ELLIPSE

EXERCISE # 1

Questions based on Equation of ellipse

Q.1 The eccentricity, foci and the length of the latus rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$ is

(A) $\left[\frac{\sqrt{3}}{2}; (1 \pm \sqrt{3}, -1); 2\right]$

(B) $\left[\frac{\sqrt{3}}{2}; (1 \pm \sqrt{3}, 1); 1\right]$

(C) $\left[\frac{\sqrt{3}}{2}; (1 \pm \sqrt{3}, -1); 1\right]$

(D) None of these

Sol. **[C]**

Given equation of ellipse is

$x^2 + 4y^2 + 8y - 2x + 1 = 0$

$\Rightarrow (x - 1)^2 + 4(y^2 + 2y) = 0$

$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+1)^2}{1} = 1$

$\therefore$ Eccentricity of ellipse is given by

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

foci of the ellipse are given by

$(1 \pm ae, -1)$

where $ae = \sqrt{a^2 - b^2}$

$\Rightarrow ae = \sqrt{4-1} = \sqrt{3}$

$\therefore$ foci are $(1 \pm \sqrt{3}, -1)$

Latus rectum of the ellipse is given by

$= \frac{2b^2}{a} = \frac{2 \times 1}{2}$

$= 1$

Q.2 The equation of the ellipse with its centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6) is -

(A) $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$

(B) $\frac{(x-1)^2}{35} + \frac{(y-2)^2}{20} = 1$

(C) $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{25} = 1$

(D) None of these

Sol. **[A]**

Centre $\equiv$ (1, 2), one focus is (6, 2)

Let equation of ellipse is

$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$ ....(1)

Since (1) passes through (4, 6)

$\Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1 \Rightarrow 9b^2 + 16a^2 = a^2b^2$ ....(2)

focus of the ellipse $\equiv (1 \pm ae, 2) \equiv (6, 2)$

$\because ae = \sqrt{a^2 - b^2} \Rightarrow 1 + ae = 6$ or $1 - ae = 6$

$\Rightarrow (ae)^2 = a^2 - b^2 \Rightarrow ae = 5$ or $ae = -5$

$\Rightarrow 25 = a^2 - b^2$

$\Rightarrow a^2 - b^2 = 25$ ....(3)

$\Rightarrow a^2 = b^2 + 25$ ....(4)

from (2) & (4) we get

$9b^2 + 16(25 + b^2) = (25 + b^2)b^2$

$9b^2 + 400 + 16b^2 = 25b^2 + b^4$

$\Rightarrow b^4 = 400 \Rightarrow b^2 = 20$

$\therefore a^2 = 25 + 20 = 45$

$\therefore$ Reqd. equation of ellipse is given by

$\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$

Q.3 If major and minor axis of an ellipse is 8 and 4 then distance between directrices of the ellipse is-

(A) $2\sqrt{3}$ (B) $16/\sqrt{3}$

(C) $16\sqrt{3}$ (D) None

Sol. **[B]**

Here given $2a = 8 \Rightarrow a = 4 \Rightarrow a^2 = 16$

and $2b = 4 \Rightarrow b = 2 \Rightarrow b^2 = 4$

$\therefore$ distance b/w directrices = 2a/e

$\therefore ae = \sqrt{a^2 - b^2}$

$\Rightarrow a^2e^2 = a^2 - b^2$

$\Rightarrow 16e^2 = 16 - 4$

$\Rightarrow 16e^2 = 12$

$\Rightarrow e^2 = \frac{12}{16} = \frac{3}{4}$

$\Rightarrow e = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\therefore \frac{2a}{e} = \frac{2\times 4}{\sqrt{3}/2} = \frac{2\times 4\times 2}{\sqrt{3}} = \frac{16}{\sqrt{3}}$

Q.4 An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, the necessary length of the string and the distance between the pins respectively in cms, are-

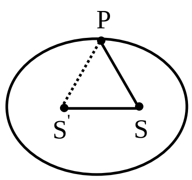
(A) 6, $2\sqrt{5}$ (B) 6, $\sqrt{5}$

(C) 4, $2\sqrt{5}$ (D) $6 + 2\sqrt{5}$, $2\sqrt{5}$

Sol. **[D]**

2a = 6 and 2b = 4 given

$\therefore$ a = 3 and b = 2



Distance between two pins = SS' = 2ae

Here $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$

$\therefore$ $2ae = 2 \times 3 \times \frac{\sqrt{5}}{3} = 2\sqrt{5}$

length of string = SPSS' = SP + S'P + SS'

= 2a + 2ae

= $6 + 2\sqrt{5}$

$\therefore$ length of string = $6 + 2\sqrt{5}$ and distance b/w pins = $2\sqrt{5}$

Questions based on Auxillary circle, point and ellipse, chord joint two point on ellipse

Q.5 P(acosθ, b sinθ) is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b). Now if this ellipse is rotated anticlockwise by 90º and P reaches P′ then co-ordinate of P′ are -

(A) (a sinθ, b cosθ) (B) (b sinθ, a cosθ)

(C) (–b sinθ, a cosθ) (D) None of these

Sol. **[C]**

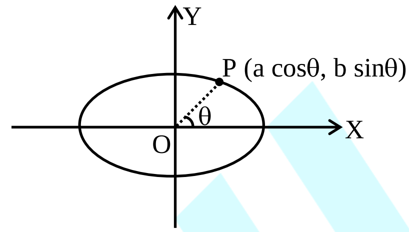


Figure-I

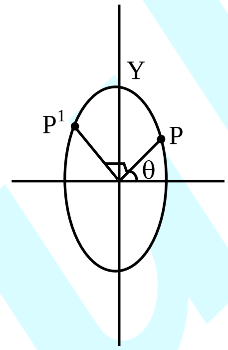


Figure-II

$P^1 = \left\{b\cos\left(\frac{\pi}{2}+\theta\right), a\sin\left(\frac{\pi}{2}+\theta\right)\right\}$

$P^1 = (-b\sin\theta, a\cos\theta)$

$\therefore P^1 \equiv (-b\sin\theta, a\cos\theta)$

Q.6 The equation $\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = 1$ will touch ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point P whose eccentric angle is -

(A) θ (B) (π – θ)

(C) (π + θ) (D) 2π – θ

Sol. **[D]**

Since $\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = 1$ touches $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

$\therefore$ point P is (a cos(–θ), b sin(–θ)}

whose eccentric angle is equal to (–θ) or (2π – θ)

Q.7 If the chord through the points whose eccentric angles are α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ passes through the focus (ae, 0), then the value of $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} =$

(A) $\frac{e+1}{e-1}$ (B) $\frac{e-1}{e+1}$ (C) $\frac{e+1}{e-2}$ (D) None

Sol. **[B]**

Since the chord passes through focus therefore it is the focal chord of ellipse. The co-ordinates of the end points of the focal chord are ($a\cos\alpha$, $b\sin\alpha$) and ($a\cos\beta$. $b\sin\beta$). therefore the equation of the focal chord is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

This passes through focus (ae, 0)

$$\therefore\ \frac{ae}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{1}{e}$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{1-e}{1+e}$$

$$\Rightarrow \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}=\frac{1-e}{1+e}$$

$$\Rightarrow -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

Questions based on Tangent & Normal and properties

Q.8 The equations of tangents to the ellipse $9x^2 + 16y^2 = 144$ which pass through the point (2, 3) is -

(A) $y = 2$ and $y = -x + 5$

(B) $y = 3$ and $y = -x + 5$

(C) $y = 3$ and $y = x - 5$

(D) None of these

Sol. **[B]**

$9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

Let equation of tangent is given by

$y = mx \pm \sqrt{a^2m^2+b^2}$ ……..(i)

Here $a^2 = 16$ and $b^2 = 9$

Since (i) passes through (2, 3) therefore

$3 = 2m \pm \sqrt{16m^2+9}$

$(3 - 2m)^2 = 16m^2 + 9$

$9 + 4m^2 - 12m = 16m^2 + 9$

$12m^2 = -12m \Rightarrow 12m(m + 1) = 0$

$m = -1, 0$

Put $m = -1$ in (i) we get

$y = -x \pm \sqrt{16+9}$

$y = -x \pm 5$

and put $m = 0$, we get

$y = 3$

$\therefore$ $y = 3$ and $y = -x + 5$

Q.9 Tangents are drawn from a point on the circle $x^2 + y^2 = 25$ to the ellipse $9x^2 + 16y^2 - 144 = 0$ then find the angle between the tangents.

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{2}$ (C) $\frac{\pi}{2}$ (D) $\frac{2\pi}{3}$

Sol. **[C]**

Given equation of ellipse is

$9x^2 + 16y^2 - 144 = 0$

$\Rightarrow 9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1 \quad \text{……..(i)}$$

Given equation of circle is

$x^2 + y^2 = 25$

which is the director circle of the ellipse, therefore tangents drawn makes angle $\frac{\pi}{2}$. Since drawn tangents are perpendicular to each other. So angle between the tangent is $\frac{\pi}{2}$.

Q.10 The points where the normals to the ellipse $x^2 + 3y^2 = 37$ are parallel to the line $6x - 5y = 2$ are -

(A) (4, 2) (– 5, – 2) (B) (5, 2) (– 5, – 3)

(C) (5, 2) (– 5, – 2) (D) None of these

Sol. **[C]**

Given ellipse is

$x^2 + 3y^2 = 27$

on differentiation

$$2x + 6y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x}{6y}=-\frac{x}{3y} = \text{slopes of tangent}$$

$$\therefore \text{Slope of normal} = \frac{3y}{x}$$

which is parallel to the line 6x – 5y = 2

$\therefore \frac{3y}{x} = \frac{6}{5}$

$\Rightarrow 6x - 15y = 0$

$\Rightarrow 2x - 5y = 0$

$\Rightarrow \frac{x}{y} = \frac{5}{2}$ ……..(i)

option (C) satisfies (i), therefore (C) is correct answer

Q.11 The distance of a point on the ellipse $x^2 + 3y^2 = 6$ from the centre is 2. Find the eccentric angle of the point in the first quadrant. Also find the equation of the tangent at the point.

(A) $\frac{\pi}{4}$, $x + \sqrt{3}\,y - 2\sqrt{3} = 0$

(B) $\frac{3\pi}{4}$, $x + \sqrt{3}\,y - 2\sqrt{3} = 0$

(C) $\frac{5\pi}{4}$, $x + 2\sqrt{3}\,y - 2\sqrt{3} = 0$

(D) None of these

Sol. **[A]**

Given ellipse is

$x^2 + 3y^2 = 6$

$\Rightarrow \frac{x^2}{6} + \frac{y^2}{2} = 1$

Any point on this ellipse is ($\sqrt{6}$ cosθ, $\sqrt{2}$ sinθ)

its distance from centre (0, 0) is 2 given

$\sqrt{6\cos^2\theta + 2\sin^2\theta} = 2$

$\Rightarrow 6\cos^2\theta + 2\sin^2\theta = 4$

$\Rightarrow 4\cos^2\theta = 2$

$\Rightarrow \cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

Now given point is $\left(\sqrt{6}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\right)$

$= (\sqrt{3}, 1)$

$\therefore$ equation of tangent is given by

$\frac{x.\sqrt{3}}{6} + \frac{y.1}{2} = 1$

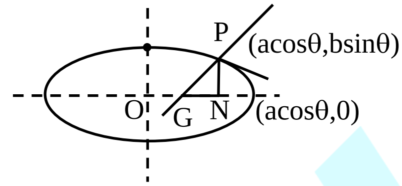
$\Rightarrow \sqrt{3}\,.\,x + 3y = 6$ or $\frac{\sqrt{3}x}{\sqrt{3}} + \frac{3}{\sqrt{3}}y = \frac{6}{\sqrt{3}}$

$\Rightarrow x + \sqrt{3}y - 2\sqrt{3} = 0$

Q.12 The normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at a point P (x_1, y_1) on it, meets the x-axis in G. PN is perpendicular to OX, where O is origin. Then value of λ (OG)/ λ(ON) is -

(A) e (B) e^2 (C) e^3 (D) $e^2 – 1$

Sol. **[B]**



Normal to the ellipse :

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2e^2$

$G\left(\frac{\cos\theta(a^2.e^2)}{a}, 0\right)$

N(a cos θ, 0)

$\frac{\lambda(OG)}{\lambda(ON)} = \frac{ae^2\cos\theta}{a\cos\theta} = e^2$

Q.13 The tangent and the normal at a point P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet its major axis in T and T' so that TT' = a then $e^2\cos^2\theta + \cos\theta$ (where e is eccentricity of the ellipse) is equal to -

(A) 1 (B) 2 (C) 3 (D) 4

Sol. **[A]**

Let the electric angle of the point P be θ so that tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ……(1)

and normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ ……(2)

There lines meet the major axis y = 0 in points T and T^1 such that $TT^1 = a$

So T is $\left(\frac{a}{\cos\theta}, 0\right)$ and T^1 is $\left(\frac{(a^2 - b^2)\cos\theta}{a}, 0\right)$

$\therefore TT^1$ is $\frac{a}{\cos\theta} - \frac{(a^2 - b^2)\cos\theta}{a} = a$ (given)

or $a^2 - a^2e^2\cos^2\theta = a^2\cos^2\theta$

or $1 - e^2\cos^2\theta = \cos\theta$

or $e^2\cos^2\theta + \cos\theta - 1 = 0$

or $e^2\cos^2\theta + \cos\theta = 1$

Q.14 P and Q are corresponding points on an ellipse and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Value of CR is –

(A) a + b (B) 2(a + b)

(C) 2a + b (D) a + 2b

Sol. [A]

Let P be (a cosθ, b sinθ) so that θ is (a cosθ, a sinθ) which lies on the auxiliary circle, C is (0, 0) equation of CQ is

$$y - 0 = \frac{a\sin\theta}{a\cos\theta}(x - 0)$$

or $y = x\tan\theta$ ....(i)

equation of normal at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad\text{........(ii)}$$

Both (i) and (ii) intersect at R.

on solving (i) and (ii), we get

$$\frac{ax}{\cos\theta} - \frac{b}{\sin\theta} . x . \frac{\sin\theta}{\cos\theta} = a^2 - b^2$$

or $\frac{x}{\cos\theta}(a - b) = a^2 - b^2$

$\therefore$ x = (a + b) cos θ

and y = x tanθ = (a + b) sin θ

Also R is [(a + b) cosθ, (a + b) sinθ] and C is (0, 0)

$\therefore CR^2 = (a + b)^2 (\cos^2\theta + \sin^2\theta)$

$\therefore CR^2 = (a + b)^2$

$\therefore$ CR = (a + b)

Questions based on — Equation of chord when mid point given, chord of contact & pair of tangent

Q.15 Angle between tangents drawn from any point on the circle $x^2 + y^2 = (a + b)^2$ to the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = (a + b)$ is-

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\tan^{-1}\frac{1}{2}$ (D) $\sin^{-1} 1$

Sol. [D]

Ellipse is $\frac{x^2}{a} + \frac{y^2}{b} = (a + b)$

$$\Rightarrow \frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$$

its director circle will be

$x^2 + y^2 = a(a + b) + b(a + b)$

$\Rightarrow x^2 + y^2 = (a + b)(a + b)$

$\Rightarrow x^2 + y^2 = (a + b)^2$

which is the given circle.

Hence angle between tangents drawn from any point on the circle $x^2 + y^2 = (a + b)^2$ to the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = a + b$ is π/2 which is equal to $\sin^{-1}$ (1).

Q.16 A line meets ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ at A and B and tangents to ellipse at A and B are perpendicular to each other, then point of intersection of tangents must be-

(A) (1, 2) (B) (2, 3)

(C) (1, 4) (D) None of these

Sol. [A]

Ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

its director circle is

$x^2 + y^2 = 4 + 1$

$\Rightarrow x^2 + y^2 = 5$

only option (A) satisfies this.

Q.17 Find the middle point of the chord intercepted on the line 2x – y + 3 = 0 by the ellipse $\frac{x^2}{10} + \frac{y^2}{6} = 1$.

