SOLVED EXAMPLES

Ex. 1 Let $f(x) = \begin{cases} \text{sgn}(x) + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \frac{\pi}{2} \\ \cos x; & \frac{\pi}{2} \le x < \infty \end{cases}$. Discuss the continuity & differentiability at $x = 0$ & $\frac{\pi}{2}$.

Sol. $f(x) = \begin{cases} -1 + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \frac{\pi}{2} \\ \cos x; & \frac{\pi}{2} \le x < \infty \end{cases}$

To check the differentiability at $x = 0$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-1 + 0 - h - (-1)}{-h} = 1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{-1 + \sin h + 1}{h} = 1$$

✈ LHD = RHD

∴ Differentiable at $x = 0$.

⇒ Continuous at $x = 0$.

To check the continuity at $x = \frac{\pi}{2}$

LHL $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (-1 + \sin x) = 0$

RHL $\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \cos x = 0$

✈ $\text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right) = 0$

∴ Continuous at $x = \frac{\pi}{2}$.

To check the differentiability at $x = \frac{\pi}{2}$

$$\text{LHD} = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h \to 0} \frac{-1 + \cos h - 0}{-h} = 0$$

$$\text{RHD} = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0} \frac{-\sin h - 0}{h} = -1$$

✈ LHD ≠ RHD

∴ not differentiable at $x = \frac{\pi}{2}$.

Ex. 2 Discuss the differentiability of $f(x) = \begin{cases} x\sin(\ln x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0.

Sol. For continuity

$f(0^+) = \lim_{h\to 0} h\sin(\ln h^2)$

$= 0 \times$ (any value between –1 and 1) = 0.

$f(0^-) = \lim_{h\to 0} (-h)\sin(\ln h^2)$

$= 0 \times$ (any value between – 1 and 1) = 0

Hence, f(x) is continuous at x = 0.

For differentiability.

$$f'(0^+) = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h\sin(\ln h^2) - 0}{h} = \lim_{h\to 0}\sin(\ln h^2)$$

= any value between – 1 and 1.

Hence, f '(0) does not take any fixed value.

Hence, f(x) is not differentiable at x = 0.

Ex. 3 Determine the values of x for which the following functions fails to be continuous or differentiable

$$f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x), & x > 2 \end{cases}$$, Justify your answer.

Sol. By the given definition it is clear that the function f is continuous and differentiable at all points except possibly at x = 1 and x = 2.

Check the differentiability at x = 1

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

✈ q = p ∴ Differentiable at x = 1. ⇒ Continuous at x = 1.

Check the differentiability at x = 2

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(2-h) - f(2)}{-h} = \lim_{h\to 0}\frac{(1-2+h)(2-2+h)-0}{-h} = 1 = \text{finite}$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h} = \lim_{h\to 0}\frac{(3-2-h)-0}{h} \to \infty \text{ (not finite)}$$

✈ $q \neq p$

∴ not differentiable at x = 2.

Now we have to check the continuity at x = 2

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0}(3-(2+h)) = 1$$

$\therefore$ LHL $\neq$ RHL

$\Rightarrow$ not continuous at x = 2.

Ex. 4 Test the continuity and differentiability of the function $f(x) = \left|\left(x+\frac{1}{2}\right)[x]\right|$ by drawing the graph of the function when $-2 \le x \le 2$, where $[\cdot]$ represents the greatest integer function.

Sol. Here, $f(x)\ \left|\left(x+\frac{1}{2}\right)[x]\right|, -2 \le x \le 2$ by

$$= \begin{cases} \left|\left(x+\frac{1}{2}\right)(-2)\right|, & -2 \le x < -1 \\ \left|\left(x+\frac{1}{2}\right)(-1)\right|, & -1 \le x < 0 \\ \left|\left(x+\frac{1}{2}\right)(0)\right|, & 0 \le x < 1 \\ \left|\left(x+\frac{1}{2}\right)(1)\right|, & 1 \le x < 2 \\ \left|\frac{3}{2}\times 2\right|, & x = 2 \end{cases}$$

$$= \begin{cases} -(2x+1), & -2 \le x < -1 \\ -\left(x+\frac{1}{2}\right), & -1 \le x < -1/2 \\ (x+1/2), & -\frac{1}{2} \le x < 0 \\ 0, & 0 \le x < 1 \\ x+\frac{1}{2}, & x = 2 \\ 3, & x = 2 \end{cases}$$

Which could be plotted as

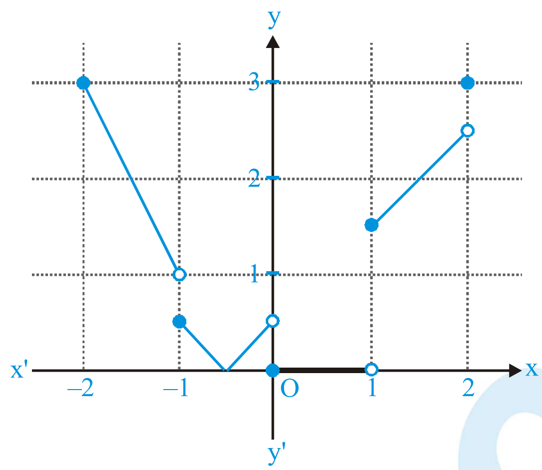


Fig. Clearly shows that f(x) is not continuous at x = {–1, 0, 1, 2} as at these points, the graph is broken. f(x) is not differentiable at $x = \left\{-1, \frac{-1}{2}, 0, 1, 2\right\}$ as at x = {–1, 0, 1, 2}, the graph is broken, and x = –1/2, there is a sharp edge.

Ex. 5 If $f(x) = \begin{cases} |x-1|([x]-x) & , \quad x \neq 1 \\ 0 & , \quad x = 1 \end{cases}$

Test the differentiability at x = 1, where [.] denotes the greatest integer function.

