DIFFERENTIATION
EXERCISE # 1

Question based on Differentiability at a point

Q.1 If $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$ then f(x) is

(A) continuous as well differentiable at x =0
(B) continuous but not differentiable at x = 0
(C) neither differentiable at x = 0 nor continuous at x = 0
(D) none of these

Sol. **[B]**

Q.2 If $f(x) = \dfrac{x}{\sqrt{x+1}-\sqrt{x}}$ be a real valued function then

(A) f(x) is continuous, but $f'(0)$ does not exist
(B) f(x) is differentiable at x = 0
(C) f(x) is not continuous at x = 0
(D) f(x) is not differentiable at x = 0

Sol. **[B]**

Q.3 The function $f(x) = \sin^{-1}(\cos x)$ is

(A) discontinuous at x = 0
(B) continuous at x = 0
(C) differentiable at x = 0
(D) none of these

Sol. **[B]**

Q.4 If $f(x) = \begin{cases} e^x & x < 1 \\ a - bx & x \geq 1 \end{cases}$ is differentiable for $x \in R$ then

(A) $a = 1, b = e^{-1}$ (B) $a = 0, b = e$
(C) $a = 0, b = -e$ (D) $a = e, b = 1$

Sol. **[C]**

Q.5 The left and right hand derivatives of $|\lambda nx|$ at x = 1 are

(A) equal
(B) 1 and –1 respectively
(C) –1 and 1 respectively
(D) None of these

Sol. **[C]**

Q.6 If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

where {x} denotes the fractional part function, then

(A) 'f ' is continuous & differentiable at x = 0
(B) 'f ' is continuous but not differentiable at x = 0
(C) 'f ' is continuous & differentiable at x =2
(D) none of these

Sol. **[D]**

Q.7 Let the function f, g and h be defined as follows

$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{for } -1 \leq x \leq 1 \text{ and } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$$g(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x}\right) & \text{for } -1 \leq x \leq 1 \text{ and } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$h(x) = |x|^3$ for $-1 \leq x \leq 1$ which of these functions are differentiable at x = 0 ?

(A) f and g only (B) f and h only
(C) g and h only (D) none

Sol. **[C]**

Question based on Differentiability over an interval

Q.8 The number of points at which the function $f(x) = \max.\{a-x, a+x, b\}, -\infty < x < \infty, 0 < a < b$ cannot be differentiable is

(A) 1 (B) 2 (C) 3 (D) none

Sol. **[C]**

Q.9 If f(x) is differentiable everywhere, then

(A) |f | is differentiable everywhere
(B) $|f|^2$ is differentiable everywhere
(C) f |f | is not differentiable at some point
(D) f + |f | is differentiable everywhere

Sol. **[B]**

Q.10 A function f defined as $f(x) = x[x]$ for $-1 \leq x \leq 3$ where [x] defines the greatest integer $\leq$ x is

(A) continuous at all points in the domain of f but non-derivable at a finite number of points

(B) discontinuous at all points & hence non-derivable at all points in the domain of f

(C) discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) discontinuous & also non-derivable at a finite number of points of f

Sol. **[D]**

$f(x) = x\,[x]\ ;\ -1 \le x \le 3.$

Since, Greatest Integral function are not continuous as well as differentiable at Integers.

$\therefore$ Option (D) is correct answer.

Question based on Differentiation of function of function

Q.11 If $2^x + 2^y = 2^{x+y}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{(2^x + 2^y)}{(2^x - 2^y)}$ (B) $\dfrac{(2^x + 2^y)}{(1 + 2^{x+y})}$

(C) $2^{x-y} \cdot \dfrac{2^y - 1}{1 - 2^x}$ (D) $\dfrac{(2^{x+y} - 2^x)}{2^y}$

Sol. **[C]**

$2^x + 2^y = 2^{x+y}$

$$\frac{dy}{dx} = -\frac{(2^x \ln 2 - 2^y 2^x \ln 2)}{(2^y \ln 2 - 2^y 2^x \ln 2)}$$

$$= -2^{x-y}\left[\frac{1-2^y}{1-2^x}\right]$$

$$\frac{dy}{dx} = 2^{x-y}\left[\frac{2^y - 1}{1-2^x}\right]$$

Q.12 The differential coefficient of $x\,|x|$ is-

(A) $2x$ (B) $-2x$

(C) $2\,|x|$ (D) None of these

Sol. **[C]**

$y = x\,|x|$

$y' = 2|x|$

Question based on Differentiation of Parametric Function

Q.13 If $x = e^{\sin^{-1} t}$, $y = \tan^{-1} t$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{1}{1+t^2} e^{-\sin^{-1} t}\sqrt{1-t^2}$

(B) $\dfrac{1}{1+t^2} e^{-\sin^{-1} t}$

(C) $(1+t^2) e^{-\sin^{-1} t}\sqrt{1-t^2}$

(D) None

Sol. **[D]**

$$\frac{dy}{dx} = \frac{1}{1+t^2}\left(\frac{\sqrt{1-t^2}}{e^{\sin^{-1} t}}\right)$$

$$= \frac{\sqrt{1-t^2}}{(1+t^2)(e^{-\sin^{-1} t})}$$

Q.14 If $x = a\,[\cos\theta + \log \tan \frac{\theta}{2}]$, $y = a \sin\theta$ then $\dfrac{dy}{dx} =$

(A) $\cos\theta$ (B) $\sin\theta$ (C) $\tan\theta$ (D) $\operatorname{cosec}\theta$

Sol. **[C]**

$$\frac{dy}{dx} = \frac{a\cos\theta}{-a\sin\theta + \dfrac{a\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}}$$

$$= \frac{a\cos\theta}{-a\sin\theta + \dfrac{a}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}}$$

$$= \frac{a\cos\theta}{\dfrac{a}{\sin\theta} - a\sin\theta}$$

$$= \frac{a\cos\theta\sin\theta}{2a - a\sin^2\theta}$$

$$= \frac{a\cos\theta\sin\theta}{a\cos^2\theta}$$

$= \tan\theta$

Question based on Differentiation by taking logrithm

Q.15 $\dfrac{d}{dx} x^{\log x} =$

(A) $x^{\log x}\,(2\log x / x)$ (B) $x^{\log x}\,(2\log x)$

(C) $x^{\log x}\,(\log x / x)$ (D) None of these

Sol. **[A]**

$y = x^{\log x}$

$$y' = \log x\,(x)^{\log x - 1} + \frac{x^{\log x}\log x}{x}$$

$$= \frac{2\log x\ x^{\log x}}{x}$$

Q.16 $\frac{d}{dx}\, x^{x^x} =$

(A) $x^{x^x}\left[x^x.\frac{1}{x} + \log x(\log x+1).x^x\right]$

(B) $\left[x^x.\frac{1}{x} + \log x(\log x+1).x^x\right]$

(C) $x^{x^x}\left[\frac{1}{x} + \log x(\log x+1).x^x\right]$

(D) None of these

Sol. **[A]**

$y = x^{x^x}$

$\lambda n\ y = x^x\ \lambda n\ x$

$\frac{dy}{dx} = y\left[\frac{x^x}{x} + x^x(1+\lambda nx)(\lambda nx)\right]$

Question based on Differentiation of function w.r.to another function

Q.17 Differentiation of $\log_e x$ w. r. t. $\log_{1/5} x$ is

(A) $\log_e 1/5$ (B) $-\log_e 1/5$

(C) $\log_{1/5} e$ (D) None

Sol. **[A]**

$\frac{(\log x)}{(\log_{1/5} x)}$

$\log_{1/5} x = \frac{\log x}{\log 1/5}$

$\frac{d}{dx}(\log_{1/5} x) = \frac{1}{\log 1/5} \times \frac{1}{x}$

$\frac{y}{x} = \log 1/5$

Q.18 The differential coefficient of

$\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. $\sin^{-1}\left[\frac{2x}{1+x^2}\right]$ is -

(A) 1 (B) 3/2 (C) 1/4 (D) –3/4

Sol. **[C]**

$\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

$x = \tan\theta$

$y = \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right]$

$y = \tan^{-1}\ \tan\frac{\theta}{2}$

$y = \frac{\theta}{2} = \frac{\tan^{-1}(x)}{2}$

$= \frac{1}{2(1+x^2)}$ ….(1)

& $z = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$

$z = 2\theta$

$z' = \frac{2}{1+x^2}$ ….(2)

$\frac{dy}{dx} = \frac{1}{4}$

Question based on Differentiation of implicit functions

Q.19 If $ax^2 + 2hxy + by^2 = 0$, then $\frac{dy}{dx} =$

(A) $-\frac{(ax+hy)}{(hx+by)}$ (B) $\frac{(ax+hy)}{(hx+by)}$

(C) $-\frac{(hx+by)}{(ax+hy)}$ (D) None

Sol. **[A]**

$ax^2 + 2hxy + by^2 = 0$

$\frac{dy}{dx} = -\left(\frac{2ax+2hy}{2hx+2by}\right)$

$= -\frac{(ax+hy)}{(hx+by)}$

Q.20 If $y = \cfrac{x}{a+\cfrac{x}{b+\cfrac{x}{a+\cfrac{x}{b+\ldots}}}}$ then $\frac{dy}{dx}$ equals

(A) $\frac{b}{a(b+2y)}$ (B) $\frac{b}{(b+2y)}$

(C) $\frac{a}{b(b+2y)}$ (D) None of these

Sol. **[A]**

$y = \cfrac{x}{a+\cfrac{x}{b+y}}$

$\Rightarrow\ y = \frac{x(b+y)}{ab+ay+x}$

$\Rightarrow$ yab + y^2a + xy = xb + xy

$\Rightarrow$ yab + y^2a – xb = 0

$$\frac{dy}{dx} = -\left(\frac{-b}{ab+2ay}\right)$$

$$\frac{dy}{dx} = \frac{b}{a(b+2y)}$$

Q.21 If 3 sin(xy) + 4 cos (xy) = 5 then $\frac{dy}{dx}$ =

(A) y/x (B) –y/x (C) –x/y (D) x/y

Sol. **[B]**

3 sin (xy) + 4 cos (xy) = 5

$$\frac{dy}{dx} = -\left(\frac{3\cos(xy)y - 4\sin xy\, y}{3\cos xy\, x - 4\sin xy\, x}\right)$$

$$= -\frac{y}{x}$$

Question based on Differentiation of Determinants

Q.22 If f(x) =

$$\begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$$

Then $f'(0)$ is equal to :

(A) 4 (B) 2 (C) 3 (D) 0

Sol. **[B]**

f(x) =

$$\begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$$

$$\Rightarrow f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$\Rightarrow$ f '(0) = 4 – 2 = 2

Q.23 If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & \cos x & 1 \\ 0 & 1 & \cos x \end{vmatrix}$ then $f'\left(\frac{\pi}{3}\right)$ is equal to:

(A) $\frac{5}{8}$ (B) $\frac{-5\sqrt{3}}{8}$ (C) $\frac{11\sqrt{3}}{8}$ (D) None

Sol. **[D]**

$$f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & \cos x & 1 \\ 0 & 1 & \cos x \end{vmatrix}$$

$$f'\left(\frac{\pi}{3}\right) = \begin{vmatrix} -\sin\frac{\pi}{3} & 0 & 0 \\ 1 & \cos\frac{\pi}{3} & 1 \\ 0 & 1 & \cos\frac{\pi}{3} \end{vmatrix} + \begin{vmatrix} \cos\frac{\pi}{3} & 1 & 0 \\ 0 & -\sin\frac{\pi}{3} & 0 \\ 0 & 1 & \cos\frac{\pi}{3} \end{vmatrix} + \begin{vmatrix} \cos\frac{\pi}{3} & 1 & 0 \\ 1 & \cos\frac{\pi}{3} & 1 \\ 0 & 1 & -\sin\frac{\pi}{3} \end{vmatrix}$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\left[\frac{1}{4}-1\right] + \left[\frac{1}{2}\left[-\frac{\sqrt{3}}{4}\right]\right] + \left[\frac{1}{2}\times\frac{-\sqrt{3}}{4}\right]$$

$$\Rightarrow \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{8} + \left(\frac{-\sqrt{3}}{8}\right)$$

$$= \frac{\sqrt{3}}{8}$$

Question based on Heigher order derivatives

Q.24 If y = x log $\frac{x}{a+bx}$, then $x^3\frac{d^2y}{dx^2}$ is equal to-

(A) $x\frac{dy}{dx} - y$ (B) $\left(x\frac{dy}{dx} - y\right)^2$

(C) $y\frac{dy}{dx} - x$ (D) $\left(y\frac{dy}{dx} - x\right)^2$

Sol. **[B]**

$$y = x\log\left(\frac{x}{a+bx}\right) = x(\ln x - \ln(a+bx))$$

$$\Rightarrow y' = \ln\left(\frac{x}{a+bx}\right) + x\left[\frac{1}{x} - \frac{b}{a+bx}\right]$$

$$\Rightarrow y' = \frac{y}{x} + 1 - \frac{bx}{a+bx}$$

$$\Rightarrow y' = \frac{y}{x} + \frac{a}{a+bx} \quad \ldots.(1)$$

$$y'' = \frac{y'x - y}{x^2} - \frac{ab}{(a+bx)^2}$$

$$= \frac{y'x - y}{x^2} - \frac{x(bx + a - a)}{x(a + bx)^2}$$

$$= \frac{y'x - y}{x^2} - \frac{a}{x}\left[\frac{1}{(a + bx)} - \frac{a}{(a + bx)^2}\right]$$

$$= \frac{y'x - y}{x^2} - \frac{a}{x(a + bx)} + \frac{a^2}{x(a + bx)^2}$$

use (1)

$$y'' = \frac{y'x - y}{x^2} - \frac{1}{x}\left\{y' - \frac{y}{x} - \left(y' - \frac{y}{x}\right)^2\right\}$$

$$= \frac{y'}{x} - \frac{y}{x^3} + \frac{y}{x} + \frac{y}{x^2} + \left(\frac{y'x - y}{x^3}\right)^2$$

$$x^3y'' = (xy' - y)^2$$

Q.25 If $x = a\cos\theta$, $y = b\sin\theta$ then $\frac{d^3y}{dx^3}$ is equal to -

(A) $\frac{-3b}{a^3}$ $\text{cosec}^4\theta\ \cot^4\theta$

(B) $\frac{3b}{a^3}$ $\text{cosec}^4\theta\ \cot\theta$

(C) $\frac{-3b}{a^3}$ $\text{cosec}^4\theta\ \cot\theta$

(D) None of these

Sol. **[C]**

$x = a\cos\theta$

$y = b\sin\theta$

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a\sin\theta(-b\sin\theta) - b\cos\theta(-a\cos\theta)}{-(a\sin\theta)^2 a\sin\theta}$$

$$= \frac{ab\sin^2\theta + ab\cos^2\theta}{(a\sin\theta)^3}$$

$$\frac{d^2y}{dx^2} = \frac{ab}{(a\sin\theta)^3}$$

$$\frac{d^3y}{dx^3} = \frac{ab[-3(a\sin\theta)^{-4} a\cos\theta]}{-a\sin\theta}$$

$$= -\frac{3b}{a^3}\text{cosec}^4\theta\cot\theta$$

Question based on Miscellaneous

Q.26 If $y = \sum_{r=1}^{x} \tan^{-1}\frac{1}{1 + r + r^2}$ then $\frac{dy}{dx}$ is equal to-

(A) $\frac{1}{1 + x^2}$ (B) $\frac{1}{1 + (1 + x)^2}$

(C) 0 (D) None of these

Sol. **[B]**

Q.27 If $\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$

$= A + Bx + Cx^2 + \ldots..$, then

(A) $\Delta(x)$ is divisible by x (B) $\Delta(x) = 0$

(C) $\Delta'(x) = 0$ (D) None of these

Sol. **[A]**

$$\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$$

$$\Delta(x) = \begin{vmatrix} x & (1+x^2) & x^3 \\ \left(x^2 - \frac{x^4}{2} + \ldots\right) & \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right) & \left(x - \frac{x^3}{3!} + \ldots\right) \\ \left(1 - \frac{x^2}{2!} + \ldots.\right) & \left(x - \frac{x^3}{3} + \ldots\right) & \left(x - \frac{x^3}{3!} + \ldots.\right)^2 \end{vmatrix}$$

$$\Delta(x) = x\begin{vmatrix} x & (1+x^2) & x^2 \\ \left(x^2 - \frac{x^4}{2} + \ldots\right) & \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots.\right) & \left(1 - \frac{x^2}{3!} + \ldots\right)^2 \\ \left(1 - \frac{x^2}{2!} + \ldots\right) & \left(x - \frac{x^3}{3} + \ldots.\right) & \left(1 - \frac{x^2}{3!} + \ldots.\right)^2 \end{vmatrix}$$

($\therefore$ x taken out from C_3 by)

Hence, $\Delta(x)$ is divisible by x.

$\therefore$ Option (A) is correct answer.

Q.28 Let $f(x) = (x^3 + 2)^{30}$. If $f^{\,n}(x)$ is a polynomial of degree 20, where $f^{\,n}(x)$ denotes the nth derivative of f(x) w.r.t. x, then the value of n is-

(A) 60 (B) 40 (C) 70 (D) None

Sol. **[C]**

$f(x) = (x^3 + 2)^{30}$

$f^n(x) = ax^{20} + \ldots\ldots\ldots$

$x = 70$

Q.29 If $x = t + 1/t$, $y = t - 1/t$ then $\frac{d^2y}{dx^2}$ is-

(A) $-4t / (t^2 - 1)$ (B) $-4t^3/(t^2 - 1)^3$

(C) $(t^2 + 1)/(t^2 - 1)$ (D) $-4t^2 / (t^2-1)^2$

Sol. **[B]**

$$x = t + \frac{1}{t},\ \ y = t - \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \left(\frac{t^2+1}{t^2-1}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(t^2-1)\,2t-(t^2+1)2t}{(t^2-1)^2\left(1-\frac{1}{t^2}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{-4t^3}{(t^2-1)^3}$$

➤ True or False type Questions

Q.30 If f(x) = cos (x^2 – 2[x]) for 0 < x < 1, where [x] denotes the greatest integer ≤ x, then f′ ($\sqrt{\pi}$ /2) is equal to – $\sqrt{\pi}$.

Sol. f(x) = cos (x^2–2[x]) for 0 < x < 1.

$$f'(\sqrt{\pi}/2) = \text{R.H.D.} \lim_{x\to\frac{\sqrt{\pi}}{2}^+} \frac{f(x)-f(\sqrt{\pi}/2)}{x-\sqrt{\pi}/2}$$

$$= \lim_{h\to 0}\frac{f(\sqrt{\pi}/2+h)-f(\sqrt{\pi}/2)}{\sqrt{\pi}/2+h-\sqrt{\pi}/2}$$

$$= \lim_{h\to 0}\frac{f(\sqrt{\pi}/2-h)-f(\sqrt{\pi}/2)}{+h}$$

$$= \lim_{h\to 0}\cos\left(\left(\frac{\sqrt{\pi}}{2}-h\right)^2-2\left[\frac{\sqrt{\pi}}{2}-h\right]\right)$$

$$= \lim_{h\to 0}\frac{-\cos\left(\frac{\pi}{4}-2\left[\sqrt{\pi}/2\right]\right)}{h}$$

Since $\frac{\sqrt{\pi}}{2} \cong \frac{1.7}{2} = 0.85$

[0.85] = 0

Also $\left[\frac{\pi}{2}-h\right] = 0$

$$= \lim_{h\to 0}\frac{\cos\left(\left(\frac{\sqrt{\pi}}{2}-h\right)^2-0\right)-\frac{1}{\sqrt{2}}}{h}$$

$$= \lim_{h\to 0}\frac{\cos\left(\frac{\sqrt{\pi}}{2}-h\right)^2-\frac{1}{\sqrt{2}}}{h}\ \left(\frac{0}{0}\text{form}\right)$$

Use L – H Rule, weget

$$= \lim_{h\to 0}\frac{-\sin\left(\left(\frac{\sqrt{\pi}}{2}-h\right)^2\right)2\left(\frac{\sqrt{\pi}}{2}-h\right)(-1)-0}{1}$$

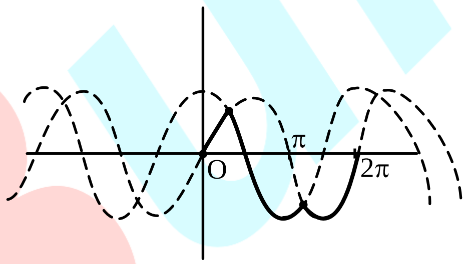
$$= -\sin p/4 \times 2\left(\frac{\sqrt{\pi}}{2}\right)(-1)$$

$$= +\frac{1}{\sqrt{2}} \times \sqrt{\pi} = \sqrt{\pi/2}\,.$$

∴ Option is False.

Q.31 Number of non differential point for f(x) = min (sinx, cosx) is 2n if x ∈ (0,2nπ) n ∈ N.

Sol.



point of discontinuity in (0, 2π) = 2

so point of discontinuity in (0, 2Nπ) = 2N (continuous)

Q.32 The function y = $e^{\sqrt{x}}$ + $e^{-\sqrt{x}}$ satisfies the relationship $xy'' + \frac{1}{2}y' - \frac{1}{4}y = 0$.

Sol. $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$

Differentiable w.r.t w, weget

$$y' = e^{\sqrt{x}}\frac{1}{2\sqrt{x}} + e^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right)$$

$$\sqrt[2]{x}\,y' = e^{\sqrt{x}} + e^{-\sqrt{x}}$$

Again differentiate w.r.t x , weget

$$2\sqrt{x}\,y'' + 2y'\frac{1}{2\sqrt{x}} = e^{\sqrt{x}}\frac{1}{2\sqrt{x}} + e^{-\sqrt{x}}\frac{1}{2\sqrt{x}}$$

$$2\sqrt{x}\left(2\sqrt{x}y''+2y'\frac{1}{2\sqrt{x}}\right) = e^{\sqrt{x}} + e^{-\sqrt{x}} = y$$

$$\Rightarrow xy'' + \frac{1}{2}y' - \frac{1}{4}y = 0$$

➤ Fill in the blanks type questions

Q.33 For the function f(x) = $|x^2 - 5x + 6|$, the derivative from the right, $f'(2^+)$ =........ and the derivative from the left, $f'(2^-)$ =.........

Sol. $f(x) = |x^2 - 5x + 6|$

$$= \begin{cases} x^2 - 5x + 6 & ; x \le 2 \text{ or } \ge 3 \\ -(x^2 - 5x + 6) & ; 2 < x < 3 \end{cases}$$

$$\underset{2-h}{|}\text{———}\underset{2}{|}\text{———}\underset{2+h}{|}$$

L.H.D.= $\lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{h\to 0} \dfrac{f(2-h)-f(2)}{2-h-2}$

$= \lim\limits_{h\to 0} \dfrac{+\left[(2-h)^2 - 5(2-h)+6\right] - (4-10+6)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{+[(2-h)2 - 5(2-h)+6] - 0}{-h}$ $\dfrac{0}{0}$ from

Apply L – H Rule , weget

$= \lim\limits_{h\to 0} \dfrac{+[2(2-h)(-1)+5+0]-0}{-1}$

$= \dfrac{+[-4+5]}{-1} = -1.$

R.H.D.= $\lim\limits_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{2+h-2}$

$= \lim\limits_{h\to 0} \dfrac{[(2+h)^2 - 5(2+h)+6] + (4-10+6)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-[(2+h)^2 - 5(2+h)+6]+0}{h}$ $\dfrac{0}{0}$ form

Apply L – H rule, weget

$= \lim\limits_{h\to 0} \dfrac{-[(2+h)\times 2 - 5 + 0]}{1}$

$= -[4-5] = 1.$

Q.34 If $f(x) = \sqrt{x^2+6x+9}$, then $f'(x)$ is equal to $(x < -3)$ = ...

Sol. $f'(x) = \sqrt{x^2+6x+9}$

$f'(x) = \dfrac{1}{2\sqrt{x^2+6x+9}} \times (2x+6)$

$f'(x) = \dfrac{(x+3)}{\sqrt{x^2+6x+9}}$.

Q.35 If $y = \sin^{-1}\left(\dfrac{\sin\alpha\ \sin x}{1-\cos\alpha\ \sin x}\right)$, then $y'(0)$ is.......

