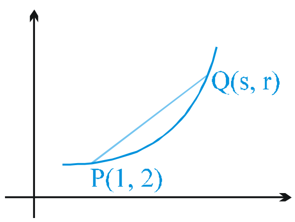
16. I By definition $f'(1)$ is the limit of the slope of the secant line when s → 1.

$$\text{Thus}\quad f'(1) = \lim_{s\to 1}\frac{s^2+2s-3}{s-1}$$

$$= \lim_{s\to 1}\frac{(s-1)(s+3)}{s-1}$$



$$= \lim_{s\to 1}(s+3) = 4 \quad \Rightarrow \text{(D)}$$

II By substituting x = s into the equation of the secant line, and cancelling by s – 1 again, we get

$y = s^2 + 2s - 1$. This is f (s), and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.]

18. f'(x)

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(h)+|x|h+xh^2}{h}$$

where $x = h$ and $y = x$

$\Rightarrow$ $f(0) = 0$; hence $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}+|x|+xh\right)$$

$$f'(x) = f'(0) + |x| = |x|$$

19. $\lim\limits_{x\to 2^-} f(x) = \frac{3}{5} = f(2) \neq \lim\limits_{x\to 2^+} f(x) = 1$

f(x) is not continous at x = 2

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3) = \frac{9}{2}$

Now LHD (x =3) is

$$\lim_{h\to 0}\frac{\frac{1}{4}((3-h)^3-(3-h)^2)-\frac{9}{2}}{-h}$$

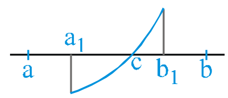
$$= \lim_{h\to 0}\frac{h^2-8h+21}{4} = \frac{21}{4}$$

and RHD (x = 3) is $\lim\limits_{h\to 0}\dfrac{\frac{9}{4}(|h-1|+|-1-h|)-\frac{9}{2}}{h} = 0$

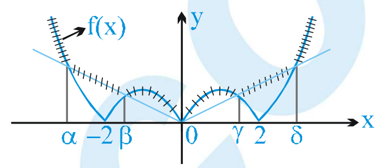
f(x) is not differentiable at x =2 and x = 3

20. I and II are false. The function f (x) = 1/x, 0 < x <1, is a counter example.

Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$



22. $f(x)$ is non differentiable at $x = \alpha, \beta, 0, \gamma, \delta$



and $g(x)$ is non differentiable at $x = \alpha, \beta, 0, -2, 2$ ⇒ **(B)**

23. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} x^2 e^{2(x-1)} = 1$

$f(1) = 1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} a\,\text{sgn}(x+1)\cos 2(x-1) + bx^2 = a.1.1 + b$

for continuity a + b = 1

LHD $(x = 1)$ is $\lim\limits_{h\to 0}\dfrac{(1-h)^2 e^{-2h}-1}{h}$

$$= \lim_{h\to 0} 2e^{-2h} + he^{-2h} + \left(\frac{e^{-2h}-1}{h}\right)$$

$$= 2 + 0 + 2 = 4$$

RHD $(x = 1)$ is $\lim\limits_{h\to 0}\dfrac{a\,\text{sgn}(2+h)\cos 2h + b(1+h)^2 - 1}{h}$

$$= \lim_{h\to 0}\frac{a\cos 2h + b + bh^2 + 2bh - (a+b)}{h}$$

$$= \lim_{h\to 0} a\left(\frac{\cos 2h - 1}{h}\right) + bh + 2b = 2b$$

f(x) is differentiable at x = 1 if 2b = 4

$\Rightarrow$ $b = 2$ $a = -1$

24. $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \geq 1 \end{cases}$

for differentiability at $x = 1$, $g'(1^+) = g'(1^{-1})$

$a = 6x - \dfrac{4}{2\sqrt{x}}$ $\Rightarrow$ $a = 6 - 2 = 4$

for continuity at $x = 1$, $g(1^+) = g(1^-)$

$a + b = 3 - 4 + 1$ $\Rightarrow$ $a + b = 0$ $\Rightarrow$ $b = -4$

$a = 4$, and $b = -4$

$$\Rightarrow \quad f'(x) = \begin{bmatrix} 3x^2 & \text{if } x > -1 \\ -3x^2 & \text{if } x < -1 \end{bmatrix}$$

$f'(-1^+) = 3; \quad f'(-1^-) = -3$

$\Rightarrow$ f is not differentiable at $x = -1$

also since f is not bijective hence it has no inverse

$\Rightarrow$ **(C)**

10. $$f(x) = \begin{cases} (x+1)(2x-1), & x < -1 \\ (x+1)(1-2x), & -1 \le x \le 0 \\ x+1, & 0 \le x < 1 \\ (x+1)(2x-1), & x \ge 1 \end{cases}$$

$$f'(x) = \begin{cases} 4x+1, & x < -1 \\ -4x-1, & -1 \le x \le 0 \\ 1, & 0 \le x < 1 \\ 4x+1, & x \ge 1 \end{cases}$$

Function is not differentiable at $x = -1, 0$ and 1.