Sol. Check the differentiability at x = 1

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \quad (\because x > 1)$$

$$= \lim_{h\to 0} \frac{|1+h-1|([1+h]-(1+h))-0}{h} = \lim_{h\to 0} \frac{h(1-1-h)}{h} = \lim_{h\to 0} \frac{h(-h)}{h} = 0$$

$$L(f'(1)) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{|1-h-1|([1-h]-(1-h))-0}{-h} = \lim_{h\to 0} \frac{h(0-1+h)}{-h} = 1$$

$Lf'(1) \neq Rf'(1)$

Hence f(x) is not differentiable at x = 1.

Ex. 6 If $f(x) = \begin{cases} e^{-|x|}, & -5 < x < 0 \\ -e^{-|x-1|} + e^{-1} + 1, & 0 \le x < 2 \\ e^{-|x-2|}, & 2 \le x < 4 \end{cases}$

Discuss the continuity and differentiability of f(x) in the interval (–5, 4).

Sol. Check the differentiability at x = 0

$$\text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{e^{-|-h|}-1}{-h} = \lim_{h\to 0}\frac{e^{-h}-1}{-h} = 1$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{-e^{-|h-1|}+e^{-1}+1-1}{h} = \lim_{h\to 0}\frac{e^{-1}(1-e^{h})}{h} = -e^{-1}$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ not differentiable at x = 0.

but f(x) is continous at x = 0, because p ≠ q and both are finite.

check the differentiability at x = 2

$$\text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{e^{-|1-h|}+e^{-1}+1-1}{-h} = \lim_{h\to 0}\frac{e^{-1}(1-e^{h})}{-h} = e^{-1}$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{e^{-|h|}-1}{h} = \lim_{h\to 0}\frac{(e^{-h}-1)}{h} = -1$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ not differentiable at x = 2.

but f(x) is continous at x = 2, because p ≠ q and both are finite.

Ex. 7 Discuss the differentiability of $f(x) = \begin{cases}(x-e)2^{-2^{\left(\frac{1}{e-x}\right)}} & x \neq e \\ 0, & x = e\end{cases}$ at x = e

Sol.
$$f(e^{+}) = \lim_{h\to 0}(e+h-e)\,2^{-2^{\frac{1}{e-(e+h)}}}$$

$$= \lim_{h\to 0}(h)\,2^{-2^{-\frac{1}{h}}}$$

$= 0 \times 1 = 0$ (as for $h \to 0,\ -\frac{1}{h} \to \infty \Rightarrow 2^{-\frac{1}{h}} \to 0$)=

$$f(e^{-}) = \lim_{h\to 0}(-h)\,2^{-2^{\frac{1}{h}}} = 0 \times 0 = 0$$

Hence, f(x) is continuous at x = e.

$$f'(e^{+}) = \lim_{h\to 0}\frac{f(e+h)-f(e)}{h} = \lim_{h\to 0}\frac{h\times 2^{-2^{-\frac{1}{h}}}-0}{h}$$

$$= \lim_{h\to 0} 2^{-2^{\frac{1}{h}}} = 1$$

$$f'(e^{-}) = \lim_{h\to 0}\frac{f(e-h)-f(0)}{-h} = \lim_{h\to 0}\frac{(-h)2^{-2^{\frac{1}{h}}}-0}{-h}$$

$$= \lim_{h\to 0} 2^{-2^{\frac{1}{h}}} = 0$$

Hence, f(x) is non-differentiable at x = e .

Ex. 8 Let f(x + y) = f(x) + f(y) – 2xy – 1 for all x and y. If f'(0) exists and f'(0) = –sinα, then find f{f'(0)}.

Sol. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\{f(x)+f(h)-2xh-1\}-f(x)}{h}$ (Using the given relation)

$= \lim_{h\to 0} -2x + \lim_{h\to 0} \frac{f(h)-1}{h} = -2x + \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

[Putting x = 0 = y in the given relation we find f(0) = f(0) + f(0) + 0 – 1 ⇒ f(0) = 1]

∴ $f'(x) = -2x + f'(0) = -2x - \sin\alpha$

⇒ $f(x) = -x^2 - (\sin\alpha).x + c$

$f(0) = -0 - 0 + c \Rightarrow c = 1$

∴ $f(x) = -x^2 - (\sin\alpha).x + 1$

So, $f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1$

∴ $f\{f'(0)\} = 1$.

Ex. 9 If f(x) is a function satisfies the relation for all x, y ∈ R, f(x + y) = f(x) + f(y) and if f′(0) = 2 and function is differentiable every where, then find f(x).

Sol. $f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0^+} \frac{f(x)+f(h)-f(x)-f(0)}{h}$ (✈ f(0) = 0)

$= \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = f'(0)$

$f'(x) = 2$

⇒ $\int f'(x)\,dx = \int 2\,dx$

$f(x) = 2x + c$

✈ $f(0) = 2.0 + c$

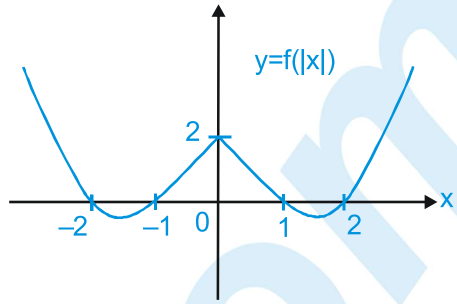
as $f(0) = 0$

∴ $c = 0$

∴ $f(x) = 2x$

Ex. 10 If $f(x) = \begin{cases} x-3 & x < 0 \\ x^2 - 3x + 2 & x \geq 0 \end{cases}$. Draw the graph of the function & discuss the continuity and differentiability of $f(|x|)$ and $|f(x)|$.

