HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. **Hint :** Distance between directrix and focus is 2a

2. The coordinates of the focus and vertex of required parabola are $S(a_1, 0)$ and $A(a, 0)$, respectively. Therefore, the distance between the vertex and the focus is $AS = a_1 - a$. So, the length of the latus rectum is $4(a_1 - a)$.

Thus, the equation of the parabola is

$$y^2 = 4(a_1 - a)(x - a)$$

4. $x = 3 \cot t,\ y = 4 \sin t$

Eliminating t, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

which is an ellipse. Therefore,

$$x^2 - 2 = 2 \cos t \text{ and } y = 4 \cos^2 \frac{t}{2}$$

or $y = 2(1 + 2 \cos t)$

and $y = 2\left(1 + \frac{x^2 - 2}{2}\right)$

which is a parabola.

$\sqrt{x} = \tan t\ ;\ \sqrt{y} = \sec t$

Eliminating t, we have

$y - x = 1$

which is a straight line.

$$x = \sqrt{1 - \sin t}$$

$$y = \sin\frac{t}{2} + \cos\frac{t}{2}$$

Eliminating t, we have

$x^2 + y^2 = 1 - \sin t + 1 + \sin t = 2$

which is a circle.

5. Given $(t^2, 2t)$ be one end of focal chord then other end be $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$

length of focal chord

$$= \sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + \left(2t + \frac{2}{t}\right)^2} = \left(t + \frac{1}{t}\right)^2$$

8. $(\sqrt{3h}, \sqrt{3k+2})$ lie on the line $x - y - 1 = 0$. Therefore,

$(\sqrt{3h})^2 = (\sqrt{3k+2} + 1)^2$

or $3h = 3k + 2 + 1 + 2\sqrt{3k+2}$

or $3^2(h - k - 1)^2 = 2^2(\sqrt{3k+2})^2$

or $9(h^2 + k^2 + 1 - 2hk - 2h + 2k) = 4(3k + 2)$

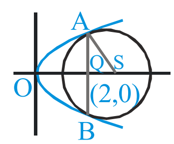
or $9(x^2 + y^2) - 18xy - 18x + 6y + 1 = 0$

Now, $h^2 = ab$ and $\Delta \neq 0$

Therefore, the locus is a parabola.

9. Focus of parabola $y^2 = 8x$ is (2, 0). Equation of circle with centre (2, 0) is

$(x - 2)^2 + y^2 = r^2$



AB is common chord

Q is mid point i.e. (1, 0)

$AQ^2 = y^2$ where $y^2 = 8 \times 1 = 8$

$\therefore\ r^2 = AQ^2 + QS^2 = 8 + 1 = 9$

so circle is $(x - 2)^2 + y^2 = 9$

10. Let the point P(h, k) on the parabola divides the line joining A(4, –6) and B(3, 1) in the ratio λ.

Then, we have

$$(h, k) \equiv \left(\frac{3\lambda + 4}{\lambda + 1}, \frac{\lambda - 6}{\lambda + 1}\right)$$

This point lies on the parabola. Therefore,

$$\left(\frac{\lambda - 6}{\lambda + 1}\right)^2 = 4\left(\frac{3\lambda + 4}{\lambda + 1}\right)$$

or $(\lambda - 6)^2 = 4(3\lambda + 4)(\lambda + 1)$

or $11\lambda^2 + 40\lambda - 20 = 0$

or $\lambda = \frac{-20 \pm 2\sqrt{155}}{11} : 1$

Then equation of QR is

$ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$

$\Rightarrow$ $-2ax + ky + 2ah - k^2 = 0$ .....(ii)

Clearly (i) and (ii) represents same line.

$$\frac{2a}{-2a} = \frac{-y_1}{k} = \frac{2ax_1}{2ah - k^2}$$

$y_1 = k$ and $2ax_1 = k^2 - 2ah$

$2ax_1 = y_1^2 - 2ah$

$2ax_1 = 4ax_1 - 2ah$

$\Rightarrow$ $x_1 = h$

$\therefore$ mid point of QR is (x_1, y_1)

EXERCISE - 2

Part # I : Multiple Choice

2. The line $y = 2x + c$ is a tangent to $x^2 + y^2 = 5$.

If $c^2 = 25$, then $c = \pm 5$

Let the equation of the parabola be $y^2 = 4ax$. Then

$$\frac{a}{2} = \pm 5$$

or $a = \pm 10$

So, the equation of the parabola is $y^2 = \pm 40x$.

Also, the equation of the directrices are $x = \pm 10$.

3. Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $(h, (h + at^2)/t)$.

The chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t}\right)y = a^2$$

or $(aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$

which is a family of straight lines passing through the point of intersection of

$ty - a = 0$ and $x + \frac{y}{t}$ 0

So, the fixed point is $(-a/t^2, a/t)$. Therefore,

$$x_2 = -\frac{a}{t^2},\ y_2 = \frac{a}{t}$$

Clearly, $x_1x_2 = -a^2$, $y_1y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$

and $\frac{y_1}{y_2} = 2t^2$

or $4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$

6. $P \equiv (\alpha, \alpha + 1)$, where $\alpha \neq 0, -1$

or $P \equiv (\alpha, \alpha - 1)$, where $\alpha \neq 0, 1$

The point $(\alpha, \alpha + 1)$ is on $y^2 = 4x + 1$. Therefore,

$(\alpha + 1)^2 = 4\alpha + 1$

or $\alpha^2 - 2\alpha = 0$

or $\alpha = 2$ $(\because \alpha \neq 0)$

$$2x = a\left(1+\frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

i.e., $2ax = a^2 + y^2$

i.e., $y^2 = 2a\left(x - \frac{a}{2}\right)$

It is a parabola with vertex at (a/2, 0) and latus rectum 2a.