(A) $\left(\frac{-30}{23}, \frac{9}{23}\right)$ (B) $\left(\frac{30}{23}, \frac{9}{23}\right)$

(C) $\left(\frac{30}{24}, \frac{9}{24}\right)$ (D) None of these

Sol. [A]

Ellipse : $\frac{x^2}{10} + \frac{y^2}{6} = 1$ ....(1)

line : 2x – y + 3 = 0 ....(2)

from (2), we can write

y = 2x + 3

put in (i) we get

$$\frac{x^2}{10} + \frac{(2x+3)^2}{6} = 1$$

$\Rightarrow 3x^2 + 5(4x^2 + 9 + 12x) = 30$

$\Rightarrow 3x^2 + 20x^2 + 60x + 15 = 0$

$\Rightarrow 23x^2 + 60x + 15 = 0$

Let its roots are x_1 & x_2 then

$x_1 + x_2 = -\frac{60}{23}$

$\therefore \frac{x_1 + x_2}{2} = -\frac{30}{23}$

similarly, we can find $\frac{y_1 + y_2}{2} = \frac{9}{23}$

Hence mid point of the chord is

$\left(\frac{-30}{23}, \frac{9}{23}\right)$

Q.18 Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points where it is intersected by the line $\lambda x + my + n = 0$. Find the point of intersection of tangents at these points.

(A) $\frac{-a^2}{n}, \frac{-b^2m}{n}$ (B) $\frac{-a^2\lambda}{n}, \frac{-b^2m}{n}$

(C) $\frac{-a^2\lambda}{n}, \frac{-b^2}{n}$ (D) None of these

Sol. **[B]**

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ……..(i)

line : $\lambda x + my + n = 0$ ……..(ii)

Let line (ii) intersect the ellipse at A & B points. tangents drawn from A and B. Let intersect at Q point. then locus of point Q is a straight line called polar and the point is called polar pole of the polar.

∴ point of intersection will be the pole.

we know that pole of a given line $\lambda x + my + n = 0$

w.r.t. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$\left(\frac{-a^2\lambda}{n}, \frac{-b^2m}{n}\right)$

Alter :

Let (x_1, y_1) be the point of intersection of tangent to the ellipse where it is intersected by the line $\lambda x + my + n = 0$ then, the equation of the chord of contact of tangents is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{……..(i)}$$

clearly, $\lambda x + my + n = 0$ and (i) represents the same line.

$$\therefore \frac{x_1}{a^2\lambda} = \frac{y_1}{b^2m} = \frac{1}{-n}$$

$$\Rightarrow x_1 = \frac{-a^2\lambda}{n} \text{ \& } y_1 = \frac{-b^2m}{n}$$

$$\therefore \left(\frac{-a^2\lambda}{n}, \frac{-b^2m}{n}\right)$$

Questions based on Pole & Polar & conjugate diameter

Q.19 If the polar with respect to $y^2 = 4x$ touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, the locus of its pole is-

(A) $\frac{x^2}{\alpha^2} - \frac{y^2}{(4\alpha^2/\beta^2)} = 1$

(B) $\frac{x^2}{\alpha^2} + \frac{\beta^2 y^2}{4\alpha^2} = 1$

(C) $\alpha^2x^2 + \beta^2y^2 = 1$

(D) None of these

Sol. **[A]**

Let P(h, k) be the pole, then the equation of the polar is

$ky = 2a(x + h)$

$\Rightarrow y = \frac{2a}{k}x + \frac{2ah}{k}$

This line touches the ellipse

$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

So, $\left(\frac{2ah}{k}\right)^2 = \alpha^2\left(\frac{2a}{k}\right)^2 + \beta^2$

$\Rightarrow 4a^2h^2 = 4a^2\alpha^2 + k^2\beta^2$

So, locus of (h, k) is

$4a^2x^2 = 4a^2\alpha^2 + \beta^2y^2$

$$\Rightarrow \frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1 \quad \text{… (1)}$$

∵ $y^2 = 4x$ is given parabola.

Here $4a = 4 \Rightarrow a = 1$

Put a = 1 in (1), we get

$$\frac{x^2}{\alpha^2} - \frac{y^2}{4\alpha^2/\beta^2} = 1$$

Q. 20 The eccentricity of an ellipse whose pair of a conjugate diameter are $y = x$ and $3y = -2x$ is-

(A) 2/3 (B) 1/3

(C) $1/\sqrt{3}$ (D) None of these

Sol. **[C]**

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since y = x and 3y = –2x is a pair of conjugate diameters, therefore

$m_1m_2 = -\frac{b^2}{a^2}$

$\Rightarrow (1)\left(-\frac{2}{3}\right) = -\frac{b^2}{a^2}$

$\Rightarrow 2a^2 = 3b^2$

$\Rightarrow 2a^2 = 3a^2(1 - e^2)$

$\Rightarrow 2 = 3(1 - e^2)$

$\Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$

Q.21 For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of the diameter conjugate to ax – by = 0 is-

(A) bx + ay = 0 (B) bx – ay = 0

(C) $a^3y + b^3x = 0$ (D) $a^3y - b^3x = 0$

Sol. **[C]**

Let $y = m_2x$ be the conjugate diameter.

Here $m_1 = \frac{a}{b}$

$\therefore m_1m_2 = -\frac{b^2}{a^2}$

$\Rightarrow \frac{a}{b} . m_2 = -\frac{b^2}{a^2} \Rightarrow m_2 = -\frac{b^3}{a^3}$

$\therefore$ equation of conjugate diameter is

$y = -\frac{b^3}{a^3}x$

$\Rightarrow a^3y + b^3x = 0$

True or false type questions

Q.22 In a triangle ABC if B (–2, 0), C (2, 0) and AB + AC = 6 then vertex A moves on an ellipse whose eccentricity is $\frac{1}{2}$.

Sol. Let vertex A is (x, y) then accordingly to

AB + AC = 6

$\Rightarrow \sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$

$\Rightarrow \sqrt{(x+2)^2 + y^2} = 6 - \sqrt{(x-2)^2 + y^2}$

$\Rightarrow$ on squaring both sides, we get

$\Rightarrow (x + 2)^2 + y^2 = 36 + (x - 2)^2 + y^2 - 12\sqrt{(x-2)^2 + y^2}$

$\Rightarrow x^2 + 4x + y + y^2 = 36 + x^2 - 4x + y + y^2 - 12\sqrt{(x-2)^2 + y^2}$

$\Rightarrow 4x + 4x - 36 = -12\sqrt{(x-2)^2 + y^2}$

$\Rightarrow 2x - 9 = -3\sqrt{(x-2)^2 + y^2}$

$\Rightarrow (2x - 9)^2 = 9\{(x - 2)^2 + y^2\}$

$\Rightarrow 4x^2 + 81 - 36x = 9x^2 - 36x + 36 + 9y^2$

$\Rightarrow 5x^2 + 9y^2 = 45$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$

$\therefore$ electricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

$\therefore e = \frac{2}{3}$

Therefore given statement is False

Q.23 For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b, sum of squares of perpendicular from (0, ae) and (0, –ae) on any tangent to it is $2a^2$.

Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b)

Let its tangent is

$y = mx + \sqrt{a^2m^2 + b^2}$

$\Rightarrow mx - y + \sqrt{a^2m^2 + b^2} = 0$

Sum of perpendiculars drawn from (0, ae) & (0, – ae) is given by

$\frac{-ae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + 1}} + \frac{ae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + 1}}$

$= \frac{2\sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + 1}}$

Here m is unknown, so this one is not equal to $2a^2$

Therefore the given statement is false

Alter :

Let equation of tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(1)

Sum of perpendicular drawn from (0, ae) & (0, – ae) is given by

$\left|\frac{\frac{ae\sin\theta}{b} - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} + \frac{\frac{-ae\sin\theta}{b} - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$

$$\Rightarrow \left| \frac{-2}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} \right|$$

$$\Rightarrow \frac{+2ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{2ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Q.24 Tangent at point $\left(\frac{1}{5}, \frac{-2}{\sqrt{5}}\right)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $\left(-1, \frac{4}{\sqrt{5}}\right)$ on the ellipse $4x^2 + 5y^2 = 20$.

Sol. Given parabola is $y^2 = 4x$

Tangent drawn from $\left(\frac{1}{5}, \frac{-2}{\sqrt{5}}\right)$ to this parabola is given by

$yy_1 = 2a(x + x_1)$

$\Rightarrow y \cdot \left(\frac{-2}{\sqrt{5}}\right) = 2.1. \left(x + \frac{1}{5}\right)$

$\Rightarrow \frac{-2y}{\sqrt{5}} = 2x + \frac{2}{5}$

$\Rightarrow \frac{-2y}{\sqrt{5}} = \frac{10x+2}{5}$

$\Rightarrow -2\sqrt{5}\,y = 10x + 2$

$\Rightarrow 10x + 2\sqrt{5}\,y + 2 = 0$

$\Rightarrow 5x + \sqrt{5}\,y + 1 = 0$ ....(i)

Given equation of ellipse is

$4x^2 + 5y^2 = 20$

$\Rightarrow \frac{x^2}{5} + \frac{y^2}{4} = 1$

Normal drawn from $\left(-1, \frac{4}{\sqrt{5}}\right)$ to the ellipse is given by

$$\frac{5(x-(-1))}{-1} = \frac{4\left(y - \frac{4}{\sqrt{5}}\right)}{4/\sqrt{5}}$$

$\Rightarrow -5(x + 1) = \frac{\sqrt{5}(\sqrt{5}y - 4)}{\sqrt{5}}$

$\Rightarrow -5x - 5 = \sqrt{5}y - 4$

$\Rightarrow 5x + \sqrt{5}y + 1 = 0$ ......(ii)

equation (i) & (ii) are the save line

$\therefore$ Given statement is True.

Q.25 For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b. maximum distance of any normal from its centre is a + b.

Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b)

its centre is (0, 0)

equation of normal to the ellipse is given by

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

its distance from it centre (0, 0) is

$$\left| \frac{-(a^2-b^2)}{\sqrt{\frac{a^2}{\cos^2\theta}+\frac{b^2}{\sin^2\theta}}} \right|$$

$$= \frac{|a^2-b^2|}{\sqrt{\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin^2\theta\cos^2\theta}}}$$

$$= \frac{|a^2-b^2|\,|\sin\theta\cos\theta|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= \frac{1}{2} \cdot \frac{(a^2-b^2).\sin^2\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

For max. distance put $\sin 2\theta = 1$, $\sin\theta = 1$

$$= \frac{1}{2} \cdot \frac{(a^2-b^2)}{a}$$

Hence it is False.

➢ Fill in the blanks type questions

Q.26 Line x = 1 meets ellipse $\frac{x^2}{4} + y^2 = 1$ at A and B, then angle subtended by AB at centre of ellipse is ....

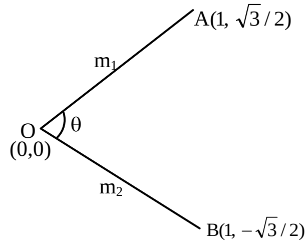
Sol. Put x = 1 in $\frac{x^2}{4} + y^2 = 1$

$\Rightarrow y^2 = \frac{3}{4}$

$\Rightarrow y = \pm \frac{\sqrt{3}}{2}$

$\therefore$ A $\left(1, \frac{\sqrt{3}}{2}\right)$ & B $\left(1, \frac{-\sqrt{3}}{2}\right)$

since centre of ellipse is (0, 0)



slope of OA = $m_1 = \frac{-\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$

and slope of OB = $\frac{-\sqrt{3}}{2}$

$\therefore \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}{1 - \frac{3}{4}}$

$\Rightarrow \tan\theta = \frac{\sqrt{3}}{1/4} = 4\sqrt{3}$

$\therefore \theta = \tan^{-1}(4\sqrt{3})$

Q.27 The point of intersection of $\frac{x^2}{4} + y^2 = 1$ and $x^2 + y^2 = 2$ lying in first quadrant is ....

Sol. $\frac{x^2}{4} + y^2 = 1$ … (1)

$x^2 + y^2 = 2$ … (2)

from (1) and (2), we get

$\frac{x^2}{4} + 2 - x^2 = 1 \Rightarrow \frac{x^2}{4} - x^2 = -1 \Rightarrow -3x^2 = -4$

$\Rightarrow 3x^2 = 4 \quad \Rightarrow x = \frac{2}{\sqrt{3}}$

$\therefore y^2 = 2 - \frac{4}{3} = \frac{2}{3}$

$\therefore y = \sqrt{2/3}$

$\therefore$ point of intersection in first quadrants $\left(\frac{2}{\sqrt{3}}, \sqrt{2/3}\right)$

Q.28 If line y = 3x + 10 touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{8^2} = 1$ then value of a is equal to ....

Sol. The line y = 3x + 10 touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{8} = 1$

If $c = \sqrt{a^2m^2 + b^2}$

i.e. $10 = \sqrt{a^2(3)^2 + 8}$

$\Rightarrow 100 = 9a^2 + 8$

$\Rightarrow 9a^2 = 92$

$\Rightarrow a = \pm\sqrt{\frac{92}{9}}$

Corrections:

Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{8^2} = 1$

In this case,

$c = \sqrt{a^2m^2 + b^2}$

$10 = \sqrt{a^2.9 + 8^2}$

$\Rightarrow 100 = 9a^2 + 64$

$\Rightarrow 9a^2 = 36$

$\Rightarrow a^2 = 4$

$\Rightarrow a = 2$

$\therefore a = 2$

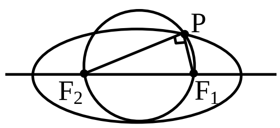
EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is -

(A) 11 (B) 12 (C) 13 (D) 15

Sol. **[C]**



$PF_1 + PF_2 = 17 = 2a$

Area of triangle = 30

$\frac{1}{2}\ .\ PF_1\ .\ PF_2 = 30$

$\frac{1}{2}\ (PF_1)\ (17\ .\ PF_1) = 30$

$PF_2 = 5,\ PF_1 = 12$

By pythagorus theorem

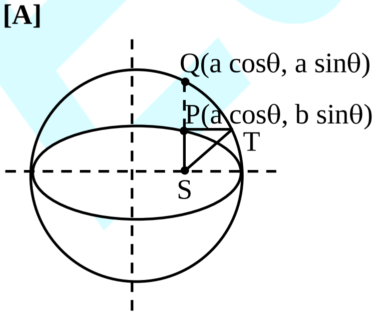
$F_1F_2 = 13$

Q.2 Q is a point on the auxiliary circle corresponding to the point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. If T is the foot of the perpendicular dropped from the focus S onto the tangent to the auxiliary circle at Q then the Δ SPT is -

(A) isosceles (B) equilateral

(C) right angled (D) right isosceles

Sol. **[A]**



P(a cosθ, b sinθ)

$SP = e\left|\frac{a}{e} - a\cos\theta\right| = |\,a - ae\cos\theta\,|$

tangent at Q

$x\cos\theta + y\sin\theta = a$

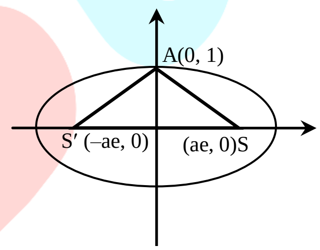
$ST = \left|\frac{ae\cos\theta + 0 - a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a\,|\,e\cos\theta - 1\,|$

ST = SP

Q.3 If S and S′ are foci and A be one end of minor axis of ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$, then area of $\Delta SAS'$ is-

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) None

Sol. **[B]**



$\because ae = \sqrt{a^2 - b^2}$

$\therefore ae = \sqrt{4-1} = \sqrt{3}$

$\therefore S = (\sqrt{3}, 0)\ \&\ S' = (-\sqrt{3}, 0),\ A(0, 1)$

$\therefore$ area of Δ SAS′ is $\frac{1}{2}\begin{vmatrix}\sqrt{3} & -\sqrt{3} & 0\\ 0 & 0 & 1\\ 1 & 1 & 1\end{vmatrix}$

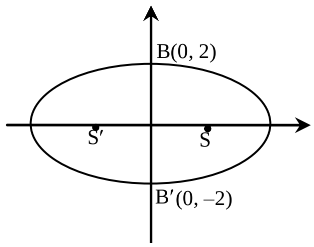
$= \frac{1}{2}(2\sqrt{3}) = \sqrt{3}$

Q.4 e $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is-

(A) 2 (B) 3 (C) $\sqrt{5}$ (D) none

Sol. **[B]**

$\frac{x^2}{9} + \frac{y^2}{4} = 1$



$e = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$

$a^2 = 9, b^2 = 4$

$\Rightarrow a = 3, b = 2$

$\therefore$ focus $S = (ae, 0) = \left(3.\frac{\sqrt{5}}{3}, 0\right) = (\sqrt{5}, 0)$

and $S' = (-ae, 0) = \left(-3.\frac{\sqrt{5}}{3}, 0\right) = (-\sqrt{5}, 0)$

ends of minor axis = B(0, 2) and B′(0, –2)

$\therefore SB = \sqrt{5+4} = 3$

$SB' = \sqrt{5+4} = 3$

$S'B = \sqrt{5+4} = 3$

$S'B' = \sqrt{5+4} = 3$

$\therefore$ 3

Q.5 The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ of eccentricity 'e' meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that -