Sol. $f(|x|)=\begin{cases}|x|-3; & |x|<0 \rightarrow \text{not possible}\\ |x|^2-3|x|+2; & |x|\geq 0\end{cases}$



$$f(|x|)=\begin{cases}x^2+3x+2, & x<0\\ x^2-3x+2, & x\geq 0\end{cases}$$

at $x = 0$

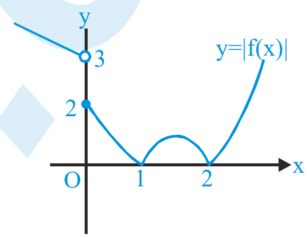
$$q = \text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{-h} = 3$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

✈ $q \neq p$

$\therefore$ not differentiable at $x = 0$. but p & q are both are finite

$\Rightarrow$ continuous at $x = 0$



$$\text{Now, } |f(x)| = \begin{cases}3-x & , \quad x<0\\ (x^2-3x+2) & , \quad 0\leq x<1\\ -(x^2-3x+2) & , \quad 1\leq x\leq 2\\ (x^2-3x+2) & , \quad x>2\end{cases}$$

To check differentiability at $x = 0$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{3+h-2}{-h} = \lim_{h\to 0}\frac{(1+h)}{-h} \rightarrow -\infty$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\Rightarrow$ not differentiable at $x = 0$.

Now to check continuity at $x = 0$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3-x = 3$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-3x+2 = 2$$

$\Rightarrow$ not continuous at $x = 0$.

To check differentiability at $x = 1$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^2-3(1-h)+2-0}{-h} = \lim_{h\to 0}\frac{h^2+h}{-h} = -1$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim_{h\to 0}\frac{-(h^2-h)}{h} = 1$$

$\Rightarrow$ not differentiable at $x = 1$.

but $|f(x)|$ is continous at $x = 1$, because $p \neq q$ and both are finite.

To check differentiability at $x = 2$

$$q = \text{LHD} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{-(4 + h^2 - 4h - 6 + 3h + 2) - 0}{-h} = \lim_{h \to 0} \frac{h^2 - h}{h} = -1$$

$$p = \text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(h^2 + 4h + 4 - 6 - 3h + 2) - 0}{h} = \lim_{h \to 0} \frac{(h^2 + h)}{h} = 1$$

$\Rightarrow$ not differentiable at $x = 2$.

but $|f(x)|$ is continous at $x = 2$, because $p \neq q$ and both are finite.

Ex. 11 A function $f(x)$ is such that $f\left(x + \frac{\pi}{2}\right) = \frac{\pi}{2} - |x| \; \forall \; x$. Find $f'\left(\frac{\pi}{2}\right)$, if it exists.

Sol. Given that $f\left(x + \frac{\pi}{2}\right) = \frac{\pi}{2} - |x|$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f\left(\frac{\pi}{2}\right)}{h} = \frac{\frac{\pi}{2} - |h| - \frac{\pi}{2}}{h} = -1$$

and, $$f'\left(\frac{\pi^-}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \frac{\frac{\pi}{2} - |-h| - \frac{\pi}{2}}{-h} = 1$$

Therefore, $f'\left(\frac{\pi}{2}\right)$ does not exist.

Ex. 12 $f(x) = 1 + 4x - x^2, \forall x \in R$ $\quad g(x) = \begin{cases} \max. \{f(t);\, x \leq t \leq (x+1);\; 0 \leq x < 3\} \\ \min. \{(x+3);\, 3 \leq x \leq 5\} \end{cases}$

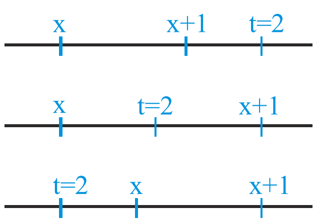
Discuss the continuity and differentiability of $g(x)$ for all $x \in [0, 5]$.

Sol. Here, $f(t) = 1 + 4t - t^2$.

$f'(t) = 4 - 2t$, when $f'(t) = 0 \Rightarrow t = 2$

at $t = 2$, $f(x)$ has a maxima.

Since, $g(x)$ max. $\{f(t)$ for $t \in [x, x+1], 0 \leq x < 3\}$



$$\therefore \quad g(x) = \begin{cases} f(x+1), & \text{if } t = 2 \text{ is on right side of } [x, x+1] \\ f(2), & \text{if } t = 2 \text{ is inside } [x, x+1] \\ f(x), & \text{if } t = 2 \text{ is on left side of } [x, x+1] \end{cases}$$

$$\therefore \quad g(x) = \begin{cases} 4+2x-x^2, & \text{if } 0 \le x < 1 \\ 5, & \text{if } 1 \le x \le 2 \\ 1+4x-x^2, & \text{if } 2 < x < 3 \\ 6, & \text{if } 3 \le x \le 5 \end{cases}$$

Which is clearly continuous for all $x \in [0, 5]$ except $x = 3$.

to check differentiability at $x = 1, 2, 3$

at $x = 1$

$$\text{LHD} = f'(1^-) = \lim_{h\to 0} \frac{f(-h) - f(1)}{-h} = \lim_{h\to 0} \frac{-(1-h)^2 + 2(1-h) + 4 - 5}{-h} = 0$$

$$\text{RHD} = f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{5-5}{h} = 0$$

$\therefore$ differentiable at $x = 1$

at $x = 2$

$$\text{LHD} = f'(2^-) = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h\to 0} \frac{5-5}{-h} = 0$$

$$\text{RHD} = f'(2^+) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{1 + 4(2+h) - (2+h)^2 - 5}{h} = \lim_{h\to 0} \frac{-h^2}{h} = 0$$

$\therefore$ differentiable at $x = 2$

Function g(x) is discontinous at x = 3 $\Rightarrow$ not differentiable at x= 3.