11. $f(x) = \left|x - \frac{1}{2}\right| + |x-1| + \tan x$

$\left|x - \frac{1}{2}\right|$ is non-differentiable at $x = \frac{1}{2}$

$\Rightarrow$ $|x-1|$ is non-differentiable at $x = 1$

$\tan x$ is non-differentiable at $x = \frac{\pi}{2}$

12. $$f'(0^+) = \lim_{h\to 0} \frac{h \ln(\cos h)}{h \ln(1+h^2)} = \lim_{h\to 0} \frac{\ln(\cos h)^{1/h^2}}{\frac{\ln(1+h^2)}{h^2}}$$

$$= \lim_{h\to 0} \frac{1}{h^2}(\cos h - 1) = -\frac{1}{2}; \quad f'(0^-) = -\frac{1}{2}$$

hence f is continuous and derivable at $x = 0$

15. $$f(x) = \begin{bmatrix} 0 & 0 < x < 1 \\ 0 & x = 0 \text{ or } 1 \text{ or } -1 \\ 0 & -1 < x < 0 \end{bmatrix}$$

$\Rightarrow$ $f(x) = 0$ for all in $[-1, 1]$

18. $f(x) = \sum_{k=0}^{n} a_k |x|^k = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 + \ldots\ldots + a_n|x|^n$

$f(0) = a_0$ we know that $\lim_{x\to 0} |x| = 0$

$\lim_{x\to 0} f(x) = a_0$

f(x) is continous for $x = 0$

$|x|^n$ is differentiable if $n \ne 1, \quad n \in N$

f(x) is not differentiable at $x = 0$, due to presence of $|x|$

If all $a_{2k+1} = 0$, f(x) does not contains $|x|$

$\Rightarrow$ f(x) is differentiable at $x = 0$

20. $$H(x) = \begin{bmatrix} \cos x \;; & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x \;; & \frac{\pi}{2} < x \le 3 \end{bmatrix}$$

$$H'\left(\frac{\pi^-}{2}\right) = -\sin x = -1 \quad \Rightarrow \quad H'\left(\frac{\pi^+}{2}\right) = -1$$

Hence H(x) is continuous and derivable in [0, 3] & has maximum value 1 in [0, 3]

Part # II : Assertion & Reason

2. $y = |\ln x|$ not differentiable at $x = 1$

$y = \big|\cos|x|\big|$ is not differentiable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$y = \cos^{-1}(\text{sgn}\, x) = \cos^{-1}(1) = 0$ differentiable $\forall\ x \in (0, 2\pi)$

3. $$f'(0^+) = \frac{h\sin h - 0}{h} = 0$$

$$f'(0^-) = \frac{h\sin(-h) - 0}{-h} = 0$$

f(x) is diff. at $x = 0$

e.g. $x|x|$ is derivable at $x = 0$

6. **Statement-1** $f(x) = \text{sgn}(\cos x)$

at $\quad x = \frac{\pi}{2}$, $\cos x = 0$

$\therefore$ f(x) is discontinuous & non differentiable at $x = \frac{\pi}{2}$

Statement - 2 $g(x) = [\cos x]$

at $\quad x = \frac{\pi}{2}$, $\cos x = 0$.

$\therefore$ g(x) is discontinuous & hencenon differentiable

at $\quad x = \frac{\pi}{2}$. **(True)**

7. Consider $g(x) = x^3$ at $x = 0$; $g(0) = 0$

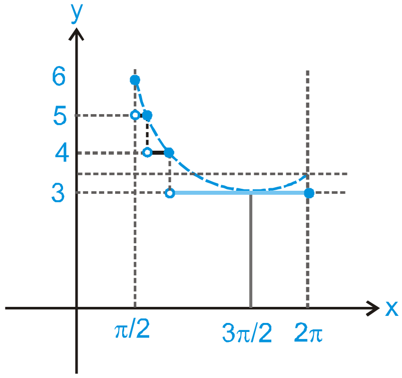
$|g(x)|$ is derivable as $x = 0$

actually nothing definite can be said. Also for

$g(x) = x - 1$ with $g(1) = 0$

then $|g(x)|$ not derivable at $x = 1$

$y = [3 \text{ cosec } x/3]$



5. **(A)** $f'(0) = \lim\limits_{h \to 0} \dfrac{\cos h - 0}{h}$ does not exist. Obviously