The directrix is

$$x - \frac{a}{2} = -\frac{a}{2}$$

i.e., $x = 0$

The focus is

$$x - \frac{a}{2} = \frac{a}{2}$$

i.e., $x = a$

i.e., (a, 0)

18. Any point on $x + y = 1$ can be taken as $(t, 1 - t)$.

The equation of chord with this as midpoint is

$$y(1 - t) - 2a(x + t) = (1 - t^2) - 4at$$

It passes through (a, 2a). So,

$$t^2 - 2t + 2a^2 - 2a + 1 = 0$$

This should have two distinct real roots. So,

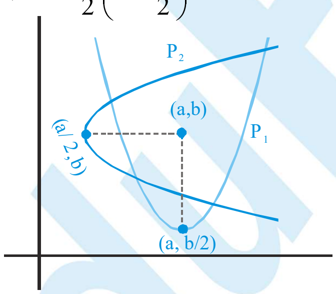
Discriminant > 0 i.e., $a^2 - a < 0$

$$0 < a < 1$$

So, length of latus rectum < 4

and $0 < \lambda < 1$

17. $P_1 \equiv (x - a)^2 = 4.\frac{b}{2}\left(y - \frac{b}{2}\right)$



$\Rightarrow$ $x^2 - 2ax + a^2 - 2yb + b^2 = 0$

Similarly

$$P_2 \equiv y^2 - 2ax - 2by + a^2 + b^2 = 0$$

Common chord is $P_1 - P_2 = 0$

$\Rightarrow$ $x^2 - y^2 = 0$

$\Rightarrow$ $(x + y)(x - y) = 0$

slope will be 1, –1

Part # II : Assertion & Reason

1. Any tangent having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or $y = mx + am + \frac{a}{m}$

This is tangent to the given parabola for all $m \in R - \{0\}$.

Hence, statement 2 is false

However, statement 1 is true as when m = 1, the tangent is $y = x + 2a$.

3. Let $P_1\ (at_1^2, 2at_1)$ & $Q_1\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$

$P_2\ (at_2^2, 2at_2)$ & $Q_2\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

on $y^2 = 4ax$

equation of P_1P_2 :

$$(t_1 + t_2)y = 2x + 2at_1t_2 \quad .....(i)$$

equation of Q_1Q_2

$$-(t_1 + t_2)y = 2x\ t_1t_2 + 2a \quad .....(ii)$$

add (i) & (ii)

$x = -a$ which is directrix of $y^2 = 4ax$

Locus of point of intersection of tangent is directrix. In case of parabola director circle is directrix

4. Statement 2 is true as it is the definition of parabola.

From statement 1, we have

$$\sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x + 4y + 5|}{5}$$

which is not a parabola as the point (1, –2) lie on the line $3x + 4y + 5 = 0$.

Hence, statement 1 is false.

6. Given C : $(y - 1)^2 = 8(x + 2)$ (which is a parabola)

Clearly, P(–4, 1) lies on the directrix x = –4.

Also, P(–4, 1) lies on the axis of the parabola,

i.e., at y = 1.

So, from any point on the directrix of the parabola, if two tangents are drawn to the parabola, then these two tangents will be mutually perpendicular.

(D) $(y-1)^2 = 2(x+2)$

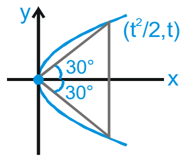
vertex is $(-2, 1)$

so equation is $(y-1)^2 = 2(x+2)$

$\Rightarrow$ $Y^2 = 2X$

Let point on $Y^2 = 2X$ is $\left(\frac{1}{2}t^2, t\right)$

From fig. $\tan 30° = \frac{2}{t}$



$\Rightarrow t = 2\sqrt{3}$

so point on parabola is $(6, 2\sqrt{3})$.

But when vertex change, distance (or length of side of equilateral triangle) remain same

$\therefore$ length of side $= \sqrt{(6)^2 + (2\sqrt{3})^2} = 4\sqrt{3}$.

Part # II : Comprehension

Comprehension # 1

1. (B), 2. (C) 3. (D)

1. **(B)** Since no point of the parabola is below the x-axis,

$D = a^2 - 4 \le 0$

Therefore, the maximum value of a is 2.

The equation of the parabola when a = 2 is

$y = x^2 + 2x + 1$

It intersects the y-axis at (0, 1)

The equation of the tangent at (0, 1) is

$y = 2x + 1$

Since $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$, we get

$r = \frac{1}{\sqrt{5}}$

2. **(C)** The equation of the tangent at (0, 1) to the parabola

$y = x^2 + ax + 1$

is $y - 1 = a(x - 0)$

or $ax - y + 1 = 0$

As it touches the circle, we get

$y = \frac{1}{\sqrt{a^2+1}}$

The radius is maximum when a = 0.

Therefore, the equation of the tangent is y = 1.

Therefore, the slope of the tangent is 0.

3. **(D)** The equation of tangent is $y = ax + 1$

The intercepts are $-1/a$ and 1.