Ex. 13 $f(x + y) = f(x) . f(y)\ \forall\ x, y \in R$ and f(x) is a differentiable function and $f'(0) = 1$, $f(x) \neq 0$ for any x. Find f(x)

Sol. f(x) is a differentiable function

$$\therefore \quad f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(x).f(h) - f(x).f(0)}{h} \qquad (\because f(0) = 1)$$

$$= \lim_{h\to 0^+} \frac{f(x).(f(h) - f(0))}{h} = f(x)\ .\ f'(0) = f(x)$$

$\therefore$ $f'(x) = f(x)$

$\therefore$ $\int \frac{f'(x)}{f(x)} dx = \int 1\, dx$

$\Rightarrow$ $\ln f(x) = x + c$ $\qquad \therefore$ $\ln 1 = 0 + c$

$\Rightarrow$ $c = 0$ $\qquad \therefore$ $\ln f(x) = x$

$\Rightarrow$ $f(x) = e^x$

Ex. 14 Discuss the differentiability of $f(x) = \sin^{-1}\dfrac{2x}{1+x^2}$.

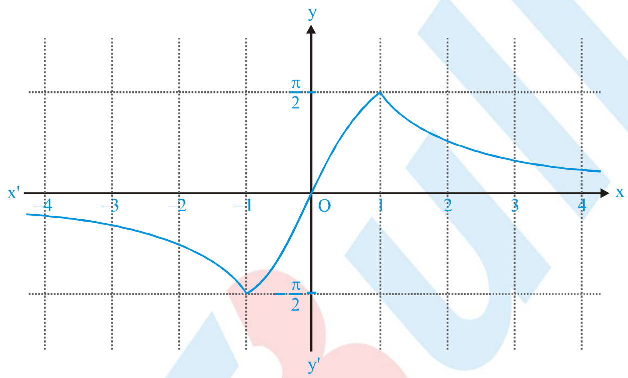
Sol. $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & x > 1 \\ -\pi - 2\tan^{-1}x, & x < -1 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} \dfrac{2}{1+x^2}, & -1 < x < 1 \\ -\dfrac{2}{1+x^2}, & x > 1 \\ -\dfrac{2}{1+x^2}, & x < -1 \end{cases}$ .....**(i)**

$\therefore \quad f'(-1^-) = -1, f'(-1^+) = 1, f'(1^-) = 1$, and $f'(1^+) = -1$

Hence, $f(x)$ is not-differentiable at $x = \pm 1$.

Figure shows that the graph of $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.



Usually it is difficult to remember all the cases of $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ in (i).

Use the following short-cut method to check the differentiability.

Differentiating f(x) w.r.t. x, we get

$$\frac{df(x)}{dx} = \frac{d\left(\dfrac{2x}{1+x^2}\right)}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} = \frac{\dfrac{2(1+x^2)-2x(2x)}{(1+x^2)}}{\sqrt{(1+x^2)^2-4x^2}}$$

$$= \frac{2(1-x^2)}{(1+x^2)|1-x^2|}$$

Clearly, $\dfrac{df(x)}{dx}$ is discontinuous at $x^2 = 1$ or $x = \pm 1$.

Hence, $f(x)$ is non-differentiable at $x = \pm 1$.

Ex. 15 Discuss the continuity and differentiability of the function $y = f(x)$ defined parametrically ; $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Sol. Here $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Now when $t < 0$;

$x = 2t - \{-(t-1)\} = 3t - 1$ and $y = 2t^2 - t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2$

when $0 \le t < 1$

$x = 2t - (-(t-1)) = 3t - 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$

When $t \ge 1$;

$x = 2t - (t-1) = t + 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$

Thus, $$y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

We have to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

$$\text{LHD} = f'(-1^-) = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h \to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = f'(-1^+) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{-h} = 0$$

Hence f(x) is differentiable at $x = -1$.

$\Rightarrow$ continuous at $x = -1$.

To check differentiability at $x = 2$;

$$\text{LHD} = f'(2^-) = \lim_{h \to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2 \quad \& \quad \text{RHD} = f'(2^+) = \lim_{h \to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

Hence f(x) is not differentiable at $x = 2$.

But continuous at $x = 2$, because LHD & RHD both are finite.

$\therefore$ f(x) is continuous for all x and differentiable for all x, except $x = 2$.

Exercise # 1

[Single Correct Choice Type Questions]

1. If both f (x) & g(x) are differentiable functions at $x = x_0$, then the function defined as, $h(x) = \text{Maximum}\{f(x), g(x)\}$

(A) is always differentiable at $x = x_0$

(B) is never differentiable at $x = x_0$

(C) is differentiable at $x = x_0$ when $f(x_0) \neq g(x_0)$

(D) cannot be differentiable at $x = x_0$ if $f(x_0) = g(x_0)$.

2. If $f(x) = \dfrac{x}{\sqrt{x+1} - \sqrt{x}}$ be a real valued function, then

(A) f(x) is continuous, but f′(0) does not exist

(B) f(x) is differentiable at x = 0

(C) f(x) is not continuous at x = 0

(D) f(x) is not differentiable at x = 0

3. For $x > 0$, let $h(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ where $p \& q > 0$ are relatively prime integers

then which one does not hold good?

(A) h(x) is discontinuous for all x in $(0, \infty)$

(B) h(x) is continuous for each irrational in $(0, \infty)$

(C) h(x) is discontinuous for each rational in $(0, \infty)$

(D) h(x) is not derivable for all x in $(0, \infty)$.

4. If f (x) is differentiable everywhere, then:

(A) $|f|$ is differentiable everywhere

(B) $|f|^2$ is differentiable everywhere

(C) $f\,|f|$ is not differentiable at some point

(D) $f + |f|$ is differentiable everywhere

5. Let $f : R \to R$ be a function such that $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$, then

(A) $\dfrac{f(x)}{x}$ is differentiable in R

(B) f(x) is continuous but not differentiable in R

(C) f(x) is continuous in R

(D) f(x) is bounded in R

6. The functions defined by $f(x) = \max\{x^2, (x-1)^2, 2x(1-x)\}$, $0 \le x \le 1$

(A) is differentiable for all x

(B) is differentiable for all x excetp at one point

(C) is differentiable for all x except at two points

(D) is not differentiable at more than two points.