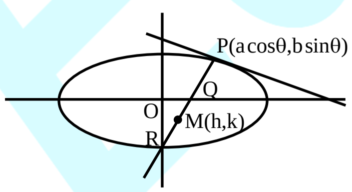
(A) e' is independent of e

(B) e' = 1

(C) e' = e

(D) e' = 1/e

Sol. **[C]**



Normal at P(a cosθ, b sinθ) is given by :

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad \text{.....(i)}$$

then, putting y = 0 in (i), we get ;

$$x = \frac{(a^2-b^2)\cos\theta}{a}$$

So, $Q \equiv \left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$

and, putting x = 0, we get ;

$R \equiv \left(0, \frac{-(a^2-b^2)\sin\theta}{b}\right)$

Let, M be the midpoint of QR, then ;

$M \equiv \left(\frac{(a^2-b^2)\cos\theta}{2a}, \frac{-(a^2-b^2)\sin\theta}{2b}\right)$

But, M ≡ (h, k)

Then, $h = \frac{(a^2-b^2)\cos\theta}{2a}$, $k = \frac{-(a^2-b^2)\sin\theta}{2b}$

$\Rightarrow \cos\theta = \frac{2ah}{(a^2-b^2)}$ .....(ii)

$\sin\theta = -\frac{2bk}{a^2-b^2}$ .....(iii)

squaring and adding (ii) and (iii), we get

$$\frac{4a^2h^2}{(a^2-b^2)^2} + \frac{4b^2k^2}{(a^2-b^2)^2} = 1$$

$$\Rightarrow \frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 1$$

[Putting h → x, k → y]

Now, $B^2 = A^2(1 - e'^2)$

$\Rightarrow e'^2 = \frac{A^2-B^2}{A^2}$

$$= 1 - \frac{B^2}{A^2} = 1 - \frac{\frac{(a^2-b^2)^2}{4b^2}}{\frac{(a^2-b^2)^2}{4a^2}}$$

$$= 1 - \frac{a^2}{b^2} = e^2$$

So, e' = e

Q.6 The locus of extremities of the latus rectum of the family of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is-

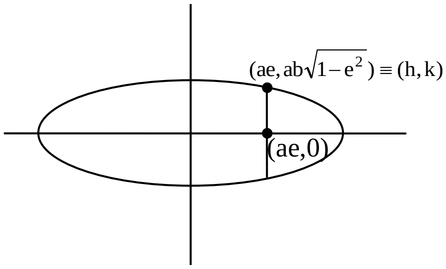
(A) $x^2 + ay = a^2$ (B) $x^2 - ay = b^2$

(C) $x^2 + y = a^2$ (D) $x^2 + ay = b^2$

Sol. **[A]**

$b^2x^2 + y^2 = a^2b^2$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{B^2} = 1$$



$$\frac{y^2}{B^2} = (1 - e^2) \qquad \begin{cases} \Theta \ B^2 = a^2(1-e^2) \\ a^2b^2 = a^2\sqrt{1-e^2} \\ b^2 = \sqrt{1-e^2} \end{cases}$$

$y^2 = B^2 (1 - e^2)$

$y = B\sqrt{1-e^2}$

$y = ab\ \sqrt{1-e^2}$

$y = ab^3$

$h = ae,\ k = ab^3$

$h = \sqrt{a^2 - B^2}\ ,\ k = b^3 a$

$\Rightarrow b^2 = \dfrac{k}{ab}$

$h^2 = a^2 - b^2a^2$

$h^2 = a^2 - \dfrac{k}{ab} . a^2 \Rightarrow ab\ h^2 = a^3b - k$

$bh^2 = a^2b - ak \Rightarrow abh^2 + k = a^3b$

$bh^2 + ak = a^2b$

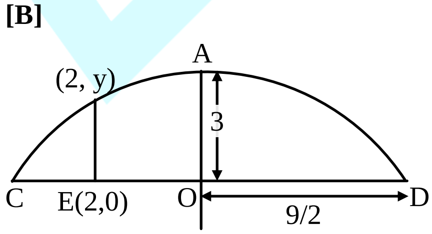
$\therefore$ locus is $bx^2 + ay = a^2b$

Q.7 An arc of a bridge is semi-elliptical with major axis horizontal. The length of the base is 9 meter and the highest part of the bridge is 3 meter from the horizontal. The best approximation of the Pillar 2 meter from the centre of the base is -

(A) 11/4 m (B) 8/3 m

(C) 7/2 m (D) 2 m

Sol. **[B]**



$a = \dfrac{9}{2}$, $b = 3$

then, $b^2 = a^2(1 - e^2)$

$\Rightarrow 9 = \dfrac{9^2}{4}\ (1 - e^2) \Rightarrow e = \dfrac{\sqrt{5}}{3}$

then, $\dfrac{4x^2}{81} + \dfrac{y^2}{9} = 1$

$\Rightarrow \dfrac{4\times(2)^2}{81} + \dfrac{y^2}{9} = 1$ [$\because$ when x = 2]

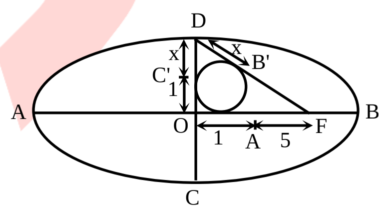
$\Rightarrow y^2 = \dfrac{65}{9}$

$\Rightarrow y \approx \dfrac{8}{3}$

Q.8 Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) is -

(A) 64 (B) 12 (C) 65 (D) 3

Sol. **[C]**



OF = 6

$\Rightarrow$ OA + AF = 6

$\Rightarrow$ 1 + AF = 6 [OA = r = 1]

$\Rightarrow$ AF = 5

Let, CD = DB = x

then, $OM^2 + OF^2 = MF^2$

$\Rightarrow (x + 1)^2 + 6^2 = (x + 5)^2$

by solving it, we get

$x = \dfrac{3}{2}$

then, OF = a [2MF = 2a]

x + 5 = a

$\Rightarrow \dfrac{3}{2} + 5 = a \Rightarrow a = \dfrac{13}{2}$

then, 2a = 13

and, 2(OD) = 2b

$\Rightarrow 2b = 2(1 + x) = 2 \times \frac{5}{2} = 5$

then, $(CD \times AB) = 2a \times 2b = 13 \times 5 = 65$

Q.9 The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having its centre (0, 3) is-

(A) 4 (B) 3 (C) $\sqrt{12}$ (D) 7/2

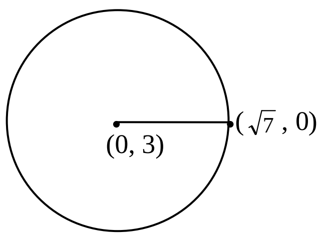
Sol. **[A]**

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

foci = $(\pm ae, 0)$

$ae = \sqrt{16-9} = \sqrt{7}$

$\therefore$ foci = $(\sqrt{7}, 0)$



radius of circle = $\sqrt{7+9} = 4$

$\therefore$ radius = 4

Q.10 The set of values of 'a' for which $(13x - 1)^2 + (13y - 2)^2 = a\,(5x + 12y - 1)^2$ represents an ellipse, is-

(A) $1 < a < 2$ (B) $0 < a < 1$

(C) $2 < a < 3$ (D) None of these

Sol. **[B]**

We have

$(13x - 1)^2 + (13y - 2)^2 = a\,(5x + 12y - 1)^2$ …(1)

Θ L.H.S. is +ve

$\therefore$ R.H.S. is also +ve then $a > 0$ … (2)

Again equation (1) can be written as

$(169-25a)x^2 + (169 - 144a)y^2 - 120\,axy + (\ldots)\,x + (\ldots)\,y + (5 - a) = 0$

It is an ellipse

$\therefore H^2 < AB$

then $3600\,a^2 < (169 - 25a)\,(169 - 144a)$ …(3)

$\Rightarrow a < 1$

from (2) and (3), we get $0 < a < 1$.

Q.11 If $\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$ then the chord joining two points θ_1 & θ_2 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will subtend a right angle at -

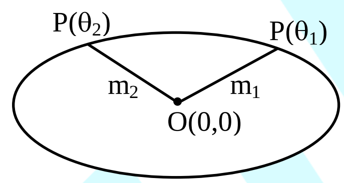
(A) focus

(B) centre

(C) end of the major axis

(D) end of the minor axis

Sol. **[B]**



$\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$

$m_1 = \frac{b\sin\theta_1}{a\cos\theta_1} = \frac{b}{a}\tan\theta_1$

$m_2 = \frac{b}{a}\tan\theta_2$

$m_1m_2 = \frac{b^2}{a^2}\tan\theta_1\tan\theta_2$

$= \frac{b^2}{a^2}\left(-\frac{a^2}{b^2}\right)$

$m_1m_2 = -1$

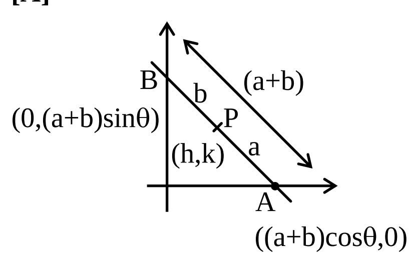
so these are $\perp^r$ lines so passes through centre (0, 0) angle subtend right angle at centre

Q.12 A line of fixed length (a + b) moves so that its ends are always on two fixed perpendicular straight lines. The locus of the point which divided this line into portions of lengths a & b is -

(A) an ellipse (B) an hyperbola

(C) a circle (D) none of these

Sol. **[A]**



$A[(a + b)\cos\theta, 0]$

B[0, (a + b) sin θ]

$$\frac{b(a+b)\cos\theta}{(a+b)} = h$$

$$\cos\theta = \frac{h}{b} \qquad ....(i)$$

$$\frac{a(a+b)\sin\theta}{a+b} = k$$

$$\sin\theta = \frac{k}{a} \qquad ....(ii)$$

squaring & adding equation (i) & (ii)

$$\frac{h^2}{b^2} + \frac{k^2}{a^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

this is the equation of ellipse

Q.13 The eccentricity of the ellipse which meets the straight line $\frac{x}{7} + \frac{y}{2} = 1$ on the axis of x and the straight line $\frac{x}{3} - \frac{y}{5} = 1$ on the axis of y and whose axis lie along the axes of coordinates, is

(A) $\frac{3\sqrt{2}}{7}$ (B) $\frac{2\sqrt{3}}{7}$ (C) $\frac{\sqrt{3}}{7}$ (D) $\frac{2\sqrt{6}}{7}$

Sol. **[D]**

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that it passes through (7, 0) and (0, –5).

Therefore $a^2 = 49$ and $b^2 = 25$

∴ Eccentricity of the ellipse is

$$e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 - \frac{25}{49}}$$

$$e = \sqrt{\frac{49-25}{49}}$$

$$e = \sqrt{\frac{24}{49}}$$

$$e = \frac{2\sqrt{6}}{7}$$

Q.14 Coordinates of the vertices B and C of a triangle ABC are (2,0) and (8, 0) respectively. The vertex A is varying in such a way that $4\tan\frac{B}{2} \cdot \tan\frac{C}{2} = 1$. Then locus of A is -

(A) $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$ (B) $\frac{(x-5)^2}{16} + \frac{y^2}{25} = 1$

(C) $\frac{(x-5)^2}{25} + \frac{y^2}{9} = 1$ (D) $\frac{(x-5)^2}{9} + \frac{y^2}{25} = 1$

Sol. [A]

Q.15 The equation of the circle passing through the points of intersection of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is -

(A) $x^2 + y^2 = a^2$ (B) $x^2 + y^2 = b^2$

(C) $x^2 + y^2 = \frac{a^2b^2}{a^2+b^2}$ (D) $x^2 + y^2 = \frac{2a^2b^2}{a^2+b^2}$

Sol. **[D]**

Equation of curve passing through the point of intersection of ellipse $E_1 = 0$ and $E_2 = 0$ is

$$E_1 + \lambda E_2 = 0$$

using λ = 1, the equation of curve is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + 1\left(\frac{x^2}{b^2} + \frac{y^2}{a^2} - 1\right) = 0$$

$$\Rightarrow x^2 + y^2 = \frac{2a^2b^2}{a^2+b^2}$$

Which is the required circle.

Q.16 If the axes of an ellipse coincide with the co-ordinate axes and it passes through the point (4, – 1) and touches the line x + 4y – 10 = 0 then its equation is -

(A) $\frac{x^2}{80} + \frac{y^2}{5/4} = 1$ (B) $\frac{x^2}{20} - \frac{y^2}{5} = 1$

(C) $\frac{x^2}{100} + \frac{y^2}{5} = 1$ (D) None of these

Sol. **[A]**

Let equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

it passes through (4, –1)

$$\Rightarrow \frac{16}{a^2} + \frac{1}{b^2} = 1 \qquad ...(1)$$

x + 4y – 10 = 0 is tangent of ellipse.

$\therefore\ y = -\dfrac{x}{4} + \dfrac{5}{2}$

Here $m = -\dfrac{1}{4}$, $c = \dfrac{5}{2}$

$\therefore\ c = \pm\sqrt{a^2m^2 + b^2}$

$\dfrac{5}{2} = \pm\sqrt{a^2.1/16 + b^2}$

$\dfrac{25}{4} = \dfrac{a^2}{16} + b^2$ …(2)

from (1), $\dfrac{16}{a^2} = 1 - \dfrac{1}{b^2} = \dfrac{b^2-1}{b^2}$

$\therefore\ \dfrac{a^2}{16} = \dfrac{b^2}{b^2-1}$ … (3)

$\therefore\ \dfrac{b^2}{b^2-1} + b^2 = \dfrac{25}{4}$

$\Rightarrow 4(b^2 + b^4 - b^2) = 25\,(b^2 - 1)$

$\Rightarrow 4b^4 - 25b^2 + 25 = 0$

$\Rightarrow 4b^4 - 20b^2 - 5b^2 + 25 = 0$

$\Rightarrow 4b^2\,(b^2 - 5) - 5\,(b^2 - 5) = 0$

$\therefore\ b^2 = 5$ or $b^2 = 5/4$

$\dfrac{25}{4} = \dfrac{a^2}{16} + \dfrac{5}{4} \Rightarrow \dfrac{a^2}{16} = 5 \Rightarrow a^2 = 80$

$\therefore$ equation of ellipse is $\dfrac{x^2}{80} + \dfrac{y^2}{5/4} = 1$

Q.17 The condition that the line $\lambda x + my = n$ may be a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

(A) $\dfrac{n^2}{(a^2-b^2)^2}\left(\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{m^2}\right) = 1$

(B) $\dfrac{n^2}{(a^2+b^2)^2}\left(\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{m^2}\right) = 1$

(C) $\dfrac{n^2}{(a^2-b^2)^2}\left(\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{m^2}\right) = 2$

(D) None of these

Sol. **[A]**

Equation of any normal to the ellipse is

$ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2$ …(1)

given line $\lambda x + my = -n$ …(2)

on comparing, we get

$\dfrac{\lambda}{a\sec\phi} = \dfrac{m}{-b\,\text{cosec}\,\phi} = \dfrac{-n}{a^2 - b^2}$

$\Rightarrow \cos\phi = \dfrac{-an}{\lambda(a^2-b^2)}$ & $\sin\phi = \dfrac{bn}{m(a^2-b^2)}$

$\Rightarrow 1 = \dfrac{a^2n^2}{\lambda^2(a^2-b^2)^2} + \dfrac{b^2n^2}{m^2(a^2-b^2)^2}$

Hence required condition will be

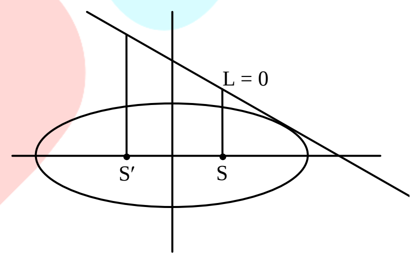
$\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2-b^2)^2}{n^2}$

$\Rightarrow \dfrac{n^2}{(a^2-b^2)^2}\left(\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{m^2}\right) = 1$

Q.18 Let L = 0 is a tangent to ellipse $\dfrac{(x-1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$ and S, S′ be its foci. If length of perpendicular from S on L = 0 is 2 then length of perpendicular from S′ on L = 0 is

(A) 3 (B) $\sqrt{3}$ (C) 2 (D) none

Sol. **[C]**



$\dfrac{(x-1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$ is given ellipse.

L = 0 is a tangent to this ellipse.