Sol. $y = \sin^{-1}\left(\dfrac{\sin\alpha \sin x}{1-\cos x \sin x}\right)$

$y' = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0}$

$= \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{0+h-0}$

$= \lim\limits_{h\to 0} \dfrac{\sin^{-1}\left(\dfrac{\sin\alpha \sin(0+h)}{1-\cos\alpha \sin(0+h)}\right) - 0}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sin^{-1}\left(\dfrac{\sin\alpha \sinh}{1-\cos\alpha \sinh}\right)}{h}$ $\dfrac{0}{0}$ form

Use L – H Rule, we get

$\lim\limits_{h\to 0} \dfrac{1}{\sqrt{1-\left(\dfrac{\sin\alpha \sinh}{1-\cos\alpha \sinh}\right)^2}} \times \dfrac{\sin\alpha \cosh(1-\cos\alpha \sinh) - \sin\alpha \sinh(-\cos\alpha \cosh)}{(1-\cos\alpha \sinh)^2}$

$= \lim\limits_{h\to 0} \dfrac{(1-\cos\alpha \sinh)}{\sqrt{(1-\cos\alpha \sinh)^2 - (\sin\alpha \sinh)^2}} \times \dfrac{\sin\alpha \cosh}{(1-\cos\alpha \sinh)^2}$

$= \lim\limits_{h\to 0} \dfrac{\sin\alpha \cosh}{\sqrt{(1-\cos\alpha \sinh)^2 - (\sin\alpha \sinh)^2}} \times \dfrac{1}{(1-\cos\alpha \sinh)^2}$

$= \dfrac{\sin\alpha.1}{1\times 1} = \sin\alpha.$

Q.36 If cos y = x cos(a + y) then $\dfrac{dy}{dx} = \dfrac{A}{1+x^2-2x\ \cos\ a}$ then the value of A is.............

Sol. cos y = x cos (a + y)

cos y = x (cosa coay – sina siny)

cosy = x cosa cosy – xsina siny

x sina siny = cosy (xcosa–1)

$\Rightarrow$ Tany = $\dfrac{x\cos a - 1}{x \sin a}$

$\Rightarrow y = \text{Tan}^{-1}\left(\dfrac{x\cos a - 1}{x\sin a}\right)$

Differentiate w.r.t x, weget

$\dfrac{dy}{dx} = \dfrac{1}{1+\left(\dfrac{x\cos a - 1}{x\sin a}\right)^2} \times \dfrac{\cos a.x\sin a - (x\cos a - 1)\times \sin a}{(x\sin a)^2}$

$= \dfrac{(x\sin a)^2}{(x\sin a)^2 + (x\cos a - 1)^2} \times \dfrac{x\sin\cos a - x\sin a\cos + \sin a}{(x\sin a)^2}$

$\dfrac{dy}{dx} = \dfrac{\sin}{x^2\sin^2 a + x^2\cos^2 a + 1 - 2x\cos a}$

$\dfrac{dy}{dx} = \dfrac{\sin a}{x^2+1-2x\cos a} = \dfrac{A}{1+x^2-2x\cos a}$

$\Rightarrow$ A = sin a.

EXERCISE # 2

Part-A Only single correct answer type questions

Q.1 Consider

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + |\sin x - \sin^3 x|}{2(\sin x - \sin^3 x) - |\sin x - \sin^3 x|}\right];$$

$x \in (0, \pi) - \{\pi/2\}$, $f(\pi/2) = 3$ where [ ] denotes the greatest integer function then-

(A) f is continuous and differentiable at $x = \pi/2$

(B) f is continuous but not differentiable at $x = \pi/2$

(C) f is neither continuous nor differentiable at $x = \pi/2$

(D) None of these

Sol. [A]

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + |\sin x - \sin^3 x|}{2(\sin x - \sin^3 x) - |\sin x - \sin^3 x|}\right]$$

$x \in (0,\pi) - \{\pi/2\}$; $f(\pi/2) = 3$.

When $\sin x - \sin^3 x \ge 0 \Rightarrow |\sin x - \sin^3 x| = (\sin x - \sin^3 x)$

$\Rightarrow \sin x \ge \sin^3 x$

$\Rightarrow \sin^2 x \le 1$

$\Rightarrow -1 \le \sin x \le 1 \Rightarrow -\pi/2 \le x \le \pi/2$.

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + (\sin x - \sin^3 x)}{2(\sin x - \sin^3 x) - (\sin x - \sin^3 x)}\right]$$

$$f(x) = \left[\frac{3(\sin x - \sin^3 x)}{(\sin x - \sin^3 x)}\right] = [3] = 3$$

hence, L.H.L. = R.H.L. = $f(\pi/2) = 3$

f(x) is continuous at $x = \pi/2$

Also be differentiable at $x = \pi/2$.

When $\sin x - \sin^3 x < 0 \Rightarrow |\sin x - \sin^3 x|$ $= -(\sin x - \sin^3 x)$

$\sin x < \sin^3 x \Rightarrow \sin^2 x > 1$

$\Rightarrow \sin x < -1$ or $\sin x > 1$

Which is not possible.

Q.2 If $f(x) = \begin{cases} x^k \sin(1/x) & , \ x \ne 0 \\ 0 & , \ x = 0 \end{cases}$ is differentiable at x = 0 then (where k is an integer)

(A) $k > 0$ (B) $k > 1$ (C) $k \ge 1$ (D) $k \ge 0$

Sol. [B]

$$f(x) = \begin{cases} x^k \sin(1/x); x \ne 0 \\ 0 \quad ; x = 0 \end{cases}$$

At x = 0

L.H.D. $= \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$

$$= \lim_{h\to 0} \frac{(0-h)^k \sin\frac{1}{0-h} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{(-1)^k h^k \sin 1/h}{h}$$

$$= (-1)^k \lim_{h\to 0} \sin 1/h . h^{k-1}$$

$$= (-1)k \lim_{h\to 0} \frac{\sin 1/h}{\left(\frac{1}{h}\right)^{k-1}}$$

If limit exist, Then k must be strictly greater than 1. i.e. k > 1.

Q.3 If $f(x) = x^n$, then the value of

$$f(1) + \frac{f'(1)}{1!} + \frac{f^2(1)}{2!} \ .....+ \frac{f^n(1)}{n!}$$

where $f^r(1)$ is the r^{th} derivative of f (x) w.r.t. x

(A) 1 (B) 0 (C) 2^n (D) n

Sol. [C]

$f(x) = x^n$

$f'(x)$ = first derivative of x w.r.t. x = nx^{n-1}

$f''(x)$= second derivative of x w.r.t.x=$n(n-1)x^{n-2}$

$f'''(x)$ = Third derivative of x w.r.t. x = $n(n-1)(n-2)\, x^{n-3}$

$\vdots$

$f^n(x)$ = nth derivative of f(x) w.r.t. x = n (n–1) (n –2) ……3.2.1

$$f(x) + \frac{f^1(x)}{1!} + \frac{f^2(x)}{2!} + \frac{f^3(x)}{3!} + \frac{f^4(x)}{4!} + \ldots + \frac{f^n(x)}{n!}$$

At x = 1

$$f(1) + \frac{f^1(1)}{1!} + \frac{f^2(1)}{2!} + \frac{f^3(1)}{3!} + \frac{f^4(1)}{4!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}$$

$+\dfrac{n(n-1)(n-2)(n-3)}{4!}+\ldots.+\dfrac{n(n-1)(n-2)\ldots3.2.1}{n!}$

$= (1 + 1)^n = 2^n$

Q.4 Let f(x) be defined in [–2, 2] by

$$f(x)=\begin{cases}\max\left(\sqrt{4-x^2},\sqrt{1+x^2}\right), & -2\le x\le 0\\ \min\left(\sqrt{4-x^2},\sqrt{1+x^2}\right), & 0<x\le 2\end{cases}$$

then f(x)

(A) is continuous at all points

(B) is not continuous at more than one point

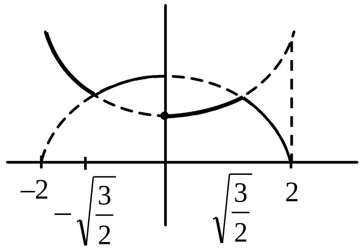
(C) is not differentiable only at one point

(D) is not differentiable at more than one point

Sol. **[D]**

$y=\sqrt{4-x^2}\ \rightarrow$ semicircle

$y=\sqrt{1+x^2}\ \rightarrow$ hyperbola



Solve : $\sqrt{4-x^2}=\sqrt{1+x^2}$

$2x^2 = 3$

$x=\sqrt{\dfrac{3}{2}}$

(non differentiable at many point)

Q.5 Let $f(x)=\begin{cases}4x^2+2[x]x & \text{if } -\frac{1}{2}\le x<0\\ ax^2-bx & \text{if } 0\le x<\frac{1}{2}\end{cases}$

where [x] denotes the greatest integer function, then

(A) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ for all a, provided b = 2

(B) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ if a = 4, b = 2

(C) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ if a = 4 and b = 0

(D) for no choice of a and b, f(x) is differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$

Q.6 If $y=\tan^{-1}\left(\dfrac{a\sin x+b\cos x}{a\cos x-b\sin x}\right)$ then the value of $\dfrac{dy}{dx}$ equals

(A) –1 (B) 0

(C) 1 (D) None of these

Sol. **[C]**

$y=\tan^{-1}\left(\dfrac{a\sin x+b\cos x}{a\cos x-b\sin x}\right)$

Let $\dfrac{a\sin x+b\cos x}{a\cos x-b\sin x}=\dfrac{\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x}{\frac{a}{\sqrt{a^2+b^2}}\cos x-\frac{b}{\sqrt{a^2+b^2}}\sin x}$

$=\dfrac{\sin x\cos\alpha+\cos x\sin\alpha}{\cos x\cos\alpha-\sin x\sin\alpha}$

$=\dfrac{\sin(x+\alpha)}{\cos(x+\alpha)}=\tan(x+\alpha)$

We as same, $\cos\alpha=\dfrac{a}{\sqrt{a^2+b^2}}$;

$\sin\alpha=\dfrac{b}{\sqrt{a^2+b^2}}$

$y=\tan^{-1}(\tan(x+\alpha))=x+\alpha$

$y=x+\cos^{-1}\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)$

Differentiate w.r.t. x, weget

$\dfrac{dy}{dx}=1$

∴ Option (C) is correct answer.

Q.7 Let $f(x) = x - x^2$ and

$$g(x)=\begin{cases}\max\{f(t),\ 0\le t\le x\}; & 0\le x\le 1\\ \sin\pi x, & x>1\end{cases}$$

Then in the interval [0, ∞]

(A) g(x) is everywhere continuous except at two points

(B) g(x) is everywhere differentiable except at two points

(C) g (x) is everywhere differentiable except at x =1

(D) none of these

Sol. **[C]**

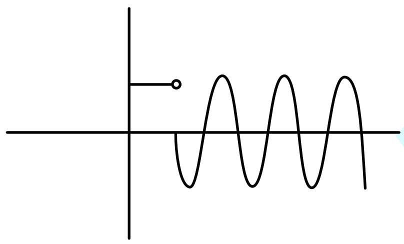
f(x) = max f(t) : $0 \le t \le 1$

$$\frac{dy}{dx} = 1 - 2x = 0$$

at $\left(x = \frac{1}{2}\right)$

$$f(x) = \frac{1}{4} \ : \ 0 \le x < 1$$

$= \sin \pi x \ : \ x > 1$



so g(x) is non diff. at x = 1

Q.8 If y = f(x) is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(A) 4 (B) 2 (C) –2 (D) 0

Sol. **[C]**

$y = f(x)$; $x \in R$

$f'(3) = -2$

Since f(x) is odd differentiable function

i.e. $f(-x) = -f(x)$

$\Rightarrow f'(-x)(-1) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$

$\Rightarrow f'(-3) = f'(3) = -2$

∴ Option (C) is correct answer.

Q.9 If $y = \tan^{-1}\frac{\log(e/x^2)}{\log(ex^2)} + \tan^{-1}\frac{3+2\log x}{1-6\log x}$, then $\frac{d^2y}{dx^2}$ is-

(A) 2 (B) 1 (C) 0 (D) –1

Sol. **[C]**

$$y = \tan^{-1}\frac{\log(e/x^2)}{\log(ex^2)} + \tan^{-1}\frac{3+2\log x}{1-6\log x}$$

$$= \tan^{-1}\frac{(\log e - \log x^2)}{(\log e + \log x^2)} + \tan^{-1}\frac{3+2\log x}{1-6\log x}$$

$$= \tan^{-1}\left(\frac{1-2\log x}{1+2\log x}\right) + \tan^{-1}3 + \tan^{-1}\log x^2$$

$= \tan^{-1}(1) - \tan^{-1}\log x^2 + \tan^{-1}3 + \tan^{-1}\log x^2$

$= \tan^{-1}(1) + \tan^{-1}(3)$

$$\frac{d^2y}{dx} = 0$$

∴ Option (C) is correct answer.

Q.10 Let P(x) be a polynomial of degree 3 with $P(0) = 4$, $P'(0) = 3$.$P''(0) = 4$,$P'''(0) = 6$ then $P'(-1) =$

(A) –10 (B) 10

(C) 2 (D) None of these

Sol. **[C]**

Let $P(x) = x^3 + bx^2 + cx + d$

$P(0) = 0 + 0 + 0 + d = 4 \Rightarrow d = 4$

$P'(x) = 3x^2 + 2bx + c$

$P'(0) = 0 + 0 + c = 3 \Rightarrow c = 3$

$P''(x) = 6x + 2b$

$P''(0) = 0 + 2b = 4 \Rightarrow b = 2$

$P'(x) = 3x^2 + 2bx + c$

$= 3x^2 + 4x + 3$

$P'(-1) = 3 - 4 + 3$

$= 2$

Q.11 The differential coefficient of f(cos x) with respect to g(sin x) at $x = \pi/3$ if $f'(1/2) = g'(\sqrt{3}/2) = -2$ is equal to-

(A) $2\sqrt{3}$ (B) $1/\sqrt{3}$ (C) $\sqrt{3}$ (D) $-\sqrt{3}$

Sol. **[D]**

$$\frac{d}{dx}(f(\cos(x))) = f'(\cos x) \times (-\sin x)$$

Differential coefficient = –sinx

$$\frac{d}{dx}(g(\sin x)) = g'(\sin x) \times \cos x$$

Differential coefficient = cosx

Hence, differential coefficient of f(cosx) w.r.t. g(sinx) is $\frac{-\sin x}{\cos x}$ at x = $\frac{\pi}{3}$

= $\frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$

∴ Option (D) is correct answer.

Q.12 If g is the inverse function of f and f ′ (x) = sin x then g′ (x) =

(A) sin g (x)
(B) $\sin^{-1}$ (x)
(C) $\frac{1}{\sqrt{1-x^2}}$
(D) cosec (g (x))

Sol. **[D]**

$g(x) = f^{-1}(x) \Rightarrow x = f(g(x))$

Differentiate above function w.r.t. x, weget

1 = f ′(g(x)) × g′(x) ⇒ g′(x) = 1/f ′(g(x))

Since f ′(x) = sin x

⇒ f ′(g(x)) = sin (g(x)).

g′(x) = 1/sin(g(x)) = cosec(g(x))

∴ Option (D) is correct answer.

Q.13 The derivative of f(tan x) w.r.t. g(sec x) at x = $\frac{\pi}{4}$, where f′(1) and g′ ($\sqrt{2}$) = 4, is-

(A) $\frac{1}{\sqrt{2}}$
(B) $\sqrt{2}$
(C) 1
(D) None of these

Sol. **[B]**

Derivative of f (tan x) w.r.t. g (secx)

is $\frac{df(\tan x)}{dg(\sec x)} = \frac{\frac{d}{dx}f(\tan x)}{\frac{d}{dx}g(\sec x)}$

$= \frac{f'(\tan x)\times \sec^2 x}{g'(\sec x)\times \sec\tan x}$

$= \frac{f'(\tan x)\times \sec x}{g'(\sec x)\times \tan x}$

$\therefore \left.\frac{d f(\tan x)}{d g(\sec x)}\right|_{x=\pi/4} = \left.\frac{f'(\tan x)\times \sec x}{g'(\sec x)\times \tan x}\right|_{x=\pi/4}$

$= \frac{f'(1)\times\sqrt{2}}{g'(\sqrt{2})\times 1} = \sqrt{2}$

Q.14 A function f(x) is so defined that for all x, $[f(x)]^n = f(nx)$. If f′(x) denotes derivative of f(x) w.r.t. x, then f′(x) · f(nx) =

(A) f(x)
(B) 0
(C) f(x) · f′(nx)
(D) None of these

Sol. **[C]**

$(f(x))^n = f(nx)$

Taking log both sides, weget

n log (f(x)) = log f(nx)

Differentiating w.r.t. x, weget

$n \cdot \frac{1}{f(x)} \times f'(x) = \frac{1}{f(nx)} \times f'(nx) \times n$

⇒ f ′(x) × f(nx) = f(x) × f ′(nx)

∴ Option (C) is correct answer.

Q.15 If $y = \left(\frac{x}{n}\right)^{nx}$ $(1+ \log \frac{x}{n})$, then y′ (n) is given by:

(A) $\frac{n^2+1}{n}$
(B) $\frac{1}{n}$
(C) $\left(\frac{1}{n}\right)^n$
(D) $\left(\frac{1}{n}\right)^n \left(\frac{n^2+1}{n}\right)$

Sol. **[A]**

$y = (x/n)^{nx}\left(1+\log\frac{x}{n}\right)$

Differentiate w.r.t. x, weget

$y'(x)=\frac{dy}{dx} = \left(1+\log\frac{x}{n}\right)\left[nx(x/n)^{nx-1}.\frac{1}{n}+\left(\frac{x}{n}\right)^{nx}\log\frac{x}{n}.n\right]$

$+ (x/n)^{nx}\left(0+\frac{1}{\frac{x}{n}}\times\frac{1}{n}\right)$

$y'(n) = (1 + 0)\,[n + 0] + 1\left(0+\frac{1}{n}\right)$

$= n + \frac{1}{n} + \frac{n^2+1}{n}$

∴ Option (A) is correct answer.

Q.16 If $\sqrt{(x^2+y^2)} = a\ e^{\tan^{-1}(y/x)}$, a > 0, y > 0 then y″(0) is-

(A) $\frac{a}{2}e^{-\pi/2}$
(B) $ae^{\pi/2}$

(C) $-\frac{2}{a}e^{-\pi/2}$ (D) does not exist

Sol. **[C]**

$\sqrt{(x^2+y^2)} = a.\ e^{\tan^{-1}(y/x)}$; a > 0, y > 0

Squaring both sides, we get

$x^2 + y^2 = a^2\ e^{2\tan^{-1} y/x}$

$$\left(\frac{x^2+y^2}{a^2}\right) = e^{2\tan^{-1} y/x}$$

Taking log both sides, we get

$$\log\left(\frac{x^2+y^2}{a^2}\right) = 2\tan^{-1}(y/x)$$

Differentiating above function w.r.t. x,

$$\frac{1}{\left(\frac{x^2+y^2}{a^2}\right)} \times \frac{2x+2y\frac{dy}{dx}}{a^2}$$

$$= 2\ \frac{1}{1+y^2/x^2}\ \frac{\frac{dy}{dx}.x - y.1}{x^2}$$

$$\frac{x+y\frac{dy}{dx}}{(x^2+y^2)} = \frac{1}{x^2+y^2}\left(\frac{dy}{dx}.x - y\right)$$

$$x + y = (x-y)\ \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

Differentiating w.r.t. x, weget

$$\frac{d^2y}{dx^2} = \frac{\left(1+\frac{dy}{dx}\right)(x-y)-(x+y)(1-dy/dx)}{(x-y)^2}$$

$$= \frac{(x-y-x-y)+\frac{dy}{dx}(x-y+x+y)}{(x-y)^2}$$

$$= \frac{-2y+2x\frac{dy}{dx}}{(x-y)^2}$$

$$= \frac{+2\left[-y+x\frac{x+y}{x-y}\right]}{(x-y)^2}$$

$$= \frac{2[-xy+y^2+x^2+xy]}{(x-y)^2(x-y)}$$

$= 2\ [x^2 + y^2]\ /\ (x-y)^3$

$$\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = y''(0) = 2(0 + y^2)/(0-y)^3$$

$= 2/(-y)$

since $\sqrt{x^2+y^2} = a\ e^{\tan^{-1} y/x}$

$= a\ e^{\pi/2}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = y''(0) = -2/ae^{\pi/2}$$

$$= -\frac{2}{a}e^{-\pi/2}$$

Q.17 For what triplets of real number (a, b, c) with $a \neq 0$; the function

$$f(x) = \begin{cases} x & \text{for } x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$$

is differentiable for all real x?

(A) {a, 1–2a, a)/ a ∈ R, a ≠ 0}

(B) {a, 1–2a, c)/a, c ∈ R, a ≠ 0}

(C) {a, b, c)/ a, b, c ∈ R, a + b + c = 1}

(D) {a, 1–2a , 0/ a ∈ R , a ≠ 0

Sol. **[A]**

$$f(x) = \begin{cases} x & \text{for} x \le 1 \\ ax^2+bx+c & \text{otherwise} \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{for} x \le 1 \\ 2ax+b & \text{otherwise} \end{cases}$$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{a(1-h)^2+b(1-h)+c-1}{-h}$$

$$\text{R.H.D.} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{a(1+h)^2+b(1+h)+c-1}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{(a+b+c-1)+ah^2+bh+2ah}{h} = 1.$$

⇒ a + b + c = 1 and 2a + b = 1 ⇒ b = 1 – 2a

⇒ b = 1 – 2a and a + 1 – 2a + c = 1

⇒ – a + c = 0

⇒ c = a

Hence, (a,1–2a, a) ; a ∈ R; a ≠ 0

∴ Option (A) is correct answer.

Part-B **One or more than one correct answer type questions**

Q.18 Given that the derivative $f'(a)$ exists. Indicate which of the following statement (s) is/are always true

(A) $f'(a) = \lim\limits_{h \to a} \dfrac{f(h) - f(a)}{h - a}$

(B) $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a) - f(a-h)}{h}$

(C) $f'(a) = \lim\limits_{t \to 0} \dfrac{f(a+2t) - f(a)}{t}$

(D) $f'(a) = \lim\limits_{t \to 0} \dfrac{f(a+2t) - f(a+t)}{2t}$

Sol. **[A,B]**

Q.19 Let f be a differentiable function on the open interval (a, b) which of the following statement must be true?

(i) f is continuous on the closed interval [a, b]

(ii) f is bounded on the open interval (a, b)

(iii) If $a < a_1 < b_1 < b$ and $f(a_1) < 0 < f(b_1)$ then there is a number c such that

$a_1 < c < b_1$ and $f(c) = 0$

(A) (i) and (ii) only (B) (i) and (iii) only

(C) (ii) and (iii) only (D) only (iii)

Sol. **[D]**

Q.20 If $f(x) = \sum\limits_{k=0}^{n} a_k |x|^k$, where a_i's are real constants, then f(x) is

(A) continuous at x = 0 for all a_i

(B) differentiable at x = 0 for all $a_i \in R$

(C) differentiable at x = 0 for all $a_{2k+1} = 0$

(D) none of these

Sol. **[A,C]**

$f(x) = a_0 + a_1 |x| + a_2 |x|^2 + \ldots\ldots\ldots$

non diff. at x = 0

if $a_1 = 0$ then f(x) is diff.

so, (A, C) correct

Q.21 The points at which the function, $f(x) = |x - 0.5| + |x - 1| + \tan x$ does not have a derivative in the interval (0, 2) are

(A) 1 (B) $\pi/2$ (C) 3 (D) $\dfrac{1}{2}$

Sol. **[A, B, D]**

Q.22 If f(x) = min {tan x, cot x} then -

(A) f is not differentiable at $x = 0, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

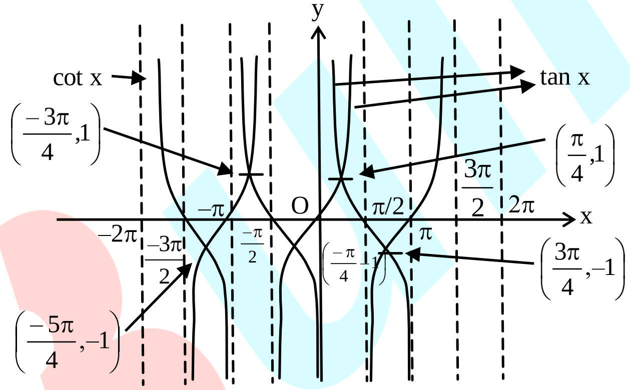
(B) f is discontinuous at $x = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

(C) $f\left(\dfrac{\pi}{4} - x\right) = f\left(\dfrac{\pi}{4} + x\right), 0 < x \le \dfrac{\pi}{4}$

(D) f is periodic with period π.

Sol. **[A, B, D]**

f(x) = min {tanx, cotx}



Darken line indicates the required area.

Hence, points of discontinuities are $\pm\pi$, $\pm\pi/2$, 0..

i.e. $\dfrac{n\pi}{2}$; $n \in I$

Points of non-differentiability are $\pm\pi, \pm\pi/2$, $3\pi/4, \ldots$

i.e. $\dfrac{n\pi}{4}$; $n \in I$

Since tanx and cot x are periodic with period π.

$\therefore$ Options (A), (B) and (D) are correct answer.