7. If $f(x) = \sin^{-1}(\sin x)$; $x \in R$ then f is

(A) continuous and differentiable for all x

(B) continuous for all x but not differentiable for all $x = (2k+1)\dfrac{\pi}{2}$, $k \in I$

(C) neither continuous nor differentiable for $x = (2k-1)\dfrac{\pi}{2}$; $k \in I$

(D) neither continuous nor differentiable for $x \in R - [-1, 1]$

8. If $f(x)=\begin{cases} x\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$ then at x = 0, f(x) is -

(A) differentiable **(B)** not differentiable **(C)** $f'(0^+)=-1$ **(D)** $f'(0^-)=1$

9. The function $f(x)=\sin^{-1}(\cos x)$ is:

(A) discontinuous at x = 0 **(B)** continuous at x = 0
(C) differentiable at x = 0 **(D)** none of these

10. Let the function *f*, *g* and *h* be defined as follows

$$f(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{for } -1\le x\le 1 \text{ and } x\neq 0 \\ 0 & \text{for } \quad x=0 \end{cases}$$

$$g(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right) & \text{for } -1\le x\le 1 \text{ and } x\neq 0 \\ 0 & \text{for } \quad x=0 \end{cases}$$

$$h(x)=|x|^3 \qquad \text{for } -1\le x\le 1$$

Which of these functions are differentiable at x = 0?

(A) *f* and *g* only **(B)** *f* and *h* only **(C)** *g* and *h* only **(D)** none

11. Given $f(x)=\begin{cases} \log_a\left(a\left|[x]+[-x]\right|\right)^x\left(\dfrac{a^{\frac{2}{\left(\frac{[x]+[-x]}{|x|}\right)}}-5}{3+a^{\frac{1}{|x|}}}\right) & \text{for } |x|\neq 0\ ;\ a>1 \\ 0 & \text{for } x=0 \end{cases}$

where [.] represents the integral part function, then:

(A) f is continuous but not differentiable at x = 0
(B) f is continuous & differentiable at x = 0
(C) the differentiability of 'f' at x = 0 depends on the value of a
(D) f is continuous & differentiable at x = 0 and for a = e only.

12. Consider $f(x)=\left[\dfrac{2\left(\sin x-\sin^3 x\right)+\left|\sin x-\sin^3 x\right|}{2\left(\sin x-\sin^3 x\right)-\left|\sin x-\sin^3 x\right|}\right],\ x\neq\dfrac{\pi}{2}$ for $x\in(0,\pi)$

$f(\pi/2)=3$ where [ ] denotes the greatest integer function then,

(A) f is continuous & differentiable at $x=\pi/2$
(B) f is continuous but not differentiable at $x=\pi/2$
(C) f is neither continuous nor differentiable at $x=\pi/2$
(D) none of these

13. Let $f(x) = x - x^2$ and $g(x) = \begin{cases} \max f(t), & 0 \le t \le x, 0 \le x \le 1 \\ \sin \pi x & , x > 1 \end{cases}$, then in the interval $[0, \infty)$

(A) g(x) is everywhere continuous except at two points
(B) g(x) is everywhere differentiable except at two points
(C) g(x) is everywhere differentiable except at x = 1
(D) none of these

14. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable ?

(A) $f(x) = x^{1/3}$ **(B)** $f(x) = \frac{|x|}{x}$ **(C)** $f(x) = e^{-x}$ **(D)** $f(x) = \tan x$

15. Let $f''(x)$ be continuous at x = 0 and $f''(0) = 4$ then value of $\lim_{x\to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is

(A) 11 **(B)** 2 **(C)** 12 **(D)** none of these

16. The graph of function f contains the point P (1, 2) and Q(s, r). The equation of the secant line through P and Q is $y = \left(\frac{s^2 + 2s - 3}{s - 1}\right)x - 1 - s$. The value of $f'(1)$, is

(A) 2 **(B)** 3 **(C)** 4 **(D)** non existent

17. Given that $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h\to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} =$

(A) does not exist **(B)** is equal to –3/2 **(C)** is equal to 3/2 **(D)** is equal to 3

18. If $f(x + y) = f(x) + f(y) + |x|y + xy^2$, $\forall$ x, y $\in$ R and $f'(0) = 0$, then

(A) f need not be differentiable at every non zero x **(B)** f is differentiable for all x $\in$ R
(C) f is twice differentiable at x = 0 **(D)** none

19. If $f(x) = \begin{cases} \dfrac{x^2 - 1}{x^2 + 1} & , \ 0 < x \le 2 \\ \dfrac{1}{4}\left(x^3 - x^2\right) & , \ 2 < x \le 3 \\ \dfrac{9}{4}\left(|x - 4| + |2 - x|\right) & , \ 3 < x < 4 \end{cases}$, then:

(A) f(x) is differentiable at x = 2 & x = 3 **(B)** f(x) is non-differentiable at x = 2 & x = 3
(C) f(x) is differentiable at x = 3 but not at x = 2 **(D)** f(x) is differentiable at x = 2 but not at x = 3.

20. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval [a, b]
II. f is bounded on the open interval (a, b)
III. If $a<a_1<b_1<b$, and $f(a_1)<0<f(b_1)$, then there is a number c such that $a_1<c<b_1$ and f(C)=0

(A) I and II only **(B)** I and III only **(C)** II and III only **(D)** only III

21. If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$, then f(x) is -

(A) discontinuous everywhere

(B) continuous as well as differentiable for all x

(C) continuous for all x but not differentiable at x = 0

(D) neither differentiable nor continuous at x = 0

22. Let $f(x) = \max.\left\{\left|x^2-2|x|\right|, |x|\right\}$ and $g(x) = \min.\left\{\left|x^2-2|x|\right|, |x|\right\}$ then

(A) both f (x) and g (x) are non differentiable at 5 points.