$\therefore$ its foci are $(1 \pm ae, 2)$

$ae = \sqrt{9-4} = \sqrt{5}$

$\therefore$ focus $(1\pm\sqrt{5}, 2)$

$\therefore\ S \equiv (1 + \sqrt{5}, 2)$ and $S' \equiv (1 - \sqrt{5}, 2)$

Any point on the ellipse = $(1 + 3\cos\theta, 2 + 2\sin\theta)$

Equation of tangent

$\dfrac{(x-1)(1+3\cos\theta)}{9} + \dfrac{(y-2)(2+2\sin\theta)}{4} - 1 = 0$

$\Rightarrow$ Length of perpendicular from S $(1+\sqrt{5}, 2)$ & $S'(1-\sqrt{5}, 2)$ is

$\dfrac{\left|\dfrac{(\sqrt{5})(1+3\cos\theta)-1}{9}\right|}{\sqrt{()^2 + ()^2}}$ & $\dfrac{\left|\dfrac{(-\sqrt{5})(1+3\cos\theta)-1}{9}\right|}{\sqrt{()^2 + ()^2}}$

$\therefore$ same length = 2

Part-B **One or more than one correct answer type questions**

Q.19 If ax + by = 1, $cx^2 + dy^2 = 1$ have only one solution, then:

(A) $\frac{a^2}{c} + \frac{b^2}{d} = 1$ (B) x = a/c

(C) y = b/d (D) $a^2 + b^2 = 1$

Sol. **[A,B,C]**

We have

ax + by = 1

$\Rightarrow y = \frac{1-ax}{b}$

putting this volume in the second equation, we get

$cx^2 + \frac{d}{b^2}(1 - ax^2) = 1$

$\Rightarrow (b^2c + a^2d)\,x^2 - 2ad\,x + d - b^2 = 0$ …. (1)

This quadratic equation will have equal roots if

$D = 4a^2d^2 - 4(b^2c + a^2d)(d - b^2) = 0$

$\Rightarrow a^2d^2 + (b^2c + a^2d)b^2 - b^2cd - a^2d^2 = 0$

$\Rightarrow b^2[b^2c + a^2d - cd] = 0$

$\Rightarrow b^2c + a^2d = cd$

$\Rightarrow \frac{b^2}{d} + \frac{a^2}{c} = 1$

Also, in this case

$x = \frac{2ad}{2(b^2c + a^2d)} = \frac{ad}{cd} = \frac{a}{c}$

$y = \frac{1-ax}{b} = \frac{1}{b}\left(1 - \frac{a^2}{c}\right)$

$= \frac{1}{b} \cdot \frac{b^2}{d} = \frac{b}{d}$

Q.20 A tangent to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T′. The circle on TT′ as diameter passes through the point -

(A) $(-\sqrt{5}, 0)$ (B) $(\sqrt{5}, 0)$

(C) (0, 0) (D) (3, 2)

Sol. **[A,B]**

We know that any tangent to the ellipse is cut by the tangents at the ends of the major axis in T and T′, then the circle TT′ as diameter will pass through the foci.

Given ellipse is

$4x^2 + 9y^2 = 36$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

its foci are (±ae, 0)

Here $ae = \sqrt{9-4} = \sqrt{5}$

$\therefore$ foci are $(\pm\sqrt{5}, 0)$

i.e. $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$

Q.21 The equations of the common tangents to the ellipse, $x^2 + 4y^2 = 8$ & the parabola $y^2 = 4x$ are

(A) 2y – x = 4 (B) 2y + x = 4

(C) 2y + x + 4 = 0 (D) 2y + x = 0

Sol. **[A,C]**

Tangent to ellipse : $\frac{x^2}{8} + \frac{y^2}{2} = 1$

$\frac{x}{2\sqrt{2}}\cos\theta + \frac{y}{\sqrt{2}}\sin\theta = 1$ …..(i)

tangent to hyperbola : $y^2 = 4x$

$y = mx + \frac{1}{m}$

$mx - y = -\frac{1}{m}$ …..(ii)

Comparing (i) & (ii)

$\frac{\cos\theta}{m\,2\sqrt{2}} = -\frac{\sin\theta}{\sqrt{2}} = -m$

$\cos\theta = -2\sqrt{2}\,m^2$, $\sin\theta = \sqrt{2}m$

$8m^4 + 2m^2 = 1$

$8m^4 + 4m^2 - 2m^2 - 1 = 0$

$m = \frac{1}{2}, -\frac{1}{2}$

common tangent

x – 2y + 4 = 0

2y + x + 4 = 0

Part-C **Assertion-Reason type questions**

The following questions consist of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true.

Q.22 **Assertion :** The angle between ellipse $\frac{x^2}{4} + y^2 = 1$ and circle $x^2 + y^2 = 2$ is θ given by $\tan\theta = \frac{1}{\sqrt{2}}$.

Reason : The angle between two intersected curves will be angle between tangents to both curve at their point of intersection.

Sol. [A]

Q.23 **Assertion :** If (α – 1) and (α – 2) are the length of perpendicular from foci of ellipse $\frac{x^2}{9} + \frac{y^2}{6} = 1$ at its any tangent then $\alpha^2 = 16$.

Reason : Product of perpendiculars from foci on any tangent to ellipse is constant and equal to (semi minor axis)^2.

Sol. **[A]**

Product of length of perpendicular from foci of ellipse = b^2

So, (α – 1) (α – 2) = 6

$\alpha^2 - 2\alpha - \alpha + 2 = 6$

$\alpha^2 - 3\alpha - 4 = 0$

$\alpha = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3+5}{2} = 4,\ -1$ [Referred]

$\alpha^2 = 16$

Q.24 **Assertion :** Let S_1 be the circle drawn on focal distance of an ellipse on diameter and S_2 is its auxiliary circle, then S_1 always touches S_2.

Reason : Perpendicular distance from centre of ellipse on the radical axis of S_1 and S_2 is equal to semi major axis.

Sol. **[A]**

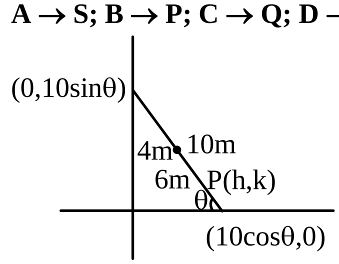
Part-D Column Matching type questions

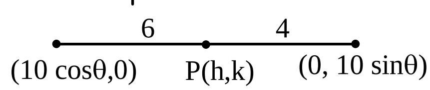
Q.25

Column I	Column II
(A) Stick of length 10 meter slides on co-ordinate axes, then locus of a point dividing this stick reckoning from x axis in the ratio of 6 : 4 is a curve whose eccentricity is e then e is equal to	(P) $\sqrt{6}$
(B) AA' is major axis of an ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ & P is a variable point on it then greatest area of triangle APA' is	(Q) $2\sqrt{7}$
(C) Distance between foci of the curve represented by the equation $x = 1 + 4\cos\theta,\ y = 2 + 3\sin\theta$ is	(R) $\frac{128}{3}$
(D) Tangents are Drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at end points of latus-rectum. The area of quadrilateral so formed is	(S) $\frac{\sqrt{5}}{3}$

Sol. **A → S; B → P; C → Q; D → R**

(A)





$h = \frac{40\cos\theta}{10} = 4\cos\theta,\quad k = \frac{60\sin\theta}{10} = 6\sin\theta$

$\cos\theta = \frac{h}{4} \qquad \sin\theta = \frac{k}{6}$

$\frac{h^2}{4^2} + \frac{k^2}{6^2} = 1$

Hence the locus is

$\frac{x^2}{16} + \frac{y^2}{36} = 1$

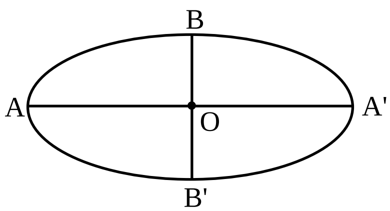
$16 = 36(1 - e^2)$

$$\frac{16}{36} = 1 - e^2$$

$$e^2 = \frac{20}{36}$$

$$e = \frac{\sqrt{20}}{6} = \frac{\sqrt{5}}{3}$$

(B)



$3x^2 + 2y^2 + 6x - 4y - 1 = 0$

$3(x^2 + 2x + 1) + 2(y^2 - 2y + 1) - 6 = 0$

$$\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$$

$a = \sqrt{3}$, $b = \sqrt{2}$

$AA' \Rightarrow 2a = 2\sqrt{3}$

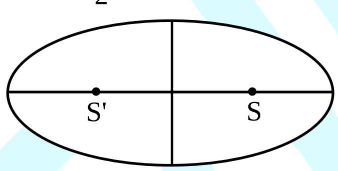
Base is same, so area will be maximum.
at maximum height.

max. height $\Rightarrow OB \Rightarrow b = \sqrt{2}$

$$\text{Area} = \frac{1}{2} \times AA' \times OB$$

$$= \frac{1}{2} \times 2\sqrt{3} \times \sqrt{2} = \sqrt{6}$$

(C)



$x = 1 + 4\cos\theta \Rightarrow \cos\theta = \frac{x-1}{4}$

$y = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{y-2}{3}$

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

$b^2 = a^2(1 - e^2)$

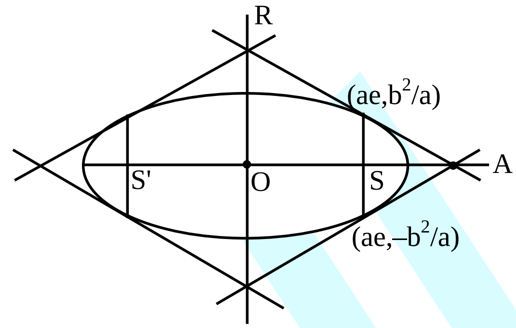
$9 = 16(1 - e^2)$

$$\frac{9}{16} = (1 - e^2)$$

$$e^2 = \frac{7}{16} \Rightarrow e = \frac{\sqrt{7}}{4}$$

$$2ae = 2 \times 4 \times \frac{\sqrt{7}}{4} = 2\sqrt{7}$$

(D)



Tangent at end of latus rectum

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$$

$$\frac{x\,ae}{a^2} + \frac{yb^2}{b^2 a} = 1$$

$$\frac{xe}{a} + \frac{y}{a} = 1 \quad ....(1)$$

and other will be

$$\frac{xe}{a} - \frac{y}{a} = 1 \quad .....(2)$$

Point R $\to x = 0$

$$\frac{xe}{a} + \frac{y}{a} = 1 \Rightarrow y = a$$

Point A $\to$ Add (1) & (2)

$$\frac{2xe}{a} = x \Rightarrow x = \frac{a}{e}$$

$y = 0$

Area of quadrilateral = 4 × Area of triangle AOR

$$= 4 \times \frac{1}{2} \times a \times \frac{a}{e}$$

$$= \frac{2a^2}{e} = \frac{2 \times 16 \times 4}{3}$$

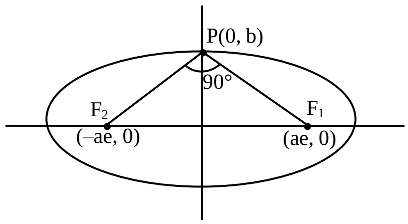
$$= \frac{128}{3}$$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 If the angle between the lines joining the foci of an ellipse to an extremity of the minor axis is 90°, find the eccentricity. Find also the equation of the ellipse if the major axis is $2\sqrt{2}$ units in length.

Sol. Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Since lines PF_1 and PF_2 are perpendicular lines. Let slope of PF_1 is m_1 and of PF_2 is m_2 then

$m_1m_2 = -1$

$\Rightarrow \frac{b}{-ae}\cdot\frac{b}{ae} = -1 \Rightarrow \frac{b^2}{a^2e^2} = 1$

$\Rightarrow \frac{a^2(1-e^2)}{a^2e^2} = 1$ $\quad \left\{\begin{array}{l}\because a^2e^2 = a^2 - b^2\\ \therefore b^2 = a^2(1-e^2)\end{array}\right.$

$\Rightarrow 2e^2 = 1$ $\qquad \Rightarrow e = \frac{1}{\sqrt{2}}$ **Ans.**

Since length of major axis $= 2a = 2\sqrt{2}$

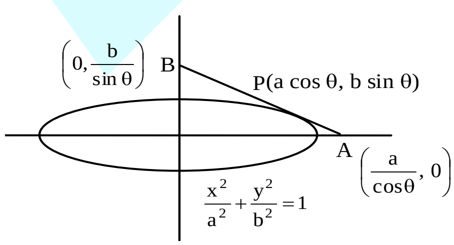
$\Rightarrow a = \sqrt{2}$

$\therefore b^2 = 2\left(1-\frac{1}{2}\right) = 1 \Rightarrow b = 1$

$\therefore$ equation of ellipse is $\frac{x^2}{2}+\frac{y^2}{1} = 1$ **Ans.**

Q.2 A tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ cuts the coordinate axes at A and B and touches the ellipse in the first quadrant at a point which bisects AB. Find the equation of the tangent.

Sol.



equation of tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

it cuts coordinate axes at A and B.

since P is mid-point of AB

$\therefore \frac{a}{2\cos\theta} = a\cos\theta$ and $\frac{b}{2\sin\theta} = b\sin\theta$

$\Rightarrow 2\cos^2\theta = 1$ and $2\sin^2\theta = 1$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ $\qquad \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\therefore$ equation of tangent is given by

$\frac{x}{a}\cdot\frac{1}{\sqrt{2}} + \frac{y}{b}\cdot\frac{1}{\sqrt{2}} = 1$

$\frac{x}{a} + \frac{y}{b} = \sqrt{2}$

Q.3 Prove that the straight line $\frac{ax}{3} + \frac{by}{4} = c \ \forall\ c > 0$ will be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, if $5c = a^2e^2$.

Sol. Equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

at (x_1, y_1) is given by $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

given normal is $\frac{ax}{3} + \frac{by}{4} = c$

$\therefore \frac{a^2}{x_1} = \frac{a}{3}$ and $\frac{b^2}{y_1} = \frac{-b}{4}$

Also $a^2 - b^2 = c$

$\therefore x_1 = 3a$ and $y_1 = -4b$

The line $\left(\frac{a}{3}\right)x + \left(\frac{b}{4}\right)y - c = 0$ is a normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ if

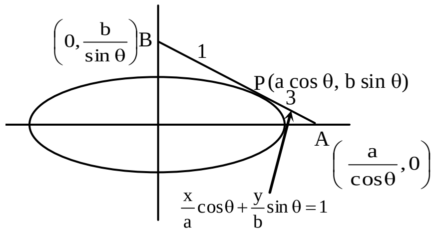
$\frac{a^2}{\left(\frac{a}{3}\right)^2} + \frac{b^2}{\left(\frac{b}{4}\right)^2} = \frac{(a^2-b^2)^2}{(-c)^2}$

$\Rightarrow 9 + 16 = \frac{(a^2e^2)^2}{c^2}$ $\Rightarrow 25c^2 = (a^2e^2)^2$

$\Rightarrow (a^2e^2)^2 = 25c^2 \Rightarrow a^2e^2 = 5c$

Q.4 A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the coordinates axes in A and B respectively. If P divides AB in ratio 3 : 1, find the equation of the tangent.

Sol.



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent at P is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

P divides AB in 3 : 1

$\therefore$ coordinates of P is given by

$$\left\{\frac{(1)\frac{a}{\cos\theta} + 3(0)}{4}, \frac{(1)(0) + 3\left(\frac{b}{\sin\theta}\right)}{4}\right\}$$

$$= \left(\frac{a}{4\cos\theta}, \frac{3b}{4\sin\theta}\right)$$

This must be equal to P(a cos θ, b sin θ)

$\therefore \frac{a}{4\cos\theta} = a\cos\theta$ and $\frac{3b}{4\sin\theta} = b\sin\theta$

$\Rightarrow 4\cos^2\theta = 1$ and $4\sin^2\theta = 3$

$\Rightarrow \cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ equation of tangent is

$$\frac{x}{a}.\frac{1}{2} + \frac{y}{b}.\frac{\sqrt{3}}{2} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y\sqrt{3}}{b} = 2 \Rightarrow bx + a\sqrt{3}\,y = 2ab$$

Q.5 Common tangents are drawn to the parabola $y^2 = 4x$ and the ellipse $3x^2 + 8y^2 = 48$ touching the parabola at A and B and the ellipse at C and D. Find the area of the quadrilateral.

