then find $\left[\lim_{h\to 0}\frac{f\left(\frac{1}{2}+6h\right) - f\left(\frac{1}{2}\right)}{6h}\right] =$

(C) If $\cos^{-1}(4x^3 - 3x) = a + b\cos^{-1} x$ for $-1 < x < \frac{-1}{2}$, then $[a + b + 2] =$ **(r)** 4

(D) If $f(x) = \cos^{-1}(4x^3 - 3x)$ and $\lim_{x\to\frac{1}{2}^+} f'(x) = a$ and **(s)** -2

$\lim_{x\to\frac{1}{2}^-} f'(x) = b$, then $a + b - 3 =$ **(t)** -3

SECTION - V : COMPREHENSION TYPE

23. Read the following comprehension carefully and answer the questions.

$$\tan^{-1}(\tan\theta)=\begin{cases}\pi+\theta & , -\frac{3\pi}{2}<\theta<-\frac{\pi}{2}\\ \theta & , -\frac{\pi}{2}<\theta<\frac{\pi}{2}\\ -\pi+\theta & , \frac{\pi}{2}<\theta<\frac{3\pi}{2}\end{cases},\ \sin^{-1}(\sin\theta)=\begin{cases}-\pi-\theta & , -\frac{3\pi}{2}\le\theta<-\frac{\pi}{2}\\ \theta & , -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ \pi-\theta & , \frac{\pi}{2}<\theta\le\frac{3\pi}{2}\end{cases},$$

$$\cos^{-1}(\cos\theta)=\begin{cases}-\theta & , -\pi\le\theta<0\\ \theta & , 0\le\theta\le\pi\\ 2\pi-\theta & , \pi<\theta\le2\pi\end{cases}$$

Based on the above results, answer each of the following :

1. $\cos^{-1}x$ is equal to

(A) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<1$ **(B)** $-\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(C) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$ **(D)** $\sin^{-1}\sqrt{1-x^2}$ if $0<x<1$

2. $\sin^{-1}x$ is equal to

(A) $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<0$ **(B)** $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<1$

(C) $\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$ **(D)** $-\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$

3. $\cos^{-1}x$ is equal to

(A) $-\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$ **(B)** $\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

24. Read the following comprehension carefully and answer the questions.

Given that $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\begin{cases}2\tan^{-1}x & , |x|<1\\ -\pi+2\tan^{-1}x & , x>1\\ \pi+2\tan^{-1}x & , x<-1\end{cases}$

$\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x & \text{is } |x|\le1\\ \pi-2\tan^{-1}x & \text{is } x>1\\ -(\pi+2\tan^{-1}x) & \text{is } x<-1\end{cases}$ and $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le1$

1. $\sin^{-1}\left(\frac{4x}{x^2+4}\right)+2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent from x then :

(A) $x\in[-3,4]$ **(B)** $x\in[-2,2]$ **(C)** $x\in[-1,1]$ **(D)** $x\in[1,\infty]$

2. If $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$, then $x \in$

(A) $\left(\frac{1}{3}, \infty\right)$ (B) $(-1, \infty)$ (C) $(-\infty, -1)$ (D) none of these

3. If $(x-1)(x^2+1) > 0$, then $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} - \tan^{-1}x\right) =$

(A) 1 (B) $\frac{1}{\sqrt{2}}$ (C) -1 (D) none of there

25. **Read the following comprehension carefully and answer the questions.**

It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$ where $x > 0$ and $B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$ where $t \in \left[0, \frac{1}{\sqrt{2}}\right]$, and $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \le x \le 1$ and $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in R$.

1. The interval in which A lies is

(A) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right)$ (B) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$ (C) $\left(\frac{\pi^3}{40}, \frac{\pi^3}{10}\right)$ (D) none of these

2. The maximum value of B is

(A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{16}$ (C) $\frac{\pi^2}{4}$ (D) none of these

3. If least value of A is λ and maximum value of B is μ, then $\cot^{-1}\cot\left(\frac{\lambda-\mu\pi}{\mu}\right) =$

(A) $\frac{\pi}{8}$ (B) $-\frac{\pi}{8}$ (C) $\frac{7\pi}{8}$ (D) $-\frac{7\pi}{8}$

SECTION - VI : INTEGER TYPE

26. Find number of solution of the equation $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$.

27. $\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right] + \tan^{-1}\left[\frac{\tan\alpha}{4}\right] = \lambda\alpha$, then find the value of λ, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

28. If $\tan\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5}\right)$ is expressed as a rational $\frac{a}{b}$ in lowest form then find $(a-b)$.

29. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then find $(-a+b+c)$.

30. The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right) + \cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$.

ANSWER KEY

EXERCISE - 1

1. A **2.** A **3.** C **4.** D **5.** B **6.** D **7.** D **8.** B **9.** D **10.** C **11.** D **12.** A **13.** B
14. D **15.** A **16.** C **17.** D **18.** B **19.** B **20.** B **21.** B **22.** C **23.** C **24.** D **25.** A **26.** B
27. C **28.** D **29.** C

EXERCISE - 2 : PART # I

1. ABC **2.** BCD **3.** BCD **4.** ABC **5.** ABC **6.** ACD **7.** BC **8.** D **9.** CD
10. AD **11.** ABCD **12.** BD **13.** B **14.** ABC **15.** C

PART - II

1. D **2.** C **3.** A **4.** D **5.** A **6.** A **7.** D **8.** B

EXERCISE - 3 : PART # I

1. $A \to q$ **B** $\to s$ **C** $\to q$ **D** $\to r$
2. **A** $\to q$ **B** $\to r$ **C** $\to p$ **D** $\to p$
3. **A** $\to p$ **B** $\to q,r$ **C** $\to p$ **D** $\to q,s$
4. **A** $\to p$ **B** $\to p$ **C** $\to r$ **D** $\to s$

PART - II

Comprehension # 1 : **1.** B **2.** D **3.** A, C **Comprehension # 2 :** **1.** A **2.** C **3.** A
Comprehension # 3 : **1.** A **2.** C **3.** B

EXERCISE - 5 : PART # I

1. A, B **2.** A **3.** C **4.** C **5.** B **6.** D **7.** A

PART - II

1. C **2.** B **4.** D **5.** $A \to p$ $B \to q$ $C \to p$ $D \to s$ **6.** C **7.** 1 **8.** B **9.** B

MOCK TEST

1. D **2.** D **3.** B **4.** C **5.** A **6.** C **7.** B **8.** C **9.** A
10. C **11.** AC **12.** AC **13.** CD **14.** ABC **15.** AB **16.** A **17.** B **18.** A
19. A **20.** A
21. $A \to r$ $B \to s$ $C \to p$ $D \to q$ **22.** $A \to q,s$ $B \to r,s,t$ $C \to r,s$ $D \to t,q$
23. **1.** C **2.** A **3.** B **24.** **1.** B **2.** A **3.** C **25.** **1.** B **2.** C **3.** A
26. 1 **27.** 1 **28.** 9 **29.** 2 **30.** 1