(B) f (x) is not differentiable at 5 points whether g (x) is non differentiable at 7 points.

(C) number of points of non differentiblity for f (x) and g (x) are 7 and 5 respectively.

(D) both f (x) and g (x) are non differentiable at 3 and 5 points respectively.

23. Given $f(x)=\begin{cases} x^2 e^{2(x-1)} & \text{for } 0 \le x \le 1 \\ a \operatorname{sgn}(x+1) \cos(2x-2) + bx^2 & \text{for } 1 < x \le 2 \end{cases}$. f(x) is differentiable at x = 1 provided:

(A) a = – 1, b = 2 **(B)** a = 1, b = – 2 **(C)** a = – 3, b = 4 **(D)** a = 3, b = – 4

24. Let $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \ge 1 \end{cases}$.

If g (x) is the continuous and differentiable for all numbers in its domain then

(A) a = b = 4 **(B)** a = b = – 4

(C) a = 4 and b = – 4 **(D)** a = – 4 and b = 4

25. Let f(x) be continuous and differentiable function for all reals.

$f(x+y) = f(x) - 3xy + f(y)$. If $\lim_{h\to 0}\frac{f(h)}{h} = 7$, then the value of $f'(x)$ is

(A) – 3x **(B)** 7 **(C)** – 3x + 7 **(D)** $2f(x) + 7$

26. If a differentiable function f satisfies $f\left(\frac{x+y}{3}\right) = \frac{4-2(f(x)+f(y))}{3}$ ∀ x, y ∈ R, then f(x) is equal to

(A) $\frac{1}{7}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{8}{7}$ **(D)** $\frac{4}{7}$

27. Let [x] be the greatest integer function and $f(x) = \frac{\sin\frac{1}{4}\pi[x]}{[x]}$. Then which one of the following does not hold good?

(A) not continuous at any point **(B)** continuous at 3/2

(C) discontinuous at 2 **(D)** differentiable at 4/3

28. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in Z$, p is a prime number and [x] is greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is

(A) p **(B)** $p - 1$ **(C)** $2p + 1$ **(D)** $2p - 1$

29. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then f(1) equals

(A) 1 **(B)** 2 **(C)** 0 **(D)** –1

30. Number of points where the function $f(x) = (x^2 - 1)|x^2 - x - 2| + \sin(|x|)$ is not differentiable, is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

Exercise # 2 Part # I [Multiple Correct Choice Type Questions]

1. Let $f(x)=\begin{cases}2x+3; & -3\le x<-2\\ x+1; & -2\le x<0\\ x+2; & 0\le x\le 1\end{cases}$. At what points the function is/are not differentiable in the interval [–3, 1]

(A) –2 **(B)** 0 **(C)** 1 **(D)** 1/2

2. The function $1+|\sin x|$ is

(A) continuous everywhere **(B)** differentiable nowhere

(C) not differentiable at $x=0$ **(D)** not differentiable at infinite no. of points

3. Which one of the following statements is not correct ?

(A) The derivative of a differentiable periodic function is a periodic function with the same period.

(B) If f(x) and g(x) both are defined on the entire number line and are aperiodic then the function F(x) =f(x).g(x) can not be periodic.

(C) Derivative of an even differentiable function is an odd function and derivative of an odd differentiable function is an even function.

(D) Every function f(x) can be represented as the sum of an even and an odd function.

4. Given that the derivative $f'(a)$ exists. Indicate which of the following statement(s) is/are always True

(A) $f'(a)=\underset{h\to a}{\text{Limit}}\dfrac{f(h)-f(a)}{h-a}$ **(B)** $f'(a)=\underset{h\to 0}{\text{Limit}}\dfrac{f(a)-f(a-h)}{h}$

(C) $f'(a)=\underset{t\to 0}{\text{Limit}}\dfrac{f(a+2t)-f(a)}{t}$ **(D)** $f'(a)=\underset{t\to 0}{\text{Limit}}\dfrac{f(a+2t)-f(a+t)}{2t}$

5. The function $f(x)=\begin{cases}|x-3| & , x\ge 1\\ \left(\dfrac{x^2}{4}\right)-\left(\dfrac{3x}{2}\right)+\left(\dfrac{13}{4}\right) & , x<1\end{cases}$ is:

(A) continuous at x = 1 **(B)** differentiable at x = 1

(C) continuous at x = 3 **(D)** differentiable at x = 3

6. Let [x] be the greatest integer function $f(x)=\dfrac{\sin\frac{1}{4}\pi[x]}{[x]}$ is -

(A) not continuous at any point **(B)** continuous at $\dfrac{3}{2}$

(C) discontinuous at 2 **(D)** differentiable at $\dfrac{4}{3}$

7. The function $f(x) = \sqrt{1-\sqrt{1-x^2}}$

(A) has its domain $-1 \le x \le 1$.
(B) has finite one sided derivates at the point x = 0.
(C) is continuous and differentiable at x = 0.
(D) is continuous but not differentiable at x = 0.

8. If $f(x) = \cos \pi(|x|+[x])$, then f(x) is/are (where [.] denotes greatest integer function)

(A) continuous at $x = \frac{1}{2}$
(B) continuous at x = 0
(C) differentiable in (2, 4)
(D) differentiable in (0, 1)

9. Consider the function $f(x) = |x^3 + 1|$ then

(A) Domain of f $x \in R$
(B) Range of f is R^+
(C) f has no inverse.
(D) f is continuous and differentiable for every $x \in R$.