Q.23 If $f(x) = \begin{cases} (\sin^{-1} x)^2 \cos(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then -

(A) f(x) is continuous everywhere in $x \in [-1, 1]$

(B) f(x) is continuous nowhere in $x \in [-1, 1]$

(C) f(x) is differentiable everywhere in $x \in (-1,1)$

(D) f(x) is differentiable nowhere in $x \in [-1, 1]$

Sol. **[A,C]**

$f(x) = \begin{cases} (\sin^{-1} x)^2 \cos(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$\Rightarrow$ continuous

$f(x) = (\sin^{-1}(x))^2 \cos(1/x)$

$$= \frac{(\sin^{-1}(x))^2}{x^2}\left(\cos\frac{1}{x}\right)x^2$$

$$f(x) = \left(\cos\frac{1}{x}\right)x^2$$

f(x) = 0

differentiable

$$f(0+) = \frac{\sin^{-1}(h)^2\cos(1/h) - 0}{h} = 0$$

Q.24 If $f(x) = \tan^{-1}(\cot x)$, then-

(A) f(x) is periodic with period π

(B) f(x) is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

(C) f(x) is not differentiable at $x = \pi, 99\pi, 100\pi$

(D) $f(x) = -1$ for $2n\pi \le x \le (2n+1)\pi$

Sol. $f(x) = \tan^{-1}(\cot x)$

Cot is periodic with π

Differentiability at $x = \pi$

L.H.D.$= \lim_{x\to\pi^-}\frac{f(x)-f(\pi)}{x-\pi}$

$$= \lim_{h\to 0}\frac{f(\pi-h)-f(\pi)}{\pi-h-\pi}$$

$$= \lim_{h\to 0}\frac{\text{Tan}^{-1}(\cot(0-h)) - \tan^{-1}(\cot\pi)}{-h}$$

$$= \lim_{h\to 0}\frac{\text{Tan}^{-1}(-\cot h) - \tan^{-1}(-\infty)}{-h}\quad\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{h\to 0}\frac{\text{Tan}^{-1}(-\cot h) - \pi/2}{-h}$$

Use L – H Rule, we get

$$= \lim_{h\to 0}\frac{1}{1+\cot^2 h}\frac{(\text{cosec}^2 h)+0}{(-1)}$$

$$= \lim_{h\to 0}(-1)\frac{\text{cosec}^2 h}{1+\cot^2 h}$$

$$= \lim_{h\to 0}(-1)\frac{1/\sin^2 h}{1+\tan^2 h}\times \text{Tan}^2 h$$

$$= \lim_{h\to 0}(-1)\times\frac{1}{\cos^2 h(1+\text{Tan}^2 h)}$$

$$= (-1)\times\frac{1}{1(1+0)} = -1.$$

R.H.D.$= \lim_{x\to\pi^+} f(x)\frac{f(x)-f(\pi)}{x-\pi} = \lim_{h\to 0}\frac{f(\pi+h)-f(\pi)}{\pi+h-\pi}$

$$= \lim_{h\to 0}\frac{f(\pi+h)-f(\pi)}{h}$$

$$= \lim_{h\to 0}\frac{\tan{-}1(\cot(\pi+h))+\pi/2}{h}$$

$$= \lim_{h\to 0}\frac{\pi/2+\pi/2}{h}$$

$$= \lim_{h\to 0}\left(\frac{\pi}{h}\right)$$

= Not differentiable

Q.25 If $f(x) = x^2 - 3x + 2$, $g(x) = f(|x|)$, $h(x) = |f(x)|$ and $I(x) = |f(|x|)|$ then-

(A) Number of non differential points for I(x) is 5

(B) Number of non differential points for f(x), g(x), and h(x) is greater than number of non differential points for I(x)

(C) Number of solution of I(x) = 1/4 is 6

(D) Infinite number of solution exist for equation f(x) + I(x) = 0

Sol. [C] $f(x) = x^2 - 3x + 2$

$f(x) = f(|x|)$, $h(x) = |f(x)|$ and $I(x) = |f(|x|)|$

Q.26 If $\begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, the value of $f^{(n)}(x)$ at x = 0 for n = 2m +1 is -

(A) –1 (B) 0

(C) 1 (D) Independent of a

Sol. [B] $f(x) = \begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$

Differentiate w. r.t. x , we get

$$f(x) = \begin{vmatrix} nx^{n+1} & \cos x & +\sin x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} + 0 + 0$$

When we Differentiate 2nd row and third row, It becomes zero, because in these rows no x-terms are involved.

Again differentiate f'(x) w.r.t x and continues successively until n times

$$f''(x) = \begin{vmatrix} n(n-1)x^{n-2} & -\sin x & \cos x \\ n! & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} + 0 + 0$$

$$f^{111}(x) = \begin{vmatrix} n(n-1)(n-2)x^{n-3} & -\cos x & -\sin x \\ n! & \sin\frac{\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} +0+0$$

$$f^{n}(x) = \begin{vmatrix} n(n-1)(n-2)(n-3)\ldots 2.1 & -\cos x & \sin x \\ n! & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} +0+0$$

$$= \begin{vmatrix} n! & \cos x & \sin x \\ n! & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$$

At x = 0

$$f^{n}(0) = \begin{vmatrix} n! & 1 & 0 \\ n! & 1 & 0 \\ a & a^2 & a^3 \end{vmatrix}$$

$= 0 - 0 + 0^3 (n! - n!)$

Q.27 For the function $f(x) = (\pi - x)\dfrac{\cos x}{|\sin x|}$; $x \neq \pi$, $f(\pi) = 1$, which of the following statements are true ?

(A) $f(\pi^-) = -1$

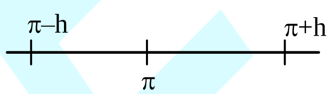
(B) $f(\pi^+) = 1$

(C) f(x) is continuous at $x = \pi$

(D) f(x) is differentiable at $x = \pi$

Sol. [B, C] $f(x) = (\pi - x)\dfrac{\cos x}{|\sin x|}$; $x \neq p$

$$f(x) = \begin{cases} (\pi - x)\dfrac{\cos x}{\sin x}; x \geq \pi \\ -(\pi - x)\dfrac{\cos x}{\sin x}; x < \pi \end{cases}$$



L.H.L. = $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{h\to 0} f(\pi - h)$

$= + \lim\limits_{h\to 0} h \times \dfrac{\cosh}{\sinh}$

$= \lim\limits_{h\to 0} \cosh \times \lim\limits_{h\to 0} \left(\dfrac{h}{\sinh}\right)$

$= 1 \times 1$

$= 1.$

R.H.L. = $\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{h\to 0} f(\pi + h)$

$= \lim\limits_{h\to 0} (\pi - \pi - h) \dfrac{\cos(\pi + h)}{\sin(\pi + h)}$

$= \lim\limits_{h\to 0} h \times \left(\dfrac{+\cosh}{+\sinh}\right)$

Part-C Assertion-Reason type Questions

The following questions 28 to 30 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.28 **Assertion:** $f(x) = |x| \cos x$ is not differentiable at x = 0.

Reason: Every absolute value functions are not differentiable.

Sol. [C]

Q.29 **Assertion:** f(x) = Sgn (cosx) is not differentiable at $x = \pi/2$

Reason: g(x) = [cosx] is not differentiable at $x = \dfrac{\pi}{2}$ ∀ [x] is greatest integer less than or equal to x.

Sol. [B] f(x) = sgn (cos x)

f(x) is discontinuous at $x = \dfrac{\pi}{2}$

as $f(\pi/2 +) = -1$

$f(\pi/2 -) = 1$

so non-differentiable

g(x) = [cos x]

non-differentiable at $\pi/2$ as discontinuous

Q.30 **Assertion:** f(x) = |x|. sinx is differentiable at x = 0

Reason: If f(x) is not differentiable and g(x) is differentiable at x = a, then f(x). g(x) will be differentiable at x = a

Sol. [C]

Part-D Column Matching type questions

Q.31 **Column I** **Column II**

(A) $f(x) = |x^3|$ is (P) continuous in (–1, 1)

(B) $f(x) = \sqrt{|x|}$ (Q) differentiable in (–1, 1)

(C) $f(x) = |\sin^{-1}x|$ is (R) differentiable in (0, 1)

(D) $f(x) = \cos^{-1}|x|$ is (S) not differentiable at least at one point in (–1, 1)

Sol.

(A) $f(x) = |x^3|$

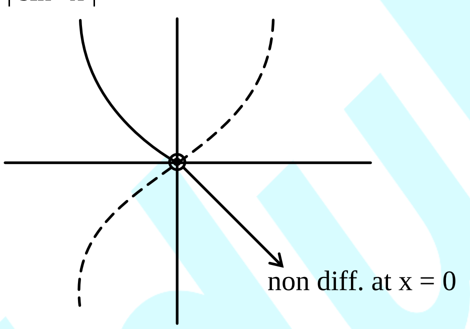
$f'(x) = -3x$

at x = 0 it is differentiable

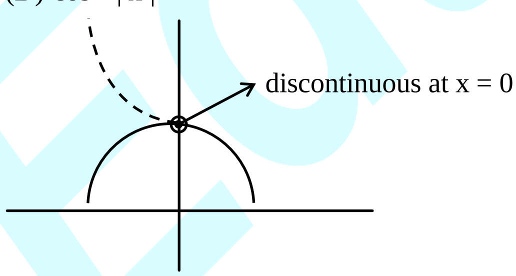
(B) $f(x) = \sqrt{|x|}$ is continuous in (–1, 1)

& non differentiable at x = 0

(C) $|\sin^{-1}x|$



(D) $\cos^{-1}|x|$



Q.32 **Column I**

(A) $f(x) = \begin{cases} 1-x & ; \quad x<1 \\ (1-x)(2-x) & ; \quad 1\le x\le 2 \\ 3-x & ; \quad x>2 \end{cases}$

(B) $f(x) = [x] + \sqrt{x-[x]}$; where [x] is greatest integer function

(C) y = f(x) where $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$

(D) $f(x) = |x-1| \cos\left(\frac{\pi x}{2}\right)$

Column - II

(P) f(x) is differentiable at x = 1

(Q) f(x) is non differentiable at x = 2

(R) f(x) is non- differentiable at x =1

(S) f(x)is differentiable at x = 2

Sol.

(A) $f(x) = 1 - x \quad : \quad x < 1$

$= 2 + x^2 - 3x \quad : \quad 1 \le x \le 2$

$= 3 - x \quad : \quad x > 2$

f '(x) at x = 1

$\Rightarrow$ diff. & continuous

f '(x) at x = 2

discontinuous

(B) $f(x) = [x] + \sqrt{x-[x]}$

$\Rightarrow$ non diff. at all integer

(C) $\mathbf{0 \le x < 1}$

$y = 3t^2,\ x = 3t - 1$

$3t^2 = 3t - 1$

$3t^2 - 3t + 1 = 0$ ….(1)

$\mathbf{1 \le x}$

$y = 3t^2,\ x = (t + 1)$

$3t^2 = t + 1$

$3t^2 - t - 1 = 0$ …(2)

so it is diff. at x = 1 & x = 2

(D) $f(x) = |x-1| \cos\left(\frac{\pi x}{2}\right)$

diff. at x = 1, x = 2

Q.33 **Column 1**

(A) Number of points where the function

$$f(x) = \begin{cases} 1+\left[\cos\frac{\pi x}{2}\right] & ; \quad 1<x\le 2 \\ 1-\{x\} & ; \quad 0\le x<1 \\ |\sin \pi x| & ; \quad -1\le x<0 \\ 0 & ; \quad x=1 \end{cases}$$

is continuous but not differentiable where [ ] denote greatest integer & { } denote fractional part function

(B) $f(x) = \begin{cases} x^2 e^{1/x} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$ then $f'(0^-) =$

(C) The number of points at which

$$g(x) = \frac{1}{1+\frac{2}{f(x)}} \text{ is not}$$

differentiable where $f(x) = \frac{1}{1+\frac{1}{x}}$, is

(D) Number of points where tangent does not exist for the curve $y = \text{sgn}(x^2 - 1)$

Column II

(P) 0
(Q) 1
(R) 2
(S) 3

Sol. (B) $f'(0) = \frac{h^2 e^{-1/h} - 0}{-h} = \frac{-h}{e^{1/h}} = 0$

(C) $f(x) = \frac{x}{1+x}$, $x = 1$

$$g(x) = \frac{1}{1+\frac{2(1+x)}{x}} = \frac{1}{3x+2}$$

$$x = \frac{2}{3}, \quad x = 0$$

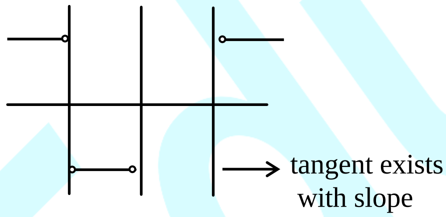
(D) $y = \text{sgn}((x-1)(x+1))$

$\text{sgn}(x^2 - 1)$

$= 1 \; ; \; x^2 > 1$

$= 0 \; ; \; x^2 = 1$

$= -1 \; ; \; x^2 < 1$



EXERCISE # 3

Part-A Subjective Type Questions

Q.1 (a) If $x = \sin^{-1} t$ and $y = t^3$

prove that $dy/dx = 3\sqrt{y(y^{1/3} - y)}$

(b) If $y = (1 + 1/x)^x + x^{(1 + 1/x)}$ find dy/dx.

(c) If $y = \log_u |\cos 4x| + |\sin x|$, where $u = \sec 2x$

find dy/dx at $x = -\pi/6$.

Sol. (a) $x = \sin^{-1}(t)$, $t = y^{1/3}$, $y = t^3$

$$\Rightarrow \frac{dy}{dx} = \frac{3t^2}{\sqrt{1-t^2}}$$

$t = y^{1/3}$

$\Rightarrow x = \sin^{-1}(y^{1/3})$

$\Rightarrow x - \sin^{-1}(y^{1/3}) = 0$

$$\frac{dy}{dx} = \left(\frac{-3\sqrt{1-y^{2/3}}}{y^{-2/3}}\right)$$

$$\frac{dy}{dx} = 3\sqrt{y(y^{1/3} - y)}$$

(b) $y = \left(1+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Differentiate w.r.t x, we get

$$\frac{dy}{dx} = x\left(1+\frac{1}{x}\right)^{x-1}\left(-\frac{1}{x^2}\right) + \left(1+\frac{1}{x}\right)^x \log\left(1+\frac{1}{x}\right)$$

$$+ \left(1+\frac{1}{x}\right) x^{\left(1+\frac{1}{x}-1\right)} + x^{\left(1+\frac{1}{x}\right)} \log x \left(-\frac{1}{x^2}\right)$$

$$= \left(1+\frac{1}{x}\right)^{x-1}\left(\frac{-1}{x}\right) + \left(1+\frac{1}{x}\right)^x \log\left(1+\frac{1}{x}\right) +$$

$$\left(1+\frac{1}{x}\right) x^{\left(1+\frac{1}{x}-1\right)} + x^{\left(1+\frac{1}{x}\right)} \log x\left(-\frac{1}{x^2}\right)$$

$$= \left(1+\frac{1}{x}\right)^x \left[-\frac{1}{x}\times\frac{1}{1+\frac{1}{x}} + \log\left(1+\frac{1}{x}\right)\right] +$$

$$x^{\left(1+\frac{1}{x}\right)} \left[\left(1+\frac{1}{x}\right)\times\frac{1}{x} - \frac{\log x}{x^2}\right]$$

$$= \left(1+\frac{1}{x}\right)^x \left[\log\left(1+\frac{1}{x}\right) - \frac{1}{x}\times\frac{x}{1+x}\right]$$

$$+ x^{\left(1+\frac{1}{x}\right)} \times \left[\frac{x+1}{x^2} - \frac{\log x}{x^2}\right]$$

$$= \left(1+\frac{1}{x}\right)^x \left[\log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right]$$

$$+ x^{\left(1+\frac{1}{x}\right)} \left[\frac{x+1-\log x}{x^2}\right]$$

(c) $y = \log_u |\cos 4x| + |\sin x|$

$u = \sec 2x$ at $x = -\pi/6$

Since, cos 4x will be +ve in 4th quadrant and sinx will be negative in 4th quadrant –

$$y = \frac{\log_e|\cos 4x|}{\log_e \sec 2x} + |\sin x|$$

$$y = \frac{\log_e \cos 4x}{\log_e \sec 2x} - \sin x$$

(y_1) (y_2) Assume

Differentiate w.r.t. x, weget

$$\frac{dy_1}{dx} =$$

$$\frac{\frac{1}{\cos 4x}(-4\sin 4x)\times\log_e \sec 2x - \log_e \cos 4x \times \frac{1\times\sec 2x\times\tan 2x}{\sec 2x}\times 2}{(\log_e \sec 2x)^2}$$

$$\frac{dy_1}{dx} = \frac{-4\tan 4x\times\log_e \sec 2x - 2\tan 2x\times\log_e \cos 4x}{(\log_e \sec 2x)^2}$$

$$\left.\frac{dy_1}{dx}\right|_{x=\frac{-\pi}{6}} = \frac{-4\tan(-120)\times\log_e 2 - 2\tan(-60)\times\log_e\frac{1}{2}}{(\log_e 2)^2}$$

$$= \frac{4\tan(90+30)\times\log_e 2 + 2\tan 60(-\log 2)}{(\log_e 2)^2}$$

$$= \frac{-4\times\sqrt{3}\times\log 2 - 2\sqrt{3}\log 2}{\log 2}$$

$= -6\sqrt{3}/\log_e 2$

$dy_2/dx = -\cos x$

$$\left.\frac{dy_2}{dx}\right|_{x=\frac{-\pi}{6}} = -\sqrt{3}/2$$

Hence, $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$

$$\frac{dy}{dx} = \frac{-6\sqrt{3}}{\log_e 2} - \sqrt{3}/2$$

$$= \frac{-\sqrt{3}(12+\log_e 2)}{2\log_e 2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}(12+\log_e 2)}{\log_e 4}$$

Q.2 (a) If f(x) is derivable at x = 3 and $f'(3) = 2$ then, find the value of

$$\lim_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2}$$

(b) if $f'(a) = \frac{1}{4}$ then find the value of

$$\lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{h^2}$$

Sol. Given that $f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$

$$= \lim_{h\to 0} \frac{f(3-h)-f(3)}{-h}$$

Replace h by h^2

$$f'(3) = \lim_{h\to 0} \frac{f(3+h^2)-f(3)}{h^2} \quad \ldots(i)$$

$$= \lim_{h\to 0} \frac{f(3-h^2)-f(3)}{-h^2} \quad \ldots.(ii)$$

(A) We have to find $\lim_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2}$

Adding (i) and (ii), we get

$$\lim_{h\to 0} \frac{f(3+h^2)-f(3)}{h^2} - \lim_{h\to 0} \frac{f(3-h^2)-f(3)}{h^2}$$

= 2 + 2 = 4

$$\lim_{h\to 0} \frac{f(3+h^2)-f(3)-f(3-h^2)+f(3)}{h^2}$$

$$= \lim_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{h^2} = 4.$$

$$\therefore \lim_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2} = 2.$$

(B) If f '(a) = 1/4 .

we have to find $\lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{h^2}$

$$f'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Replace h by $2h^2$, weget

$$f'(a) = \lim_{h\to 0} \frac{f(a-2h^2)-f(a)}{-2h^2}$$

$$= \lim_{h\to 0} \frac{f(a+2h^2)-f(a)}{2h^2} = 1/4$$

$$= \lim_{h\to 0} \frac{f(a+2h^2)-f(a)}{2h^2} - \lim_{h\to 0} \frac{f(a-2h^2)-f(a)}{2h^2} = 1/2$$

$$= \lim_{h\to 0} \frac{f(a+2h^2)-f(a)-f(a-2h^2)+f(a)}{2h^2} = 1/2$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{2h^2} = 1/2$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{h^2} = 1$$

Q.3 A function f satisfies the two conditions for all x, y ∈ R

(a) f(x + y) = f(x) . f(y)

(b) f(x) = 1 + x . g(x) where $\lim_{x\to 0} g(x) = 1$

Prove that the derivative $f'(x)$ exists and $f'(x) = f(x)$.

Sol. (A) f (x + y) = f(x). f(y) ; x,y ∈ R.

(B) f(x) = 1 + x. g(x) where $\lim_{x\to 0} g(x) = 1$.

$$f'(x) = \textbf{L.H.D.} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{x-h-x}$$

$$= \lim_{h\to 0} \frac{f(x).f(-h)-f(x)}{-h}$$

$$= f(x) . \lim_{h\to 0} \frac{f(-h)-1}{-h}$$

Use given condition, f(x) = 1 + x. g(x)

$$= g(x) = \frac{f(x)-1}{x}$$

Put x = – h

$$\Rightarrow g(-h) = \frac{f(-h)-1}{-h}$$

$$\Rightarrow \lim_{h\to 0} g(-h) = \lim_{h\to 0} \frac{f(-h)-1}{-h}$$

= 1.

∴ f ' (x) = L.H.D. = f(x)

$$\text{R.H.D.} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{x+h-x}$$

$$= \lim_{h\to 0} \frac{f(x).f(h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{f(h)-1}{h}$$

Use again given condition, weget

f '(x) = R.H.D. = f(x)

Since, L.H.D.= R.H.D. = f(x)

Hence, f(x) exists and f '(x) = f(x)

Q.4 Let R be the set of real numbers and $f : R \to R$ be such that for all x & y in R, $|f(x) - f(y)| \le |x - y|^3$. Prove that f(x) is constant.

Sol. Given condition $|f(x)-f(y)| \le |x-y|^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{x+h-x}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Use above condition

$|f(x+h) - f(x)| \le |x+h-x|^3$

$\Rightarrow |f(x+h) - f(x)| \le |h|^3$

$$\therefore\ f'(x) = \lim_{h\to 0} \frac{|h|^3}{(-h)} = 0$$

$f'(x) = 0$

Integrate w.r.t. x, weget

f(x) = constant.

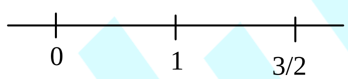
Q.5 Discuss the derivability in [–2, 2] of

$$f(x) = \begin{cases} \sqrt{4x^2-12x+9}\ \{x\}\ ;\ x \ge 1 \\ \cos\left(\frac{\pi}{2}(|x|-\{x\})\right)\ ;\ x < 1 \end{cases}$$

where {x} denotes the fractional part of x.

Sol.

$$f(x) = \begin{cases} \sqrt{4x^2-12x+9}\ \{x\} \text{ for } x \ge 1 \\ \cos\left(\frac{\pi}{2}(|x|-\{x\})\right) \text{ for } x < 1 \end{cases}$$



When $x < 0$, $f(x) = \cos\left(\frac{\pi}{2}(-x-x+[x])\right)$

$$= \cos\left(\frac{\pi}{2}([x]-2x)\right)$$

When $0 \le x < 1$, $f(x) = \cos\left(\frac{\pi}{2}(x-x+[x])\right)$

$$= \cos\left(\frac{\pi}{2}[x]\right)$$

When $1 \le x < 3/2$; $f(x) = (3-2x)(x-[x])$

When $x \ge 3/2$; $f(x) = (2x-3)(x-[x])$

$$f(x) = \begin{cases} \cos\left(\frac{\pi}{2}([x]-2x)\right) ; x < 0 \\ \cos\frac{\pi}{2}[x] \qquad ; 0 \le x < 1 \\ (3-2x)(x-[x])\ ; 1 \le x < 3/2 \\ (2x-3)(x-[x])\ ;\ x \ge 3/2 \end{cases}$$

Differentiability at x = 0

L.H.D. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}([0-h]-2(0-h))-1\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}(-1+2h)\right)-1}{-h}$$

$$= \lim_{h\to 0} \frac{\cos(-\pi/2)-1}{-h}$$

$$= \lim_{h\to 0} \frac{0-1}{-h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right) \to \text{Non-Differentiable}$$

$\therefore$ f(x) is not differentiable at x = 0

Differentiablity at x = 1

L.H.D. $= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$

$$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{1-h-1}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}[1-h]\right)-0}{-h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}.0\right)-0}{-h}$$

$$= \lim_{h\to 0} \frac{1-0}{-h}$$

$$= \lim_{h\to 0} \frac{1}{-h} \to \text{Non-differentiable}$$

$\therefore$ f(x) is not differentiable at x = 1.