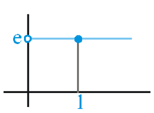
$f(0) = f(0^-) = f(0^+) = 1$

Hence continuous and not derivable

(B) $g(x) = 0$ for all x, hence continuous and derivable

(C) as $0 \le \{f(x)\} < 1$, hence $h(x) = \sqrt{\{x\}^2} = \{x\}$ which is discontinuous hence non derivable all $x \in I$

(D) $\lim\limits_{x \to 1} x^{\frac{1}{\ln x}} = \lim\limits_{x \to 1} x^{\log_x e} = e = f(1)$



Hence k (x) is constant for all x > 0 hence continuous and differentiable at x = 1.

6. **(A)** $f(x) = \begin{cases} 1 - 1 = 0 & ; \quad 1 < x \le 2 \\ 0 & ; \quad x = 1 \\ 1 - x & ; \quad 0 \le x < 1 \\ -\sin \pi x & ; \quad -1 \le x < 0 \end{cases}$

at x = 0, f(x) is not continuous & not differentiable

at x = 1, f(x) is continuous & not differentiable

at x = 2 and – 1, f(x) is continuous & differentiable

(C) $f(x) = \dfrac{x}{x+1}$, not defined at $x = -1$

$g(x) = \dfrac{f(x)}{f(x)+2}$

g(x) is not defined at $f(x) = -2$

$\dfrac{x}{x+1} = -2 \quad \Rightarrow x = \dfrac{-2}{3}$

Also x = 0 is not in the domain of f(x)

So, at 3 points g(x) is not differentiable.

Part # II : Comprehension

Comprehension–2

$$\text{LHD} = \lim_{h \to 0} \frac{\dfrac{-\sinh + \tanh + \cosh - 1}{2h^2 + \ln(2-h) - \tanh} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{\dfrac{\sinh}{h} - \dfrac{\tanh}{h} + \dfrac{1-\cosh}{h^2} \times h}{2h^2 + \ln(2-h) - \tanh} = 0$$

$$f'(0^+) = \text{RHD} = \sqrt{2} = h \times \frac{e^{h^2} - 1}{h^2} = 0$$

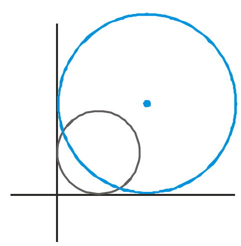
$L_1 \equiv y = 0 \quad$ and $\quad L_2 \equiv x = 0$

1. $(x-r)^2 + (y-r)^2 = r^2$ (family of circle)

$x^2 + y^2 - 2rx - 2ry + r^2 = 0$

$2(r_1r_2 + r_1r_2) = r_1^2 + r_2^2 \quad$ or $\quad 4r_1r_2 = r_1^2 + r_2^2$

$$\left(\frac{r_2}{r_1}\right)^2 - 4\left(\frac{r_2}{r_1}\right) + 1 = 0$$



$$\frac{r_2}{r_1} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

2. $2[\Delta_1 + \Delta_2 + \Delta_3]$

$$\Delta = 2 \times \frac{1}{2}\left(\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cot\frac{\theta}{2} + 1\right) \qquad \left[\text{using } \frac{1}{2}ab\right]$$

$$\Delta = \frac{\cos\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)}{\sin\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)} + \frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} + 1$$

8. $(fog)(x) = x + 1$ for $-2 \le x \le -1$, $-(x+1)$ for $-1 < x \le 0$ & $x - 1$ for $0 < x \le 2$. $(fog)(x)$ is continuous at $x = -1$, $(gof)(x) = x + 1$ for $-1 \le x \le 1$ & $3 - x$ for $1 < x \le 3$. $(gof)(x)$ is not differentiable at $x = 1$

9. **(A)** $f'(0) = \dfrac{h^m \sin\left(\frac{1}{h}\right) - 0}{h} = h^{m-1} \sin\left(\frac{1}{h}\right)$

$\Rightarrow$ $m - 1 > 0$ for derivable

$f'(x) = mx^{m-1} \sin\left(\frac{1}{x}\right) - x^{m-2} \cos\left(\frac{1}{x}\right)$

$f'(x)$ to be discontinuous at $x = 0$, $m \in (1, 2]$

(B) Clearly for $f(x)$ to be derivable, & its derivative continuous at $x = 0$, $m \in (2, \infty)$

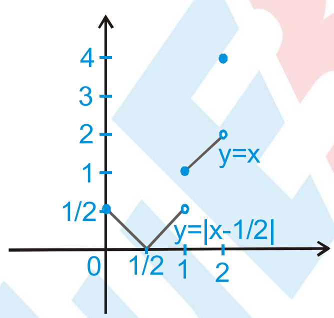
10. $f(x)$ is continuous but not differentiable at $x = n\pi$, $n \in I$, $f(x)$ is not periodic.