Therefore, the area of the triangle bounded by the tangent and the axes is

$\frac{1}{2}\left|-\frac{1}{a}, 1\right| = \frac{1}{2|a|}$

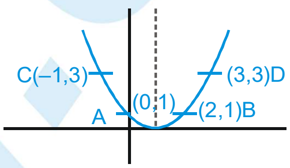
It is minimum when a = 2. Therefore,

Minimum area $= \frac{1}{4}$

Comprehension # 2

Axis of parabola is bisector of parallel chord A B & CD are parallel chord.

so axis x = 1



equation of parabola is

$(x-1)^2 = ay + b$

It passing (0, 1) & (3, 3)

So $1 = a + b$ .....**(i)**

$4 = 3a + b$ .....**(ii)**

from **(i)** & **(ii)**

$a = \frac{3}{2}$ & $b = -\frac{1}{2}$

$(x-1)^2 = \frac{3}{2}(y - \frac{1}{3})$

1. Vertex $(1, \frac{1}{3})$

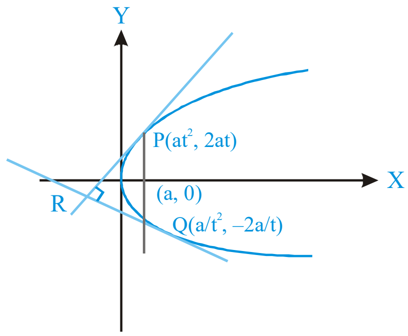
2. $a = \frac{3}{8}$

directrix of $x^2 = 4ay$ is $y = -a$

$y - \frac{1}{3} = -\frac{3}{8}$

$\Rightarrow y = \frac{1}{3} - \frac{3}{8}$

$y + \frac{1}{24} = 0$



Hence, the point of intersection of tangent at point P(t) and Q(–1/t) is (–a, a{t – (1/t)}) and the coordinates of the centroid (G) is ((a/3) {t^2 – (1/t^2) – 1}, a{t – (1/t)}). Hence, the slope of line RG is 0 (R is the orthocenter).

4. Equation of tangent of

$y^2 = 4ax$ in slope form at(x_1, y_1) is

$y_1 = mx_1 + \frac{a}{m}$ .....**(i)**

equation of normal at $(2bt_1, bt_1^2)$ on $x^2 = 4by$

$x + t_1y = 2bt_1 + bt_1^3$

It passes through (x_1, y_1)

$\therefore\ x_1 + t_1y_1 = 2bt_1 + bt_1^3$ .....**(ii)**

(i) & **(ii)** are same equation so compare

$$\frac{1}{t_1} = -\frac{m}{1} = \frac{a}{m(2bt_1 + bt_1^3)}$$

$t_1m = -1$

$-m^2t_1\,(2b + bt_1^2) = a$

$\Rightarrow\ m\,(2b + bt_1^2) = a$ .....**(iii)**

Put $m = -\frac{1}{t_1}$ in equation **(iii)**

$2b + bt_1^2 = -at_1$

$bt_1^2 + at_1 + 2b = 0$

t_1 will be real

$a^2 > 8b^2$

5. $x^2 = y$ .....**(i)**

Let equation of OP $y = mx$ .....**(ii)**

equation of OQ $y = \frac{-1}{m}x$ .....**(iii)**

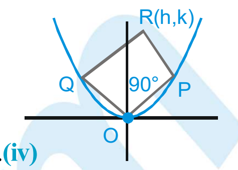
from (1) & (2) we get $P(m, m^2)$

from (1) & (3) we get $Q\left(\frac{-1}{m}, \frac{1}{m^2}\right)$

equation of PR

$$y - m^2 = -\frac{1}{m}(x - m)$$

$y + \frac{1}{m}x = m^2 + 1$ .....**(iv)**



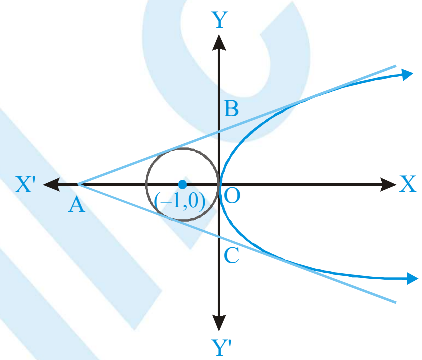
equation of QR is

$$y - \frac{1}{m^2} = m\left(x + \frac{1}{m}\right)$$

$y - mx = 1 + \frac{1}{m^2}$ .....**(v)**

Locus of R solving (4) & (5) & eliminating m we get $x^2 = y - 2$

6.



As the circle is $(x + 1)^2 + y^2 = 1$, one of the common tangent is along the y-axis.

Let the other common tangent has slope m.

Then, its equation is

$$y = mx + \frac{1}{m}$$

Solving it with the equation of circle, we get

$$x^2 + \left(mx + \frac{1}{m}\right)^2 + 2x = 0$$

or $(1 + m^2)x^2 + 4x + \frac{1}{m^2} = 0$

As the line touches the circle,

$D = 0$

or $16 - \frac{1}{m^2}(1 + m^2) = 0$

or $4m^2 = 1 + m^2$

or $m = \pm\frac{1}{\sqrt{3}}$

i.e., $\angle BOA = \angle OAC = \frac{\pi}{6}$

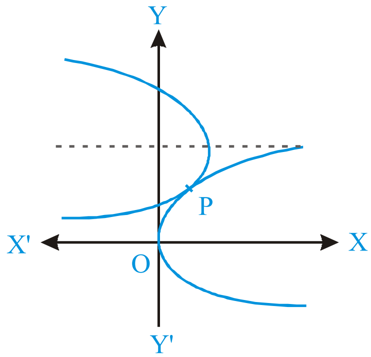
Hence, the triangle is equilateral.

on solving **(i)** and **(ii)**

$$(y-k)^2 = -4a\left(\frac{y^2}{4a} - h\right)$$

or $y^2 - 2ky + k^2 = -y^2 + 4ah$

or $2y^2 - 2ky + k^2 - 4ah = 0$



Since the two parabola touch each other, D = 0

i.e., $4k^2 - 8(k^2 - 4ah) = 0$

or $-4k^2 + 32ah = 0$

or $k^2 = 8ah$

Therefore, the locus of the vertex of the moving parabola is $y^2 = 8ax$.