Sol. Let $y = mx + \frac{1}{m}$ be a tangent to the parabola $y^2 = 4x$,

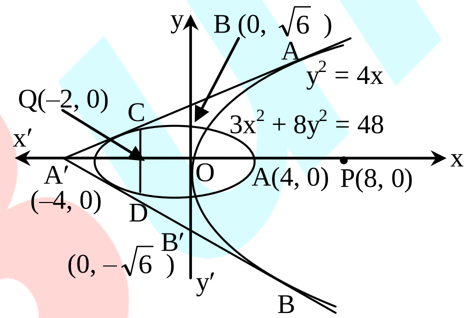
it will touch the ellipse $\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{6})^2} = 1$, if

$\frac{1}{m^2} = 16m^2 + 6$

$\Rightarrow 16m^4 + 6m^2 - 1 = 0$ [Use: $c^2 = a^2m^2 + b^2$]

$\Rightarrow (8m^2 - 1)(2m^2 + 1) = 0$

$\Rightarrow m = \pm\frac{1}{2\sqrt{2}}$



We know that a tangent of slope m touches the parabola $y^2 = 4ax$ at $(a/m^2, 2a/m)$ so the coordinates of the points of contact of the common tangents of slope $m = \pm\frac{1}{2\sqrt{2}}$ to the parabola $y^2 = 4x$ are A $(8, 4\sqrt{2})$ and B$(8, -4\sqrt{2})$. We also know that a tangent of slope m touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\mu\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$. Therefore, the coordinate of the points of contact of common tangents of slope $m = \pm\frac{1}{2\sqrt{2}}$ to the ellipse are $C\left(-2, \frac{3}{\sqrt{2}}\right)$ and $D\left(-2, \frac{-3}{\sqrt{2}}\right)$. Clearly, AB || CD so the quadrilateral ABCD is a trapezium. We have, $AB = 8\sqrt{2}$, CD = $\frac{6}{\sqrt{2}}$ and distance between AB and CD is

PQ = 8 + 2 = 10

∴ Area of quadrilateral ABCD

$= \frac{1}{2}(AB + CD) \times PQ$

$= \frac{1}{2}(8\sqrt{2} + \frac{6}{\sqrt{2}}) \times 10$

$= 55\sqrt{2}$ sq. units.

Q.6 If p is length of the perpendicular from the focus 'S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on any tangent at 'P' then show that $\frac{b^2}{p^2} = \frac{2a}{\lambda(SP)} - 1$.

Sol. Tangent at P(a cos θ, b sin θ) is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Also SP = a – ex = a(1 – e cos θ)

p is length of perpendicular from focus S(ae, 0)

$$p = \frac{e\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$$

$$\therefore b^2.\frac{1}{p^2} = \frac{b^2}{a^2b^2}\frac{(b^2\cos^2\theta + a^2\sin^2\theta)}{(e\cos\theta - 1)^2}$$

$$\therefore \frac{b^2}{p^2} = \frac{1}{a^2}\left[\frac{a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta)}{(1-e\cos\theta)^2}\right]$$

$$= \frac{(1-e^2\cos^2\theta)}{(1-e\cos\theta)^2}$$

or $\frac{b^2}{p^2} = \frac{1+e\cos\theta}{1-e\cos\theta}$

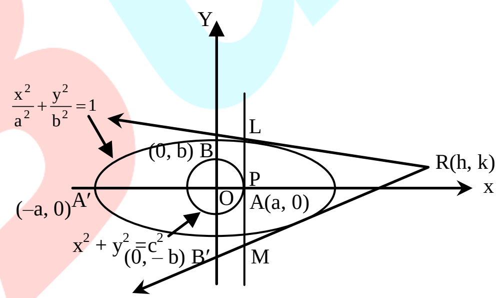
Again, $\frac{2a}{SP} - 1 = \frac{2a}{a(1-e\cos\theta)} - 1$

$$= \frac{1+e\cos\theta}{1-e\cos\theta}$$

$\therefore \frac{b^2}{p^2} = \frac{2a}{SP} - 1$ (proved)

Q.7 A circle is inscribed in an ellipse of eccentricity $1/\sqrt{2}$ and both are concentric. Now tangent at a point P on the circle cuts the ellipse at A and B. Prove that the locus of the point of intersection of tangent at A and B on the ellipse is again an ellipse. Find the eccentricity.

Sol. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse and $x^2 + y^2 = c^2$ be the circle inscribed in it such that the tangent at point P to the circle $x^2 + y^2 = c^2$ cuts the ellipse at L and M. Let R(h, k) be the point of intersection of tangents at L and M. Then LM is the chord of contact of tangents drawn from P(h, k) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Therefore, equation of LM is

$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$ or $y = \left(\frac{-b^2h}{a^2k}\right)x + \frac{b^2}{k}$

it touches the circle $x^2 + y^2 = c^2$, therefore,

$\frac{b^4}{k^2} = c^2\left(1 + \frac{b^4h^2}{a^4k^2}\right) \Rightarrow a^4b^4 = c^2a^4k^2 + c^2b^4h^2$

Hence the locus of (h, k) is

$c^2b^4x^2 + c^2a^4y^2 = a^4b^4$ or $\frac{x^2}{a^4/c^2} + \frac{y^2}{b^4/c^2} = 1$

clearly, it is a concentric ellipse with eccentricity e_1 is given by

$$e_1 = \sqrt{1 - \frac{b^4/c^2}{a^4/c^2}} = \sqrt{1 - \frac{b^4}{a^4}}$$

But, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse of eccentricity

$e = \frac{1}{\sqrt{2}}$

Therefore $e^2 = 1 - \frac{b^2}{a^2}$

$\Rightarrow \frac{1}{2} = 1 - \frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$

substituting the value of $\frac{b^2}{a^2}$ in $e_1 = \sqrt{1-\frac{b^4}{a^4}}$

we get,

$e_1 = \sqrt{1-\frac{1}{4}} \Rightarrow e_1 = \sqrt{3/4} \Rightarrow e_1 = \frac{\sqrt{3}}{2}$

Q.8 If the normal at any point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ with centre C meets the major and minor axes in G and g respectively, and if CF be perpendicular to the normal, prove that

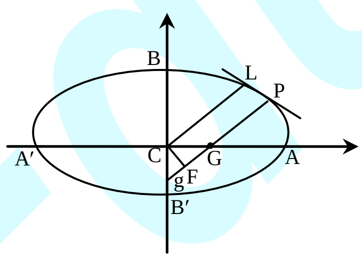
(i) $PF.PG = b^2$ and

(ii) $PF.Pg = a^2$. Where 'a' is the semi-major axis and 'b' is the semi-minor axis.

Sol. Let P be (a cos θ, b sin θ) and the points G and g are

$$G = \left(\frac{a^2-b^2}{a}\cos\theta,\ 0\right)$$

$$\& \ g = \left(0,\ -\frac{a^2-b^2}{a}\sin\theta\right)$$



$$\therefore PG^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

$$= \frac{b^4\cos^2\theta}{a^2} + b^2\sin^2\theta \quad \ldots(1)$$

$$= \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$Pg^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta) \quad \ldots(2)$$

Again CF is perpendicular from C on the normal and CL is perpendicular from C on the tangent

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

clearly $PF = CL = \dfrac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$

$$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \quad \ldots(3)$$

Now $PF^2.PG^2 = \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta) \times$

$$\frac{a^2b^2}{(b^2\cos^2\theta + a^2\sin^2\theta)} = b^4$$

$PF^2.Pg^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta) \times$

$$\frac{a^2b^2}{(b^2\cos^2\theta + a^2\sin^2\theta)} = a^4$$

$\therefore$ PF. PG = b^2 and PF. Pg = a^2 (Proved)

Q.9 If the normal at the point P(θ) to the ellipse $x^2/14 + y^2/5 = 1$, intersects it again at the point Q(2θ), show that cosθ = –2/3.

Sol. Normal at P(a cos θ, b sin θ) is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

where $a^2 = 14$, $b^2 = 5$

it meets the curve again at Q (2θ)

i.e. (a cos 2θ, b sin 2θ)

$$\therefore \frac{a}{\cos\theta}.\ a\cos 2\theta - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$$

or $\frac{14}{\cos\theta}(2\cos^2\theta - 1) - \frac{5}{\sin\theta}.(2\sin\theta\cos\theta)$

$$= 14 - 5$$

or $28\cos^2\theta - 14 - 10\cos^2\theta = 9\cos\theta$

or $18\cos^2\theta - 9\cos\theta - 14 = 0$

Now $14 \times 18 = 7 \times 2 \times 6 \times 3 = 21 \times 12$

$\therefore 18\cos^2\theta - 21\cos\theta + 12\cos\theta - 14 = 0$

or $(6\cos\theta - 7)(3\cos\theta + 2) = 0$

$\therefore \cos\theta = \frac{7}{6}$ and $\cos\theta = -\frac{2}{3}$

$\because \cos\theta = \frac{7}{6}$ (> 1) not possible

$\therefore \cos\theta = -\frac{2}{3}$

Q.10 A normal inclined at 45° to the axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is drawn. It meets the x-axis and the y-axis in P and Q respectively. If C is the centre of the ellipse, show that the area of triangle CPQ is : $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq. units.

Sol. Let R $(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that the normal at R cuts the major and minor axes at P and Q respectively and is inclined at an angle of 45° to the x- axis.

The equation of the normal at R is

$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$

It cuts the major and minor axes at $P\left(\frac{a^2-b^2}{a\sec\theta}, 0\right)$

and $Q\left(0, -\frac{a^2-b^2}{b\,\text{cosec}\,\theta}\right)$ respectively.

$\therefore CP = \left|\frac{a^2-b^2}{a\sec\theta}\right|$

$= \frac{a^2-b^2}{a}|\cos\theta|$

and $CQ = \left|\frac{a^2-b^2}{-b\,\text{cosec}\,\theta}\right|$

$= \frac{a^2-b^2}{b}|\sin\theta|$

$\therefore$ Area of $\Delta CPQ = \frac{1}{2}$. CP. CQ

$= \frac{1}{2}\cdot\left(\frac{a^2-b^2}{a}\right)|\cos\theta| \times \left(\frac{a^2-b^2}{b}\right)|\sin\theta|$

$= \frac{1}{2}\frac{(a^2-b^2)^2}{ab}|\sin\theta\cos\theta|$

$= \frac{1}{4}\frac{(a^2-b^2)^2}{ab}|\sin 2\theta|$

It is given that the normal at R is inclined at an angle of 45° with x- axis. Therefore, slope of normal at R = tan 45°

$\Rightarrow \frac{a\sec\theta}{b\,\text{cosec}\,\theta} = 1 \Rightarrow \tan\theta = \frac{b}{a}$

$\therefore$ Area of $\Delta CPQ = \frac{1}{4}\cdot\frac{(a^2-b^2)^2}{ab}\times\frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{1}{4}\cdot\frac{(a^2-b^2)^2}{ab}\times\frac{2b/a}{1+b^2/a^2}$

$= \frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq. units.

Q.11 If ϕ is the angle between the normals at θ and $\left(\frac{\pi}{2}+\theta\right)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, prove that the eccentricity e of the ellipse is given by $2\sqrt{1-e^2} = e^2\sin 2\theta\tan\phi$.

Sol. Normal at θ, $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Its slope is $\frac{a\sin\theta}{b\cos\theta} = m_1$

slope of normal at $\left(\frac{\pi}{2}+\theta\right)$ is $-\frac{a\cos\theta}{b\sin\theta} = m_2$

$\therefore \tan\phi = \frac{m_1 - m_2}{1+m_1m_2}$

$$= \frac{\left(\frac{a}{b}\cdot\frac{\sin^2\theta+\cos^2\theta}{\sin\theta.\cos\theta}\right)}{\frac{a^2-b^2}{b^2}}$$

$$= \frac{2ab}{a^2e^2}\cdot\frac{1}{\sin 2\theta}$$

$$\therefore\ e^2 \sin 2\theta \tan\phi = \frac{2a.a\sqrt{1-e^2}}{a^2} = 2\sqrt{1-e^2}$$

Q.12 Find the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ which are farthest from its centre and hence find the farthest distance.

Sol. Let P (a cos θ, b sin θ) be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. The equation of the normal to the ellipse at P is

$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$ …(1)

Its distance from the centre O(0, 0) is given by

$$\lambda = \frac{|a^2-b^2|}{\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}} \quad \dots (2)$$

Clearly, λ will be greatest,

if $a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta$ is least.

Let $z = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta$ then

$$\frac{dz}{d\theta} = 2a^2 \sec^2\theta \tan\theta - 2b^2 \operatorname{cosec}^2\theta \cot\theta$$

and, $$\frac{d^2z}{d\theta^2} = 2(a^2 \sec^4\theta + b^2 \operatorname{cosec}^4\theta)$$

$$+ 4(a^2 \sec^2\theta \tan^2\theta + b^2 \operatorname{cosec}^2\theta \cot^2\theta)$$

For maximum/minimum value of z, we must have

$$\frac{dz}{d\theta} = 0$$

$\Rightarrow a^2 \sec^2\theta \tan\theta - b^2\operatorname{cosec}^2\theta \cot\theta = 0$

$\Rightarrow a^2 \sin^4\theta - b^2 \cos^4\theta = 0$

$$\Rightarrow \tan^4\theta = \frac{b^2}{a^2} \Rightarrow \tan^2\theta = \frac{b}{a} \Rightarrow \tan\theta = \pm\sqrt{b/a}$$

Clearly, $\frac{d^2z}{d\theta^2} > 0$ for $\tan\theta = \pm\sqrt{b/a}$

Thus, z is minimum for $\tan\theta = \pm\sqrt{b/a}$

or $\theta = \tan^{-1}\left(\pm\sqrt{\frac{b}{a}}\right)$

Putting $\tan\theta = \pm\sqrt{\frac{b}{a}}$ in (1) and (2), we obtain that the equations of the normals are

$\pm\sqrt{a}\,x\ \mu\ \sqrt{b}\,y = (a-b)\sqrt{a+b}$

These normals are at a distance

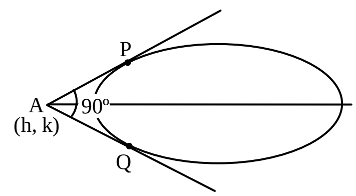
$$\lambda = \frac{|a^2-b^2|}{\sqrt{a^2\left(1+\frac{b}{a}\right)+b^2\left(1+\frac{a}{b}\right)}} = |a-b|$$

$\therefore\ \lambda = |a - b|$

from the centre of the ellipse.

Q.13 Prove that the tangents drawn to the ellipse $\frac{x^2}{a}+\frac{y^2}{b} = (a+b)$ at the points where it is cut by any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ intersect at right angles.

Sol. Let PQ be any tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$



which cuts the ellipse.

$$\frac{x^2}{a(a+b)}+\frac{y^2}{b(a+b)} = 1 \quad \dots(1)$$

at P and Q. The tangents at P and Q are at right angles and hence the point A(h, k) will lie on the director circle of 2nd ellipse.

$\therefore$ (h, k) lies on $x^2 + y^2 = A^2 + B^2$

$x^2 + y^2 = a(a + b) + b(a + b) = (a + b)^2$

since (h, k) lies on it.

$\therefore\ h^2 + k^2 = (a + b)^2$ …(2)

Again PQ is chord of contact of (h, k) w.r.t. (1)

$$\therefore\ \frac{hx}{a(a+b)}+\frac{ky}{b(a+b)} = 1$$

It will be a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If $a^2\lambda^2 + b^2m^2 = n^2$

$$a^2.\frac{h^2}{a^2(a+b)^2} + b^2.\frac{k^2}{b^2(a+b)^2} = 1$$

or $h^2 + k^2 = (a + b)^2$ which is true by (2)

Q.14 Find the locus of the point, tangents from which to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are inclined at an angle of α to each other.

Sol. The equation of any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If it passes through P(h, k), then

$$k = mh + \sqrt{a^2m^2 + b^2}$$

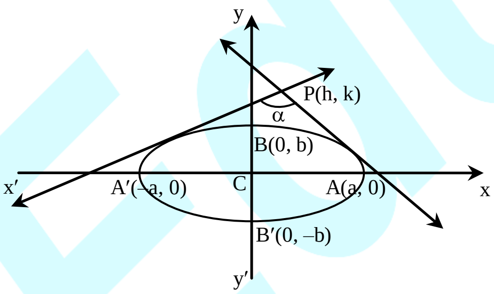
$\Rightarrow (k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow m^2 (h^2 - a^2) - 2khm + k^2 - b^2 = 0$ ...(1)

This is a quadratic equation in m, so it gives two values of m corresponding to each value of m there is a tangent passing through (h, k). Let m_1 and m_2 be the roots of equation (i), Then

$m_1 + m_2 = \frac{2hk}{h^2 - a^2}$

and $m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2}$



If α is the angle between the tangents, then

$$\tan\alpha = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

$$\Rightarrow \tan\alpha = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$\Rightarrow \tan\alpha = \pm\frac{\sqrt{4h^2k^2 - 4(k^2 - b^2)(h^2 - a^2)}}{h^2 - a^2 + k^2 - b^2}$$

$$\Rightarrow \tan\alpha = \pm\frac{2\sqrt{a^2k^2 + b^2h^2 - a^2b^2}}{h^2 + k^2 - (a^2 + b^2)}$$

$\Rightarrow \tan^2\alpha\ \{h^2 + k^2 - a^2 - b^2\}^2$

$= 4\ \{a^2k^2 + b^2h^2 - a^2\ b^2\}$

Hence the locus of (h, k) is

$\{x^2 + y^2 - a^2 - b^2\}^2 \tan^2\alpha$

$= 4\{b^2x^2 + a^2y^2 - a^2b^2\}$

Q.15 Find the locus of the middle points of chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the tangents at the extremities of which intersect at right angles.