11. f is discontinuous at $x = 2$, f is not differentiable at $x = 1, 3/2, 2$

12. $2x\,e^x$

13. f is continuous but not derivable at $x = 1/2$, f is neither differentiable nor continuous at $x = 1$ & $x = 2$

$y = f(x)$



14. $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} \Rightarrow 1 = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$

$\therefore \lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1$; $\lim\limits_{x\to 0} \dfrac{f\left(\frac{x}{2}\right)}{\frac{x}{2} \times 2} = \dfrac{1}{2}$

and similarly so on.

On substituting value we get required result.

15. $f(x + y^n) = f(x) + (f(y))^n$

$f(0+0) = f(0) + (f(0))^n \Rightarrow f(0) = 0$

also $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$

Let $I = f'(0) = \lim\limits_{h\to 0} \dfrac{f(0 + (h^{1/n})^n) - f(0)}{(h^{1/n})^n}$

$= \lim\limits_{h\to 0} \dfrac{f((h^{1/n}))^n}{(h^{1/n})^n} = \lim\limits_{h\to 0} \left(\dfrac{f(h^{1/n})}{h^{1/n}}\right)^n = I^n$

$\Rightarrow$ $I = I^n$ or $I = 0, 1, -1$

Since $f'(0) \ge 0$ & $f(x)$ is not identically zero

So $I = 1$ $\therefore$ $f'(0) = 1$ .....(i)

Thus $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(x + (h^{1/n})^n) - f(x)}{(h^{1/n})^n}$

$= \lim\limits_{h\to 0} \dfrac{f(x) + (f(h^{1/n}))^n - f(x)}{(h^{1/n})^n}$

$= \lim\limits_{h\to 0} \left(\dfrac{f(h^{1/n})}{h^{1/n}}\right)^n = (f'(0))^n$

$\Rightarrow$ $f'(x) = 1$ (using **(i)**)

Integrating both side

$f(x) = x + c$

$f(x) = x$ $[f(0) = 0]$

$f(10) = 10$

16. $y = f(x) = x\sin 1/x.\ \sin \dfrac{1}{x\sin 1/x}$

when $x \ne 0$, $\dfrac{1}{r\pi}$, $r = 1,2,3$

$y = 0$, $x = 0$, $\dfrac{1}{r\pi}$ where $r = 1,2,3,$...............

Let $t = x\sin 1/x$ as $x \to 0^+$, $t \to 0$

and as $x \to \dfrac{1}{r\pi}$, $t \to 0$

$y = t\sin 1/t$

$\lim\limits_{x\to 0} y = \lim\limits_{t\to 0} t\sin t = 0 = f(0)$

$$f'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \le \lim_{h\to 0} \frac{\ln\left(\frac{x-h}{x}\right) + x - h - x}{-h}$$

$$\le \lim_{h\to 0} \frac{\ln\left(1 - \frac{h}{x}\right)}{-h} + 1 \le \frac{1}{x} + 1 \qquad \text{.....(ii)}$$

from **(i)** and **(ii)**

$$\Rightarrow \quad f'(x) = \frac{1}{x} + 1$$

$$\therefore \sum_{n=1}^{100} g\left(\frac{1}{n}\right) = g\left(\frac{1}{1}\right) + g\left(\frac{1}{2}\right) + ... + g\left(\frac{1}{100}\right)$$

$$= (1 + 2 + 3 + ... 100) + 100 = 5150$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. $f(x+y) = f(x) \,.\, f(y) \quad \forall\, x, y$

$\therefore \quad f(5+0) = f(5) \,.\, f(0) \quad \{\because f(5) = 2\}$

$\therefore \quad f(0) = 1$

Now $f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$

$= \lim_{h\to 0} \frac{f(5)f(h) - f(5)}{h}$

$= f(5) \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

$= f(5)\, f'(0) = 2 \times 3 \Rightarrow 6$

4. Apply L Hospital rule

$$\lim_{h\to 0} \frac{f'(1+h)}{1} = 5 \quad \Rightarrow \quad f'(1) = 5$$

5. $|f(x) - f(y)| \le |x - y|^2$

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \le |x - y|$$

$$\Rightarrow \lim_{x\to y} \left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{x\to y} |x - y|$$