13. Let parabola is $y^2 = 4ax$

equation of normal at $(am^2, 2am)$

$y + mx = 2am + am^3$

it passes through (h, k)

$am^3 + m(2a - h) - k = 0$

its root are m_1, m_2 & m_3

$$\Sigma m_1 = 0,\ \Sigma m_1 m_2 = \frac{2a-h}{a}$$

$$m_1 m_2 m_3 = \frac{k}{a}$$

let equation of circle be

$x^2 + y^2 + 2gx + 2fy + C = 0$

It passes $(am^2, 2am)$

$a^2m^4 + 4a^2m^2 + 2agm^2 + 4afm + C = 0$

$a^2m^4 + m^2(4a^2 + 2ag) + 4afm + C = 0$

its roots m_1, m_2, m_3 & m_4

$m_1 + m_2 + m_3 + m_4 = 0,$

$\Rightarrow$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $m_4 = 0$ $\Rightarrow$ circle passes (0, 0)

$m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 + m_1m_3 + m_2m_4$

$$= \frac{4a^2 + 2ag}{a^2}$$

$$\Rightarrow \frac{2a-h}{a} = \frac{4a^2 + 2ag}{a^2}$$

$\Rightarrow$ $2a - h = 4a + 2g$

$$\Rightarrow g = \frac{-h-2a}{2}$$

$$m_1m_2m_3 + m_2m_3m_4 + m_3m_4m_1 + m_4m_1m_2 = \frac{-4af}{a^2}$$

$$\Rightarrow \frac{k}{a} = \frac{-4af}{a^2}$$

$$\Rightarrow f = \frac{-k}{4}$$

equation of circle

$$x^2 + y^2 - (h + 2a)x + \frac{k}{2}y = 0$$

14. Let point on $y^2 = 4ax$ be $P(at^2, 2at)$

equation of tangent of P

$ty = x + at^2$ .....**(i)**

It intersect the directrix $x = -a$ .....**(ii)**

point of intersection of (1) & (2)

is $A\left(-a, a\left(t - \frac{1}{t}\right)\right)$

Let mid point of PA is (h, k)

$2h = at^2 - a$ .....**(iii)**

$2k = 2at + a\left(t - \frac{1}{t}\right)$ .....**(iv)**

from (3) & (4) eliminating t & replace $h \to x$ & $y \to k$ we get

$y^2(2x + a) = a(3x + a)^2$

16. Normal at $P(am^2, 2am)$ on $y^2 = 4ax$

$y + mx = 2am + am^3$ .....**(i)**

$G(2a + am^2, 0)$

Equation of QG is $x = 2a + am^2$

Solving with parabola we get

$$y = \pm 2a\sqrt{2 + m^2}$$

$QG^2 - PG^2 =$

$4a^2(2 + m^2) - (am^2 - am^2 - 2a)^2 - (2am)^2$

24. Let the point P be (p, 0) and the equation of the chord through P be

$$\frac{x-p}{\cos\theta}=\frac{y-0}{\sin\theta}=r \qquad (r\in R) \qquad ....(i)$$

Therefore,

$(r\cos\theta + p, r\sin\theta)$ lies on the parabola $y^2 = 4ax$.

So, $r^2\sin^2\theta - 4ar\cos\theta - 4ap = 0$ ....(ii)

If AP = r_1 and BP = $-r_2$, then r_1 and r_2 are the roots of (ii).

Therefore,

$$r_1+r_2=\frac{4a\cos\theta}{\sin^2\theta},\ r_1r_2=\frac{-4ap}{\sin^2\theta}$$

Now, $\frac{1}{AP^2}+\frac{1}{BP^2}=\frac{1}{r_2^2}+\frac{1}{r_2^2}$

$$=\frac{(r_1+r_2)^2-2r_1r_2}{r_1^2r_2^2}$$

$$=\frac{\cos^2\theta}{p^2}+\frac{\sin^2\theta}{2ap}$$

Since $\frac{1}{AP^2}+\frac{1}{BP^2}$ should be independent of θ, we take p = 2a. Then,

$$\frac{1}{AP^2}+\frac{1}{BP^2}=\frac{1}{4a^2}(\cos^2\theta+\sin^2\theta)=\frac{1}{4a^2}$$

Hence, $\frac{1}{AP^2}+\frac{1}{BP^2}$ is independent of θ for all the position of the chord if P ≡ (2a, 0).

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

3. It is a fundamental theorem.

4. Given parabolas are

$y^2 = 4ax$ ..... (i)

$x^2 = 4ay$ ..... (ii)

Putting the value of y from (ii) in (i), we get

$$\frac{x^2}{16a^2}=4ax \Rightarrow x(x^3-64a^3)=0$$

$\Rightarrow$ x = 0, 4a

from (ii), y = 0, 4a. Let A ≡ (0, 0); B ≡ (4a, 4a)

Since, given line 2bx + 3cy + 4d = 0 passes through A and B,

∴ d = 0 and 8ab + 12ac = 0

$\Rightarrow$ 2b + 3c = 0. (∵ a ≠ 0)

Obviously, $d^2 + (2b + 3c)^2 = 0$

5. $y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$

$$\Rightarrow y=\frac{a^3}{3}\left[x^2+\frac{x}{2}\times\frac{3}{a}\times\frac{2}{2}\right]-2a$$

$$\Rightarrow y=\frac{a^3}{3}\left[\left(x+\frac{3}{4a}\right)^2\right]-\frac{3a}{16}-2a$$

$$\Rightarrow y+\frac{35a}{16}=\frac{4a^3}{12}\left(x+\frac{3}{4a}\right)^2$$

∴ Vertices will be (α, β)

So that $\alpha=-\frac{3}{4a}$ and $\beta=-\frac{35a}{16}$

or $\alpha\beta=\left(\frac{-3}{4a}\right)\times\left(\frac{-35a}{16}\right)=\frac{105}{64}$

∴ Required locus will be $xy=\frac{105}{64}$

6. Point must be on the directrix of the parabola

Hence the point is (–2, 0)

8. Locus of point of intersection of perpendicular tangent is directrix of the parabola.

so x = –1

4. The given curves are

$y^2 = 8x$ .....**(i)**

and $xy = -1$ .....**(ii)**

If m is the slope of tangent to (1), then equation of tangent is $y = mx + 2/m$.