Sol. Let (x_1, y_1) be the point of intersection of perpendicular tangents so that (x_1, y_1) lies on director circle.

$\therefore x_1^2 + y_1^2 = a^2 + b^2$ ... (1)

Then the chord will be chord of contact of (x_1, y_1)

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ... (2)

If mid-point is (h, k), then its equation is

$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ … (3)

Compare (2) & (3) and find x_1, y_1 and put in (1)

$\therefore$ locus of (h, k) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 (a^2 + b^2)$$

$= x^2 + y^2$

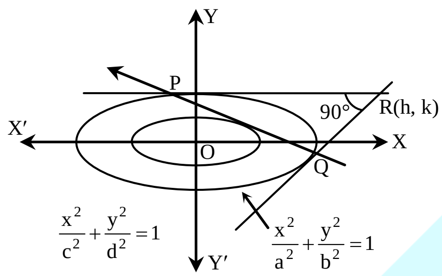
$(b^2x^2 + a^2y^2)^2\ (a^2 + b^2)$

$= a^4b^4(x^2 + y^2)$

Q.16 A tangent is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to cut the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ at the points P and Q. If tangents at P and Q to the ellipse

$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ intersect at right angle then prove that $\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$.

Sol. Suppose the tangents at P and Q to the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ intersect at right angle at point R (h, k). Then PQ is the chord of contact of tangents drawn from R to the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$



∴ The equation of PQ is

$\frac{hx}{c^2} + \frac{ky}{d^2} = 1$ … (1)

or $y = -\left(\frac{d^2h}{c^2k}\right)x + \frac{d^2}{k}$

This touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, therefore

$\frac{d^4}{k^2} = a^2\left(-\frac{d^2h}{c^2k}\right)^2 + b^2$

$\Rightarrow \frac{d^4}{k^2} = \frac{a^2d^4h^2}{c^4k^2} + b^2$

$\Rightarrow c^4 d^4 = a^2 d^4 h^2 + b^2 c^4k^2$

$\Rightarrow k^2 = \frac{c^4d^4 - a^2d^4h^2}{b^2c^4}$ … (2)

Since the tangents at P and Q to the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$, intersect at right angles at R(h, k). Therefore R lies on the director circle of the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$. Thus we must have

$h^2 + k^2 = c^2 + d^2$

Now $h^2 + k^2 = h^2 + \frac{c^4d^4 - a^2d^4h^2}{b^2c^4}$

$\Rightarrow h^2 + k^2 = \frac{h^2(b^2c^4 - a^2d^4) + c^4d^4}{b^2c^4}$

$\therefore h^2 + k^2 = c^2 + d^2$

$\Rightarrow \frac{h^2(b^2c^4 - a^2d^4) + c^4d^4}{b^2c^4} = c^2 + d^2$

$\Rightarrow h^2 (b^2c^4 - a^2d^4) + c^4d^4 - b^2c^6 - b^2c^4d^2 = 0$

This is only possible for varying h when

$b^2c^4 - a^2d^4 = 0$ and $c^4d^4 - b^2c^6 - b^2c^4d^2 = 0$

$\Rightarrow b^2c^4 = a^2d^4$ and $c^4d^4 = b^2c^6 + b^2c^4d^2$

$\Rightarrow c^4d^4 = (a^2d^4)\, c^2 + b^2c^4d^2$

$\Rightarrow c^2d^2 = a^2d^2 + b^2c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$

Q.17 Show that the tangents at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

Part-B Passage based objective questions

Passage I (Q. 18 to 20)

For the ellipse $\frac{(x+y-1)^2}{9} + \frac{(x-y+2)^2}{4} = 1$

Q.18 The equation of minor axis is -

(A) $x + y - 1 = 0$ (B) $x - y + 2 = 0$

(C) $x = \frac{1}{2}$ (D) $y = \frac{3}{2}$

Sol. **[A]**

Equation of its minor axis will be Y-axis whose equation is given by X = 0

i.e. $x + y - 1 = 0$

Q.19 Ends of minor axis are -

(A) $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{-3}{2}, \frac{5}{2}\right)$ (B) (1, 3) (–2, 0)

(C) (2, 4), (0, 2) (D) None of these

Sol. **[A]**

(X = 0, Y = 2)

$(x + y - 1 = 0,\ x - y + 2 = 2)$

$(x + y = 1,\ x - y = 0)$

solving above, we get $2x = 1$

$x = 1/2$

Θ x = y = 1/2

$\therefore$ one end is $\left(\frac{1}{2}, \frac{1}{2}\right)$

Similarly put (X = 0, Y = –2), we get other end which is $\left(-\frac{3}{2}, \frac{5}{2}\right)$

Q.20 Length of major axis and minor axis are-

(A) $3\sqrt{2}$, $2\sqrt{2}$ (B) 6, 4

(C) 9, 4 (D) 3, 2

Sol. **[A]**

Length of major axis:

Ends of major axis will be

(X = a, Y = 0) and (X = –a, Y = 0)

(x + y –1 = 3, x – y + 2 = 0)	and (x + y – 1 = –3, x – y + 2 = 0)
On solving, we find	& on solving we find
2x – 2 = 0	2x = –4
x = 1	x = –2
$\therefore$ y = 3	$\therefore$ y = 0
one end (1, 3)	other end is (–2, 0)

$\therefore$ length of major axis will be = $\sqrt{(3)^2 + (3)^2}$

= $\sqrt{9+9} = 3\sqrt{2}$

and length of minor axis = distance between $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{-3}{2}, \frac{5}{2}\right)$

= $\sqrt{4+4}$ = $\sqrt{8} = 2\sqrt{2}$

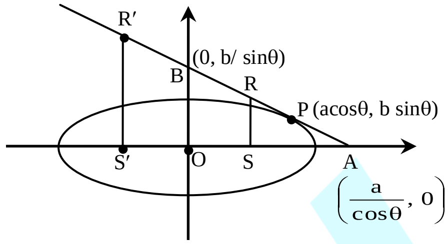
Passage II (Q. 21 to 23)

Consider an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b). Let P (a cosθ, b sinθ) be any point on the ellipse in the first quadrant at which tangent is drawn which meets the major axis and minor axis at A and B respectively. Let SR and S′R′ are perpendicular drawn from foci S and S′ on this tangent where R and R′ are feet of perpendiculars. O is the centre of ellipse. Now answer the following questions.

Q.21 The minimum area of triangle OAB is equal to-

(A) ab (B) $\frac{ab}{2}$ (C) a^2 (D) b^2

Sol. **[A]**



equation of tangent $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Area of ΔOAB = $\frac{1}{2}$. OA . OB

$= \frac{1}{2} . \frac{a}{\cos\theta} . \frac{b}{\sin\theta}$

$= \frac{1}{2} . \frac{ab}{\sin\theta\cos\theta} = \frac{ab}{\sin 2\theta}$

Minimum area of ΔOAB, when sin 2θ is maximum.

$\therefore$ Maximum value of sin 2θ = 1

$\therefore$ Minimum area of ΔOAB = ab **Ans.**

Q.22 Minimum value of AB is equal to -

(A) 2a (B) 2b (C) a + b (D) a – b

Sol. **[C]**

$AB = \sqrt{\frac{a^2}{\cos^2\theta} + \frac{b^2}{\sin^2\theta}}$

$AB = \sqrt{a^2\sec^2\theta + b^2\text{cosec}^2\theta}$

Let $Z = \sqrt{a^2\sec^2\theta + b^2\text{cosec}^2\theta}$

For max./min.

$\frac{dz}{d\theta} = \frac{1.(2a^2\sec^2\theta\tan\theta - 2b^2\text{cosec}^2\theta\cot\theta)}{2\sqrt{a^2\sec^2\theta + b^2\text{cosec}^2\theta}}$

$\frac{dz}{d\theta} = 0$

$\Rightarrow a^2\sec^2\theta\tan\theta = b^2\text{cosec}^2\theta\cot\theta$

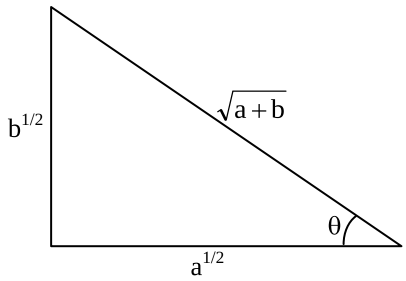
$\Rightarrow \frac{a^2}{\cos^2\theta} . \frac{\sin\theta}{\cos\theta} = \frac{b^2}{\sin^2\theta} . \frac{\cos\theta}{\sin\theta}$

$\Rightarrow \frac{a^2}{\cos^3\theta} . \sin\theta = \frac{b^2}{\sin^3\theta} . \cos\theta$

$\Rightarrow b^2\cos^4\theta = a^2\sin^4\theta \Rightarrow \tan^4\theta = \frac{b^2}{a^2}$

$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^2 . \frac{1}{4} = \left(\frac{b}{a}\right)^{1/2} = \frac{b^{1/2}}{a^{1/2}}$

$\therefore AB = \sqrt{\frac{a^2(a+b)}{a} + \frac{b^2(a+b)}{b}}$



$\therefore \sin\theta = \dfrac{b^{1/2}}{\sqrt{a+b}}$, $\cos\theta = \dfrac{a^{1/2}}{\sqrt{a+b}}$

$\therefore AB = \sqrt{(a+b)^2}$; AB = a + b **Ans.**

$\therefore$ Minimum value of AB = a + b **Ans.**

Q.23 The minimum value of OA + OB is equal to-

(A) $(a^{2/3} + b^{2/3})^{3/2}$ (B) $(a^{3/2} + b^{3/2})^{2/3}$

(C) $(a^2 + b^2)^{1/2}$ (D) None of these

Sol. **[A]**

OA + OB

$= \dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta} = a\sec\theta + b\,\text{cosec}\,\theta$

Let z = $a\sec\theta + b\,\text{cosec}\,\theta$

$\dfrac{dz}{d\theta} = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta$

For max./min. $\dfrac{dz}{d\theta} = 0$

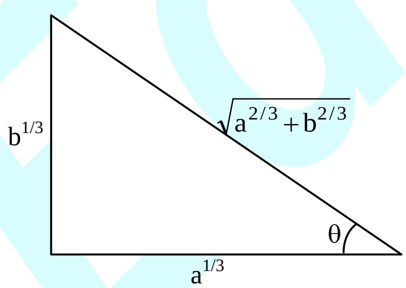
$\Rightarrow a\sec\theta\tan\theta = b\,\text{cosec}\,\theta\cot\theta$

$\Rightarrow \dfrac{a}{\cos\theta}\cdot\dfrac{\sin\theta}{\cos\theta} = \dfrac{b}{\sin\theta}\cdot\dfrac{\cos\theta}{\sin\theta}$

$\Rightarrow a\sin^3\theta = b\cos^3\theta$

$\Rightarrow \tan^3\theta = \dfrac{b}{a}$

$\Rightarrow \tan\theta = \left(\dfrac{b}{a}\right)^{1/3} = \dfrac{b^{1/3}}{a^{1/3}}$



$\therefore \sin\theta = \dfrac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

$\therefore \cos\theta = \dfrac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

$\therefore z = \dfrac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \dfrac{b\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$

$z = \sqrt{a^{2/3}+b^{2/3}}\,(a^{2/3} + b^{2/3})$

$z = (a^{2/3} + b^{2/3})^{3/2}$

Passage III (Q. 24 to 26)

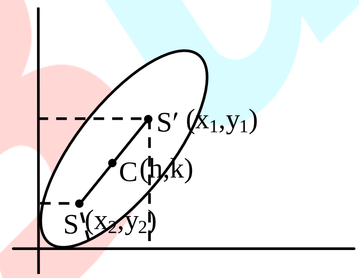
An ellipse whose major axis and minor axis length are 2a and 2b (a > b) slides between co-ordinate axes such that it always touches both axes. Let S_1 and S_2 be its foci and C be its centre.

If $S_1 \equiv (x_1, y_1)$, $S_2 \equiv (x_2, y_2)$ and $C \equiv (h, k)$ then

Q.24 $\dfrac{x_1x_2}{y_1y_2}$ is equal to-

(A) 1 (B) 2 (C) $\dfrac{a^2}{b^2}$ (D) $\dfrac{b^2}{a^2}$

Sol. **[A]**



$x_1 x_2 = b^2$

$y_1 y_2 = b^2$

$\dfrac{x_1x_2}{y_1y_2} = \dfrac{b^2}{b^2} = 1$

Q.25 The value of $(x_2 - h)^2 + (y_2 - k)^2$ is equal to-

(A) $2(a^2 - b^2)$ (B) $a^2 - b^2$

(C) $2a^2$ (D) None of these

Sol. **[B]**

$(x_2 - h)^2 + (y_2 - k)^2 = CS^2 = a^2e^2 = a^2 - b^2$

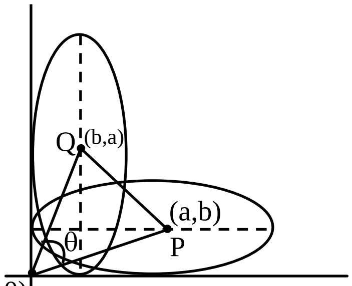
$$\left[\begin{array}{l} \because\ b^2 = a^2(1-e^2) \\ \quad b^2 = a^2 - a^2e^2 \\ \therefore\ a^2e^2 = a^2 - b^2 \end{array}\right.$$

Q.26 The locus of centre (h, k) is a part of circle i.e. an arc. The angle subtended by this arc at origin is θ then cosθ is equal to -

(A) $\dfrac{ab}{a^2+b^2}$ (B) $\dfrac{2ab}{a^2+b^2}$

(C) $\dfrac{3ab}{a^2+b^2}$ (D) $\dfrac{4ab}{a^2+b^2}$

Sol. **[B]**



$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 4(a^2 - b^2)$

$(x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4y_1y_2 = 4(a^2 - b^2)$

$4h^2 + 4k^2 - 8b^2 = 4(a^2 - b^2)$

$x^2 + y^2 = a^2 + b^2$

$$\cos\theta = \frac{(CP)^2 + (CQ)^2 - (PQ)^2}{2(CP)(CQ)}$$

$$\cos\theta = \frac{(a^2 + b^2) + (b^2 + a^2) - [(a-b)^2 + (b-a)^2]}{2(\sqrt{a^2 + b^2})\,(\sqrt{b^2 + a^2})}$$

$$\cos\theta = \frac{4ab}{2(a^2 + b^2)}$$

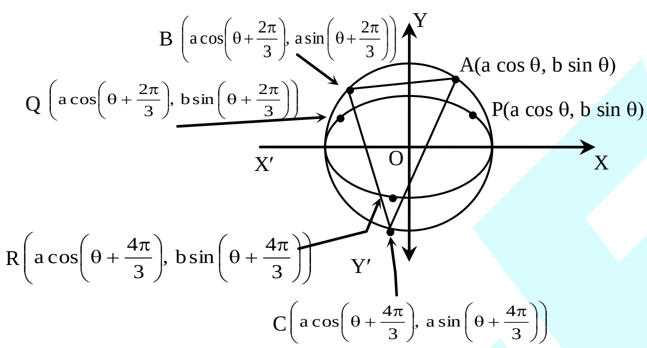
$$\cos\theta = \frac{2ab}{(a^2 + b^2)}$$

EXERCISE # 4

Old IIT-JEE questions

Q.1 Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) meet the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. **[IIT-2000]**

Sol.



Let the coordinates of the vertex A of ΔABC be (a cos θ, a sin θ). Since the triangle is equilateral. So, the coordinates of the vertices B and C are $\left(a\cos\left(\theta+\frac{2\pi}{3}\right), a\sin\left(\theta+\frac{2\pi}{3}\right)\right)$ and $\left(a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{4\pi}{3}\right)\right)$ respectively.