10. If $f(x) = |x + 1|(|x| + |x - 1|)$ then at what points the function is/are not differentiable at in the interval [–2, 2]

(A) –1
(B) 0
(C) 1
(D) 1/2

11. The points at which the function, $f(x) = |x - 0.5| + |x - 1| + \tan x$ does not have a derivative in the interval (0, 2) are:

(A) 1
(B) π/2
(C) π/4
(D) 1/2

12. If $f(x) = \begin{cases} \dfrac{x \cdot \ln(\cos x)}{\ln(1+x^2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$ then :

(A) f is continuous at x = 0
(B) f is continuous at x = 0 but not differentiable at x = 0
(C) f is differentiable at x = 0
(D) f is not continuous at x = 0.

13. Let $f(x) = \begin{bmatrix} (x-e)2^{-2^{\frac{1}{e-x}}}, & x \neq e \\ 0, & x = e \end{bmatrix}$, then -

(A) f is continuous and differentiable at x = e
(B) f is continuous but not differentiable at x = e
(C) f is neither continuous nor differentiable at x = e
(D) geometrically f has sharp corner at x = e

14. $f(x) = (\sin^{-1}x)^2 \cdot \cos(1/x)$ if $x \neq 0$; f(0) = 0, f(x) is:

(A) continuous no where in $-1 \le x \le 1$
(B) continuous everywhere in $-1 \le x \le 1$
(C) differentiable no where in $-1 \le x \le 1$
(D) differentiable everywhere in $-1 < x < 1$

15. Let [x] denote the greatest integer less than or equal to x . If $f(x) = [x \sin \pi x]$, then f(x) is:

(A) continuous at x = 0
(B) continuous in (–1, 0)
(C) differentiable at x = 1
(D) differentiable in (–1, 1)

16. Let h(x) = min {x, x^2 }, for every real number of x . Then -

(A) h is continuous for all x **(B)** h is differentiable for all x

(C) h'(x) = 1, for all x > 1 **(D)** h is not differentiable at two values of x .

17. f(x) = 1 + [cosx] x in $0 < x \le \pi/2$, where [ ] denotes greatest integer function then -

(A) it is continuous in $0 < x < \pi/2$ **(B)** it is differentiable in $0 < x < \pi/2$

(C) its maximum value is 2 **(D)** it is not differentiable in $0 < x < \pi/2$

18. If $f(x) = a_0 + \sum_{k=1}^{n} a_k |x|^k$, where a_i's are real constants, then f(x) is

(A) continuous at x = 0 for all a_i **(B)** differentiable at x = 0 for all $a_i \in R$

(C) differentiable at x = 0 for all $a_{2k-1} = 0$ **(D)** none of these

19. Select the correct statements.

(A) The function f defined by $f(x) = \begin{cases} 2x^2+3 & \text{for } x \le 1 \\ 3x+2 & \text{for } x > 1 \end{cases}$ is neither differentiable nor continuous at x = 1.

(B) The function $f(x) = x^2 |x|$ is twice differentiable at x = 0.

(C) If f is continuous at x = 5 and $f(5) = 2$ then $\lim_{x \to 2} f(4x^2 - 11)$ exists.

(D) If $\lim_{x \to a} (f(x) + g(x)) = 2$ and $\lim_{x \to a} (f(x) - g(x)) = 1$ then $\lim_{x \to a} f(x) \cdot g(x)$ need not exist.

20. Let f(x) = cos x & $H(x) = \begin{cases} \text{Min } [f(t)/\ 0 \le t \le x] & \text{for } 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} < x \le 3 \end{cases}$, then -

(A) H(x) is continuous & derivable in [0, 3]

(B) H(x) is continuous but not derivable at x = $\pi/2$

(C) H(x) is neither continuous nor derivable at x = $\pi/2$

(D) Maximum value of H(x) in [0,3] is 1

Part # II [Assertion & Reason Type Questions]

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is **NOT** a correct explanation for Statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. Consider $f(x) = ax^2 + bx + c$, a, b, c ∈ R and a ≠ 0.

Statement - I If $f(x) = 0$ has two distinct positive real roots then number of non-differentiable points of $y = |f(-|x|)|$ is 1.

Statement - II Graph of $y = f(|x|)$ is symmetrical about y-axis.

2. Let h (x) = $f_1(x) + f_2(x) + f_3(x) + ... + f_n(x)$ where $f_1(x), f_2(x), f_3(x), ... , f_n(x)$ are real valued functions of x.

Statement - I $f(x) = |\cos|x|| + \cos^{-1}(\text{sgn } x) + |\ln x|$ is not differentiable at 3 points in $(0, 2\pi)$

Statement - II Exactly one function $f_i(x)$, $i = 1, 2, ..., n$ not differentiable and the rest of the function differentiable at x = a makes h (x) not differentiable at x = a.

3. **Statement - 1** f (x) = | x | sin x is differentiable at x = 0

Statement - II If g (x) is not differentiable at x= a and h (x) is differentiable at x = a then g (x) · h (x) can not be differentiable at x = a.

4. **Statement - I** f (x) = |x| cos x is not differentiable at x = 0

Statement - II Every absolute value functions are not differentiable.

5. **Statement - I** $|x^3|$ is differentiable at x = 0

Statement - II If f(x) is differentiable at x = a then |f(x)| is also differentiable at x = a.

6. **Statement - I** f (x) = Sgn (cos x) is not differentiable at $x = \frac{\pi}{2}$

Statement - II g(x) = [cosx] is not differentiable at $x = \frac{\pi}{2}$

where [.] denotes greatest integer function

7. **Statement - I** f (x) = | cos x | is not deviable at $x = \frac{\pi}{2}$.

Statement - II If g (x) is differentiable at x = a and g (a) = 0 then | g(x) | is non-derivable at x = a.

8. Consider the function $f(x) = \cos(\sin^{-1}x)$, differentiable in (–1, 1).

Statement - I $f(x)$ is bounded in [–1, 1].