If this tangent is also a tangent to (2), then

$$x\left(mx + \frac{2}{m}\right) = -1$$

$\Rightarrow$ $mx^2 + \frac{2}{m}x + 1 = 0$

$\therefore$ $m^2x^2 + 2x + m = 0$

We should get repeated roots for this equation (conditions of tangency)

$\Rightarrow$ $D = 0$

$\therefore$ $(2)^2 - 4m^2 . m = 0$

$\Rightarrow$ $m^3 = 1$

$\Rightarrow$ $m = 1$

Hence required tangent is $y = x + 2$.

6. Let P be the point (h, k). Then equation of normal to parabola $y^2 = 4x$ from point (h, k), if m is the slope of normal, is $y = mx - 2m - m^3 = 0$

As it passes through (h, k), therefore

$mh - k - 2m - m^3 = 0$

or $m^3 + (2 - h)m + k = 0$ .....**(i)**

which is cubic in m, giving three values of m say m_1, m_2 and m_3. Then $m_1m_2m_3 = -k$ (from equation)

but given that $m_1m_2 = \alpha$

$\therefore$ We get $m_3 = -\frac{k}{\alpha}$

But m_3 must satisfy equation **(i)**

$\therefore$ $\frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$

$\Rightarrow$ $k^2 - 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$

$\therefore$ Locus of P(h, k) is $y^2 = \alpha^2x + (\alpha^3 - 2\alpha^2)$

But ATQ, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get $\alpha^2 = 4$ and $\alpha^3 - 2\alpha^2 = 0$

$\Rightarrow$ $\alpha = 2$

8. The given equation of parabola is

$y^2 - 2y - 4x + 5 = 0$ .....**(i)**

$\Rightarrow$ $(y - 1)^2 = 4(x - 1)$

Any parametric point on this parabola is

$P(t^2 + 1, 2t + 1)$

Differentiating **(i)** w.r.t. x, we get

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} - 4 = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{2}{y - 1}$

$\therefore$ Slope of tangent to **(i)** at pt.

$P(t^2 + 1, 2t + 1)$ is $m = \frac{2}{2t} = \frac{1}{t}$

$\therefore$ Equation of tangent at $P(t^2 + 1, 2t + 1)$ is

$$y - (2t + 1) = \frac{1}{t}(x - t^2 - 1)$$

$\Rightarrow$ $yt - 2t^2 - t = x - t^2 - 1$

$\Rightarrow$ $x - yt + (t^2 + t - 1) = 0$ .....**(ii)**

Now direction of given parabola is

$(x - 1) = -1 \Rightarrow x = 0$

Tangent to (2) meets directrix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Let pt. R be (h, k)

ATQ R divides the line joining QP in the ratio $\frac{1}{2} : 1$ i.e. 1 : 2 externally.

$\therefore$ $(h, k) = \left[\frac{1(1 + t^2) - 0}{-1}, \frac{t + 2t^2 - 2t^2 - 2t + 2}{-t}\right]$

$\Rightarrow$ $h = -(1 + t^2)$ and $k = \frac{t - 2}{t}$

$\Rightarrow$ $t^2 = -1 - h$ and $t = \frac{2}{1 - k}$

Eliminating t we get $\left(\frac{2}{1 - k}\right)^2 = -1 - h$

$\Rightarrow$ $4 = -(1 - k)^2(1 - h)$

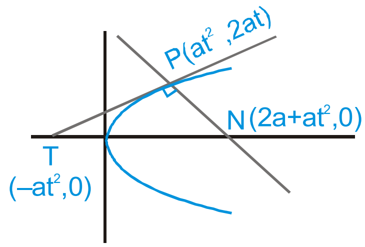
$\Rightarrow$ $(h - 1)(k - 1)^2 + 4 = 0$

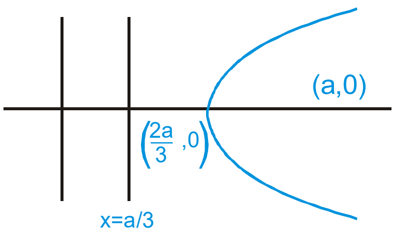
$\therefore$ locus of R(h, k) is, $(x - 1)(y - 1)^2 + 4 = 0$

21. $3h = 2a + at^2$

$3k = 2at$

$$3h = 2a + \frac{a \cdot 9k^2}{4a^2}$$





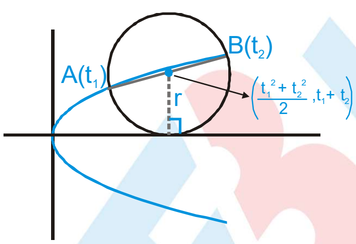
$$y^2 = \frac{4a}{9}(3x - 2a)$$

$$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$$

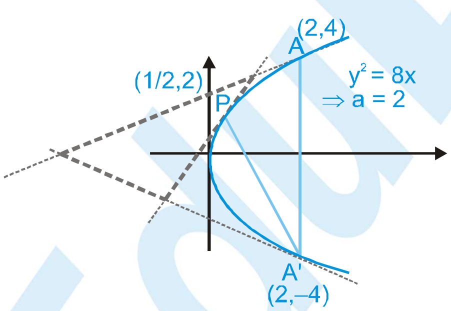
22. $t_1 + t_2 = r$

$$\frac{2}{r} = \frac{2}{t_1 + t_2}$$

similarly $-\frac{2}{r}$ is also possible



23.