Consequently, the coordinate of P, Q and R are given as :

P(a cos θ, b sin θ),

Q $\left(a\cos\left(\theta+\frac{2\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right)$ and

R $\left(a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{4\pi}{3}\right)\right)$

The equations of the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ =1 at P, Q and R are

ax sec θ – by cosec θ = $(a^2 – b^2)$

$ax\sec\left(\theta+\frac{2\pi}{3}\right) - by\,\text{cosec}\left(\theta+\frac{2\pi}{3}\right) = a^2 - b^2$

and

$ax\sec\left(\theta+\frac{4\pi}{3}\right) - by\,\text{cosec}\left(\theta+\frac{4\pi}{3}\right) = a^2 - b^2$

We have to prove that these normals are concurrent.

We have,

$$\Delta = \begin{vmatrix} a\sec\theta & -b\,\text{cosec}\theta & a^2-b^2 \\ a\sec\left(\theta+\frac{2\pi}{3}\right) & -b\,\text{cosec}\left(\theta+\frac{2\pi}{3}\right) & a^2-b^2 \\ a\sec\left(\theta+\frac{4\pi}{3}\right) & -b\,\text{cosec}\left(\theta+\frac{4\pi}{3}\right) & a^2-b^2 \end{vmatrix}$$

$$= -ab(a^2 - b^2)\begin{vmatrix} \sec\theta & \text{cosec}\theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \text{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta+\frac{4\pi}{3}\right) & \text{cosec}\left(\theta+\frac{4\pi}{3}\right) & 1 \end{vmatrix}$$

$$= \frac{-ab(a^2-b^2)}{\sin\theta\cos\theta\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)} \times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta\cos\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right) \\ \sin\left(\theta+\frac{4\pi}{3}\right) & \cos\left(\theta+\frac{4\pi}{3}\right) & \sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \frac{-4ab(a^2-b^2)}{\sin 2\theta\sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(2\theta+\frac{8\pi}{3}\right)} \times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ 2\sin\left(\theta+\frac{4\pi}{3}\right) & \cos\left(\theta+\frac{4\pi}{3}\right) & \sin\left(2\theta+\frac{8\pi}{3}\right) \end{vmatrix} \times$$

$$\frac{-4ab(a^2-b^2)}{\sin 2\theta\sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(2\theta+\frac{8\pi}{3}\right)}$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ 2\sin(\theta+\pi)\cos\frac{\pi}{3} & 2\cos(\theta+\pi)\cos\frac{\pi}{3} & 2\sin(2\theta+2\pi)\cos 2\pi/3 \end{vmatrix}$$

$$\frac{-4ab(a^2-b^2)}{\sin 2\theta \sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(2\theta+\frac{8\pi}{3}\right)} \times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ 2\sin(\theta+\pi)\cos\frac{\pi}{3} & 2\cos(\theta+\pi)\cos\frac{\pi}{3} & 2\sin(2\theta+2\pi)\cos 2\pi/3 \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$= \frac{-4ab(a^2-b^2)}{\sin 2\theta \sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(2\theta+\frac{8\pi}{3}\right)} \times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ -\sin\theta & -\cos\theta & -\sin 2\theta \end{vmatrix}$$

$$= \frac{4ab(a^2-b^2)}{\sin 2\theta \sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(2\theta+\frac{8\pi}{3}\right)} \times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\theta & \cos\theta & \sin 2\theta \end{vmatrix}$$

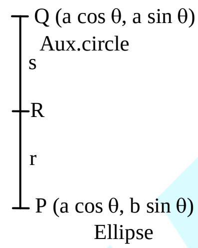
= 0 (Θ R_1 and R_3 are identical)

Hence, the normals at P, Q, R are concurrent.

Q.2 Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $0 < b < a$. Let the line parallel to y axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same sides of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse.

[IIT-2001]

Sol. P is a point on the ellipse and Q the corresponding point on the auxiliary circle where PQ is parallel to y-axis and both P and Q are on same side of x- axis. R is a point which divides PQ in the ratio r : s. If its coordinates be (x, y), then



x = $\frac{ra\cos\theta + sa\cos\theta}{r+s}$ and y = $\frac{ra\sin\theta + sb\sin\theta}{r+s}$

$\Rightarrow$ x = a cos θ, y = $\frac{ra+sb}{r+s}\sin\theta$

In order to find the locus of R, we have to eliminate the variable θ from the above two relations.

We know that $\sin^2\theta + \cos^2\theta = 1$

$\therefore \frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(ra+sb)^2} = 1$

which is the required locus.

Q.3 Prove that, in an ellipse that perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.

[IIT-2002]

Sol. Any tangent to the ellipse is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ …(i)

its point of contact is P(a cos θ, b sin θ) and its slope is $-\frac{b}{a}\cot\theta$. Also s the focus is (ae, 0). Any line through focus S and perpendicular to tangent (i) is $y - 0 = \frac{a}{b}\tan\theta\,(x - ae)$ …(ii)

Also equation of PC where c is centre (0, 0) is

$y - 0 = \frac{b}{a}\tan\theta\,(x - 0)$ …(iii)

In order to find the locus of point of intersection of the lines (ii) and (iii), we have to eliminate the variable θ between their equations.

Dividing, we get

$1 = \frac{a^2}{b^2}\,\frac{x-ae}{x}$ or $\frac{a^2 - a^2e^2}{a^2} = 1 - \frac{ae}{x}$

$\therefore -e^2 = -\frac{ae}{x}$ or $x = \frac{a}{e}$

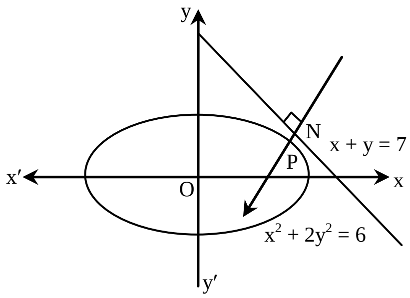
which is the equation of directrix of the ellipse

Q.4 Find the point on the ellipse $x^2 + 2y^2 = 6$ closest to the line x + y = 7. **[IIT 2003]**

Sol. We have $x^2 + 2y^2 = 6$

$\Rightarrow \frac{x^2}{6} + \frac{y^2}{3} = 1$ …(i)

Let P ($\sqrt{6}\cos\theta$, $\sqrt{3}\sin\theta$) be a point on (i) which is closest to the line x + y = 7.



Let PN be perpendicular from P on the line x + y = 7. Clearly, PN is nearest (minimum) when it is along the normal at P.

The equation of the normal at P is

$(\sqrt{6}\sec\theta)x - (\sqrt{3}\,\mathrm{cosec}\,\theta)y = 3$

it is perpendicular to x + y = 7

$\therefore \frac{\sqrt{6}\sec\theta}{\sqrt{3}\,\mathrm{cosec}\,\theta} x - 1 = -1$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$ and $\cos\theta = \sqrt{2/3}$

$\therefore$ required point is

$P\left(\sqrt{6}\times\sqrt{\frac{2}{3}}, \sqrt{3}\times\frac{1}{\sqrt{3}}\right) \equiv (2, 1)$

Q.5 The value of 'θ', θ ∈ [0, π] for which the sum of intercepts on coordinate axes cut by tangent at point ($3\sqrt{3}\cos\theta$, $\sin\theta$) to ellipse $\frac{x^2}{27} + y^2 = 1$ is minimum, is - **[IIT Scr.2003]**

(A) π/6 (B) π/3

(C) π/4 (D) π/8

Sol. **[A]**

Given tangent is drawn at ($3\sqrt{3}\cos\theta$, $\sin\theta$) to $\frac{x^2}{27} + \frac{y^2}{1} = 1$

$\Rightarrow$ equation of tangent is $\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1$

Thus sum of intercepts

$= (3\sqrt{3}\sec\theta + \mathrm{cosec}\,\theta) = f(\theta)$

(say to minimize)

$\Rightarrow f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$

put f ′(θ) = 0

$\Rightarrow \sin^3\theta = \frac{1}{3^{3/2}}\cos^3\theta$ or $\tan\theta = \frac{1}{\sqrt{3}}$

i.e. $\theta = \frac{\pi}{6}$ and at $\theta = \frac{\pi}{6}$, $f''(0) > 0$

$\therefore$ minimum at $\theta = \frac{\pi}{6}$

Q.6 Locus of middle point of segment of tangent to ellipse $x^2 + 2y^2 = 2$ which is intercepted between the coordinate axis is- **[IIT Scr.2004]**

(A) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (B) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(C) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (D) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

Sol. **[A]**

Any point on the ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is ($\sqrt{2}\cos\theta$, $\sin\theta$) tangent at which is

$\frac{x\cos\theta}{\sqrt{2}} + \frac{y\sin\theta}{1} = 1$. If (h, k) be the mid-point of the portion of tangent intercepted between the axes then $2h = \frac{\sqrt{2}}{\cos\theta}$, $2k = \frac{1}{\sin\theta}$

Eliminate θ by $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4k^2} + \frac{1}{2h^2} = 1$

$\Rightarrow \frac{1}{2h^2} + \frac{1}{4k^2} = 1$ $\therefore$ locus is $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Q.7 A tangent is drawn at some point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is intersecting to the coordinate axes at points A & B then minimum area of the Δ OAB is- **[IIT Scr.2005]**

(A) ab (B) $\frac{a^2 + b^2}{2}$

(C) $\frac{a^2 + b^2}{4}$ (D) $\frac{a^2 + b^2 - ab}{3}$

Sol. **[A]**

Let P(a cos θ, b sin θ)

Equation of tangent at this point is given by $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

it meets axes at A (0, b cosec θ), B (a sec θ, 0)

$\therefore$ area $= \frac{ab}{2\sin\theta\cos\theta}$

Area = $\frac{ab}{\sin 2\theta}$

[For area to be minimum, sin 2θ should be maximum and sin 2θ = 1]

≥ ab

∴ minimum area is equal to ab.

Q.8 Find the eqation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the coordinate axes. **[IIT - 2005]**

Q.9 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are **[IIT 2008]**

(A) $x^2 + 2\sqrt{3}\,y = 3 + \sqrt{3}$

(B) $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$

(D) $x^2 - 2\sqrt{3}\,y = 3 - \sqrt{3}$

Sol. **[B, C]**

Let ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

P and Q points are $\left(\pm\sqrt{3}, -\frac{1}{2}\right)$, focus S of the ellipse is $\left(0, -\frac{1}{2}\right)$. Now find directrix of the parabola parallel to PQ and at $\sqrt{3}$ unit distance from PQ, and then use $PS^2 = PM^2$ to find the parabola.

Q.10 The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x- axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points : **[IIT 2009]**

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Sol. **[C]**

$\frac{x^2}{16} + \frac{y^2}{4} = 1$

P ≡ (4 cos θ, 2 sin θ)

$\frac{2x}{16} + \frac{2y}{4}y' = 0$ $y' = -\frac{x}{16}\cdot\frac{4}{y}$

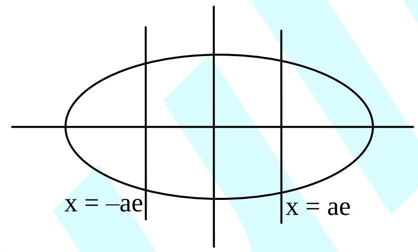
$M_N = \frac{4y}{x} = \frac{4(2\sin\theta)}{4\cos\theta} = 2\tan\theta$ (M_N = slope of normal)

y – 2 sin θ = 2 tan θ (x – 4 cos θ)

Q ≡ (4 cos θ – cos θ, 0) ⇒ Q ≡ (3 cos θ, 0)

M ≡ (h, k) ≡ (7/2 cos θ, sin θ)

$\frac{4x^2}{49} + y^2 = 1$



∵ $b^2 = a^2(1 - e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$

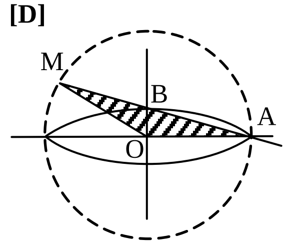
$x = 2\sqrt{3}$, $x = -2\sqrt{3}$ (equation of latus rectum)

$\frac{4.12}{49} + y^2 = 1$ $y^2 = \frac{1}{49}$ $\Rightarrow y = \pm\frac{1}{7}$

Q.11 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is- **[IIT 2009]**

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Sol. **[D]**



equation of AB is x = 3 –3y

equation of auxiliary circle

$x^2 + y^2 = 9$

on solving M = $\left(-\frac{12}{5}, \frac{9}{5}\right)$

Area of Δ OAM = $\frac{1}{2}$ (OA) (y_m)

$= \frac{1}{2}(3)\left(\frac{9}{5}\right) = \frac{27}{10}$

Passage (Q. 12 to 14)

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, touching the ellipse at points A and B. **[IIT-2010]**

Q.12 The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

Sol. **[D]**

Equation of tangent

$y = mx \pm \sqrt{9m^2+4}$

as it passes through (3, 4)

so $4 = 3m \pm \sqrt{9m^2+4}$

$m = \frac{1}{2}$ and undefined.

So equation of the tangents will be

$x - 2y + 5 = 0$ and $x = 3$

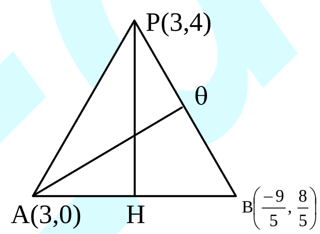
so point of contacts are (3, 0) and $\left(\frac{-9}{5}, \frac{8}{5}\right)$

Q.13 The orthocentre of the triangle PAB is

(A) $\left(5, \frac{8}{7}\right)$ (B) $\left(\frac{7}{5}, \frac{25}{8}\right)$

(C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

Sol. **[C]**



Equation of two altitudes PH and AQ are

$3x - y - 5 = 0$ and $2x + y - 6 = 0$ respectively

so orthocentre will be $\left(\frac{11}{5}, \frac{8}{5}\right)$

Q.14 The equation of the locus of the point whose distances from the point P and the line AB are equal, is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Sol. **[A]**

Equation of AB is $x + 3y - 3 = 0$

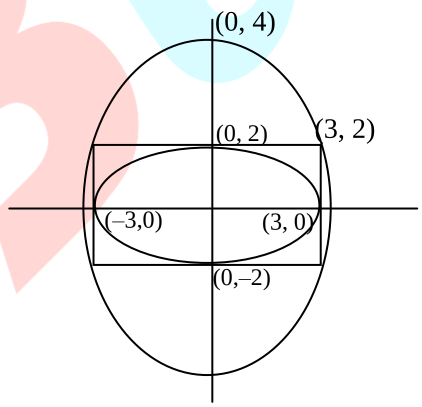
so required locus will be $(x-3)^2 + (y-4)^2 = \frac{(x+3y-3)^2}{10}$

$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Q.15 The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is **[IIT-2012]**

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Sol. [C]



Let equation of ellipse E_2 is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

it passes through (0, 4)

so $b^2 = 16$

and also passes through (3, 2)

So $\frac{9}{a^2}+\frac{4}{b^2}=1$

$\Rightarrow \frac{9}{a^2}+\frac{1}{4}=1$

$\Rightarrow a^2 = 12$

$\Rightarrow$ as $a < b$

so $12 = 16(1 - e^2)$

$\Rightarrow e^2 = \frac{1}{4}$

$\Rightarrow e = \frac{1}{2}$

EXERCISE # 5

Q.1 The tangents at a point α on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2\alpha)^{-1/2}$.

Sol. Let point α is $(a \cos \theta, b \sin \theta)$

The equation of the tangent at this point to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ...(1)

The equation of the auxiliary circle is

$x^2 + y^2 = a^2$ ...(2)

The combined equation of the lines joining the origin to be points of intersection of (1) and (2) is

$$x^2 + y^2 = a^2\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right)^2$$

$$\Rightarrow x^2(1 - \cos^2\theta) + y^2\left(1 - \frac{a^2}{b^2}\sin^2\theta\right)$$

$$-2xy\frac{a}{b}\sin\theta\cos\theta = 0$$

These two lines are mutually perpendicular.

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \sin^2\theta + 1 - \frac{a^2}{b^2}\sin^2\theta = 0 \Rightarrow \frac{a^2 - b^2}{b^2}\sin^2\theta = 1$$

$$\Rightarrow \frac{a^2e^2\sin^2\theta}{a^2(1-e^2)} = 1 \Rightarrow e^2\sin^2\theta = 1 - e^2$$

$$\Rightarrow e^2 = \frac{1}{1+\sin^2\theta} \Rightarrow e^2 = (1 + \sin^2\theta)^{-1}$$

$\Rightarrow e = (1 + \sin^2\theta)^{-1/2}$ Proved.