Differentiability at x = 3/2

L.H.D. $= \lim_{x\to 3/2^-} \frac{f(x)-f(3/2)}{x-3/2}$

$$= \lim_{h\to 0} \frac{f(3/2-h)-f(3/2)}{3/2+h-3/2}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3}{2}-h\right)-f\left(\frac{3}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{(3-2(3/2-h)(3/2-h-[3/2-h])-0}{-h}$$

$= \lim_{h\to 0} \frac{(3-3+2h)(3/2-h-[1.5-h])-0}{-h}$

$= \lim_{h\to 0} \frac{(2h)(3/2-h-1)-0}{-h}$

$= \lim_{h\to 0} \frac{(2h)(1/2-h)}{-h}$

$= \lim_{h\to 0} 2(h-1/2) = -1.$

R.H.D.$= \lim_{x\to 3/2^+} \frac{f(x)-f(3/2)}{x-3/2}$

$= \lim_{h\to 0} \frac{f(3/2+h)-f(3/2)}{3/2+h-3/2}$

$= \lim_{h\to 0} \frac{\left(2\left(\frac{3}{2}+h\right)-3\right)\left(\frac{3}{2}+h-\left[\frac{3}{2}+h\right]\right)-0}{h}$

$= \lim_{h\to 0} \frac{(3+2h-3)\left(\frac{3}{2}+h-1\right)-0}{h}$

$= \lim_{h\to 0} \frac{(2h)\left(\frac{1}{2}+h\right)-0}{h}$

$= \lim_{h\to 0} 2\left(\frac{1}{2}+h\right)$

$= 1.$

Since, L.H.D. $\neq$ R.H.D.

Hence, f(x) is not differentiable at x = 3/2.

Q.6 If $f : R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$ then show that f(2) = f(1) – f(0) and find out the function.

Sol. $f : R \to R$

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$= x^3 + ax^2 + bx + c$

Then Differentiate w.r.t x, we get

$f'(x) = 3x^2 + 2ax + b \to$ First differentiation

$f''(x) = 6x + 2a \to$ Second differentiation

$f'''(x) = 6 \to$ third Differentiation

$f'(1) = 3 + 2a + b = a \Rightarrow a + b = -3$

$f''(2) = 12 + 2a = b \Rightarrow -2a + b = 12$

$f'''(3) = 6 = c \Rightarrow c = 6$

$a + b = -3$ & $-2a + b = 12$

on solving we get a = – 5, b = 2

Hence, $f(x) = x^3 - 5x^2 + 2x + 6$

$f(2) = 8 - 20 + 4 + 6 = -2$

$f(1) = 1 - 5 + 2 + 6 = 4$

$f(0) = 6$

$f(1) - f(0) = 4 - 6 = -2 \Rightarrow f(1) - f(0) = f(2)$

Hence, Proved.

Q.7 If $y = \text{arc}\cos\sqrt{(\cos 3x)/(\cos^3 x)}$, show that $dy/dx = \sqrt{6}/\sqrt{(\cos 2x)+(\cos 4x)}$

Sol. $y = \cos^{-1}\sqrt{(\cos 3x/\cos^3 x)}$

Differentiate w.r.t. x, weget

$dy/dx = \frac{-1}{\sqrt{1-(\cos 3x/\cos^3 x)^2}} \times \frac{1}{2\sqrt{\cos 3x/\cos^3 x}}$

$\times \frac{-3\sin 3x\cos^3 x + \cos 3x\, 3\cos^2 x\sin x}{\cos^6 x}$

$= \frac{-1\times\cos^3 x}{\sqrt{\cos^6 x - \cos^2 3x}} \times \frac{\sqrt{\cos^3 x}}{2\sqrt{\cos 3x}}$

$\times \frac{3\cos^2 x(\sin x\cos 3x - \cos x.\sin 3x)}{\cos^6 x}$

$= \frac{-1\times 3\sqrt{\cos^3 x}}{\sqrt{\cos 3x(\cos^6 x - \cos^2 3x)}} \times \frac{\sin(-2x)}{2\cos x}$

$= \frac{3\sqrt{\cos^3 x}\times 2\sin x\times\cos x}{2\cos x\sqrt{\cos 3x(\cos^3 x - \cos 3x)(\cos^3 x + \cos 3x)}}$

$= \frac{3\sqrt{\cos^3 x}\times\sin x\times\cos x}{\cos x\times\sqrt{\cos 3x(\cos^3 x - 4\cos^3 x + 3\cos x)\times(5\cos^3 x - 3\cos x)}}$

$= \frac{3\sqrt{\cos^3 x}\times\sin x\times\cos x}{\cos x\times\sqrt{\cos 3x.3\cos x.(1-\cos^2 x)(5\cos^3 x - 3\cos x)}}$

$= \frac{3\sqrt{\cos^3 x}\times\sin x\times\cos x}{\cos x\times\sqrt{3}\sqrt{\cos x}\times\sin x\sqrt{\cos 3x(5\cos^3 x - 3\cos x)}}$

$= \frac{\sqrt{3}\times\sqrt{\cos x}\times\cos x\times\sin x\times\cos x}{\cos^2 x\times\sqrt{\cos x}\times\sin x\sqrt{\cos 3x(5\cos^2 x - 3)}}$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{\cos 3x(5\cos^2 x - 3)}}$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{(4\cos^3 x - 3\cos x)(5\cos^2 x - 3)}}$$

Q.8 If $y = 1+\frac{C_1}{(x-C_1)}+\frac{C_2x}{(x-C_1)(x-C_2)}$ $+\frac{C_3x^2}{(x-C_1)(x-C_2)(x-C_3)}$ .....to $(n+1)^{th}$ terms, then prove that

$$\frac{dy}{dx}=\frac{y}{x}\left[\frac{C_1}{C_1-x}+\frac{C_2}{C_2-x}+........+\frac{C_n}{(C_n-x)}\right]$$

Sol. $y = 1 + \frac{a}{x-a} + \frac{c_2x}{(x-a)(x-a_2)}$ $+\frac{C_3x^2}{(x-a)(x-a_2)(x-a_3)}+\ldots\ldots. + (n+1)^{th}$

Q.9 If $f(x) = \begin{cases} 3-x^2, & -1 \le x < 2 \\ 2x-4, & 2 \le x \le 4 \end{cases}$, then find fof(x) and discuss its continuity and differentiability.

Sol. Non continuous and hence non differentiable at x = 1, 2, 3

Q.10 If $x = at^3$ and $y = bt^2$ where t is a parameter then prove that $-d^3y/dx^3 = 8b/27a^3t^7$.

Sol. $x = at^3$ and $y = bt^2$

Differentiate above function w.r.t.t, weget

$\frac{dx}{dt} = 3at^2$ and $dy/dt = 2bt$

$$\frac{dy}{dx}=\frac{2bt}{3at^2}=2b/3at$$

Differentiating w.r.t.x, weget

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{2b}{3at}\right)=-\frac{2b}{3a}\times\frac{1}{t^2}\times\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{2b}{3a}\times\frac{1}{t^2}\times\frac{1}{3at^2}$$

$$=-\frac{2b}{9a^2}\times t^{-4}$$

Again Differentiating, w.r.t. x, weget

$$\frac{d^3y}{dx^3}=-\frac{2b}{9a^2}\frac{d}{dx}(t^{-4})$$

$$=-\frac{2b}{9a^2}\times(-4\times t^{-5})\times\frac{dt}{dx}$$

$$=\frac{8b}{9a^2}\times\frac{1}{t^5}\times\frac{dt}{dx}=\frac{8b}{9a^2}\times\frac{1}{t^5}\times\frac{1}{3at^2}$$

$d^3y/dx^3 = \frac{8b}{27a^3}\times\frac{1}{t^7}$. Hence Proved.

Q.11 The function f is defined by y = f(x). Where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$. Draw the graph of f for the interval $-1 \le x \le 1$. Also discuss its continuity and differentiability at x = 0.

Sol. y = f(x)

$x = 2t - |t|$; $y = t^2 + t|t|$; $t \in R$

when $t \ge 0$, $x = 2t - t \Rightarrow x = t \Rightarrow x \ge 0$

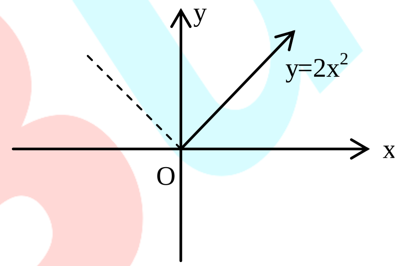
$y = t^2 + t^2 = 2t^2 \Rightarrow t^2 = \frac{y}{2} = x^2$

$y = 2x^2$; $x \ge 0$

when $t < 0$, $x = 2t + t = 3t \Rightarrow t = x/3$

$y = t^2 - t^2 = 0$

$$f(x)=\begin{cases}2x^2 & ;x\ge 0\\ 0 & ;x<0\end{cases}$$



continuity at x = 0

L.H.L.$=\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$=\lim\limits_{h\to 0} 2(0-h)^2$

$=\lim\limits_{h\to 0} 2h^2 = 0$

R.H.L.$=\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$=\lim\limits_{h\to 0} 2(0+h)^2$

$=\lim\limits_{h\to 0} 2h^2 = 0$

$f(0) = \lim\limits_{x\to 0} f(x) = 0$

Since, L.H.L. = R.H.L = f(0) = 0.

$\therefore$ f(x) is continuous at x = 0

differentiability at x = 0

L.H.D. $=\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim\limits_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$

$=\lim\limits_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$=\lim\limits_{h\to 0}\frac{2(0-h)^2-0}{-h}=\lim\limits_{h\to 0}\frac{2h^2-0}{h}=0$

R.H.D.$=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{2(0+h)^2-0}{h}$

$$= \lim_{h\to 0} \frac{2h^2 - 0}{h} = 0.$$

Since, L.H.D.=R.H.D.

Hence, f(x) is differentiable at x = 0.

Q.12 Find $f''(0)$ if $f(x) = 2^{\sin x}\cos(\sin x)$.

Sol. $f(x) = 2^{\sin x} . \cos(\sin x)$.

Differentiate w.r.t. x, weget

$f'(x) = 2^{\sin x} \cdot \log 2.\cos x.\cos(\sin x) +$

$2^{\sin x}(-\sin(\sin x))\cos x$

$f'(x) = 2^{\sin x} \cdot \log 2.\cos x.\cos(\sin x) -$

$2^{\sin x}\cos x.\sin(\sin x)$

f '(x)=2sinx[log2.cosx.cos(sinx)–cosx. sin(sinx)]

Again differentiating w.r.t x, weget

$f''(x) = 2^{\sin x} . \log 2.\cos x.[\log 2.\cos x.\cos(\sin x) - \cos x.\sin(\sin x)] + 2^{\sin x}$

$[\log 2.(-\sin x)\cos(\sin x)+\log 2.\cos x$

$(-\sin(\sin x))\times\cos x+\sin x.\sin(\sin x)-$

$\cos x.\cos(\sin x)\times\cos x]$

$f''(x)|_{x=0} = f''(0) = 1\cdot \log 2.\ 1[\log 2.1.-0] + 1\ [0-0 + 0 - 1]$

$= (\log 2)^2 - 1$

$f''(0) = (\log 2)^2 - 1$

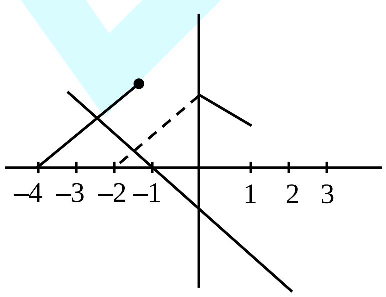
Q.13 If $f(x) = \begin{cases} 2x^2+12x+16 & -4\le x\le -2 \\ 2-|x| & -2<x\le 1 \\ 4x-x^2-2 & 1<x\le 3 \end{cases}$ then

comment on continuity and differentiability

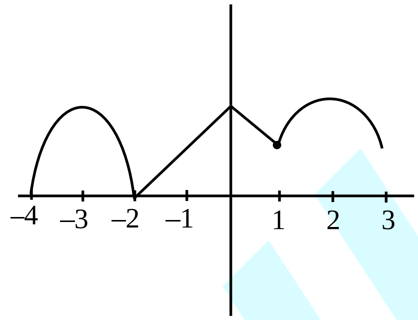
(i) |f(x)| (ii) f(|x|)

Sol.

$$f(x) = \begin{cases} 2x^2+12x+16 : -4\le x\le -2 \\ 2+x \quad :-2<x<0 \\ 2-x \quad : \ 0\le x\le 1 \\ 4x-x^2-2 \ : \ 1<x\le 3 \end{cases}$$



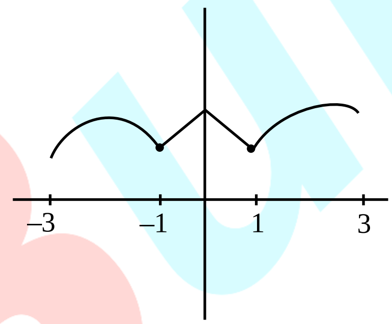
| f(x) | = f(x)



Continuity

⇒ continuous every where

f |x|



⇒ continuous

⇒ Non-differentiable

| f(x) | = –2, 0, 1

f | x | = –1, 0, 1

Part-B Passage based objective questions

Passage I (Question 14 to 16)

Left hand derivative and right hand derivative of a function f(x) at a point x = a are defined as

$$f'(a^-) = \lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$$

$$\& \ f'(a^+) = \lim_{h\to 0+}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0-}\frac{f(a)-f(a-h)}{h}$$

Respectively. Let f be a twice differentiable function.

On the basis of above information, answer the following questions.

Q.14 If f is odd, which of the following is left hand derivative of f at x = –a

(A) $\lim_{h\to 0^-}\dfrac{f(a-h)-f(a)}{-h}$

(B) $\lim_{h \to 0^-} \frac{f(h-a)-f(a)}{h}$

(C) $\lim_{h \to 0^+} \frac{f(a)+f(a-h)}{-h}$

(D) $\lim_{h \to 0^-} \frac{f(-a)-f(-a-h)}{-h}$

Sol. **[A]**

f(x) is odd

$\Rightarrow f(-x) = -f(x)$

LHD

$f(-a-) = \lim_{x \to 0^-} \frac{f(-a+h)-f(-a)}{h}$

$= \lim_{x \to 0^-} \frac{-f(a-h)+f(a)}{h}$

LHD $= \lim_{x \to 0^-} \frac{f(a-h)-f(a)}{-h}$

Q.15 If f is even which of the following is Right hand derivative of f' at x = a

(A) $\lim_{h \to 0^-} \frac{f'(a)+f'(-a+h)}{h}$

(B) $\lim_{h \to 0^+} \frac{f'(a)+f'(-a-h)}{-h}$

(C) $\lim_{h \to 0^-} \frac{-f'(-a)+f'(-a-h)}{-h}$

(D) $\lim_{h \to 0^+} \frac{f'(a)+f'(-a+h)}{-h}$

Sol. **[A,B]**

$f(-x) = f(x)$

$\Rightarrow -f'(-x) = f'(x)$

$\Rightarrow -f'(-x) = +f'(x)$

RHD : at x = a

$\lim_{x \to 0^+} \frac{f'(a+h)-f'(a)}{h}$

$= \lim_{h \to 0^+} \frac{f'(a)+f'(a+h)}{-h}$

$\Rightarrow \lim_{x \to 0^+} \frac{-f'(-a-h)+f'(-a)}{h}$

or $\lim_{h \to 0^-} \frac{f'(a)-f'(a-h)}{h}$

$\lim_{h \to 0^-} \frac{f'(a)+f'(-a+h)}{h}$

Q.16 The statement

$\lim_{h \to 0} \frac{f(-x)-f(-x-h)}{h} = \lim_{h \to 0} \frac{f(x)-f(x-h)}{-h}$

implies that

(A) f is odd

(B) f is even

(C) f is neither odd nor even

(D) nothing can be concluded

Sol. **[B]**

$\lim_{h \to 0} \frac{f(-x)-f(-x-h)}{h}$

$= \lim_{h \to 0} \frac{f(x)-f(x-h)}{-h}$

RHS

$\lim_{h \to 0} \frac{f(x)-f(x-h)}{-h} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{-h}$

$= \lim_{h \to 0} \frac{f(x)-f(x+h)}{h}$

for even $f(-x) = f(x)$

so this satisfies

$\lim_{h \to 0} \frac{f(-x)-f(-x-h)}{h}$

So f(x) is even.

Passage II (Question 17 to 19)

Let us define here two functions

$f(x) = ax^2 + bx + c$; a, b, c $\in$ R, $a \neq 0$ & $c \neq 0$

$g(x) = dx^2 + ex + f$; d, e, f $\in$ R, $d \neq 0$

f(x) is symmetrical about the y- axis and

g(x) = 0 has complex roots.

Q.17 If h(x) = |f(x)| has zero non differential point, then f(x) = 0 has-

(A) two real and different roots

(B) two real and equal roots

(C) Two purely imaginary complex roots

(D) None of these

Sol. **[C]**

h(x) = | f(x) | is zero

non diff point so f(x) must not cut x-axis. Thus

f(x) = 0 has 2 complex roots.

Q.18 If f(x) has two real roots α and β, then

h(x) = |f (|x|)| has-

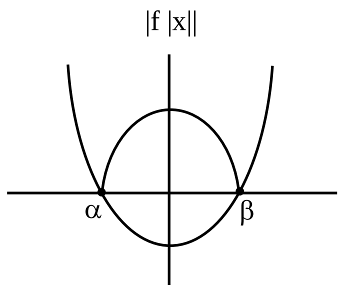
(A) zero non differential point

(B) one non differential point, x = 0

(C) two non differential point, x = α, β

(D) None of these

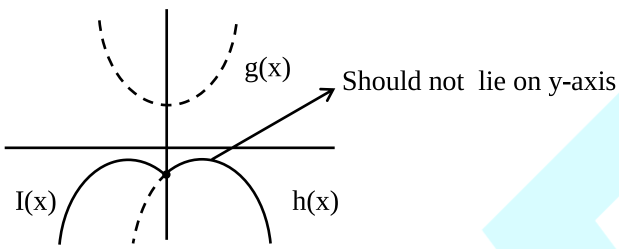
Sol. **[C]**



2 point of non differentiable

Q.19 If graph of h(x) function is reflection of g(x) with respect to x - axis and I(x) = h(|x|) has one non differential point, then-
(A) e may be zero (B) e never be zero
(C) f may be zero (D) None of these

Sol. **[D]**



(a) x = 0, g(x) must have some value on y-axis so, $f \neq 0$.

Passage III (Question 20 to 22)

If $f(x) = x^2 - 2|x|$ and

$$g(x) = \begin{cases} \min.\{f(t) : -2 \le t \le x,\ -2 \le x < 0\} \\ \max.\{f(t) : 0 \le t \le x,\ 0 \le x \le 3\} \end{cases}$$

then

Q.20 The function f(x) is continuous for -
(A) R (B) R – {0}
(C) R – {0, –2, 2} (D) None of these

Sol. **[C]**

$f(x) = x^2 - 2|x|$

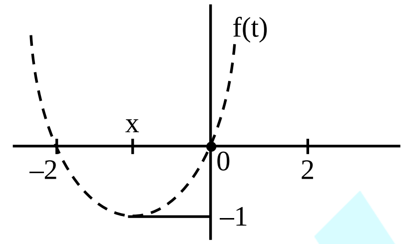
$\Rightarrow$ continuous for all R.

(1) $g(x) = \min(f(t) : -2 \le t \le x) : -2 \le x < 0$

$f(t) = t^2 - 2|t|$

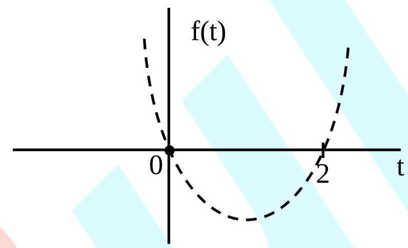
$= |t|(|t| - 2)$ $(x^2 + 2x)$

$\Rightarrow -(4)$



$$\left.\begin{aligned} g(x) &= x^2 + 2x : -2 \le x < -1 \\ &= -1 \quad : -1 \le x < 0 \end{aligned}\right] \quad \ldots..(1)$$

$g(x) = \max(f(t) : 0 \le t \le x : 0 \le x \le 3)$



Q.21 The function g(x) is continuous for -
(A) $x \in R - \{0\}$ (B) $x \in R - \{-1, 0, 1\}$
(C) $x \in R - \{0, 2\}$ (D) None of these

Sol. **[A]**

$g(x) = 0 \quad : \quad 0 \le x \le 2$

$= x^2 - 2x \quad : \quad 2 < x \le 3$

for g(x)

$g(x) = x^2 + 2x \quad : \quad -2 \le x < -1$

$= -1 \quad : \quad -1 \le x < 0$

$= 0 \quad : \quad 0 \le x \le 2$

$= x^2 - 2x \quad : \quad 2 < x \le 3$

g(x) is continuous for R – {0}

Q.22 Function fog(x) is discontinuous for -
(A) x = 0, 2 (B) x = –1, 0
(C) x = –1, 0, 2 (D) x = 2

Sol. **[B]**

$fog(x) = (g(x))^2 - 2|g(x)|$

(1) $= x^4 + 4x^2 + 4x^3 + 2x^2 + 4x$

$= x^4 + 4x^3 + 6x^2 + 4x \quad (-2 \le x < -1)$

(3) $= (-1)^2 + 2$

$= 3 \quad : \quad -1 \le x < 0$

(3) $= 0 : \quad 0 \le x < 2$

(4) $x^4 + 4x^3 + 4x^2 - 2x^2 - 4x$

$= x^4 + 4x^2 + 2x^2 - 4x \qquad (2 \le x < 0)$

so fog (x) is discontinuous at

x = –1, 0

EXERCISE # 4

Old IIT-JEE Questions

Q.1 Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$.

If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$.

Prove that $|a_1 + 2a_2 + \ldots\ldots\ldots + na_n| \le 1$.

[IIT 2000]

Sol. $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.+n\,x^n$

Given $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$.

we have to prove that

$|a_1 + 2a_2 + 3a_3 + \ldots..+n\,a_n| \le 1$.

from given condition $|p(x)| \le |e^{x-1}-1|$

differentiating w.r.t. x, we get

$|p'(x)| \le |e^{x-1}-0|$

$|p'(x)| \le |e^{x-1}|$ ……..(i)

Since, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.+ a_nx^n$

differentiating w.r.t. x, we get

$p'(x) = 0+a_1+2a_2x + 3x^2a_3 + \ldots.+a_n.nx^{n-1}$ …(ii)

Put x = 1 in equations (1) and (2), we get

$|p'(1)| \le |1| \Rightarrow |p'| \le 1$

and $p'(1) = a_1 + 2a_2 + 3a_3 + \ldots.+na_n$

$\Rightarrow |a_1 + 2a_2 + 3a_3 + \ldots..+na_n| \le 1$. Hence, Proved.

Q.2 Let $f : R \to R$ is a function which is defined by $f(x) = \max\{x, x^3\}$ set of points on which f (x) is not differentiable is **[IIT Scr. 2001]**

(A) {–1, 1} (B) {–1, 0}

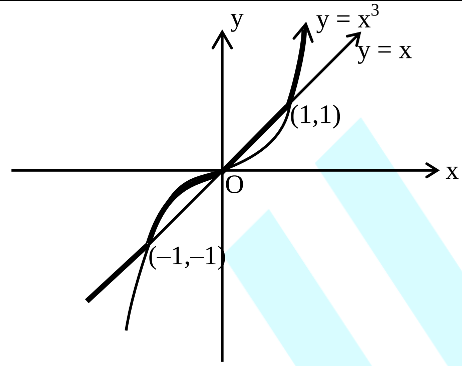
(C) {0, 1} (D) {–1, 0, 1}

Sol.[D] $f : R \to R$

$f(x) = \max\{x, x^3\}$

Darken line indicates, portion of curve will be above point of Intersection.

$$f(x) = \begin{cases} x \,;\, x < -1 \\ x^3 ;\, -1 \le x < 0 \\ x \,;\, 0 \le x < 1 \\ x^3 ;\, x \ge 1 \end{cases}$$



Differentiability at x = 1.

L.H.D.$= \lim\limits_{x\to -1^-} \dfrac{f(x)-f(-1)}{x+1} = \lim\limits_{h\to 0} \dfrac{f(-1-h)-f(-1)}{-1-h+1}$

$= \lim\limits_{h\to 0} \dfrac{f(-1-h)-f(-1)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{(-1-h)+1}{-h} = 1.$

R.H.D.$= \lim\limits_{x\to -1^+} \dfrac{f(x)-f(-1)}{x+1} = \lim\limits_{h\to 0} \dfrac{f(-1+h)-f(-1)}{-1+h+1}$

$= \lim\limits_{h\to 0} \dfrac{(-1+h)^3+1}{h} \left(\dfrac{0}{0}\text{form}\right)$

$= \lim\limits_{h\to 0} \dfrac{3(-1+h)^2+0}{1} = 3$

Since, L.H.D. $\ne$ R.H.D. Hence f(x) is not differentiable at x = –1.

Differentiability at x = 0

L.H.D.$= \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{0-h+0}$

$= \lim\limits_{h\to 0} \dfrac{(0-h)^3-0}{-h}$

$= \lim\limits_{h\to 0} (-h^2) = 0.$

R.H.D.$= \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{0+h-0}$

$= \lim\limits_{h\to 0} \dfrac{(0+h)-0}{h}$

= 1.