Δ_1 = area of Δ PAA' $= \frac{1}{2}.8.\frac{3}{2} = 6$

$$\Delta_2 = \frac{1}{2}(\Delta_1)$$

(Using property : Area of triangle formed by tangents is always half of original triangle)

$$\Rightarrow \frac{\Delta_1}{\Delta_2} = 2$$

24. Let P be (h, k)

on using section formula $P\left(\frac{x}{4}, \frac{y}{4}\right)$

$\therefore$ $h = \frac{x}{4}$ and $k = \frac{y}{4}$

$\Rightarrow$ x = 4h and y = 4k

$\because$ (x, y) lies on $y^2 = 4x$

$\therefore$ $16k^2 = 16h$ $\Rightarrow$ $k^2 = h$

Locus of point P is $y^2 = x$.

25. Equation of normal is $y = mx - 2m - m^3$

It passes through the point (9, 6) then

$6 = 9m - 2m - m^3$

$\Rightarrow$ $m^3 - 7m + 6 = 0$

$\Rightarrow$ $(m - 1)(m - 2)(m + 3) = 0$

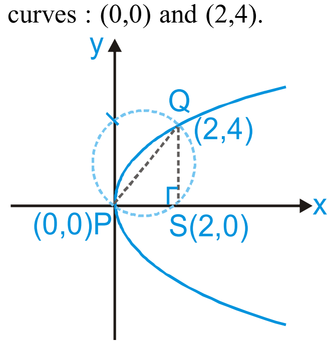
$\Rightarrow$ m = 1, 2, –3

Equations of normals are

$y - x + 3 = 0$, $y + 3x - 33 = 0$

& $y - 2x + 12 = 0$

26. Focus of parabola S(2,0) points of intersection of given curves : (0,0) and (2,4).



Area (ΔPSQ) $= \frac{1}{2}.2.4 = 4$ sq. units

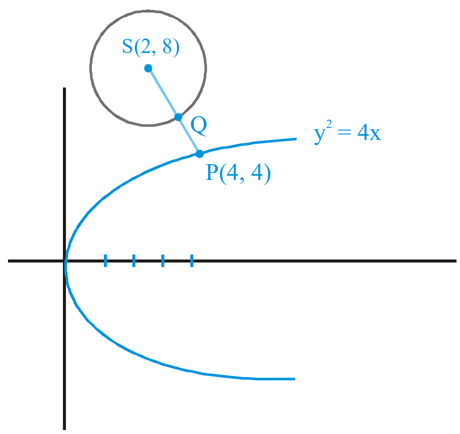
36. $x^2 + y^2 - 4x - 16y + 64 = 0$

Centre $S \equiv (2, 8)$

$r = \sqrt{4 + 64 - 64} = 2$

Normal $y = mx - 2m - m^3$

As shortest distance $\Rightarrow$ common normal



$\Rightarrow$ It passes S(2, 8)

$\Rightarrow$ $8 = 2m - 2m - m^3$

$\Rightarrow$ $m = -2$

Normal at P $y = -2x + 12$

Point $P \equiv (am^2, -2am) \equiv (4, 4)$

$SP = \sqrt{(4-2)^2 + (8-4)^2} = 2\sqrt{5}$

$SQ : QP = 2 : \left(2\sqrt{5} - 2\right)$

Slope of tangent at Q is $= \frac{1}{2}$

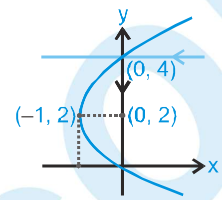
MOCK TEST

1. (A)

Given curve is $(y-2)^2 = 4(x+1)$

focus $(x + 1 = 1, y - 2 = 0) \Rightarrow (0, 2)$

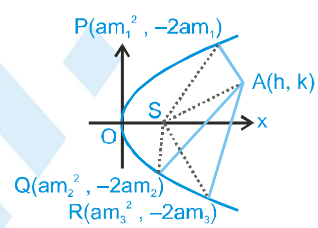
Point of intersection of the curve and



$y = 4$ is (0,4) from the reflection property of parabola reflected ray passes through the focus.

$\therefore$ $x = 0$ is required line

3. (C)



|SP|. |SQ|. |SR|

$= a^3 (1 + m_1^2)(1 + m_2^2)(1 + m_3^2)$

$= a^3 |1 + (\sum m_1)^2 - 2\sum m_1 m_2 + (\sum m_1 m_2)^2 - 2m_1 m_2 m_3 \sum m_1 + (m_1 m_2 m_3)^2|$

$= a^3 \left|1 + 0 + \frac{2(h-2a)}{a} + \frac{(h-2a)^2}{a^2} - 0 + \frac{k^2}{a^2}\right|$

$= a\left|k^2 + (h-a)^2\right| = a\,(SA)^2$

5. (C)

Since $(4, -4)$ and $(9, 6)$ lie on $y^2 = 4a(x - b)$

$\therefore$ $4 = a(4 - b)$ and $9 = a(9 - b)$

$\therefore$ $a = 1$ and $b = 0$

$\therefore$ the parabola is $y^2 = 4x$

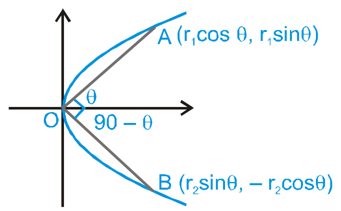
let the point R be $(t^2, 2t)$

$\therefore$ Area of Δ PRQ

$$= \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 9 & 6 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$$

from **(iii)** and **(ii)**

$$\sin^3\theta = \frac{64\,a^3}{r_1^2\,r_2} \qquad ......\textbf{(iv)}$$



$$\text{Similarly } \cos^3\theta = \frac{64\,a^3}{r_1\,r_2^2} \qquad ......\textbf{(v)}$$