Q.2 The tangent and the normal to the ellipse $x^2 + 4y^2 = 4$ at a point $P(\theta)$ on it meet the major axis in Q and R respectively. If QR = 2, show that the eccentric angle 'θ' of P is given by $\cos\theta = \pm 2/3$.

Sol. $\frac{x^2}{4} + \frac{y^2}{1} = 1$ $\quad P(\theta) = (2\cos\theta, \sin\theta)$

equation of tangent at $P(\theta)$ is

$$\frac{x\cos\theta}{2} + \frac{y\sin\theta}{1} = 1 \quad \text{...(1)}$$

equation of normal at $P(\theta)$ is

$$\frac{4x}{2\cos\theta} - \frac{y}{\sin\theta} = 3$$

$$\Rightarrow \frac{2x}{\cos\theta} - \frac{y}{\sin\theta} = 3 \quad \text{... (2)}$$

at major axis i.e. at x-axis

from (1) Q = $(2/\cos\theta, 0)$

from (2) R = $\left(\frac{3\cos\theta}{2}, 0\right)$

$$\therefore QR = \sqrt{\left(\frac{2}{\cos\theta} - \frac{3\cos\theta}{2}\right)^2}$$

$$\therefore QR = \frac{2}{\cos\theta} - \frac{3\cos\theta}{2}$$

Θ QR = 2 given

$$\therefore \frac{2}{\cos\theta} - \frac{3\cos\theta}{2} = 2$$

$\Rightarrow 4 - 3\cos^2\theta = 4\cos\theta \Rightarrow 3\cos^2\theta + 4\cos\theta - 4 = 0$

$\Rightarrow 3\cos^2\theta + 6\cos\theta - 2\cos\theta - 4 = 0$

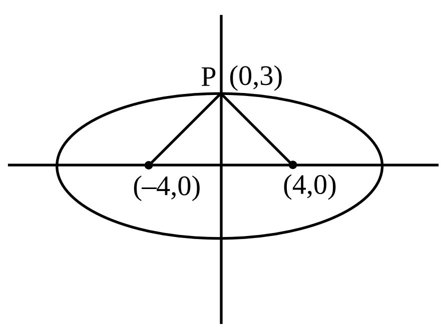
$\Rightarrow 3\cos\theta(\cos\theta + 2) - 2(\cos\theta + 2) = 0$

$\Rightarrow (\cos\theta + 2)(3\cos\theta - 2) = 0$

$\Rightarrow \cos\theta = 2/3$ Proved.

Q.3 A ray emanating from the point (–4, 0) is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with ordinate 3. Find the equation of the reflected ray after first reflection.

Sol.



Ellipse :

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$b^2 = a^2(1 - e^2)$

$e^2 = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}}$

$e = \frac{4}{5}$

foci : S (4, 0)

S' (– 4, 0)

∴ Equation of reflected ray

$(y - 0) = \frac{3}{-4}(x - 4)$

$-4y = 3x - 12$

$3x + 4y = 12$

Q.4 The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, r > 1 represents **[IIT 1981]**

(A) an ellipse (B) a hyperbola

(C) a circle (D) None of these

Sol. **[D]**

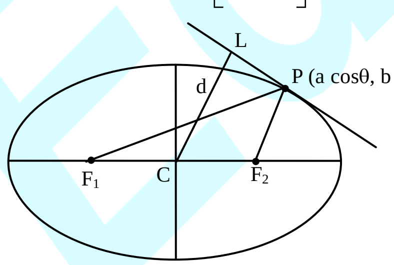
$\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, r > 1

∵ r > 1 given

∴ 1 – r is negative but for a conic namely for ellipse or hyperbola it must be positive. Therefore given equation neither show circle nor ellipse or hyperbola, so option (D) is correct.

Q.5 Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse. If F_1 & F_2 are the two foci of the ellipse , then show that $(PF_1 - PF_2)^2 = 4a^2\left[1 - \frac{b^2}{d^2}\right]$. **[IIT-95]**

Sol.



$PF_1 = a + ex$ and $PF_2 = a - ex$

$PF_1 - PF_2 = 2ex = 2e. a \cos\theta$

∴ $(PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta$ …(i)

Tangent at P(θ) is given by

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$\therefore d = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$

$\therefore \frac{1}{d^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$

or $\frac{b^2}{d^2} = \frac{b^2}{a^2}\cos^2\theta + \sin^2\theta$

$\therefore 1 - \frac{b^2}{d^2} = 1 - \frac{b^2}{a^2}\cos^2\theta - \sin^2\theta$

$= \cos^2\theta\left(1 - \frac{b^2}{a^2}\right) = \cos^2\theta. e^2$

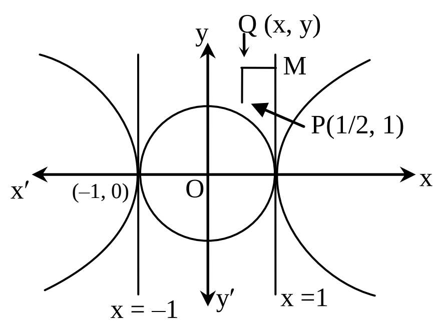
$\Rightarrow 4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2\cos^2\theta. e^2$ …(ii)

from (i) and (ii), we get

$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$ Proved.

Q.6 An ellipse has eccentricity 1/2 and focus at the point P (1/2, 1). Its one directrix is the common tangent nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. Find the equation of the ellipse, in the standard form. **[IIT-96]**

Sol.



It is evident from the diagram that x = 1 is the common tangent to the circle and hyperbola which is nearer to P. Let Q(x,y) be any point on the ellipse, then

$QP = \frac{1}{2}QM$ [∵ e = 1/2]

$\Rightarrow \sqrt{\left(x - \frac{1}{2}\right)^2 + (y-1)^2} = \frac{1}{2}(1-x)$

$\Rightarrow \left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \left(\frac{1-x}{2}\right)^2$

$\Rightarrow 3x^2 - 2x + 4y^2 - 8y + 4 = 0$

$\Rightarrow 3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 4(y^2 - 2y + 1)$

$= \frac{1}{3} + 4 - 4$

$\Rightarrow 3\left(x - \frac{1}{3}\right)^2 + 4(y-1)^2 = \frac{1}{3}$

$\Rightarrow 9\left(x - \frac{1}{3}\right)^2 + 12(y-1)^2 = 1$

which is the required ellipse. In standard form, it is given by

$$\frac{\left(x - \frac{1}{3}\right)^2}{\left(\frac{1}{3}\right)^2} + \frac{(y-1)^2}{\left(\frac{1}{2\sqrt{3}}\right)^2} = 1$$

Q.7 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **[IIT-97]**

Sol. $\frac{x^2}{4} + \frac{y^2}{1} = 1$ ..(i)

and $\frac{x^2}{6} + \frac{y^2}{3} = 1$ ..(ii)

Any tangent to (i) is

$\frac{x\cos\theta}{2} + \frac{y\sin\theta}{1} = 1$ .. (iii)

It cuts (ii) in P and Q tangents at which meet at (h, k) say, then it is chord of contact of (h, k) w.r.t. ellipse (ii).

Hence the equation is

$\frac{hx}{6} + \frac{ky}{3} = 1$ .. (iv)

Comparing (iii) and (iv) as they represents the same line PQ.

$\Rightarrow \frac{\cos\theta}{h/3} = \frac{\sin\theta}{k/3} = 1$

Eliminating θ, we get $\frac{h^2}{9} + \frac{k^2}{9} = 1$

$\therefore$ locus of (h, k) is $x^2 + y^2 = 9 = 6 + 3$

$\Rightarrow x^2 + y^2 = 6 + 3 = a^2 + b^2$

i.e. director circle of second ellipse.

Hence the tangents are at right angles.

Q.8 The number of values of c such that the straight line $y = 4x + c$ touches the curve $(x^2/4) + y^2 = 1$, is - **[IIT-98]**

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. **[C]**

For ellipse condition of tangency is

$c = \pm\sqrt{a^2m^2 + b^2} \Rightarrow c = \pm\sqrt{4.4^2 + 1}$

$\Rightarrow c = \pm\sqrt{65}$

Therefore there are two values of c.

Q.9 If P = (x, y), F_1 = (3, 0), F_2 = (–3, 0) and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals- **[IIT-98]**

(A) 8 (B) 6 (C) 10 (D) 12

Sol. **[C]**

$16x^2 + 25y^2 = 400$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow a^2 = 25$ and $b^2 = 16$

But $b^2 = a^2(1 - e^2)$

$\Rightarrow 16 = 25(1 - e^2)$

$\Rightarrow \frac{16}{25} = 1 - e^2 \Rightarrow e^2 = 1 - \frac{16}{25}$

$\Rightarrow e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

Now foci of the ellipse are $(\pm ae, 0) \equiv (\pm 3, 0)$

we have $3 = a.\frac{3}{5}$ $\Rightarrow a = 5$

Now, $PF_1 + PF_1$ = focal distance

$= 2a = 2 \times 5 = 10$

Q.10 If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) - **[IIT-99]**

(A) lie on a straight line

(B) lie on an ellipse

(C) lie on a circle

(D) are vertices of a triangle

Sol. **[A]**

Θ x_1, x_2, x_3 are in G.P.

$\Rightarrow$ Let common ratio is r.

$\therefore x_1, x_1r, x_1r^2$ are in G.P.

Similarly y_1, y_2, y_3 are in G.P. with common ratio = r

y_1, y_1r, y_1r^2 are in G.P.

$$\therefore \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_1r & x_1r^2 \\ y_1 & y_1r & y_1r^2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}x_1y_1\begin{vmatrix} 1 & r & r^2 \\ 1 & r & r^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

[Θ R_1 & R_2 are Identical]

$\therefore$ Given points lie on a straight line.

Q.11 On the ellipse, $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are- **[IIT-99]**

(A) $\left(\frac{2}{5},\frac{1}{5}\right)$ (B) $\left(-\frac{2}{5},\frac{1}{5}\right)$

(C) $\left(-\frac{2}{5},-\frac{1}{5}\right)$ (D) $\left(\frac{2}{5},-\frac{1}{5}\right)$

Sol. **[B, D]**

$4x^2 + 9y^2 = 1$

$\Rightarrow \frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

We have $a^2 = \frac{1}{4}$, $b^2 = \frac{1}{9}$, $m = \frac{8}{9}$

$\therefore$ The required points are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\ \mu\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

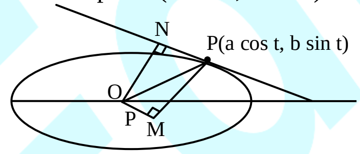
$= \left(\pm\frac{2}{5},\ \mu\frac{1}{5}\right)$

i.e. $\left(\frac{2}{5}, -\frac{1}{5}\right)$ and $\left(-\frac{2}{5}, \frac{1}{5}\right)$

$\therefore$ required points are $\left(\frac{2}{5}, -\frac{1}{5}\right)$ and $\left(-\frac{2}{5}, \frac{1}{5}\right)$

Q.12 Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P. **[IIT-99]**

Sol. Let P be the point (a cos t, b sin t)



Tangent and normal at P are

$\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1 \Rightarrow \frac{ax}{\cos t} - \frac{by}{\sin t} = a^2 - b^2$

Δ OPN $= \frac{1}{2}$ ON. NP $= \frac{1}{2} p_1p_2$

where p_1 and p_2 are perpendicular distances of O(0, 0) from T and N. Δ is maximum when p_1p_2 is maximum.

$p_1 = \frac{1}{\sqrt{\frac{\cos^2 t}{a^2}+\frac{\sin^2 t}{b^2}}}$ and $p_2 = \frac{a^2-b^2}{\sqrt{\frac{a^2}{\cos^2 t}+\frac{b^2}{\sin^2 t}}}$

$\therefore p_1p_2 = \frac{a^2-b^2}{1+1+\frac{a^2}{b^2}\tan^2 t+\frac{b^2}{a^2}\cot^2 t}$

$= \frac{a^2-b^2}{4+\left(\frac{a}{b}\tan t-\frac{b}{a}\cot t\right)^2}$

It will be maximum when denominator is minimum

i.e. $\frac{a}{b}\tan t - \frac{b}{a}\cot t = 0 \Rightarrow \tan^2 t = \frac{b^2}{a^2}$

$\therefore \tan t = \pm\frac{b}{a}$

$\Rightarrow \sin t = \pm\frac{b}{\sqrt{a^2+b^2}}$ and $\cos t = \pm\frac{a}{\sqrt{a^2+b^2}}$

Hence the point P is $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

Q.13 Consider the family of circles $x^2 + y^2 = r^2$; $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at A and B, then find the equation of the locus of the mid point of AB. **[IIT-99]**

Sol. For ellipse, $a^2 = 25$, $b^2 = 4$

$$\left[\Theta\ \frac{x^2}{25}+\frac{y^2}{4}=1 \text{ ellipse}\right]$$

Any tangent to ellipse is given by

$y = mx + \sqrt{25m^2+4}$ …(i)

it meets the axes in A and B whose mid-point is (h, k) say

$2h = \frac{-\sqrt{25m^2+4}}{m}$, $2k = \sqrt{25m^2+4}$

$\therefore \frac{h}{k} = -\frac{1}{m}$ … (ii)

Again (i) is a tangent to circle $x^2 + y^2 = r^2$, therefore the condition of tangency p = r

$\Rightarrow \frac{\sqrt{25m^2+4}}{\sqrt{1+m^2}} = r \Rightarrow (25m^2 + 4) = r^2 (1 + m^2)$

$\Rightarrow m^2 (25 - r^2) = r^2 - 4 \Rightarrow \frac{k^2}{h^2}(25 - r^2) = r^2 - 4$

$\Rightarrow k^2 = h^2\left(\frac{r^2-4}{25-r^2}\right) \Rightarrow k = \pm h\sqrt{\frac{r^2-4}{25-r^2}}$

$\therefore$ locus is $y = \pm x\sqrt{\frac{r^2-4}{25-r^2}}$

where r is given.

ANSWERKEY

EXERCISE # 1

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	A	B	D	C	D	B	B	C	C	A	B	A	A	D
Qus.	16	17	18	19	20	21									
Ans.	A	A	B	A	C	C									

22. False **23.** True **24.** True **25.** False **26.** $\tan^{-1}4\sqrt{3}$ **27.** $\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}$ **28.** 2

EXERCISE # 2

PART - A

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ans.	C	A	B	B	C	A	B	C	A	B	B	A	D	A	D	A	A	C

PART-B

Qus.	19	20	21
Ans.	A,B,C	A,B	A,C

PART-C

Qus.	22	23	24
Ans.	A	A	A

PART-D

25. A $\rightarrow$ S; B $\rightarrow$ P; C $\rightarrow$ Q; D $\rightarrow$ R

EXERCISE # 3

1. $e = \frac{1}{\sqrt{2}}; \left(\frac{x^2}{2}\right) + y^2 = 1$ **2.** $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ **4.** $bx + a\sqrt{3}\,y = 2ab$ **5.** $55\sqrt{2}$ sq. unit

7. $\frac{x^2}{\frac{a^4}{c^2}} + \frac{y^2}{\frac{b^4}{c^2}} = 1;\ e = \frac{\sqrt{3}}{2}$ **12.** $\pm\sqrt{a}\ x\ \mu\ \sqrt{b}\ y = (a - b)\ \sqrt{a + b}\ ;\ |a - b|$

14. $(x^2 + y^2 - a^2 - b^2)\tan^2\alpha = 4(b^2x^2 + a^2y^2 - a^2b^2)$ **15.** $a^4b^4(x^2 + y^2) = (a^2 + b^2)(b^2x^2 + a^2y^2)^2$

18. (A) **19.** (A) **20.** (A) **21.** (A) **22.** (C) **23.** (A)

24. (A) **25.** (B) **26.** (B)

EXERCISE # 4

2. $\frac{x^2}{a^2} + \frac{(r+s)^2 y^2}{(ar+bs)^2} = 1$

4. (2, 1)

5. (A)

6. (A)

7. (A)

8. Length of x -intercept = $2\sqrt{7}$ and length of y-intercept = $4\sqrt{\frac{7}{3}}$

9. (B,C)

10. (C)

11. (D)

12. (D)

13. (C)

14. (A)

15. (C)

EXERCISE # 5

3. $3x + 4y - 12 = 0$

4. (D)

6. $9(x - 1/3)^2 + 12(y - 1)^2 = 1$

8. (C)

9. (C)

10. (A)

11. (B,D)

12. $\left(\frac{\pm a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$

13. $25y^2 + 4x^2 = 4x^2y^2$