Since, L.H.D. $\ne$ R.H.D.

$\therefore$ f(x) is not differentiable at x = 0

Differentiability at x = 1.

L.H.D.$= \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{1-h-1}$

$= \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{1-h-1}{-h} = 1.$

R.H.D.$= \lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{1+h-1}$

$= \lim\limits_{h\to 0} \dfrac{(1+h)^3-1}{h} \left(\dfrac{0}{0}\text{form}\right)$

use L–H Rule, we get

$= \lim\limits_{h\to 0} \dfrac{3(1+h)^2-0}{1} = 3$

Since , L.H.D. $\neq$ R.H.D

Hence f(x) is not differentiable at x = 1.

Option (D) is correct answer.

Q.3 Let, $\alpha \in R$, prove that a function $f : R \to R$ is differentiable at α iff there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. **[IIT 2001]**

Sol. Given : $\alpha \in R$

$g : R \to R$ is continuous at $x = \alpha$

$f(x) - f(\alpha) = g(x)(x-\alpha)$

$\Rightarrow g(x) = \dfrac{f(x)-f(\alpha)}{x-\alpha}$

Since, g(x) is continuous at $x = \alpha$

L.H.L. $= \lim\limits_{x\to \alpha^-} g(x) = \lim\limits_{h\to 0} g(\alpha-h)$

R.H.L.$= \lim\limits_{x\to \alpha^+} g(x) = \lim\limits_{h\to 0} g(\alpha+h)$

$g(\alpha) = \lim\limits_{x\to \alpha} g(x)$

$\therefore \lim\limits_{h\to 0} g(\alpha-h) = \lim\limits_{h\to 0} g(\alpha+h) = g(\alpha)$

We have to prove that $f : R \to R$ is differentiable at $x = \alpha$

L.H.D. $= \lim\limits_{x\to \alpha^-} \dfrac{f(x)-f(\alpha)}{x-\alpha}$

$= \lim\limits_{h\to 0} \dfrac{f(\alpha-h)-f(\alpha)}{\alpha-h-\alpha}$

$= \lim\limits_{h\to 0} \dfrac{f(\alpha-h)-f(\alpha)}{\alpha-h-\alpha}$

$= \lim\limits_{h\to 0} \dfrac{f(\alpha-h)-f(\alpha)}{-h}$

Using given condition, $g(x) = \dfrac{f(x)-f(\alpha)}{x-\alpha}$

Put $x = \alpha - h$

$g(\alpha-h) = \dfrac{f(\alpha-h)-f(\alpha)}{\alpha-h-\alpha}$

$g(\alpha-h) = \dfrac{f(\alpha-h)-f(\alpha)}{-h}$

i.e. $\lim\limits_{h\to 0} g(\alpha-h) = \lim\limits_{h\to 0} \dfrac{f(\alpha-h)-f(\alpha)}{-h} = g(\alpha)$

$\therefore$ L.H.D. $= \lim\limits_{h\to 0} \dfrac{f(\alpha-h)-f(\alpha)}{-h} = g(\alpha)$

R.H.D.$= \lim\limits_{x\to \alpha^+} \dfrac{f(x)-f(\alpha)}{x-\alpha} = \lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{\alpha+h-\alpha}$

$= \lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{h}$

Using given condition, $g(x) = \dfrac{f(x)-f(\alpha)}{x-\alpha}$

Put $x = \alpha + h$

$g(\alpha+h) = \dfrac{f(\alpha+h)-f(\alpha)}{\alpha+h-\alpha}$

$\lim\limits_{h\to 0} g(\alpha+h) = \lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{h} = g(\alpha)$

$\therefore$ R.H.D. $= \lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{h} = g(\alpha)$

Hence, f(x) is differentiable function at $x = \alpha$.

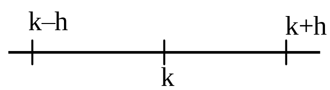
Q.4 Find left hand derivative at $x = k$, $k \in I$. $f(x) = [x]\sin(\pi x)$ **[IIT Scr. 2001]**

(A) $(-1)^k (k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$

(C) $(-1)^k (k-1)k\pi$ (D) $(-1)^{k-1}(k-1)k\pi$

Sol.[A] $f(x) = [x]\sin(\pi x)$

At $x = k$; $k \in I$



L.H.D.$= \lim\limits_{x\to k^-} \dfrac{f(x)-f(k)}{x-k} = \lim\limits_{h\to 0} \dfrac{f(k-h)-f(k)}{k-h-k}$

$= \lim\limits_{h\to 0} \dfrac{f(k-h)-f(k)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{[k-h]\sin(\pi(k-h)) - k\sin \pi k}{-h}$

$= \lim\limits_{h\to 0} \dfrac{+(k-1)\sin(\pi(k-h))-0}{-h} \left(\dfrac{0}{0}\text{form}\right)$

Use L – H Rule we get

$= \lim\limits_{h\to 0} \dfrac{(k-1)\cos\pi(k-h)\times(-\pi)}{-1}$

$= \pi(k-1)\cos k\pi$

$= (-1)^k \pi(k-1)$

$\therefore$ Option (A) is correct answer.

Q.5 Which of the following functions is differentiable at x = 0? **[IIT Scr. 2001]**

(A) cos (|x|) + |x| (B) cos (|x|) – |x|

(C) sin (|x|) + |x| (D) sin (|x|) – |x|

Sol.[D] We have to check for every for every options

For A : f(x) = cos (|x|) + |x|

L.H.D. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0} \frac{\cos(|0-h|)+|0-h|-1}{-h}$

$= \lim_{h\to 0} \frac{\cos h + h - 1}{-h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{-\sin h + 1}{-1}$

$= \frac{1-\sin 0}{-1} = -1$

R.H.D. $= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$

$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{\cos|0+h|+|0+h|-1}{h}$

$= \lim_{h\to 0} \frac{\cos h + h - 1}{h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{-\sin h + 1 - 1}{1}$

= 1.

Since, L.H.D. ≠ R..H.D.

Hence, f(x) is not differentiable at x = 0

∴ Option (A) is not correct answer.

For B : f(x) = cos (|x|) + |x|

L.H.D. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0} \frac{\cos|0-h|-|0-h|-1-0}{-h}$

$= \lim_{h\to 0} \frac{\cos h - h - 1}{-h} \left(\frac{0}{0}\text{form}\right)$

use L–H Rule, weget

$= \lim_{h\to 0} \frac{-\sin h - 1 - 0}{-1} = 1.$

R.H.D.=

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\cos|0+h|-|0+h|-1}{h}$

$= \lim_{h\to 0} \frac{\cos h - h - 1}{h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{-\sin h - 1 - 0}{1}$

= – 1.

∴ (B) is not differentiable at x = 0.

For C : f(x) = sin(|x|) + |x|

L.H.D. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0} \frac{\sin|0-h|+|0-h|-0}{-h}$

$= \lim_{h\to 0} \frac{\sin h + h}{-h} \left(\frac{0}{0}\text{form}\right)$

= – 2.

R.H.D. $= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{\sin|0+h|+|0+h|-0}{h}$

$= \lim_{h\to 0} \frac{\sin h + h}{h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{\cos h + 1}{1} = 2$

∴ (C) is not differentiable at x = 0.

For D : f(x) = sin(|x|) – |x|

L.H.D. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \frac{\sin|0-h|-|0-h|-0}{-h}$

$= \lim_{h\to 0} \frac{\sin h - h}{-h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{\cos h - 1}{-1} = 0$

R.H.D. $= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$

$= \lim_{h\to 0} \frac{\sin|0+h|-|0+h|-0}{h}$

$= \lim_{h\to 0} \frac{\sin h - h}{h} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to 0} \frac{\cos h - 1}{1} = 0.$

∴ (D) is differentiable at x = 0

∴ Option (D) is correct answer.

Q.6 The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$ is- **[IIT 2002]**

(A) R – {0} (B) R – {1}
(C) R – {– 1} (D) R – {– 1, 1}

Sol.[D] $$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{1+x^2}; & \text{if } -1 \le x \le 1 \\ \frac{1}{2} & ;\text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Hence, Domain of derivative of f(x) is R –[–1,1]

∴ Option (D) is correct answer.

Q.7 Let $f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and

$g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$,

where a and b non-negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x. Determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0? Justify your answer. **[IIT 2002]**

Sol. $f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and

$g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$

$$\text{gof}(x) = \text{gf}(x) = \begin{cases} f(x)+1 & \text{if } f(x) < 0 \\ (f(x)-1)^2 + b & \text{if } f(x) \ge 0 \end{cases}$$

$$= \begin{cases} x+a+1 & \text{if } x+a < 0 \\ (x+a-1)^2 + b & \text{if } x+a \ge 0 \\ |x-1|+1 & \text{if } |x-1| < 0 \text{ Not possible} \\ (|x-1|-1)^2 + b & \text{if } |x-1| \ge 0 \end{cases}$$

$$= \begin{cases} x+a+1 & \text{if } x < -a \\ (x+a-1)^2 + b & \text{if } -a \le x < 0 \\ |(x-1-1)^2 + b & \text{if } x-1 \ge 0 \text{ or } x \ge 1 \\ (1-x-1)^2 + b & \text{if } x-1 < 0 \;\&\; x \ge 0 \\ & \text{i.e. } 0 \le x < 1 \end{cases}$$

$$= \begin{cases} x+a+1 & \text{if } x < -a \\ (x+a-1)^2 + b & \text{if } -a \le x < 0 \\ (x-2)^2 + b & \text{if } x \ge 1 \\ x^2 + b & \text{if } 0 \le x < 1 \end{cases}$$

Continuity at x = 0

L.H.L. = $\lim\limits_{x \to 0^-}$ go f(x)= $\lim\limits_{h \to 0}$ go f(0–h)

$= \lim\limits_{h \to 0} (0-h+a-1)^2 + b$

$= (a-1)^2 + b.$

R.H.L.= $\lim\limits_{x \to 0^+}$ gof(x) = $\lim\limits_{h \to 0}$ gof(0 + h)

$= \lim\limits_{h \to 0} ((0+h)^2 + b)$

$= \lim\limits_{h \to 0} (h^2 + b)$

= b.

Since, gof(x) is continuous for all real x,

Hence, $(a-1)^2 + b = b$. It holds for all $b \in R$ at a = 1.

i.e. a = 1 ; b ∈ R.

differentiability at x = 0 .

L.H.D. = $\lim\limits_{x \to 0^-} \frac{\text{gof}(x) - \text{gof}(0)}{x-0}$

$= \lim\limits_{h \to 0} \frac{\text{gof}(0-h) - \text{gof}(0)}{-h}$

$= \lim\limits_{h \to 0} \frac{(0-h)^2 + b - b}{0-h}$

$= \lim\limits_{h \to 0} \frac{h^2 + b - b}{-h}$

= 0.

R.H.D.= $\lim\limits_{x \to 0^+} \frac{\text{gof}(x) - \text{gof}(0)}{x-0}$

$= \lim\limits_{h \to 0} \frac{\text{gof}(0+h) - \text{gof}(0)}{0+h-0}$

$= \lim\limits_{h \to 0} \frac{(0+h)^2 + b - b}{h}$

= 0

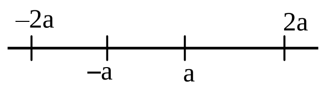
Hence, gof(x) is differentiable at x = 0.

Q.8 If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and left hand derivative at $x = a$ is 0, then find the left hand derivative at $x = -a$. **[IIT 2003]**

Sol. $f : [-2a, 2a] \to R$

$f(-x) = -f(x)$

$f(x) = f(2a - x)$ for $x \in [a, 2a]$



L.H.D. $= \lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a}$

$= \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{a - h - a}$

$= \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

Using given condition as, $f(x) = f(2a - x)$

Put $x = a - h$

$f(a-h) = f(2a - a + h)$

$f(a - h) = f(a + h)$

L.H.D. $= \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{-h} = 0$ ……(i)

Left hand derivative at $x = -a$

$= \lim\limits_{x \to a^-} \dfrac{f(x) - f(-a)}{x + a}$

$= \lim\limits_{h \to 0} \dfrac{f(-a-h) - f(-a)}{-a - h + a}$

$= \lim\limits_{h \to 0} \dfrac{f(-a-h) - f(-a)}{-h}$

Since, $f(-x) = -f(x)$

$f(-a - h) = -f(a + h)$

$f(-a) = -f(a)$

Hence, LH.D. at $x = -a = = \lim\limits_{h \to 0} \dfrac{-f(a+h) + f(a)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = 0.$

L.H.D. at $x = -a$ is zero.

Q.9 If $f(x)$ is a differentiable function and $f'(2) = 6$, $f'(1) = 4$, $f'(c)$ represents the differentiation of $f(x)$ at $x = c$, then $\lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{f(1 + h^2 + h) - f(1)}$ **[IIT 2003]**

(A) is equal to 3 (B) will not exist

(C) may exist (D) is equal to –3

Sol.[A] $f'(2) = 6;\ f'(1) = 4,$

$$\lim_{h \to 0} \frac{f(2 + 2h + h^2) - f(2)}{f(1 + h^2 + h) - f(1)}$$

$f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{2 + h - 2}$

$= \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ … (i)

$f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{1 + h - 1}$

$= \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$ … (ii)

Replace h by $2h + h^2$ in (i) and h by $(h^2 + h)$ in (ii), we get

$f'(2) = \lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{(h^2 + 2h)} = 6$

$f'(1) = \lim\limits_{h \to 0} \dfrac{f(1 + h^2 + h) - f(1)}{h^2 + h} = 4$

$$\frac{f'(2)}{f'(1)} = \frac{\lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{(h^2 + 2h)}}{\lim\limits_{h \to 0} \dfrac{f(1 + h^2 + h) - f(1)}{h^2 + h}} = \frac{6}{4}$$

$= \lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{f(1 + h^2 + h) - f(1)} \times \dfrac{h^2 + h}{h^2 + 2h}$

$= 3/2$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{f(1 + h^2 + h) - f(1)}$

$= \lim\limits_{h \to 0} \dfrac{3}{2} \times \dfrac{h^2 + 2h}{h^2 + h}$

$= \lim\limits_{h \to 0} \dfrac{3}{2} \times \dfrac{h^2 + 2h}{h^2 + h}$

$= \lim\limits_{h \to 0} \dfrac{3}{2} \times \dfrac{(h + 2)}{(h + 1)}$

$= \dfrac{3}{2} \times 2 = 3.$

$\therefore$ Option (A) is correct answer.

Q.10 Let y be a function of x, such that log (x + y) – 2xy = 0, then y′ (0) is-

[IIT 2004]

(A) 0 (B) 1

(C) 1/2 (D) 3/2

Sol.[B] log (x + y) – 2xy = 0

Differentiate w.r.t x, we get

$$\frac{1}{(x+y)}(1+dy/dx) = 2x\frac{dy}{dx} + 2y.1$$

$$1 + \frac{dy}{dx} = \left(2x\frac{dy}{dx}+2y\right)(x + y)$$

y' = first differentiation w.r.t. x = dy/dx

1 + y '(x) = (2x y'(x)+ 2y) (x + y)

Put x = 0 ; 1 + y'(0) = (0 + 2y) (0 + y)

$1 + y'(0) = 2y^2$

$\Rightarrow y'(0) = 2y^2–1$

from log (x+y) = 2xy

Put x = 0 also, log(0 + y) = 0 ⇒ y = 1

$y'(0) = 2(1)^2 – 1$

y'(0) = 1

∴ Option (B) is correct answer.

Q.11 $$f(x) = \begin{cases} b\sin^{-1}\left(x+\frac{c}{2}\right), & -\frac{1}{2} < x < 0 \\ \frac{1}{2}, & x = 0 \\ \frac{e^{\frac{ax}{2}}-1}{x}, & 0 < x < \frac{1}{2} \end{cases}$$

If f(x) is differentiable at x = 0 and |c| < 1/2, then find the value of a and prove that $64b^2 = \left(1-\frac{c^2}{4}\right)$ **[IIT 2004]**

Sol. $$f(x) = \begin{cases} b\sin^{-1}\left(x+\frac{c}{2}\right), & -\frac{1}{2} < x < 0 \\ \frac{1}{2}, & x = 0 \\ \frac{e^{a/2x}-1}{x}, & 0 < x < \frac{1}{2} \end{cases}$$

Differentiability at x = 0

L.H.D. = $\lim\limits_{x\to0^-}\frac{f(x)-f(0)}{x-0} = \lim\limits_{h\to0}\frac{f(0-h)-f(0)}{-h}$

$$= \lim_{h\to0}\frac{b\sin^{-1}(0-h+c/2)-1/2}{-h}$$

Since f(x) is differentiable at x = 0

L.H.D. = $$\lim_{h\to0}\frac{b\sin^{-1}\left(\frac{c}{2}-h\right)-1/2}{-h}$$

$$= \lim_{h\to0}\frac{b\frac{1}{\sqrt{1-\left(\frac{c}{2}-h\right)^2}}\times(-1)-0}{-1}$$

$$= \lim_{h\to0} b\frac{1}{\sqrt{1-c^2/4}} = \frac{2b}{\sqrt{4-c^2}}$$

R.H.D. = $$\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to0}\frac{\frac{e^{a/2(0+h)}-1}{0+h}-\frac{1}{2}}{h}$$

$$= \lim_{h\to0}\frac{\frac{e^{a/2h}-1}{+h}-\frac{1}{2}}{h}$$

Since, limit exist then $\lim\limits_{h\to0}\frac{e^{a/2h}-1}{h} = \frac{1}{2}$

$$= \lim_{h\to0}\frac{e^{a/2h}-1}{h} = \frac{1}{2}$$

$$\Rightarrow \lim_{h\to0}\frac{\left(1+\frac{a}{2h}+(a/2h)^2\times\frac{1}{2!}+.....\right)-1}{h} = 1/2$$

$$\Rightarrow \lim_{h\to0}\frac{\left(\frac{a}{2h}+(a/2h)^2\times\frac{1}{2!}+.....\right)}{h} = 1/2$$

if it exists, there must be a = h^2, a → 0.

$$\lim_{h\to0}\frac{\left(\frac{h}{2}+\left(\frac{h}{2}\right)^2\times\frac{1}{2!}+.....\right)}{h} = 1/2.$$

R.H.D. = $$\lim_{h\to0}\frac{\frac{\left(\frac{h}{2}+\left(\frac{h}{2}\right)^2\times\frac{1}{2!}+.....\right)}{h}-1/2}{h}$$

$$= \lim_{h\to0}\frac{\frac{h}{2}+\frac{h^2}{8}+.......-h/2}{h^2}$$

$= \frac{1}{8}$

Since, f(x) is differentiable at x = 0,
L.H.D. = R.H.D.

$$\frac{2b}{\sqrt{4-c^2}} = \frac{1}{8}$$

$\Rightarrow$ 16b= $\sqrt{4-c^2}$ $\Rightarrow$ $(4–c^2)$ = 256.Hence, proved.

Q.12 If $f : [–1, 1] \to R$ and $f'(0) = \lim_{n\to\infty} nf\left(\frac{1}{n}\right)$ and $f(0) = 0$. Find the value of :

$\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$ given that

$0 < \left|\lim_{n\to\infty}\cos^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$. **[IIT 2004]**

Sol. To find, $\lim_{n\to\infty}\left[(n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - n\right]$

$$= \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - 1\right]$$

$$= \lim_{n\to\infty} n\, f\left(\frac{1}{n}\right)$$

where $f(x) = \left[(1+x)\frac{2}{\pi}\cos^{-1}x - 1\right]$ s.t.

$$f(0) = \left[(1+0)\frac{2}{\pi}\cos^{-1}0 - 1\right]$$

$$= \frac{2}{\pi}.\frac{\pi}{2} - 1 = 0$$

$\therefore$ Using given relation as $\lim_{n\to\infty} nf\left(\frac{1}{n}\right) = f'(0)$

then given limit becomes

$$= f'(0) = \frac{d}{dx}\left[(1+x)\frac{2}{\pi}\cos^{-1}x - 1\right]\Bigg|_{x=0}$$

$$\frac{2}{\pi}\left[\cos^{-1}x - \frac{1+x}{\sqrt{1-x^2}}\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 1\right] = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$$

Q.13 If two functions 'f' and 'g' satisfying given conditions for $\forall$ x, y $\in$ R. $f(x – y) = f(x) g(y) – f(y).g(x)$ and $g(x – y) = g(x) . g(y) + f(x) f(y)$. If right hand derivative at x = 0 exists for f(x) then find the derivative of g(x) at x = 0.
[IIT 2005]

Sol. Given that,

$f(x – y) = f(x) . g(y) – f(y) . g(x)$ ….(i)

$g(x – y) = g(x) . g(y) + f(x) f(y)$ ….(ii)

In eqn. (i) putting x = y we get

$f(0) = f(x) g(x) – f(x) g(x)$

$f(0) = 0$

Putting y = 0 in eqn. (i), we get

$f(x) = f(x) g(0) – f(0) g(x)$

$\Rightarrow$ $f(x) = f(x) g(0)$ [using f(0) = 0]

$\Rightarrow$ $g(0) = 1$

Putting x = y in eqn. (ii), we get

$g(0) = g(x) g(x) + f(x) f(x)$

$\Rightarrow$ $1 = [g(x)]^2 + [f(x)]^2$ [using g(0) = 1]

$\Rightarrow$ $[g(x)]^2 = 1 – [f(x)]^2$ ….(iii)

Clearly g(x) will be differentiable only if f(x) is differentiable.

$\therefore$ First we will check the differentiability of f(x)

Given that Rf ′ (0) exists

i.e., $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists

i.e., $\lim_{h\to 0}\frac{f(0)g(-h)-f(-h)g(0)}{h}$ exists

i.e., $\lim_{h\to 0}\frac{-f(-h)}{h}$ exists

[using f(0) = 0 and g(0) = 1]

Which can be written as,

$$\lim_{h\to 0}\frac{f(0)-f(-h)}{-h} = Lf'(0)$$

$\Rightarrow$ Lf ′ (0) = Rf ′ (0)

$\therefore$ f is differentiable, at x = 0

Differentiating equation (iii) we get

$2g(x) . g'(x) = – 2f(x) . f'(x)$

For x = 0

$\Rightarrow$ $g(0) . g'(0) = – f(0) f'(0)$

$\Rightarrow$ $g'(0) = 0$ [using f(0) = 0 and g(0) = 1]

Q.14 If x cos y + y cos x = π, then y ″ (0) =
[IIT 2005]

(A) π (B) – π
(C) 0 (D) 1

Sol.[A] $x \cos y + y \cos x = \pi$

Differentiating w.r.tx, weget

$1.\cos y + x(-\sin y)\dfrac{dy}{dx} + y(-\sin x) + \dfrac{dy}{dx}\cos x = 0$

$\cos y - x\sin y\dfrac{dy}{dx} - y\sin x + dy/dx.\cos x = 0$

$\Rightarrow \dfrac{dy}{dx}(\cos x - x\sin y) = y\sin x - \cos y$ ….(i)

$\dfrac{dy}{dx} = y' =$ first derivative

Put x = 0

$y'(0)(1-0) = y.0 - \cos y \Rightarrow y'(0) = -\cos y$

Again differentiating (i) w.r.t. x, weget

$\dfrac{d^2y}{dx^2}(\cos x - x\sin y) + \dfrac{dy}{dx}\times(-\sin x - x\cos y - \sin y)$

$= y\cos x + \sin x\,dy/dx + \sin y$…(ii)

Put x = 0 in (ii), weget

$y''(0)(1-0) + y'(0)(-0-0-\sin y) = y + 0 + \sin y$

$y''(0) = y'(0)\sin y + y + \sin y$

$y''(0) = -\cos y \sin y + y + \sin y$

Also $0.\cos y + y.1 = \pi \Rightarrow y = \pi$

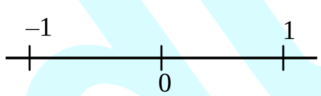
$y''(0) = 0 + \pi + 0 \Rightarrow y''(0) = \pi$

$\therefore$ Option (A) is correct answer.