15. (A,B,C,D)

Obvious

16. (A)

Statement- 1 : $y^2 = 4x$

Clearly $x = 0$ is tangent to the parabola at (0,0)

$y^2 = 4(-y-1) \Rightarrow (y+2)^2 = 0$

$\therefore$ $x + y + 1 = 0$ is a tangent to the parabola

Again $y^2 = 4(y-1)$ i.e. $y^2 - 4y + 4 = 0$ i.e. $(y-2)^2 = 0$

$\therefore$ $x - y + 1 = 0$ is a tangent to the parabola

(1, 0) is the focus.

$\therefore$ Statement- 1 true

Consider a parabola $y^2 = 4\,ax$

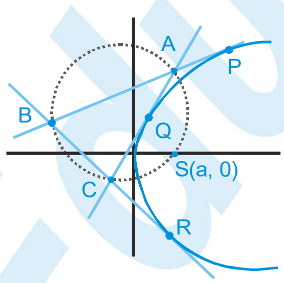
Let P (t_1), Q (t_2) and R(t_3) be these point on it.

Tangents are drawn at these points which

intersect at $A \equiv (a\,t_1\,t_2\,,\,a\,(t_1 + t_2))$

$B \equiv (a\,t_1\,t_3\,,\,a\,(t_1 + t_3))$

$C \equiv (a\,t_2\,t_3\,,\,a\,(t_2 + t_3))$



Let $\angle SAC = \alpha$ & $\angle SBC = \beta$

$$\Rightarrow \tan\alpha = \left|\frac{\dfrac{1}{t_2} - \dfrac{t_1+t_2}{t_1t_2-1}}{1 + \dfrac{1}{t_2}\left(\dfrac{t_1+t_2}{t_1t_2-1}\right)}\right| = \left|\frac{1}{t_1}\right|$$

$$\text{similarly } \tan\beta = \left|\frac{1}{t_1}\right|$$

$\Rightarrow$ $\alpha = \beta$ or $\alpha + \beta = \pi$

$\Rightarrow$ A, B, C and S are concyclic.

17. Given $C : (y-1)^2 = 8(x+2)$ (which is a parabola.)

Clearly $P(-4, 1)$ lies on directrix $x = -4$.

Also $P(-4, 1)$ lies on axis of parabola i.e., $y = 1$.

So from any point on directrix of parabola, if two tangents are drawn to the parabola then these two tangents will be mutually perpendicular.

18. (C)

Statement- 2 : Area of triangle formed by these tangents and their corresponding chord of contact is

$$\frac{(y_1^2 - 4ax_1)^{\frac{3}{2}}}{2|a|}$$

$\therefore$ Statement is false.

Statement - 1 : $x_1 = 12$, $y_1 = 8$

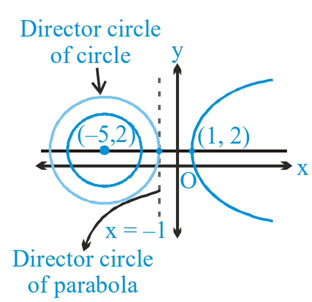
$$\text{Area} = \frac{\left(y_1^{\,2} - 4x_1\right)^{3/2}}{2} = \frac{(64-48)^{3/2}}{2} = 32$$

Statement is true.

19. Equation of director circle of $(x+5)^2 + (y-2)^2 = 8$ will be

$(x+5)^2 + (y-2)^2 = 16$ and

of $(y-2)^2 = 8(x-1)$ is $x = -1$.

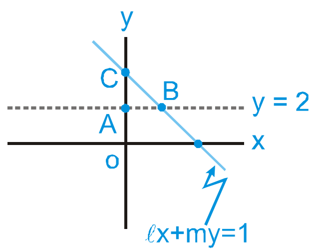


Clearly the line $x = -1$ touches $(x+5)^2 + (y-2)^2 = 16$

Hence only one such point exist.

✈ $C \equiv \left(0, \frac{1}{m}\right)$

$B \equiv \left(\frac{1-2m}{1}, 2\right)$, A (0, 2)



Let (h, k) be the circumcentre of ΔABC

$\therefore \quad h = \frac{1-2m}{2l} \quad ; \quad k = \frac{1+2m}{2m}$

$\therefore \quad 2h = \frac{1-2m}{1} \quad ; \quad k = 1 + \frac{1}{2m}$

$\therefore \quad m = \frac{1}{2k-2} \quad ; \quad \bullet = \frac{k-2}{2h\,(k-1)}$

✈ $(\bullet, m)$ lies on $y^2 = 4ax$

$\therefore \quad m^2 = 4a\bullet$

$\Rightarrow \quad \left(\frac{1}{2k-2}\right)^2 = 4a\left\{\frac{k-2}{2h(k-1)}\right\}$

$h = 8a\,(k^2 - 3k + 2)$

$\therefore$ locus of (h, k) is

$x = 8a\,(y^2 - 3y + 2)$

$\Rightarrow \quad \left(y - \frac{3}{2}\right)^2 = \frac{1}{8a}\,(x + 2a)$

$\therefore$ vertex is $\left(-2a, \frac{3}{2}\right)$

✈ Length of smallest focal chord = length of latus rectum $= \frac{1}{8a}$

From the equation of curve C it is clear that it is symmetric about line $y = \frac{3}{2}$.