$\Rightarrow \quad f'(x) \le 0 \quad \Rightarrow \quad f'(x) = 0$

$\Rightarrow \quad f(x)$ is continuous function

$\therefore \quad f(1) = 0 = f(0)$

6. $f(x) = \frac{x}{1+|x|}$ is differentiable

$$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ 0, & x = 0 \\ \frac{x}{1+x}, & x > 0 \end{cases}$$

L.H.D. $= \lim_{h\to 0} \frac{f(0-h) - f(0)}{h}$

$$\text{L.H.D.} = \frac{\frac{-h}{1+h} - 0}{-h} = 1$$

5. Given that

$$f(x)=\begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x\ge 0 \end{cases} = \begin{cases} x+a & \text{if } x<0 \\ 1-x & \text{if } 0\le x<1 \\ x-1 & \text{if } x\ge 1 \end{cases}$$

and $g(x)=\begin{cases} (x+1) & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\ge 0 \end{cases}$

where $a, b \ge 0$

Then $(gof)(x)=g[f(x)]$

$$=\begin{cases} f(x)+1 & \text{if } f(x)<0 \\ [f(x)-1]^2+b & \text{if } f(x)\ge 0 \end{cases}$$

(Using definition of $g(x)$)

Now, $f(x)<0$ when $x+a<0$ i.e. $x<-a$

$f(x)=0$ when $x=-a$ or $x=1$

$f(x)>0$ when $-a<x<1$ or $x>1$

$$g(f(x))=\begin{cases} f(x)+1 & \text{if } x<-a \\ [f(x)-1]^2+b & \text{if } x=-a \text{ or } x=1 \\ [f(x)-1]^2+b & \text{if } -a<x<0 \\ [f(x)-1]^2+b & \text{if } 0\le x<1 \\ [f(x)-1]^2+b & \text{if } x>1 \end{cases}$$

[Keeping in mind that $x=0$ and 1 are also the breaking pt's because of definition of $f(x)$]

$$\therefore g[f(x)]=\begin{cases} x+a+1 & \text{if } x<-a \\ (x+a-1)^2+b & \text{if } -a\le x<0 \\ ((1-x)-1)^2+b & \text{if } 0\le x\le 1 \\ (x-1-1)^2+b & x>1 \end{cases}$$

(Substituting the value of $f(x)$ under different conditions)

$$\therefore g[f(x)]=\begin{cases} x+a+1 & \text{if } x<-a \\ (x+a-1)^2+b & \text{if } -a\le x<0 = F(x)(\text{say}) \\ x^2+b & \text{if } 0\le x\le 1 \\ (x-2)^2+b & \text{if } x>1 \end{cases}$$

Now given that $gof(x)\equiv F(x)$ is continuous for all real numbers, therefore it will be continuous at $-a$.

$\Rightarrow$ L.H.L. = R.H.L. = $f(-a)$

$$\Rightarrow \lim_{h\to 0} F(-a-h)=\lim_{h\to 0} F(-a+h)=F(-a)$$

Now, $\lim_{h\to 0} F(-a-h)=\lim_{h\to 0} a-h+a+1=1$

$$\lim_{h\to 0} F(-a+h)=\lim_{h\to 0}(-a+h+a-1)^2+b=1+b$$

$F(-a)=1+b$

Thus we should have $1=1+b \Rightarrow b=0$

Again for continuity at $x=0$

L.H.L. $=f(0)$

$$\Rightarrow \lim_{h\to 0} f(0-h)=f(0)$$

$$\Rightarrow \lim_{h\to 0}(-h+a-1)^2+b=b$$

$\Rightarrow (a-1)^2=0 \Rightarrow a=1$

For $a=1$ and $b=0$, gof becomes

$$gof(x)=\begin{cases} x+2, & x<-1 \\ x^2, & -1\le x\le 1 \\ (x-2)^2 & x>1 \end{cases}$$

Now to check differentiability of $gof(x)$ at $x=0$

We see $gof(x)=x^2=F(x)$

$\Rightarrow F'(x)=2x$ which exists clearly at $x=0$

Hence gof is differentiable at $x=0$.

6. Given that $f:[-2a, 2a]\to R$

f is an odd function.

Lf' at $x=a$ is 0.

$$\Rightarrow \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}=0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a-h)-f(a)}{h}=0 \quad \text{....(1)}$$

To find Lf' at $x=-a$ which is given by

$$\lim_{h\to 0}\frac{f(-a-h)-f(-a)}{-h}=\lim_{h\to 0}\frac{-f(a+h)+f(a)}{-h}$$

[$\because$ $f(-x)=-f(x)$]

$$= -4\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2+h}\right).\left(-\frac{\pi}{(2+h)^2}\right)}{1} = \pi$$

LHD : $\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0}\frac{(2-h)^2\left(-\cos\left(\frac{\pi}{2-h}\right)\right)}{-h}$$

$$= \lim_{h\to 0}\frac{4\left(\sin\frac{\pi}{(2-h)}\right)\left(\frac{\pi}{(2-h)^2}\right)}{-1} = -\pi.$$

LHD $\neq$ RHD at x = 2

$\therefore$ Not differentiable at x = 2.