Q.15 $f(x) = |\,|x| - 1|$ is not differentiable at x = **[IIT 2005]**

(A) 0, ± 1 (B) ± 1 (C) 0 (D) 1

Sol.[A] $f(x) = ||x| - 1|$



when $x < -1 \Rightarrow x + 1 < 0$

$\Rightarrow -x - 1 > 0$

$f(x) = |-x-1| = -x - 1$

when $-1 \le x < 0 \Rightarrow x + 1 \ge 0$

$|-x-1| = x + 1$

when $0 \le x < 1$; $x - 1 < 0$

$|x - 1| = 1 - x$

when $x \ge 1$; $x - 1 \ge 0$

$|x - 1| = x - 1$

$$f(x) = \begin{cases} -x-1; & x < -1 \\ x+1 \;; & -1 \le x < 0 \\ 1-x \;; & 0 \le x < 1 \\ x-1 \;; & x \ge 1 \end{cases}$$

Differentiability at x = –1

L.H.D.$= \lim\limits_{x\to -1^-}\dfrac{f(x)-f(-1)}{x+1} = \lim\limits_{h\to 0}\dfrac{f(-1-h)-f(-1)}{-1-h+1}$

$= \lim\limits_{h\to 0}\dfrac{f(-1-h)-f(-1)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{-(-1-h)-1-0}{-h}$

$= \lim\limits_{h\to 0}\dfrac{1+h-1}{-h} = -1.$

R.H.D. $= \lim\limits_{h\to 0}\dfrac{-1+h-1-0}{h} = 1.$

Hence f(x) is not differentiable at x = –1

Differentiability at x = 0

L.H.D. $= \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{0-h-0}$

$= \lim\limits_{h\to 0}\dfrac{0-h+1-1}{-h}$

= 1.

R.H.D.$= \lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{0+h-0}$

$= \lim\limits_{h\to 0}\dfrac{1-(0+h)-1}{h} = -1$

f(x) is not differentiable at x = 0

Differentiability at x = 1.

L.H.D.$= \lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{1-h-1}$

$= \lim\limits_{h\to 0}\dfrac{1-(1-h)-0}{-h}$

$= \lim\limits_{h\to 0}\dfrac{h-0}{-h} = -1.$

R.H.D. $= \lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{1+h-1}$

$= \lim\limits_{h\to 0}\dfrac{1+h-1-0}{h} = 1.$

$\therefore$ Not differentiable at x = 1.

$\therefore$ Option (A) is correct answer.

Q.16 If f(1) = 1 ; f (2) = 4 , f (3) = 9 & f is twice differentiable then **[IIT 2005]**

(A) $f''(x) = 2$ for atleast $x \in [1, 3]$

(B) $f''(x) = f'(x) = 5$; $x \in [1, 3]$

(C) $f''(x) = 2$ for only $x \in [1, 3]$

(D) $f''(x) = 3$, for $x \in (1, 3)$

Sol.[A] f(1) = 1 ; f(2) = 4 ; f(3) = 9 let $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 1$ ....(i)

$f(2) = 4a + 2b + c = 4$ ....(ii)

$f(3) = 9a + 3b + c = 9$ .....(iii)

solving (i), (ii), (iii), weget

$c = 1–a–b = 4 – 4a – 2b \Rightarrow 3a + b = 3$ ...(iv)

$c = 4 – 4a – 2b = 9 – 9a – 3b$

$\Rightarrow 5a + b = 5$ ......(v)

On solving (iv) & (v) we get

$a = 1$ and $b = 0$

$c = 1 – a – b = 1 – 1 – 0 = 0$

$f(x) = 1.x^2 + 0 + 0 \Rightarrow f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$ for all $x \in [1, 3]$

Q.17 If f is a differentiable function satisfying $f\left(\frac{1}{n}\right) = 0$ for all $n \geq 1$, $n \in I$, then-

(A) $f(x) = 0, x \in (0, 1]$

(B) $f'(0) = 0 = f(0)$

(C) $f(0) = 0$ but $f'(0)$ not necessarily zero

(D) $|f(x)| \leq 1, x \in (0, 1]$ **[IIT 2005]**

Sol.[B] Since $f\left(\frac{1}{n}\right) = 0$ for all $n \geq 1$, $n \in I$

f is a differentiable function

Replace, n by $\frac{1}{x}$ as follows.

$f(x) = 0$ for all $\frac{1}{x} \geq 1$. i.e. $x \leq 1$

$f'(x) = 0$

Put $x = 0$

$f'(0) = 0$. Also $f(0) = 0$ for $x \leq 1$.

Hence, (B) Option is correct answer.

Q.18 If $f''(x) = –f(x)$ and $g(x) = f'(x)$ and

$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ & given that $F(5) = 5$,

then F (10) is– **[IIT 2006]**

(A) 15 (B) 0

(C) 5 (D) 10

Sol. **[C]**

Given $f''(x) = –f(x)$ and $g(x) = f'(x)$

and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ ...(1)

$F(5) = 5$

Differentiating (1) w.r.t x, we get

$F'(x) = 2f(x/2).f'(x/2)\frac{1}{2} + 2g(x/2).g'(x/2)\frac{1}{2}$ ..(2)

$f''(x) = – f(x)$ and $g(x) = f'(x) \Rightarrow g'(x) = f''(x)$

$\Rightarrow g'(x) = – f(x)$

$\Rightarrow g(x) = f'(x)$ or $g(x/2) = f'(x/2)$

$\Rightarrow g'(x) = – f(x)$ or $g'(x/2) = – f(x/2)$

From (2), we get

$F'(x) = 2f(x/2).\ g(x/2)\frac{1}{2} + 2g(x/2)\ (–f(x/2))\ \frac{1}{2}$

$= 2f(x/2).\ g(x/2)\ \frac{1}{2} – 2f(x/2)\ .\ g(x/2)\ \frac{1}{2}$

$F'(x) = 0$

It means F(x) would be a constant function

Since $F(5) = 5$

$F(x) = C \Rightarrow F(x) = F(5) = C = 5 \Rightarrow C = 5$

$\therefore F(x) = 5$ $\therefore F(10) = 5$

Q.19 If $f(x) = \min\{1, x^2, x^3\}$, then **[IIT 2006]**

(A) $f'(x) > 0\ \forall\ x \in R$

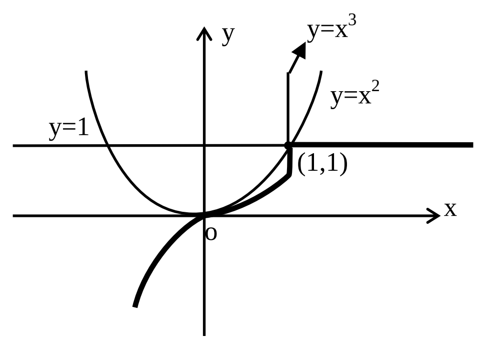
(B) f(x) is continuous $\forall\ x \in R$

(C) f(x) is not differentiable for two values of x

(D) f(x) is not differentiable but continuous $\forall\ x \in R$

Sol.[B, D] $f(x) = \min\{1, x^2, x^3\}$

$$f(x) = \begin{cases} x^3;\ 0 \leq x < 1 \\ 1\ ;\ x \geq 1. \\ x^3;\ x < 0 \end{cases}$$



Differentiability at x = 0

L.H.D. $= \lim_{x \to 0^-} \frac{f(x) – f(0)}{x – 0} = \lim_{h \to 0} \frac{f(0 – h) – f(0)}{0 – h – 0}$

$= \lim_{h \to 0} \frac{(0 – h)^3 – 0}{– h}$

$= \lim_{h \to 0} h^2 = 0$

R.H.D.= $\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{h\to0}\frac{f(0+h)-f(0)}{0+h-0}$

$=\lim_{h\to0}\frac{h^3-0}{h}=0$

f(x) is differentiable at x = 0

Differentiability at x = 1

L.H.D.= $\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{h\to0}\frac{f(1-h)-f(1)}{1-h-1}$

$=\lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$

$=\lim_{h\to0}\frac{(1-h)^3-1}{-h}$

$=\lim_{h\to0}\frac{3(1-h)^2(-1)-0}{-1}=3.$

R.H.D.= $\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{h\to0}\frac{f(1+h)-f(1)}{1+h-1}$

$=\lim_{h\to0}\frac{1-1}{h}=0$

Hence, f(x) is not differentiable at x = 1

Continuity at x = 1.

L.H.L. = $\lim_{x\to1^-}f(x)=\lim_{h\to0}f(1-h)$

$=\lim_{h\to0}(1-h)^3=1.$

R.H.L.= $\lim_{x\to1^+}f(x)=1.$

$f(1)=\lim_{x\to1}f(x)=1.$

Hence, f(x) is continuous at all $x\in R$.

∴ Option (B) and (D) are correct answers.

Q.20 $\frac{d^2x}{dy^2}$ equals **[IIT 2007]**

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Sol. **[D]**

We have $\frac{dx}{dy}=\frac{1}{dy/dx}$

Now, $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{1}{dy/dx}\right)$

$=\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^{-1}\right)\frac{dx}{dy}$

$=-\left(\frac{dy}{dx}\right)^{-2}\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-1}$

$=-\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}$

Q.21 Let $g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)}$; $0<x<2$, m and n are integers, $m\neq0$, $n>0$, and let p be the left hand derivative of $|x-1|$ at x = 1. If $\lim_{x\to1+}g(x)=p$, then **[IIT 2008]**

(A) n = 1, m = 1 (B) n = 1, m = –1

(C) n = 2, m = 2 (D) n > 2, m = n

Sol. **[C]**

p = left hand derivative of $|x-1|=-1$

$\lim_{x\to1^+}g(x)=p=-1$

$\lim_{x\to1^+}\frac{(x-1)^n}{\log\cos^m(x-1)}=-1$

$\lim_{x\to1^+}\frac{n(x-1)^{n-1}}{\frac{-m\cos^{m-1}(x-1).\sin(x-1)}{\cos^m(x-1)}}=-1$

$\lim_{x\to1^+}\frac{n(x-1)^{n-1}}{-m\tan(x-1)}=-1$

if $n-1=1\Rightarrow n=2$ and m = 2.

Q.22 Let f and g be real valued functions defined on interval (–1, 1) such that $g''(x)$ is continuous, $g(0)\neq0$, $g'(0)=0$, $g''(0)\neq0$, and $f(x)=g(x)\sin x$.

STATEMENT – 1 **[IIT-2008]**

$\frac{1}{2}[g(x)\cot x-g(0)\operatorname{cosec}x]=f''(0)$

STATEMENT - 2

$f'(0)=g(0)$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(C) Statement-1 is true, Statement-2 is false

(D) Statement-1 is false, Statement-2 is true.

Sol. **[B]**

Statement 1. $\lim\limits_{x\to 0}\dfrac{g(x)\cos x - g(0)}{\sin x}$ =

$\lim\limits_{x\to 0}\dfrac{g'(x)\cos x - g(x)\sin x}{\cos x}$ (Applying L – H rule)

= g´(0) – 0 = 0 = f´´(0) (True)

Statement 2

f ´(x) = g (x) cos x + g´(x) sin x

f ´(0) = g(0)

(Ture)

Q.23 Let g(x) = log f(x) where f(x) is a twice differentiable positive function on (0, ∞) such that f(x + 1) = x f(x). Then, for N = 1, 2, 3, …

$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$ **[IIT 2008]**

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+....+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+....+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

Sol. **[A]**

g(x) = log f(x)

& f(x + 1) = x f(x)

Replece x by $x - \frac{1}{2}$

So, $f(x + \frac{1}{2}) = (x - \frac{1}{2}).\ f(x - \frac{1}{2})$

$\therefore\ g(x + \frac{1}{2}) = \log(x - \frac{1}{2}) + g(x - \frac{1}{2})$

$\Rightarrow g(x + \frac{1}{2}) - g(x - \frac{1}{2}) = \log(x - \frac{1}{2})$

$\Rightarrow g'(x + \frac{1}{2}) - g'(x - \frac{1}{2}) = \dfrac{1}{\left(x - \frac{1}{2}\right)}$

$\Rightarrow g''(x + \frac{1}{2}) - g''(x - \frac{1}{2}) = \dfrac{-1}{\left(x - \frac{1}{2}\right)^2}$

$\Rightarrow g''\left(\frac{3}{2}\right) - g''\left(\frac{1}{2}\right)\ \dfrac{-1}{\left(\frac{1}{4}\right)}$

$\Rightarrow g''\left(\frac{5}{2}\right) - g''\left(\frac{3}{2}\right) = \dfrac{-1}{\left(\frac{9}{4}\right)}$

$g''\left(N+\frac{1}{2}\right) - g''\left(N-\frac{1}{2}\right) = \dfrac{-4}{(2n-1)^2}$

Adding above n equations.

$\Rightarrow$ So, $g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left[1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right]$

Q.24 If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then find the value of g′ (1) **[IIT 2009]**

Sol. **[2]**

$g(x) = f^{-1}(x)$

f(g(x)) = x

$\Rightarrow g(f(x)) = x \Rightarrow g'(f(x))\ f'(x) = 1$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

put x = 0

$\Rightarrow g'(1) = \dfrac{1}{f'(0)} \Rightarrow$ $f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$\therefore\ f'(0) = \frac{1}{2} \Rightarrow$ $\therefore\ g'(1) = 2$

Q.25 Let f be a real-valued function defined on the interval (–1, 1) such that

$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all x ∈ (–1, 1),

and let f^{-1} be the inverse function of f. Then $(f^{-1})'$ (2) is equal to- **[IIT 2010]**

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

Sol. **[B]**

$f(x) = 2e^x + e^x\int_0^x \sqrt{t^4+1}\,dt$

x = 0 ; f(0) = 2

$$f'(x) = 2e^x + e^x \int_0^x \sqrt{t^4+1}\,dt + e^x\sqrt{1+x^4}$$

$f'(0) = 2 + 1 = 3$ $\qquad (f^{-1})'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$

Q.26 Let $f : R \to R$ be a function such that
$f(x + y) = f(x) + f(y), \forall x, y \in R$
If $f(x)$ is differentiable at $x = 0$, then

[IIT 2011]

(A) $f(x)$ is differentiable only in a finite interval containing zero
(B) $f(x)$ is continuous $\forall\ x \in R$
(C) $f'(x)$ is constant $\forall\ x \in R$
(D) $f(x)$ is differentiable except at finitely many points

Sol. **[B,C] OR [B,C,D]**
$f(x + y) = f(x) + f(y)$
By Partial differentiation with respect to x
$f'(x + y) = f'(x)$
$f'(y) = f'(0)$
$f(y) = (f'(0))y + c$
$f(y) = ky + c$
$\therefore\ f(y) = ky$ $\qquad$ as $f(0) = 0$
$\therefore\ f(x) = kx$
Alternate

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h} = \lim_{h\to 0}\frac{f(h)}{h}$$

$= \lambda$ (let)
$f(x) = \lambda x + c$ As $f(0) = 0 \Rightarrow c = 0$
$f(x) = \lambda x$

Q.27 If $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x - 1 & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$, then

[IIT 2011]

(A) $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$
(B) $f(x)$ is not differentiable at $x = 0$
(C) $f(x)$ is differentiable at $x = 1$
(D) $f(x)$ is differentiable at $x = -\dfrac{3}{2}$

Sol. [A, B, C, D]

At $x = -\dfrac{\pi}{2}$

LHL = 0, RHL = 0, $f\left(-\dfrac{\pi}{2}\right) = 0$, So $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$

At $x = 0$
LHD = 0; $\qquad$ RHD = 1
So $f(x)$ is not differentiable at $x = 0$
At $x = 1$
LHD = 1, $\qquad$ RHD = 1
So $f(x)$ is differentiable at $x = 1$

in $\left(-\dfrac{\pi}{2}, 0\right]$; $f(x) = -\cos x$

so $f(x)$ is differentiable at $x = -\dfrac{3}{2}$

Q.28 Let $f : (0, 1) \to R$ be defined by $f(x) = \dfrac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

[IIT 2011]

(A) f is not invertible on (0, 1)
(B) $f \ne f^{-1}$ on (0, 1) & $f'(b) = \dfrac{1}{f'(0)}$
(C) $f = f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$
(D) f^{-1} is differentiable on (0, 1)

Sol. [A] $f : (0, 1) \to R$

$f(x) = \dfrac{b-x}{1-bx}$ $\qquad \forall\ b \in (0, 1)$

$f'(x) = \dfrac{b^2-1}{(1-bx)^2} = (-)$ ve

So $f(x)$ is monotonically decreasing for $x \in (0, 1)$
so for $x \in (0, 1)$
$f(x) \in (f(1), f(0))$
$f(x) \in (-1, b)$
so $f(x)$ is not onto.
so $f(x)$ is not invertible function.

Q.29 Let f(x) = $\begin{cases} x^2\left|\cos\dfrac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$, $x \in IR$,

then f is **[IIT 2012]**
(A) differentiable both at x = 0 and at x = 2
(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

Sol. [B] $f'(0+h) = \lim_{h\to 0} \dfrac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h-0} = 0$

$f'(0-h) = \lim_{h\to 0} \dfrac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{-h} = 0$

$\because f'(0^+) = f'(0^-) = 0 =$ finite

So f(x) is differentiable at x = 0

$f'(2+h) = \lim_{h\to 0} \dfrac{(2+h)^2\left|\cos\left(\frac{\pi}{2+h}\right)\right| - 0}{h} = \pi$

$f'(2-h) = \lim_{h\to 0} \dfrac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right| - 0}{-h} = -\pi$

$\because f'(2^+) \neq f'(2^-)$ but both are finite so f(x) is not differentiable at x = 2 but continuous at x = 2

EXERCISE # 5

Q.1 If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then-

(A) f(x) is continuous but not differentiable at x = 0

(B) f(x) is differentiable at x = 0

(C) f(x) is not differentiable at x = 0

(D) None of these **[IIT-1985]**

Sol. **[B]**

We have $f(x) = x(\sqrt{x} - \sqrt{x+1})$

Let us check differentiability of f(x) at x = 0

$$Lf'(0) = \lim_{h\to 0} \frac{(0-h)[\sqrt{0-h} - \sqrt{0-h+1}] - 0}{-h}$$

$$\lim_{h\to 0} [\sqrt{-h} - \sqrt{-h+1}] = 0 - \sqrt{+1} = -1$$

$$Rf'(0) = \lim_{h\to 0} \frac{(0+h)[\sqrt{0+h} - \sqrt{0+h+1}] - 0}{h}$$

$$= \lim_{h\to 0} \sqrt{h} - \sqrt{h+1} = -1$$

Since Lf ' (0) = Rf ' (0)

$\therefore$ f is differentiable at x = 0

Q. 2 Let $f(x) = x^3 - x^2 + x + 1$ and

$g(x) = \max.\{f(t); 0 \le t \le x\}, 0 \le x \le 1$

$= 3 - x, \quad 1 < x \le 2$

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

[IIT-1985]

Sol. We are given $f(x) = x^3 - x^2 + x + 1$

$$\therefore \quad f'(x) = 3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right)$$

$$= 3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{1}{3}\right]$$

$$= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right] > 0 \quad \forall\, x \in R.$$

Hence f (x) is an increasing function of x for all real values of x.

Now max $[f(t) : 0 \le t \le x]$ means the greatest value of f(t) in $0 \le t \le x$ which is obtained at t = x, since f(t) is increasing for all t.

$\therefore$ max. $[f(t) ; 0 \le t \le x] = x^3 - x^2 + x + 1$

Hence the function g is defined as follows :

$g(x) = x^3 - x^2 + x + 1 \quad$ when $0 \le x \le 1$

$= 3 - x \quad$ when $1 < x \le 2$

Now it is sufficient to discuss the continuity and differentiability of g(x) at x = 1 Since for all other values of x, g(x) is clearly continuous and differentiable, being a polynomial function of x.

We have, $g(1) = 2$

$g(1-0) = \lim_{h\to 0} [(1-h)^3 - (1-h)^2 + (1-h) + 1] =$ 2

$g(1+0) = \lim_{h\to 0} [3 - (1+h)] = 2$

Hence g(x) is continuous at x = 1

Now,

$$Lg'(1) = \lim_{h\to 0} \frac{[(1-h)^3 - (1-h)^2 + (1-h) + 1] - 2}{-h}$$

$$= \lim_{h\to 0} \frac{1 - 3h + 3h^2 - h^2 - 1 + 2h - h^2 + 1 - h + 1 - 2}{-h}$$

$$= \lim_{h\to 0} \frac{-2h + 2h^2 - h^3}{-h}$$

$$= \lim_{h\to 0} [2 - 2h + h^2] = 2$$

$$Rg'(1) = \lim_{h\to 0} \frac{[3 - (1+h) - 2]}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

Since $Lg'(1) \ne Rg'(1)$, the function g(x) is not differentiable at x = 1.

Hence g(x) is continous on (0, 2). It is also differentiable on (0, 2) except at x = 1, where it is non-differentiable

Q.3 If $f(x) = \log_x(\lambda nx)$, then $f'(x)$ at x = e is.........

[IIT-1985]

Sol. Given that

$$f(x) = \log_x(\ln x) = \frac{\log_e(\log x)}{(\log_e x)}$$

$$f'(x) = \frac{\frac{1}{\log_e x} \times \frac{1}{x} \times \log_e x - \frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\frac{1}{x}[1 - \log_e(\log_e x)]}{(\log_x x)^2}$$

At $x = e$, $f'(e) = \dfrac{\frac{1}{e}[1-\log_e(\log_e e)]}{(\log_e e)^2}$

$= \dfrac{\frac{1}{e}[1-\log_e 1]}{(1)^2}$

$= \dfrac{1}{e}(1-0) = \dfrac{1}{e}$

Q.4 The derivative of $\sec^{-1}\left(-\dfrac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \dfrac{1}{2}$ is ....

[IIT-1986]

Sol. Let $u = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$; $v = \sqrt{1-x^2}$

Then to find $\left.\dfrac{du}{dv}\right|_{x=1/2}$

We have, $u = \cos^{-1}(2x^2-1) = 2\cos^{-1}x$

$\therefore \dfrac{du}{dx} = \dfrac{-2}{\sqrt{1-x^2}}$ and $v = \sqrt{1-x^2}$

$\therefore \dfrac{dv}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$ $\quad\therefore \dfrac{du}{dv} = \dfrac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \dfrac{2}{x}$

$\therefore \left.\dfrac{du}{dv}\right|_{x=1/2} = 4$

Q.5 Let $f(x)$ be defined in the interval $[-2, 2]$ such that $f(x) = -1, -2 \le x \le 0$

$= x - 1, 0 < x \le 2$ and

$g(x) = f(|x|) + |f(x)|$ Test the differentiability of $g(x)$ in $(-2, 2)$ **[IIT-1986]**

Sol. We have $f(x) = -1, -2 \le x \le 0$

$= x - 1, 0 < x \le 2$

and $g(x) = f(|x|) + |f(x)|$

Hence $g(x)$ involves $|x|$ and $|x-1|$ or $|-1| = 1$

Therefore we should divide the given interval $(-2, 2)$ into the following intervals.

	I_1	I_2	I_3
	$[-2, 2] = [-2, 0)$	$[0, 1)$	$[1, 2]$
	$x = -$ ve	+ve	+ve
	$\|x\| = +$ve	+ve	+ve
$f(x) =$	-1	-1	$x-1$ $\;x-1$
$f(\|x\|) =$	$\|x\| - 1$	$\|x\| - 1$	$\|x\| - 1$
	$= -x - 1$	$= x - 1$	$= x - 1$
$\|f(x)\| =$	$\|-1\|$	$\|x - 1\|$	$\|x - 1\|$
	$= 1$	$= -(x-1)$	$= x - 1$

$\therefore$ Using above we get

$g(x) = f|x| + |f(x)|$

$= -x - 1 + 1 = -x$ in I_1

in $I_2 = x - 1 - (x - 1) = 0$

in $I_3 = (x - 1) + (x - 1) = 2(x - 1)$

Hence $g(x)$ is defined as follows :

$g(x) = -x, -2 \le x \le 0 = 0, 0 < x < 1$

$= 2(x - 1), 1 \le x \le 2$

Clearly $g'(x) = -1, 0, 2$ respectively as $g(x)$ is polynomial in x.

Also $Lg'(0) = -1$; $Rg'(0) = 0$ (not equal)

$Lg'(1) = 0$; $Rg'(1) = 2$ (not equal)

Hence $g(x)$ is not differentiable both at $x = 0$ and $x = 1$.

Q.6 The function $f(x) = 1 + |\sin x|$ is-

(A) continuous no where

(B) continuous everywhere

(C) not differentiable at $x = 0$

(D) not differentiable at infinite number of points **[IIT-1986]**

Sol. [B, C, D] We have, $f(x) = 1 + |\sin x| = \begin{cases} 1+\sin x & \text{if } \sin x \ge 0 \\ 1-\sin x & \text{if } \sin x \le 0 \end{cases}$

We know that $g(x) = \sin x$ is continuous every where on R also $h(x) = 1 + |x|$ is continuous every where on R

$\therefore$ h o g(x) must be continuous every where.

i.e., $1 + |\sin x|$ is continuous every where.

Alternatively for any real number a,

$f(a + 0) = \lim\limits_{h\to 0} [1 + |\sin(a + h)|] = 1 + |\sin a|$

$f(a - 0) = \lim\limits_{h\to 0} [1 + |\sin(a - h)|] = 1 + |\sin a|$

$f(a) = 1 + |\sin a|$

$\therefore f(a + 0) = f(a - 0) = f(a)$, for every real no $a \in R$.