25.

1. (A)

minimum distance is along common normal

so firstly find out the common normal

Length of LR 4a = 1 $\quad \therefore \quad a = \frac{1}{4}$

Equation of normal for $y^2 = x - 1$ at point P

$y = mx - \frac{3m}{2} - \frac{m^3}{4}$ ....(i)

Equation of normal for $x^2 = y - 1$ at point Q

$y = mx + \frac{3}{2} + \frac{1}{4m^2}$ ....(ii)

(i) and (ii) are similar so compare coefficient

$1 = \dfrac{\frac{-3m}{2} - \frac{m^3}{4}}{\frac{3}{2} + \frac{1}{4m^2}} \Rightarrow \frac{3}{2} + \frac{1}{4m^2} = -\frac{3}{2}m - \frac{m^3}{4}$

$\Rightarrow \quad m^5 + 6m^3 + 6m^2 + 1 = 0$

Its one root is m = – 1 and remaining roots are imaginary

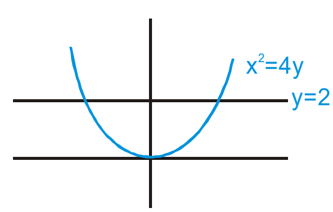
so coordinates of $P \equiv \left(1 + \frac{1}{4}, 2 \times \frac{1}{4}\right) \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

$Q \equiv \left(-\frac{2 \times \frac{1}{4}}{-1}, 1 + \frac{1}{4}\right) = \left(\frac{1}{2}, \frac{5}{4}\right)$

Distance $= \sqrt{\left(\frac{5}{4} - \frac{1}{2}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2}$ is $\left(\frac{5}{4} - \frac{1}{2}\right)\sqrt{2}$

$= \frac{5-2}{4}\sqrt{2}$ is $\frac{3\sqrt{2}}{4}$

2. (B)



Let any point on parabola $x^2 = 4y \Rightarrow (2t, t^2)$

slope of tangent is ‘t’

slope of normal is $-\frac{1}{t}$

equation of normal

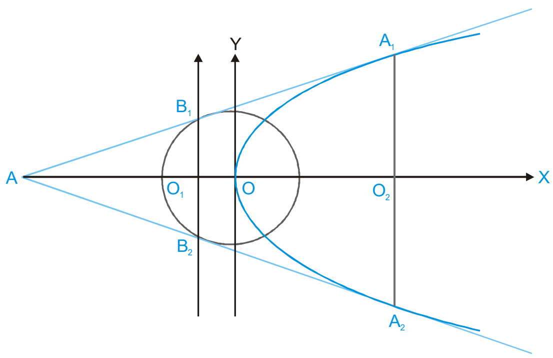
$t^3 - (y - 2)t - x = 0$ ⟶ t_1, t_2, t_3 ....(i)

$t_1 + t_2 + t_3 = 0, \; t_1.t_2.t_3 = x$

29. Let a common tangent through A meet the circle at

$B_1\left(\frac{a}{\sqrt{2}}\cos\theta, \frac{a}{\sqrt{2}}\sin\theta\right)$ and the parabola at

A_1 (at^2, 2at) (figure).



Equation of the tangent to the parabola at A_1 is

$t\,y = x + at^2$ ...... **(i)**

Equation of the tangent to the circle at

B_1 is $x\cos\theta + y\sin\theta = \frac{a}{\sqrt{2}}$ ..... **(ii)**

Since **(i)** and **(ii)** represent the same line.

$$-\frac{1}{\cos\theta} = \frac{t}{\sin\theta} = \sqrt{2}\ t^2 \quad \text{..... (iii)}$$

$\Rightarrow \frac{1}{2t^4} + \frac{1}{2t^2} = 1 \quad \Rightarrow \quad 1 + t^2 = 2t^4$

$\Rightarrow 2t^4 - t^2 - 1 = 0$

$\Rightarrow (t^2 - 1)(2t^2 + 1) = 0$

which gives two real values of t, equal to ± 1 giving two common tangents through A to the given circle and the parabola. Let the other common tangent meet the circle at B_2 and the parabola at A_2.

$\Rightarrow$ coordinate of A_1 are (a, 2a) and coordinate of A_2 are (a, –2a)

$\Rightarrow A_1A_2 = 4a$.

From **(iii)** we get coordinate of B_1 are $\left(-\frac{a}{2}, \frac{a}{2}\right)$ and

the coordinate of B_2 as $\left(-\frac{a}{2}, -\frac{a}{2}\right)$

$\Rightarrow B_1B_2 = a$.

The quadrilateral $A_1B_1B_2A_2$ formed by the common tangents and the chords of contact B_1B_2 of the circle and A_1A_2 of the parabola is a trapezium whose area.

$$= \frac{1}{2}(A_1A_2 + B_1B_2) \times \left(\frac{a}{2} + a\right)$$

$$= \frac{1}{2} \times 5a \times \frac{3a}{2} = \frac{15a^2}{4}.$$

30. Equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$ .....**(i)**

If the slope $m = \sqrt{2}$ then the equation **(i)** becomes

$y = \sqrt{2}x - 2a.\sqrt{2} - a(\sqrt{2})^3$

$\Rightarrow y = \sqrt{2}x - 2a\sqrt{2} - 2a\sqrt{2}$

$\Rightarrow y - x\sqrt{2} + 4a\sqrt{2} = 0$

So the given line is a normal to the parabola.

Length of chord $= \frac{4}{m^2}\sqrt{(1+m^2)(a-mc)}$

$= \frac{4}{2}\sqrt{(1+2)(a+8a)}$

$= 6\sqrt{3a}$