19. $f(x) = a\cos(|x^3-x|) + b|x|\sin(|x^3+x|)$

(A) If a = 0, b = 1, $\quad f(x) = |x|\sin(|x^3+x|)$

$\Rightarrow f(x) = x\sin(x^3+x) \quad \forall x \in R$

Hence f(x) is differentiable.

(B, C) If a = 1, b = 0, $\quad f(x) = \cos(|x^3-x|)$

$f(x) = \cos(x^3-x)$

Which is differentiable at x = 1 and x = 0.

(D) If a = 1, b = 1 $f(x) = \cos(x^3-x) + |x|\sin(|x^3+x|)$

$= \cos(x^3-x) + x\sin(x^3+x)$

Which is differentiable at x = 1

20. if $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$

if $\lim_{x\to 2}\frac{f'(x)g(x)+g'(x)f(x)}{f''(x)g'(x)+f'(x)g''(x)} = 1$

As Limit 1 $\Rightarrow \frac{f'(2)g(2)+g'(2)f(2)}{f''(2)g'(2)+f'(x)g''(2)} = 1$

$\Rightarrow \frac{g'(2).f(2)}{f''(2)g'(2)} = 1 \quad \Rightarrow \quad f''(2) = f(2)$

Hence option (D)

As $f''(2) = f(2)$ and range of $f(x) \in (0,\infty)$

$\Rightarrow f''(2) > 0$

$\Rightarrow$ f has local min. at x = 2

Hence (A)

21. $f(x) = [x^2-3] = [x^2] - 3$

f(x) is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$g(x) = (|x| + |4x-7|)([x^2]-3)$

$$g(x) = \begin{cases} 15x-21 & x < 0 \\ 9x-21 & 0 \le x \le 1 \\ 6x-14 & 1 \le x < \sqrt{2} \\ 3x-7 & \sqrt{2} \le x < \sqrt{3} \\ 0 & \sqrt{3} \le x < 2 \\ 3 & x = 2 \end{cases}$$

$\therefore$ g(x) is not differentiable,

at $x = 0, 1, \sqrt{2}, \sqrt{3}$

Therefore,

$$f(x)=\begin{cases}(x^2-1)(x-1)(x-2)+\cos x, & \text{if } -\infty<x<1\\ -(x^2-1)(x-1)(x-2)+\cos x, & \text{if } 1\le x<2\\ (x^2-1)\,(x-1)(x-2)+\cos x, & \text{if } 2\le x<\infty\end{cases}$$

Now, $x = 1, 2$ are critical point for differentiability Because $f(x)$ is differentiable on other points in its domain

Differentiability at $x = 1$

$$L\,f'(1)=\lim_{x\to 1-0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1-0}\left[(x^2-1)(x-2)+\frac{\cos x-\cos 1}{x-1}\right]$$

$= 0 - \sin 1 = -\sin 1$

$$(\therefore \lim_{x\to 1-0}\frac{\cos x-\cos 1}{x-1}=\frac{d}{dx}(\cos x)$$

at $x = 1 - 0$

$= -\sin x$ at $x = 1 - 0 = -\sin x$ at $x = 1$ $= -\sin 1$

$$\text{and } R\,f'(1)=\lim_{x\to 1+0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1+0}\left[-(x^2-1)(x-2)+\frac{\cos x-\cos 1}{x-1}\right]$$

$= 0 - \sin 1 = -\sin 1$ (same approach)

$\Rightarrow$ $Lf'(1) = Rf'(1)$.

Therefore, function is differentiable at $x = 1$.

$$\text{Again } Lf'(2)=\lim_{x\to 2-0}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to 2-0}\left[-(x^2-1)(x-1)+\frac{\cos x-\cos 2}{x-2}\right]$$

$= -(4-1)(2-1) - \sin 2 = -3 - \sin 2$

$$\text{and } R\,f'(2)=\lim_{x\to 2+0}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to 2+0}\left[(x^2-1)(x-1)+\frac{\cos x-\cos 2}{x-2}\right]$$

$= (2^2 - 1) - \sin 2 = 3 - \sin 2$

So $L\,f'(2) \ne R\,f'(2)$, f is not differentiable at $x = 2$

Therefore, (d) is the answer.