$\therefore f(x)$ is continuous every where.

Differentiability :

We have $f(x) = \begin{cases} 1-\sin x & \sin x < 0 \\ 1+\sin x & \sin x \ge 0 \end{cases}$

Clearly f(x) is differentiable at any non critical pt i.e. at all real numbers except possibly at critical pts $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots\ldots$

Let us check the differentiability of f(x) at $x = n\pi$.

We have $f(n\pi) = 1 + |\sin n\pi| = 1 + |0| = 1$

Now, $Rf'(n\pi) = \lim_{h\to 0} \dfrac{1+|\sin(n\pi+h)|-1}{h}$

$= \lim_{h\to 0} \dfrac{|\pm\sin h|}{h} = \lim_{h\to 0} \dfrac{\sin h}{h} = 1$

and $Lf'(n\pi) = \lim_{h\to 0} \dfrac{1+|\sin(n\pi-h)|-1}{-h}$

$= \lim_{h\to 0} \dfrac{|\pm\sin h|}{-h} = \lim_{h\to 0} \dfrac{\sin h}{-h} = -1$

Since $Rf'(n\pi) \neq Lf'(n\pi)$

$\therefore$ f(x) is non differentiable at $x = n\pi$,

$n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$

$\therefore$ (B) and (D) are the correct answers.

Q.7 Let [x] denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then f(x) is-

(A) continuous at x = 0 **[IIT-1986]**

(B) continuous in (–1, 0)

(C) differentiable at x = 1

(D) differentiable in (–1, 1)

Sol. **[A, B, D]**

We have, for $-1 \leq x \leq 1$

$\Rightarrow 0 \leq x \sin \pi x \leq 1/2$

$\therefore f(x) = [x \sin \pi x] = 0$

Also $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by definition of [x],

$f(x) = [x \sin \pi x] = -1$ when $1 < x < 1 + h$

Thus f(x) is constant and equal to 0 in the closed interval [–1, 1] and so f(x) is continuous and differentiable in the open interval (–1, 1).

At x = 1, f(x) is clearly discontinuous, since f(1 – 0) = 0 and f(1 + 0) = –1 and f(x) is non-differentiable at x = 1.

Hence (A), (B) and (D) are correct answers.

Q.8 The set of all points where the function $f(x) = \dfrac{x}{(1+|x|)}$ is differentiable, is-

(A) $(-\infty, \infty)$ (B) $[0, \infty)$

(C) $(-\infty, 0) \cup (0, \infty)$

(D) $(0, \infty)$ **[IIT-1987]**

Sol. **[A]**

The given function is,

$$f(x) = \frac{x}{1+|x|} = \begin{cases} \dfrac{x}{1-x} & x < 0 \\ \dfrac{x}{1+x} & x \geq 0 \end{cases}$$

For $x < 0$, $f'(x) = \dfrac{1(1-x)-(-1)x}{(1-x^2)} = \dfrac{1}{(1-x)^2}$,

defined $\forall$ $x < 0$

For $x > 0$, $f'(x) = \dfrac{1(1+x)-1\,(x)}{(1+x)^2} = \dfrac{1}{(1+x)^2}$,

defined $\forall$ $x > 0$

For x = 0, $Lf'(0) = \lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \dfrac{\frac{-h}{1+h} - 0}{-h} = 1$

$Rf'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{h}{1+h} - 0}{h} = 1$

$Lf'(0) = Rf'(0)$

$\Rightarrow$ f is differentiable at x = 0

Hence f is differentiable in $(-\infty, \infty)$

Q.9 Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, find its value. **[IIT-1987]**

Sol. Given that f(x) is a function satisfying

$f(-x) = f(x), \forall x \in R$ ……(1)

Also $f'(0)$ exists

$\Rightarrow$ $f'(0) = Rf'(0) = Lf'(0)$

Now, $Rf'(0) = f'(0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = f'(0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = f'(0)$ ……(2)

Again $Lf'(0) = f'(0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = f'(0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(-h)-f(0)}{-h} = f'(0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = f'(0)$ ……(3)

[Using eq. (1)]

From equations (2) and (3) we get,

$\Rightarrow \quad f'(0) = -f'(0)$

$\Rightarrow \quad 2f'(0) = 0$

$\Rightarrow \quad f'(0) = 0$

Q.10 Let g(x) be a polynomial of degree one and f(x) be defined by

$$f(x) = \begin{cases} g(x), & x \le 0 \\ \left[\dfrac{(1+x)}{(2+x)}\right]^{1/x}, & x > 0 \end{cases}$$

Find the continuous function f(x) satisfying $f'(1) = f(-1)$. **[IIT-1987]**

Sol. Let $g(x) = ax + b$

Then $f(x) = \begin{cases} ax+b, & x \le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Since f(x) is continuous, it is continuous at x = 0

Now, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (ax + b) = b$

and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left(\dfrac{1+x}{2+x}\right)^{1/x}$

$$= \lim_{x\to 0^+} \frac{(1+x)^{1/x}}{2^{1/x}\left[\left(1+\dfrac{x}{2}\right)^{2/x}\right]^{1/2}}.$$

$$= \frac{e}{e^{1/2}} \lim_{x\to 0^+} \frac{1}{2^{1/x}} = 0$$

For continuity of f(x) at x = 0, we must have

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 0 \Rightarrow b = 0$

We have $f(-1) = b - a = -a \qquad (\because b = 0)$

Let us now find $f'(1)$

Let $\quad y = \left(\dfrac{1+x}{2+x}\right)^{1/x}, \ x > 0$

Then, $\log y = + \dfrac{1}{x} [\log (1 + x) - \log (2 + x)]$

$$\Rightarrow \frac{1}{Y}\frac{dy}{dx} = -\frac{1}{x^2} [\log (1 + x) - \log (2 + x)] + \frac{1}{x}\left[\frac{1}{1+x} - \frac{1}{2+x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+x}{2-x}\right)^{1/x} \left[-\frac{1}{x^2}\{\log(1+x) - \log(2+x)\} + \frac{1}{x}\left\{\frac{1}{1+x} - \frac{1}{2+x}\right\}\right]$$

Now, $f'(1) = \left.\dfrac{dy}{dx}\right|_{x=1}$

$$= \left(\frac{1+1}{2+1}\right)^{1/1}\left[-(\log 2 - \log 3) + \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$

$$= \frac{2}{3}\left[\log 3 - \log 2 + \frac{1}{6}\right] = \frac{2}{3}\log\frac{3}{2} = \frac{1}{9}$$

Since, $\quad f'(1) = f(-1) \Rightarrow \dfrac{2}{3}\log\dfrac{3}{2} + \dfrac{1}{9} = -a$

$\therefore$ The required function is

$$f(x) = \begin{cases} \left(-\dfrac{2}{3}\log\dfrac{3}{2} + \dfrac{1}{9}\right)x, & x \le 0 \\ \left[\dfrac{1+x}{2+x}\right]^{1/x}, & x > 0 \end{cases}$$

Q.11 If $y^2 = P(x)$, a polynomial of degree 3, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right)$ equals- **[IIT-1988]**

(A) $P'''(x) + P'(x)$ (B) $P''(x)\,P'''(x)$

(C) $P(x)\,P'''(x)$ (D) a constant

Sol. **[C]**

We have $y^2 = P(x)$, where P(x) is a polynomial of degree 3 and hence thrice differentiable

Then $\qquad y^2 = P(x)$

Differentiating (1) w.r. to x, we get

$$2y\frac{dy}{dx} = P'(x)$$

Again differentiating with respect to x, we get

$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

$$\frac{[P'(x)]^2}{2y^2} + 2y\frac{d^2y}{dx^2} = P''(x) \qquad \text{[Using (2)]}$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2y^2 P''(x) - [P'(x)]^2$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2\,P(x)\,P''(x) - [P'(x)]^2$$

[Using (1)]

$$\Rightarrow 2y^3\frac{d^2y}{dx^2} = P(x)\,P''(x) - \frac{1}{2}[P'(x)]^2$$

Again differentiating with respect to x, we get

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'''(x)\,P(x) + P''(x)\,P'(x) - P'(x)\,P'''(x) = P'''(x)\,P(x)$$

Q.12 The function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is-

[IIT-1988]

(A) continuous at x = 1

(B) differentiable at x = 1

(C) continuous at x = 3

(D) differentiable at x = 3

Sol. **[A, B, C]**

$$f(x) = \begin{cases} |x-3| & , \quad x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & , \quad x < 1 \end{cases}$$

$$= \begin{cases} \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & , \quad x < 1 \\ 3-x & , \quad 1 \le x < 3 \\ x-3 & , \quad x \ge 3 \end{cases}$$

$$Lf'(1) = \left(\frac{2x}{4} - \frac{3}{2}\right) = -1$$

$Rf'(1) = -1$. Thus $Lf'(1) = Rf'(1)$

$\therefore$ f is differentiable at x = 1 and hence continuous at x = 1.

Again $Lf'(3) = -1$ and $Rf'(3) = 1$

$\Rightarrow Lf'(3) \ne Rf'(3)$

$\Rightarrow$ f is not differentiable at x = 3

Let us now check the continuity at x = 3

$$\text{LHL} = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} 3-(3-h) = 0$$

$$\text{RHL} = \lim_{h\to 0} f(3+h) = \lim_{h\to 0} 3+h-3 = 0$$

$$f(3) = 0$$

$\therefore$ f is continuous at x = 3

Thus (A), (B) and (C) are correct options

Q.13 If f(9) = 9, f ′(9) = 4, then $\lim\limits_{x\to 9} \dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ equals.............. **[IIT 1988]**

Sol. **[4]**

f(9) = 9, f ′(9) = 4

$$\lim_{x\to 9} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3} \left(\frac{0}{0}\right)$$

$$\lim_{x\to 9} \frac{f'(x)/2\sqrt{f(x)}}{1/2\sqrt{x}} = \frac{4/3}{1/3} = 4$$

Q.14 Draw a graph of the function

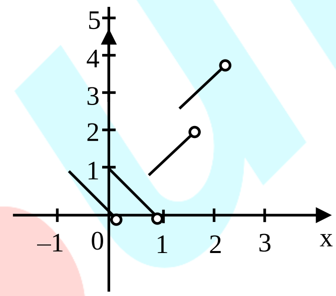
$y = [x] + |1 - x|, -1 \le x \le 3.$

Determine the points, if any, where this function is not differentiable.**[IIT-1989]**

Sol. We have, $y = [x] + |1-x|, \quad -1 \le x \le 3$

$$\text{or} \quad y = \begin{cases} -1+1-x & , \quad -1 \le x \le 3 \\ 0+1-x & , \quad 0 \le x < 1 \\ 1-1+x & , \quad 1 \le x < 2 \\ 2-1+x & , \quad 2 \le x < 3 \\ 3-1+x & , \quad x = 3 \end{cases}$$

$$\text{or} \quad y = \begin{cases} -x & , \quad -1 \le x < 0 \\ 1-x & , \quad 0 \le x < 1 \\ x & , \quad 1 \le x < 2 \\ 1+x & , \quad 2 \le x < 3 \\ -2+x & , \quad x = 3 \end{cases}$$



$\therefore$ The graph of the given function is as shown From graph we can that given function is not differentiable at x = 0, 1, 2, 3.

Q.15 If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that $(x^2 + 4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2 + 4)$

[IIT 1989]

Sol. We have, $x = \sec\theta - \cos\theta$

$y = \sec^n\theta - \cos^n\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$

$$= \sec\theta\tan\theta + \tan\theta\cos\theta$$

$$= \tan\theta(\sec\theta + \cos\theta)$$

and $\dfrac{dy}{d\theta} = n\ \sec^{n-1}\theta.\ \sec\theta\ \tan\theta - n\cos^{n-1}\theta(-\sin\theta)$

$$= n\sec^n\theta\tan\theta + n\cos^n\theta\tan\theta$$

$$= n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{n(\sec^n\theta + \cos^n\theta)}{(\sec\theta + \cos\theta)} \qquad .....(1)$$

Also $x^2 + 4 = (\sec\theta - \cos\theta)^2 + 4$

$= \sec^2\theta + \cos^2\theta - 2\sec\theta\cos\theta + 4$
$= \sec^2\theta + \cos^2\theta + 2$
$= (\sec\theta + \cos\theta)^2$ .....(2)

and $y^2 + 4 = (\sec^n\theta - \cos^n\theta)^2 + 4$
$= \sec^{2n}\theta + \cos^{2n}\theta - 2\sec^n\theta\cos^n\theta + 4$
$= \sec^{2n}\theta + \cos^{2n}\theta + 2$
$= (\sec^n\theta + \cos^n\theta)^2$ ......(3)

Now we have to prove

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

LHS = $(\sec\theta + \cos\theta)^2$.

$\dfrac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2}$ [Using (1) and (2)]

$= n^2(\sec^n\theta + \cos^n\theta)^2$
$= n^2(y^2+4)$ [From eq. (3)]
= RHS

Q.16 A function $f : R \to R$ satisfies the equation $f(x+y) = f(x)f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at x = 0 and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence, determine f(x). **[IIT-1990]**

Sol. We are given $f(x+y) = f(x)f(y)$, $\forall$ x, y $\in$ R
f is differentiable at x = 0
$f'(0) = 2$
To prove that $f'(x) = 2f(x)$, $\forall$ x $\in$ R and to find f(x)
We have for $x = y = 0$
$f(0+0) = f(0)f(0)$
$\Rightarrow$ $f(0) = [f(0)]^2 \Rightarrow f(0) = 1$
[as $f(0) \neq 0$]

Again $f'(0) = 2$

$\Rightarrow \lim\limits_{h\to0}\dfrac{f(0+h)-f(0)}{h} = 2$

$\Rightarrow \lim\limits_{h\to0}\dfrac{f(0)f(h)-f(0)}{h} = 2$

$\Rightarrow \lim\limits_{h\to0}\dfrac{f(0)[f(h)-1]}{h} = 2$

$\Rightarrow \lim\limits_{h\to0}\dfrac{f(h)-1}{h} = 2$ ......(1)

[Using f(0) = 1]

Now $f'(x) = \lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to0}\dfrac{f(x)f(h)-f(x)}{h}$

$= \lim\limits_{h\to0} f(x)\left(\dfrac{f(h)-1}{h}\right)$

$= f(x)\lim\limits_{h\to0}\left[\dfrac{f(h)-1}{h}\right]$

$= f(x) . 2$ [Using eq. (1)]
$= 2f(x)$

Also,

$\Rightarrow \dfrac{f'(x)}{f(x)} = 2$

Integrating on both sides with respect to x, we get

$\log|f(x)| = 2x + C$

At x = 0, $\log f(0) = C \Rightarrow C = \log 1 = 0$

$\therefore$ $\log|f(x)| = 2x$

$\Rightarrow$ $f(x) = e^{2x}$

Q.17 If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then $g'(x)$ =............for x > 20. **[IIT-1990]**

Sol. $f(x) = |x-2|$
$g(x) = f(f(x)) = |f(x) - 2|$
$= \big|\,|x-2|-2\,\big| = |x-2-2|$ as x > 20
$= |x-4| = x-4$ as x > 20
$\therefore$ $g'(x) = 1$

Q.18 Find $\dfrac{dy}{dx}$ at x = –1, when **[IIT-1991]**

$$(\sin y)^{\sin\left(\frac{\pi}{2}x\right)} + \frac{\sqrt3}{2}\sec^{-1}(2x) + 2^x\tan(\lambda n(x+2)) = 0.$$

Sol. We are given the function

$$(\sin y)^{\sin\left(\frac{\pi x}{2}\right)} + \frac{\sqrt3}{2}\sec^{-1}(2x) + 2^x\tan[\log(x+2)] = 0 \quad ....(1)$$

For x = –1,we have

$$(\sin y)^{\sin\left(\frac{-\pi}{2}\right)} + \frac{\sqrt3}{2}\sec^{-1}(-2) + 2^{-1}\tan[\log(-1+2)] = 0$$

$\Rightarrow$ $(\sin y)^{-1} + \dfrac{\sqrt3}{2}\left(\dfrac{2\pi}{3}\right) + \dfrac12\tan 0 = 0$

$\Rightarrow \dfrac{1}{\sin y} = -\dfrac{\pi}{\sqrt3} \Rightarrow \sin y = -\dfrac{\sqrt3}{\pi}$, when x = –1 ...(2)

Now Let $u = (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$

Taking log on both sides we get

$$\log u = \sin\left(\frac{\pi x}{2}\right) \log \sin y$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log \sin y + \cot y \frac{dy}{dx}\sin\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \frac{du}{dx} =$$

$$(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y + \sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right]$$

…..(3)

Now differentiating eq. (1), we get

$$\frac{d}{dx}\left[(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\right] + \frac{\sqrt{3}}{2}\frac{1}{2x\sqrt{4x^2-1}} \cdot 2 + 2^x \log$$

$$2 \tan ([\log (x + 2)]) + 2^x \sec^2 [\log (x + 2)] \frac{1}{x+2} = 0$$

$$\Rightarrow (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y + \sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right]$$

$$+ \frac{\sqrt{3}}{2x\sqrt{4x^2-1}} + 2^x \log 2 \tan (\log (x + 2)) + \frac{2^x \sec^2[\log(x+2)]}{x+2} = 0$$

At $x = -1$ and $\sin y = -\frac{\sqrt{3}}{\pi}$, we get

$$\Rightarrow \left(-\frac{\sqrt{3}}{\pi}\right)^{-1}\left[0 + (-1)\sqrt{\frac{\pi^2}{3}-1}\left(\frac{dy}{dx}\right)_{x=-1}\right] + \frac{\sqrt{3}}{-2\sqrt{3}} + 0 + 2^{-1} = 0$$

$$\Rightarrow \frac{\pi}{\sqrt{3}\sqrt{3}}\sqrt{\pi^2-3}\left(\frac{dy}{dx}\right)_{x=-1} - \frac{1}{2} + \frac{1}{2} = 0 \Rightarrow \frac{dy}{dx} = 0$$

Q.19 Let f(x) = x |x|. The set of points where f(x) is twice differentiable is........ **[IIT-1992]**

Sol. We have, $f(x) = x\,|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$$

$$f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \ge 0 \end{cases}$$

Clearly $f''(x)$ exists at every pt. except at x = 0

Thus f(x) is twice differentiable on R – {0}.

Q.20 Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then- **[IIT-1993]**

(A) $\lim_{x\to 0} f(x)$ does not exist

(B) f(x) is continuous at x = 0

(C) f(x) is not differentiable at x = 0

(D) $f'(0) = 1$

Sol. **[B]**

We have $f(x) = [\tan^2 x]$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0} [\tan^2 x] = 0$$

and f(0) = 0

∴ f is continous at x = 0

Q.21 Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real x & y.

If f'(0) exists & equals –1 & f(0) = 1. Find f (2). **[IIT 1995]**

Sol. $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all x , y ∈ R.

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{0+2h}{2}\right)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(0)+f(2h)}{2}-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{f(0)+f(2h)-2f(0)}{2h}$$

$$= \lim_{h\to 0}\frac{f(2h)-f(0)}{2h}$$

$$= \lim_{h\to 0}\frac{f(2h)-1}{2h} = -1.$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(2x)+f(2h)-2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{f(2x)+f(2h)-2f(x)}{2h}$$

use, $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for x, y $\in R$

$$f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} = \frac{f(x)+1}{2}$$

2f(x/2) – 1 = f(x)

Replace x by 2x, weget

2f(x) – 1 = f(2x)

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(2x)+f(2h)-2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{2f(x)-1+f(2h)-2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{f(2h)-1}{2h} = -1.$$

$\therefore$ f '(x) = – 1 $\Rightarrow$ f(x) = –x + c, where c is

Integration constant

f(x) = – x + c

Put x = 0 ; f(0) = – 0 + c = 1 $\Rightarrow$ c = 1

f(x) = – x + 1

f(2) = – 2 + 1

f(2) = – 1

Q.22 Find the differential coefficient of $\log_{(1-\sqrt{x})} \sin^{-1}(1-\sqrt{x})$ with respect to $2^{2(1-\sqrt{x})}$ and also find its value at x = 1/4.

[IIT 1996]

Sol.

$$\frac{d\left\{\log_{(1-\sqrt{x})}\sin^{-1}(1-\sqrt{x})\right\}}{d\left\{2^{2(1-\sqrt{x})}\right\}}$$

$$= \frac{\frac{d}{dx}\left\{\log_{(1-\sqrt{x})}\sin^{-1}(1-\sqrt{x})\right\}}{\frac{d}{dx}\left\{2^{2(1-\sqrt{x})}\right\}}$$

$$= \frac{d}{dx}\left\{\frac{\log_e \sin^{-1}(1-\sqrt{x})}{\log_e(1-\sqrt{x})}\right\} \bigg/ \frac{d}{dx}\left\{2^{2(1-\sqrt{x})}\right\}$$

$$\frac{1}{\sin^{-1}(1-\sqrt{x})} \times \frac{1}{\sqrt{1-(1-\sqrt{x})^2}}\left(-\frac{1}{2\sqrt{x}}\right) \times \log_e(1-\sqrt{x})$$

$$= \frac{\dfrac{-\log_e \sin^{-1}(1-\sqrt{x}) \times \frac{1}{(1-\sqrt{x})} \times \left(-\frac{1}{2\sqrt{x}}\right)}{\left(\log_e(1-\sqrt{x})\right)^2}}{2^{2(1-\sqrt{x})} \times \log 4 \times \left(-\frac{1}{2\sqrt{x}}\right)}$$

$$\frac{1}{\sin^{-1}(1-\sqrt{x})} \times \frac{1}{\sqrt{1-(1-\sqrt{x})^2}} \times \log_e(1-\sqrt{x})$$

$$= \frac{-\log_e \sin^{-1}(1-\sqrt{x}) \times \frac{1}{(1-\sqrt{x})}}{\left(\log_e(1-\sqrt{x})\right)^2 \times 2^{2(1-\sqrt{x})} \times \log 4}$$

At x = 1/4

$$= \frac{\frac{1}{\sin^{-1}\left(\frac{1}{2}\right)} \times \frac{1}{\sqrt{1-\frac{1}{4}}} \times \log(1-1/2) - \log_e \sin^{-1}(1/2) \times \frac{1}{1/2}}{\left(\log_e \frac{1}{2}\right)^2 \times 2^{2\left(1-\frac{1}{2}\right)} \times \log 4}$$

$$= \frac{\frac{1}{\pi/6} \times \frac{1}{\sqrt{3}/2} \times (-\log 2) - \log \pi/6 \times 2}{(\log 2)^2 \times 2 \times \log 4}$$

$$= -\frac{\left[\frac{1}{\sqrt{3}} \times \frac{12}{\pi} \times \log 2 + 2 \times \log \pi/6\right]}{2 \times \log 4 \times (\log 2)^2}$$

$$= -2\frac{\left[\frac{1}{\sqrt{3}} \times \frac{6}{\pi} \times \log 2 + \log \pi/6\right]}{2 \times \log 4 \times (\log 2)^2}$$

$$= \frac{-2\left[\frac{2\sqrt{3}}{\pi}\log 2 + \log \pi/6\right]}{2 \times \log 4 \times (\log 2)^2}$$

$$= -\frac{\left[\frac{2\sqrt{3}}{\pi}\log 2 + \log \pi/6\right]}{(\log 4) \times (\log 2)^2}$$

$$= -\frac{\left[\frac{2\sqrt{3}}{\pi} + \frac{\log \pi/6}{\log 2}\right]}{\log 4 \times \log 2} = \frac{-2\left[\frac{2\sqrt{3}}{\pi} + \frac{\log \pi/6}{\log 2}\right]}{2 \times \log 2 \times \log 4}$$

$$= \frac{-2}{(\log 4)^2} \times \left[\frac{2\sqrt{3}}{\pi} + \frac{\log \pi/6}{\log 2}\right]$$

Q.23 Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ test whether

(a) f (x) is continuous at x = 0

(b) f (x) is differentiable at x = 0

[IIT 1997]

Sol. $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

continuity at x = 0

$$\underset{0-h}{+}\!\!-\!\!\underset{0}{+}\!\!-\!\!\underset{0+h}{+}$$

L.H.L. $= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} (0-h)\, e^{-\left(\frac{1}{|0-h|}+\frac{1}{0-h}\right)}$

$= \lim_{h\to 0} (-h)\, e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} = \lim_{h\to 0} (-h) \times e^{-(0)}$

$= 0.$

R.H.L. $= \lim_{h\to 0} (0 + h)\, e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}$

$= \lim_{h\to 0} h.\ e^{-(2/h)} = \lim_{h\to 0} h \times e^{-\infty} = 0.$

$f(0) = \lim_{x\to 0} f(x) = 0$.

$\therefore$ f(x) is continuous at x = 0.