9. (B)

S_1 : $\lim_{x\to a} f(x)=f(a)$

$\therefore$ $\lim_{x\to a}[f(x)]$ is an integer and

$\lim_{x\to a}[f(x)]=\lim_{x\to a}f(x)=f(a)$

$\therefore$ S_1 is true

S_2 : Derivative of $\cos|x|$ at $x = 0$ is 0 but derivative of $|x|$ does not exist at $x = 0$

$\therefore$ S_2 is false

S_3 : False : Consider the function $x^{1/3}$

S_4 : Let $f(x)=\begin{cases}1 & , & x\in Q\\ -1 & , & x\in R\sim Q\end{cases}$ and

$$g(x)=\begin{cases}0 & , & x\in Q\\ 1 & , & x\in R\sim Q\end{cases}$$

then $gof(x) = 0$, $\forall$ $x \in R$

$\therefore$ S_4 is true

10. (A)

S_1 : $\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$ exists finitely

$$\therefore \lim_{h\to 0^+} f(a+h)-f(a)=\lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)h=0$$

$\Rightarrow$ $\lim_{h\to 0^+} f(a+h)=f(a)$ Similarly $\lim_{h\to 0^-} f(a+h)=f(a)$

$\therefore$ f is continuous at $x = a$

S_2 : Function is not differentaible at $5x=(2n+1)\frac{\pi}{2}$ only, which are not in its domain.

S_3 : Let $f(x)=\frac{1}{x^2}$ & $g(x)=-\frac{1}{x^2}$, $\lim_{x\to 0}(f(x)+g(x))$ exists whenever $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

S_4 : Not necessary.

17. (A)

$f(x) = |x| \sin x$

L.H.D $= \lim\limits_{h \to 0} \dfrac{|0-h|\sin(0-h) - 0}{h}$

$= \lim\limits_{h \to 0} \dfrac{-h \sin h}{h} = 0$

R.H.D $= \lim\limits_{h \to 0} \dfrac{|0+h|\sin(0+h) - 0}{h}$

$f(x)$ is differentiable at $x = 0$

18. (A)

$f(2) = 4$

$f(2^-) = \lim\limits_{x \to 2^-} |[x]\,x| = 2$

Discontinuous $\Rightarrow$ Non. differentiable

19. (A)

Statement-I $f(x) = \begin{cases} x^2 - 5x + 6 & , \quad x \le 2 \\ -x^2 + 5x - 6 & , \quad 2 \le x \le 3 \\ x^2 - 5x + 6 & , \quad x \ge 3 \end{cases}$

$f'(x) = \begin{cases} 2x - 5 & , \quad x < 2 \\ -2x + 5 & , \quad 2 < x < 3 \\ 2x - 5 & , \quad x > 3 \end{cases}$

$f'(2^-) + f'(2^+) = -1 + 1 = 0$

Statement-II $f(x) = \begin{cases} (x-a)(x-b) & , \quad x < a \\ -(x-a)(x-a) & , \quad a \le x \le b \\ (x-a)(x-b) & , \quad x > b \end{cases}$

$f'(x) = \begin{cases} 2x - a - b & , \quad x < a \\ -2x + a + b & , \quad a < x < b \\ 2x - a - b & , \quad x > b \end{cases}$

$\therefore \quad f'(a-) = a - b,\ f'(a+) = -a + b$

$\therefore \quad f'(a-) + f'(a+) = 0$

statement-2 explains statement–1.

20. (D)

Statement-I : $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$,

Since $\lim\limits_{x \to 0} f(x) = 0$, therefore, $f(x)$ is continuous

$f'(0) = \lim\limits_{x \to 0} \dfrac{x^2 \sin \frac{1}{x} - 0}{x} = 0$

$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$,

which is clearly not continuous at $x = 0$.

$\therefore$ statement is false

Statement-II : is true (standard result)

21. (A) $\to$ (q), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (p)

(A) $f(x) = \begin{cases} 0 & , \quad 1 < x \le 2 \\ 1 - x & , \quad 0 \le x < 1 \\ -\sin \pi x & , \quad -1 \le x < 0 \end{cases}$

continuous at $x = 1$ but not differentiable

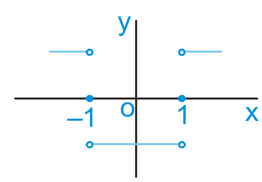
(B) $f'(0^-) = \lim\limits_{h \to 0^-} \dfrac{h^2 e^{-1/h} - 0}{-h} = \lim\limits_{h \to 0^-} (-he^{-1/h}) = 0$

(C) $g(x) = \dfrac{1}{1 + \frac{1}{x}(2 + 2x)} = \dfrac{x}{3x + 2}$

thus the points where $g(x)$ is not differentiable are

$x = 0, -1, -\dfrac{2}{3}$

(D) vertical tangents exist at $x = 1$ and $x = -1$ else where horizontal



tangents exist.

ॐ number of points where tangent does not exist is 0

$$= \lim_{h \to 0^+} \frac{(a-1)(3+h)+2a-3-8b}{h}$$

Since f is differentiable at x = 3

$\therefore \quad \lim_{h \to 0^+} (a-1)(3+h)+2a-3-8b = 0$

i.e. $5a - 8b - 6 = 0$

$\therefore \quad f'(3^+) = a - 1$

thus $\quad a - 1 = 2b \qquad$ ....**(ii)**

from (i) and (ii), we get $a = 2, b = \frac{1}{2}$

25.

1. (A)

$$\text{L.H.D.} = \lim_{h\to 0^-} \frac{f(-a+h)-f(-a)}{h}$$

$$= \lim_{h\to 0^-} \frac{-f(a-h)+f(a)}{h} = \lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h}$$

2. (A)

If f is even, then $f'(-x) = -f'(x)$

$$\therefore \quad f'(a^+) = \lim_{h\to 0^-} \frac{f'(a-h)-f'(a)}{-h}$$

$$= \lim_{h\to 0^-} \frac{f'(a)-f'(a-h)}{h} = \lim_{h\to 0^-} \frac{f'(a)+f'(h-a)}{h}$$

3. (B)

$$\lim_{h\to 0} \frac{f(-x)-f(-x-h)}{h} = f'(-x) \quad \text{and}$$

$$\lim_{h\to 0} \frac{f(x)-f(x-h)}{-h} = -f'(x)$$

$\because \quad f'(-x) = -f'(x) \quad \Rightarrow \quad f'(x)$ is an odd function

$\Rightarrow \quad$ f is an even function

26. (180)

$$f(x) = \begin{cases} ax^3 + b & , \quad 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x & , \quad 1 < x \le 2 \end{cases}$$

as $\tan [x^2] = \tan n, n \in I$

$$f'(x) = \begin{cases} 3ax^2 & , \quad 0 < x < 1 \\ -2\pi \sin \pi x + \dfrac{1}{1+x^2} & , \quad 1 < x < 2 \end{cases}$$

As the function is differentiable in [0, 2]

$\Rightarrow \quad$ function is differentiable at x = 1

$\therefore \quad f'(1^-) = f'(1^+)$

$\Rightarrow \quad 3a = \frac{1}{2} \qquad \Rightarrow \quad a = \frac{1}{6}$

Function will also be continuous at x = 1

$\therefore \quad \lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \quad a + b = -2 + \frac{\pi}{4}$

$\therefore \quad b = -2 - \frac{1}{6} + \frac{\pi}{4} = \frac{\pi}{4} - \frac{13}{6}$

$\Rightarrow \quad k_1 = 6 \qquad \& \qquad k_2 = 12$

$\Rightarrow \quad k_1^2 + k_2^2 = 180 \qquad$ **Ans.**

27. (1)

$$\lim_{x\to 0^+} \frac{|x|^p \sin\frac{1}{x} + x\,|\tan x|^q - 0}{x}$$

$$= \lim_{x\to 0^+} \left(x^{p-1} \sin\frac{1}{x} + |\tan x|^q \right) = 0$$

If $\quad p - 1 > 0 \quad$ and $\quad q > 0 \qquad$ .....**(i)**

$$\lim_{x\to 0^-} \frac{|x|^p \sin\frac{1}{x} + x\,|\tan x|^q - 0}{x}$$

$$= \lim_{x\to 0^-} \left((-1)^p x^{p-1} \sin\frac{1}{x} + |\tan x|^q \right) = 0$$

if $\quad p - 1 > 0 \quad$ and $\quad q > 0 \qquad$ .....**(ii)**

$\therefore \quad$ f(x) is differentiable if $p > 1$ and $q > 0$

i.e. $\quad p + q > 1$

$\therefore \quad$ least possible value of $[p + q] = 1$

28. Given $f(x + y^3) = f(x) + [f(y)]^3 \quad$ and $\quad f'(0) \ge 0$

putting $\quad x = y = 0$, we get

$f(0) = f(0) + (f(0))^3 \qquad \Rightarrow \quad f(0) = 0$

also $f'(0) = \operatorname*{limit}_{h\to 0} \frac{f(0+h)-f(0)}{h} = \operatorname*{limit}_{h\to 0} \frac{f(h)}{h}$

Let $L = f'(0) = \operatorname*{limit}_{h\to 0} \frac{f(0+(h^{1/3})^3) - f(0)}{(h^{1/3})^3}$