Differentiability at x = 0

L.H.D. $= \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0} \frac{(0-h)e^{-\left(\frac{1}{|0-h|}+\frac{1}{0-h}\right)}-0}{-h}$

$= \lim_{h\to 0} \frac{(-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}-0}{-h}$

$=e^{-0} = 1.$

R.H.D.$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{(0+h)e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}-0}{h}$

$= \lim_{h\to 0} \frac{h\, e^{-2/h}-0}{h} = 0$

$= \lim_{h\to 0} e^{-2/h} = e^{-\infty} = 0$

$\therefore$ f(x) is not differentiable at x= 0 .

Q.24 Determine the values of x for which the following functions fails to be continuous or differentiable **[IIT 1997]**

$f(x) = \begin{cases} 1-x & ; \quad x<1 \\ (1-x)(2-x) & ; \quad 1\le x\le 2 \\ 3-x & ; \quad x>2 \end{cases}$

Sol. $f(x) = \begin{cases} 1-x & ; \quad x<1 \\ (1-x)(2-x) & ; \quad 1\le x\le 2 \\ 3-x & ; \quad x>2 \end{cases}$

Continuity at x = 1

L.H.L. $= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} (1-(1-h)) = 0.$

R.H.L.$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} (1-(1+h))(2-1-h)$

$= \lim_{h\to 0} (-h)(1-h)$

$= 0.$

$f(1) = \lim_{x\to 1} f(x) = 0.$

Since, L.H.L. = R.H.L. = f(1) = 0.

$\therefore$ f(x) is continuous at x = 1.

continuity at x = 2.

L.H.L. $= \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0} (1-2+h)(2-2+h)$

$= \lim_{h\to 0} (h-1)(h) = 0.$

R.H.L.$= \lim_{x\to 2^+} f(x) = \lim_{h\to 0} (2+h)$

$= \lim_{h\to 0} (3-2-h)$

$= 1.$

Since, L.H.L. $\neq$ R.H.L.

Henec,f(x) is not continuous at x = 2.

Differentiability at x = 1.

L.H.D. $= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0} \frac{(1-1+h)-0}{-h}$

$= -1.$

R.H.D.$= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{1+h-1}$

$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{(1-1-h)(2-1-h)-0}{h}$

$= \lim_{h\to 0} \frac{(-h)(1-h)-0}{h}$

$= \lim_{h\to 0} \frac{(h-1)-0}{1} = -1.$

Since, L.H.D. = R.H.D. = –1.

Henec,f(x) is differentiable at x = 1.

Differentiability at x = 2

L.H.D. = $\lim\limits_{x\to 2^-}\dfrac{f(x)-f(2)}{x-2}$ = $\lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{2-h-2}$

$$= \lim_{h\to 0}\frac{(1-2+h)(2-2+h)-0}{-h}$$

$$= \lim_{h\to 0}\frac{(h-1)h-0}{-h} = 1.$$

R.H.D.= $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$$= \lim_{h\to 0}\frac{(3-2-h)-0}{h}$$

$$= \lim_{h\to 0}\frac{(1-h)}{h} \to \text{Not Differentiable}$$

$\therefore$ f(x) is not differentiable at x = 2.

Q.25 If F(x) = f(x) . g(x). h(x) for all real x where f(x), g(x), h(x) are differentiable functions. At some point x_0,

$F'(x_0) = 21F(x_0)$, $f'(x_0) = 4 f(x_0)$,

$g'(x_0) = -7 g(x_0)$ and $h'(x_0) = k\, h(x_0)$.

Then find the value of k. **[IIT 1997]**

Sol. F(x) = f(x) . g(x). h(x) for all x $\in$ R.

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = 4f(x_0)$$

$$g'(x_0) = \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h} = -7g(x_0)$$

$$h'(x_0) = \lim_{h\to 0}\frac{h(x_0+h)-h(x_0)}{h} = kh(x_0).$$

$$F'(x_0)= \lim_{h\to 0}\frac{f(x_0+h).g(x_0+h)h(x_0+h)-f(x_0)g(x_0).h(x_0)}{h}$$

$= 21\, f(x_0) . g(x_0).h(x_0)$

$\Rightarrow \lim\limits_{h\to 0} f(x_0+h).g(x_0+h).h(x_0+h)-f(x_0).g(x_0).h(x_0)$

$= 21 \lim\limits_{h\to 0} 1 \times h \times f(x_0) \times g(x_0) \times h(x_0).$

$\Rightarrow \lim\limits_{h\to 0} f(x_0+h).g(x_0+h).h(x_0+h)=f(x_0).g(x_0).h(x_0)) + 21 \lim\limits_{h\to 0} h \times f(x_0) \times g(x_0) \times h(x_0)$.......(i)

$\lim\limits_{h\to 0} f(x_0+h) = f(x_0) + \lim\limits_{h\to 0} h\times 4f(x_0)$.......(ii)

$\lim\limits_{h\to 0} g(x_0+h) = g(x_0)+ \lim\limits_{h\to 0} h\times(-7g(x_0))$.......(iii)

$\lim\limits_{h\to 0} h(x_0+h) = h(x_0) + \lim\limits_{h\to 0} h\times kh(x_0)$...(iv)

multiplying (2), (3) and (4), weget

$\lim\limits_{h\to 0} f(x_0+h).g(x_0+ h).h(x_0 + h) =$

$[f(x_0)+ \lim\limits_{h\to 0} h\times 4f(x_0)]\ [g(x_0)+ \lim\limits_{h\to 0} h\times(-7g(x_0)]$

$[h(x_0)+ \lim\limits_{h\to 0} h\times k.h(x_0)]$ .....(v)

from (i) and (v) weget

$f(x_0).g(x_0).\ h(x_0) + 21 \lim\limits_{h\to 0} h \times f(x_0).g(x_0).h(x_0)=$

$[f(x_0)+ \lim\limits_{h\to 0} h\times 4\, f(x_0)]\ [g(x_0)+ \lim\limits_{h\to 0} h\times(-7\, g(x_0))]$

$[h(x_0)+ \lim\limits_{h\to 0} h\times k.h(x_0)]$

on comparing terms containing $\lim\limits_{h\to 0}$ h only, weget

$21 \lim\limits_{h\to 0} h\times f(x_0)\times g(x_0)\times h(x_0)=4 \lim\limits_{h\to 0} h\times f(x_0)g(x_0) \times h(x_0) -7 \lim\limits_{h\to 0} h\times g(x_0)\times f(x_0)\times h(x_0)+ k \lim\limits_{h\to 0} f(x_0)g(x_0)h(x_0).$

$\Rightarrow 21 \lim\limits_{h\to 0} h\times f(x_0)\times g(x_0)\times h(x_0)=(4-7+k)\times \lim\limits_{h\to 0} h \times f(x_0)\times g(x_0).h(x_0)$

$\Rightarrow 21 = 4 - 7 + k \Rightarrow k = 21 + 3$

k = 24

Q.26 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where P is a constant. Then $\dfrac{d^3}{dx^3}$ [f(x)] at x = 0 is **[IIT 1997]**

(A) p (B) $p + p^3$

(C) $p + p^2$ (D) Independent of p

Sol.[D] $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

Differentiating w.r.t. x, weget

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$+\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Again Differentiating w.r.t. x, weget

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$+\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Again Differentiating w.r.t x, weget

$$f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$$

$$f'''(x)|_{x=0} = \frac{d^3}{dx^3} f(x)|_{x=0} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

= Independent of p.

$\therefore$ Option (D) is correct answer.

Q.27 Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then **[IIT 1998]**

(A) h is continuous for all x

(B) h is differentiable for all x

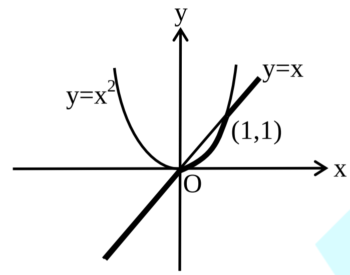
(C) $h'(x) = 1$, for all $x > 1$

(D) h is not differentiable at two values of x

Sol.[A, C, D] $h(x) = \min.\{x, x^2\}$

first we have to draw $y = x$ and $y = x^2$

Pointof intersection of $y = x$ and $y = x^2$



$x = x^2$

$\Rightarrow x(1-x) = 0 \Rightarrow x = 0, 1$

Darken line indicates the required portion. Because for min h(x), Below point of intersection would be required portion.

$$h(x) = \begin{cases} x^2; 0 < x < 1 \\ x; x \le 0 \text{ or } x \ge 1 \end{cases}$$

Differentiability at x = 0

L.H.D. $= \lim_{x\to0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to0} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to0} \frac{(0-h)-0}{-h}$

$= 1.$

R.H.D.$= \lim_{x\to0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to0} \frac{f(0+h)-f(0)}{0+h-0}$

$= \lim_{h\to0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{x\to0} \frac{(0+h)^2-0}{h} = \lim_{h\to0} \frac{h^2-0}{h}$

$= 0.$

Since L.H.D. $\neq$ R.H.D.

Hence f(x) is not differentiable at x = 0

Differentiability at x = 1

L.H.D.$= \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h\to0} \frac{f(1-h)-f(1)}{1-h-1}$

$= \lim_{h\to0} \frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to0} \frac{(1-h)^2-1}{-h} \left(\frac{0}{0}\text{form}\right)$

Use L – H Rule, weget

$= \lim_{h\to0} \frac{2(1-h)(-1)-0}{-1} = 2$

R.H.D.$= \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to0} \frac{f(1+h)-f(1)}{1+h-1}$

$= \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \dfrac{(1+h)-1}{h}$

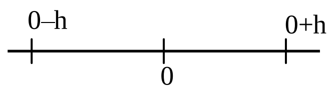
$= \lim_{h\to 0} \dfrac{h}{h} = 1$

Since, L.H.D. $\neq$ R.H.D.

Hence, f(x) is not differentiable at x = 1.

Options (C) and (D) are correct.

continuity at x = 0



L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} (0-h)$

$= 0.$

R.H.L.= $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} (0+h)^2$

$= 0.$

$f(0) = \lim_{x\to 0} f(x) = 0$

$\therefore$ f(x) is continuous at x = 0

Cotinuity at x = 1.

L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} (1-h)^2 = 1.$

R.H.L.= $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} (1+h) = 1.$

$f(1) = \lim_{x\to 1} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1.

$\therefore$ f(x) is continuous for all x

$\therefore$ Option (A) is also correct answer.

$\therefore$Options (A), (C), (D) are correct answers.

Q.28 If $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1,$

Prove that $\dfrac{y'}{y} = \dfrac{1}{x}\left[\dfrac{a}{a-x} + \dfrac{b}{b-x} + \dfrac{c}{c-x}\right]$

[IIT 1998]

Sol. $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$

$= \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c}$

$= \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \left(\dfrac{b}{x-b}+1\right)\dfrac{x}{x-c}$

$= \dfrac{ax^2}{(x-a)(x-b)(x-x)} + \dfrac{x^2}{(x-b)(x-c)}$

$= \left(\dfrac{a}{x-a}+1\right)\dfrac{x^2}{(x-b)(x-c)}$

$= \dfrac{x^3}{(x-a)(x-b)(x-c)}$

$\Rightarrow \log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$

$\Rightarrow \dfrac{y'}{y} = \dfrac{3}{x} - \dfrac{1}{x-a} - \dfrac{1}{x-b} - \dfrac{1}{x-c}$

$= \left(\dfrac{1}{x} - \dfrac{1}{x-a}\right) + \left(\dfrac{1}{x} - \dfrac{1}{x-b}\right) + \left(\dfrac{1}{x} - \dfrac{1}{x-c}\right)$

$= \dfrac{a}{x(a-x)} + \dfrac{b}{x(b-x)} + \dfrac{c}{x(c-x)}$

$= \dfrac{1}{x}\left[\dfrac{a}{a-x} + \dfrac{b}{b-x} + \dfrac{c}{c-x}\right]$

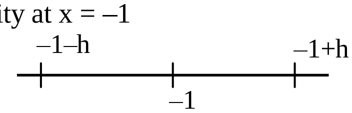
Q.29 Discuss the continuity and differentiability of the function

$f(x) = \begin{cases} 2+\sqrt{1-x^2} & ;|x|\le 1 \\ 2e^{(1-x)^2} & ;|x|>1 \end{cases}$

[IIT 1998]

Sol. $f(x) = \begin{cases} 2+\sqrt{1-x^2} & ;|x|\le 1 \text{ or } -1\le x\le 1 \\ 2e^{(1-x)^2} & ;|x|>1 \text{ or } x<-1 \text{ or } x>1 \end{cases}$

Continuity at x = –1



L.H.L. = $\lim_{x\to -1^-} f(x) = \lim_{h\to 0} f(-1-h)$

$= \lim_{h\to 0} 2\, e^{(1+1+h)^2}$

$= 2e^4.$

R.H.L.= $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{h\to 0} f(-1+h)$

$= \lim\limits_{h\to 0}\left(2+\sqrt{1-(-1-h)^2}\right)$

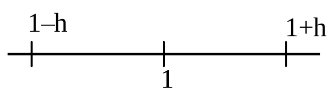
$= \lim\limits_{h\to 0}\left(2+\sqrt{1-(1+h)^2}\right)$

=2.

Since, L.H.L. $\neq$ R.H.L.

Hence, f(x) is not continuous at x = –1.

Continuity at x = 1.



L.H.L. = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h)$

$= \lim\limits_{h\to 0}\left(2+\sqrt{1-(1-h)^2}\right)$

= 2.

R.H.L.= $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h)$

$= \lim\limits_{h\to 0} 2\,e^{(1-1-h)^2}$

$= \lim\limits_{h\to 0} 2\,e^{h^2}$

$= 2\,e^0 = 2.$

$f(1) = \lim\limits_{x\to 1} f(x) = 2$

Hence, f(x) is continuous at x = 1.

Differentiability at x= –1.

L.H.D.= $\lim\limits_{x\to -1^-} \dfrac{f(x)-f(-1)}{x+1}$

$= \lim\limits_{h\to 0} \dfrac{f(-1-h)-f(-1)}{-1-h+1}$

$= \lim\limits_{h\to 0} \dfrac{2e^{(1+1+h)^2}-2}{-h}$

= Not differentiable

$\therefore$ f(x) is not diffrentiable at x = –1

Differentiability at x = 1

L.H.D. = $\lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{1-h-1}$

$= \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{2+\sqrt{1-(1-h)^2}-2}{-h}\left(\dfrac{0}{0}\text{form}\right)$

$= \lim\limits_{h\to 0} \dfrac{0+\dfrac{1(+2(1-h))}{2\sqrt{1-(1-h)^2}}-0}{-1}$

$= \lim\limits_{h\to 0} \dfrac{-2(1-h)}{2\sqrt{1-(1-h)^2}}$

Not differentiable

$\therefore$ f(x) is not differentiable at x = 1.

Q.30 The function

$f(x) = (x^2-1)\,|x^2-3x+2| + \cos(|x|)$

is not differentiable at **[IIT 1999]**

(A) –1 (B) 0

(C) 1 (D) 2

Sol.[D] $f(x) = (x^2-1)\,|x^2-3x+2| + \cos(|x|)$

Since, cos (|x|) is differentiable for all x

$|x^2-3x+2| = \begin{cases} x^2-3x+2 & ; x\le 1 \text{ or } x\ge 2 \\ -(x^2-3x+2); & 1<x<2 \end{cases}$

Differentiability at x = 1.

L.H.D. = $\lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{1-h-1}$

$= \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{((1-h)^2-1)((1-h)^2-3(1-h)+2) + \cos|1-h| - \cos 1}{-h}$

$= \lim\limits_{h\to 0} \dfrac{2(1-h)(-1)\times[(1-h)^2-3(1-h)+2] + [(1-h)^2-1]\times[2(1-h)(-1)+3]+\sin(1-h)-0}{-1}$

= – sin 1.

R.H.D.= $\lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{1+h-1}$

$= \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-[(1+h)^2-1][(1+h)^2-3(1+h)+2] + \cos|1+h| - \cos 1}{h}\left(\dfrac{0}{0}\text{form}\right)$

$= \lim\limits_{h\to 0} \dfrac{-[(1+h)^2-1][2(1+h)-3]-2(1+h)[(1+h)^2-3(1+h)+2]-\sin(1+h)-0}{1}$

= – sin 1.

Since, L.H.D = R.H.D. = – sin1

Hence, f(x) is differentiable at x = 1.

Differentiability at x = 2.

L.H.D. = $\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2} = \lim_{h\to0}\frac{f(2-h)-f(2)}{2-h-2}$

$= \lim_{h\to0}\frac{f(2-h)-f(2)}{-h}$

$= \lim_{h\to0}\frac{[(2-h)^2-1][(2-h)^2-3(2-h)+2] + \cos|2-h| - \cos 2}{-h}\left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to0}\frac{2[(2-h)(-1)][(2-h)^2-3(2-h)+2] + [(2-h)^2-1][2(2-h)(-1)+3] - \sin(2-h)(-1) - 0}{-1}$

$= \lim_{h\to0}\frac{0+(3)(-1)+\sin 2}{-1} = 3 - \sin 2$

R.H.D.= $\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2} = \lim_{h\to0}\frac{f(2+h)-f(2)}{2+h-2}$

$= \lim_{h\to0}\frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to0}\frac{-[(2+h)^2-1][(2+h)^2-3(2+h)+2] + \cos|2+h| - \cos 2}{h}\left(\frac{0}{0}\text{form}\right)$

$= \lim_{h\to0}\frac{-2(2+h)[(2+h)^2-3(2+h)+2] - [(2+h)^2-1][2(2+h)-3] - \sin(2+h) - 0}{1}$

$= -3 - \sin 2.$

Since, L.H.D. $\neq$ R.H.D.

Hence, f(x) is not differentiable at x = 2

$\therefore$ Option (D) is correct answer.

Q.31 If $x^2 + y^2 = 1$, then **[IIT Scr. 2000]**

(A) $yy'' - 2(y')^2 + 1 = 0$

(B) $yy'' + (y')^2 + 1 = 0$

(C) $yy'' - (y')^2 - 1 = 0$

(D) $yy'' + 2(y')^2 + 1 = 0$

Sol.[B] $x^2 + y^2 = 1.$

differentiate w.r.t. x, we get

$2x + 2y\,.\,\frac{dy}{dx} = 0$

$x + y\frac{dy}{dx} = 0$

Again differentiating w.r.t. we get

$1 + y\frac{d^2y}{dx} + (dy/dx)^2 = 0.$

$y\frac{d^2y}{dx^2} + (dy/dx)^2 + 1 = 0$

or $y.\, y'' + (y')^2 + 1 = 0$

y'' = second differentiation w.r.t. x = d^2y/dx

y' = first differentiation w.r.t. x = dy/dx

$\therefore$ Option (B) is correct answer.

Q.32 Let f : R $\to$ R be any function. Define g : R $\to$ R by g (x) = |f (x)| for all x. Then g is **[IIT Scr. 2000]**

(A) onto if f is onto

(B) one-one if f is one-one

(C) continuous if f is continuous

(D) differentiable if f is differentiable

Sol.[C] f : R $\to$ R and g : R $\to$ R

g(x) = |f(x)| for all x.

Since f(x) is continuous then |f(x)| must be continuous but inverse is not true.

Hence, Option (C) is correct answer.

Passage (Question 33 to 35)

Let f : R $\to$ R be a differential function satisfying $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ $\forall$ x, y $\in$ R.

If $f'(0)$ exists and equal to – 1 and f(0) = 1, Then answer the following questions-

Q.33 If g(x) = |f (|x|)| for all x $\in$ R, then for g(x) -

(A) 3 non differential point

(B) 2 non differential point

(C) 4 non differential point

(D) None of these

Sol. **[A]**

f(x) = –x + 1

Q.34 If g(x) = f(sin x), h(x) = |g(|x|)| and I(x) = g(|x|) then-

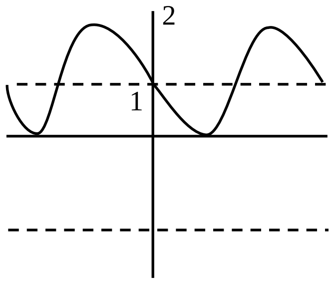
(A) number of non differential points for both function h(x) and I(x) is different

(B) range of h(x) and I(x) is different

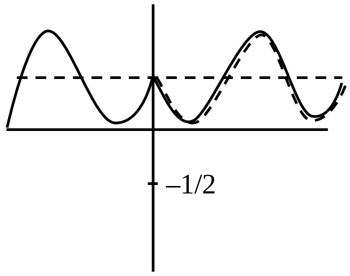
(C) solution of $I(x) + \frac{1}{2} = 0$ is infinite

(D) None of these

Sol. **[D]**



I(x) = h(x)



so answer : (D)

Q.35 Range of f(|x|) is-

(A) $(-\infty, 0)$ (B) $(-\infty, 0]$

(C) $(-\infty, 1]$ (D) None of these

Sol/ **[C]**

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	B	C	C	D	C	B	B	D	C	C	A	C	A	A	A	C	A	A
Q.No.	21	22	23	24	25	26	27	28	29											
Ans.	B	B	D	B	C	B	A	C	B											

30. False **31.** True **32.** True **33.** 1, –1 **34.** –1 **35.** sin α **36.** sin a

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Ans.	A	B	C	A	C	D	C	C	C	C	D	D	B	C	A	D	A

(PART-B)

Q.No.	18	19	20	21	22	23	24	25	26	27
Ans.	A,B	D	A,C	A,B,D	A,B,C,D	A,C	A,C	A,C,D	B,D	A,B

(PART-C)

Q.No.	28	29	30
Ans.	C	B	C

(PART-D)

31. (A) $\to$ P, Q, R (B) $\to$ P, R, S (C) $\to$ P, R, S (D) $\to$ P, R, S

32. (A) $\to$ P, Q (B) $\to$ Q, R (C) $\to$ P, S (D) $\to$ P, S

33. (A) $\to$ Q; (B) $\to$ P; (C) $\to$ S; (D) $\to$ P

EXERCISE # 3

1. **(b)** $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]$ + [x + 1 – log x] $x^{\left(\frac{1}{x}-1\right)}$ **(c)** $\frac{-\sqrt{3}}{\log_e 4}\times(12+\log 2)$

2. (a) 2, (b) 1

5. Function is not differentiable at x = – 1, 0, 1, 3/2,2

6. $x^3 - 5x^2 + 2x + 6$

9. Non continuous and hence non differentiable at x = 1, 2, 3

11. $f(x) = 2x^2$ for $0 \le x \le 1$ and $f(0) = 0$ for $-1 \le x < 0$, f is differentiable and hence continuous at x = 0

12. $(\log 2)^2 - 1$

13. (i) continuous every where in its domain, (ii) continuous everywhere in its domain

Q.No.	14	15	16	17	18	19	20	21	22
Ans.	A	A,B	B	C	C	B	A	A	B

EXERCISE # 4

2. D | **4.** A | **5. D** | **6. D** | **7.** a = 1, b = 0; differentiable | **8.** 0

9. A | **10.** B | **12.** $1-\frac{2}{\pi}$ | **13.** $g'(0) = 0$ | **14.** A | **15.** A | **16.** A

17. B | **18.** C | **19.** A, D | **20.** D | **21.** C | **22.** B | **23.** A

24. $g'(1) = 2$ | **25.** B | **26.** B, C, D | **27.** A, B, C, D | **28.** A | **29.** B

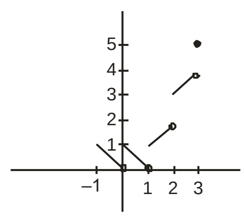
EXERCISE # 5

(1) B **(2)** Continuous on (0, 2) and differentiable on (0, 2) – {1} **(3)** 1/e

(4) 4 **(5)** Not differentiable at x = 0, 1 **(6)** B, C, D **(7)** A, B, D

(8) A **(9)** 0 **(10)** $f(x) = \begin{cases} -\left(\frac{2}{3}\log(3/2)+\frac{1}{9}\right)x, & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

(11) C **(12)** A, B, C **(13)** 4 **(14)** x = 0, 1, 2, 3



(17) 1 **(18)** $\frac{3}{\pi\sqrt{\pi^2-3}}$ **(19)** R – {0} **(20)** B

(21) – 1 **(22)** $\frac{1}{\log t}\left[\frac{1}{\sin^{-1}t}\cdot\frac{1}{\sqrt{1-t^2}}-\frac{y}{t}\right]\frac{1}{4^t\log 4}$ and $-\frac{2}{(\log 4)^2}\left[\frac{2\sqrt{3}}{\pi}+\frac{\log\frac{\pi}{6}}{\log 2}\right]$, where $t = (1-\sqrt{x})$

(23) Continuous for x ∈ R but not differentiable at x = 0

(24) f(x) is not continuous and thus not differentiable at x = 2

(25) 24 **(26)** D **(27)** A, C, D

(29) f(x) is discontinuous at x = –1 and not differentiable at |x| = 1

(30) D **(31)** B **(32)** C **(33)** A **(34)** D **(35)** C