Statement - II If a function is differentiable in (a, b), then it is bounded in [a, b].

Exercise # 3 Part # I [Matrix Match Type Questions]

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with one or more statement(s) in **Column-II**.

1. **Column - I** **Column - II**

(A) The number of the values of x in $(0, 2\pi)$, where the function $f(x) = \frac{\tan x + \cot x}{2} - \left|\frac{\tan x - \cot x}{2}\right|$ is continuous but not differentiable is

(B) The number of points where the function $f(x) = \min\{1, 1 + x^3, x^2 - 3x + 3\}$ is non-derivable

(C) The number of points where $f(x) = (x+4)^{1/3}$ is non-differentiable is

(D) Consider $f(x) = \begin{cases} -\frac{\pi}{2}\ln\left(\frac{x.2}{\pi}\right) + \frac{\pi}{2}, & 0 < x \le \frac{\pi}{2} \\ \sin^{-1}\sin x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$. Number of points in $\left(0, \frac{3\pi}{2}\right)$, where $f(x)$ is non-differentiable is

(p) 2

(q) 0

(r) 4

(s) 1

2. Let [.] denotes the greatest integer function.

Column – I

(A) If $P(x) = [2\cos x]$, $x \in [-\pi, \pi]$, then P(x)

(B) If $Q(x) = [2\sin x]$, $x \in [-\pi, \pi]$, then Q(x)

(C) If $R(x) = [2\tan x/2]$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then R(x)

(D) If $S(x) = \left[3\operatorname{cosec}\frac{x}{3}\right]$, $x \in \left[\frac{\pi}{2}, 2\pi\right]$, then S(x)

Column – II

(p) is discontinuous at exactly 7 points

(q) is discontinuous at exactly 4 points

(r) is non differentiable at some points

(s) is continuous at infinitely many values

3. **Column-I** contains 4 functions and **column-II** contains comments w.r.t their continuity and differentiability at $x = 0$. Note that **column-I** may have more than one matching options in **column-II**.

Column-I

(A) $f(x) = [x] + |1 - x|$

[ ] denotes the greatest integer function

(B) $g(x) = |x - 2| + |x|$

(C) $h(x) = [\tan^2 x]$

[ ] denotes the greatest integer function

(D) $l(x) = \begin{cases} \frac{x(3e^{1/x} + 4)}{(2 - e^{1/x})} & x \ne 0 \\ 0 & x = 0 \end{cases}$

Column-II

(p) continuous

(q) differentiable

(r) discontinuous

(s) non derivable

4. **Column – I** | **Column – II**

(A) $f(x) = |x^3|$ is — **(p)** continuous in $(-1, 1)$

(B) $f(x) = \sqrt{|x|}$ is — **(q)** differentiable in $(-1, 1)$

(C) $f(x) = |\sin^{-1} x|$ is — **(r)** differentiable in $(0, 1)$

(D) $f(x) = \cos^{-1} |x|$ is — **(s)** not differentiable atleast at one point in $(-1, 1)$

5. **Column-I** | **Column-II**

(A) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$ at $x = 0$ is — **(p)** continuous

(B) For every $x \in R$ the function — **(q)** differentiability

$g(x) = \dfrac{\sin(\pi[x-\pi])}{1+[x]^2}$ — **(r)** discontinuous

where [x] denotes the greatest integer function is — **(s)** non derivable

(C) $h(x) = \sqrt{\{x\}^2}$ where {x} denotes fractional part function for all $x \in I$, is

(D) $k(x) = \begin{cases} x^{\frac{1}{\ln x}} & \text{if } x \neq 1 \\ e & \text{if } x = 1 \end{cases}$ at $x = 1$ is

6. **Column - I** | **Column - II**

(A) Number of points where the function — **(p)** 0

$$f(x) = \begin{cases} 1 + \left[\cos\dfrac{\pi x}{2}\right] & 1 < x \leq 2 \\ 1 - \{x\} & 0 \leq x < 1 \\ |\sin \pi x| & -1 \leq x < 0 \end{cases}$$ and $f(1) = 0$ is

continuous but non-differentiable — **(q)** 1

where [ ] denote greatest integer and { } denote fractional part function

(B) $f(x) = \begin{cases} x^2 e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f'(0^-)$ is equal to

(C) The number of points at which $g(x) = \dfrac{1}{1+\dfrac{2}{f(x)}}$ — **(r)** 2

is not differentiable where $f(x) = \dfrac{1}{1+\dfrac{1}{x}}$, is

(D) Number of points where tangent does not exist for the curve $y = \text{sgn}(x^2 - 1)$ — **(s)** 3

Part # II [Comprehension Type Questions]

Comprehension # 1

$$f(x)=\begin{cases}2+(x-1)^2 & \text{if } x<1\\ 2 & \text{if } x\in[1,3]\\ 2-(x-3)^2 & \text{if } x>3\end{cases} \qquad g(x)=\begin{cases}2+\sqrt{-x} & \text{if } x<0\\ x+2 & \text{if } x\in[0,\ 4]\\ 3x-6 & \text{if } x\in(4,\ \infty)\end{cases}$$

$$h(x)=\begin{cases}4+ae^x & \text{if } x<0\\ x+2 & \text{if } x\in[0,3]\\ b^2-7b+18-\dfrac{3}{x} & \text{if } x>3\end{cases}$$

$$k(x)=\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}},\ x>0$$

On the basis of above information, answer the following questions :

1. Which of the following is continuous at each point of its domain -

(A) $f(x)$ **(B)** $g(x)$ **(C)** $k(x)$ **(D)** all three f, g, k

2. Value of (a, b) for which h(x) is continuous $\forall x \in R$:

(A) (4, 3) **(B)** (–2, 3) **(C)** (3, 4) **(D)** none of these

3. Which of the following function is not differentiable at exactly two points of its domain -

(A) $f(x)$ **(B)** $g(x)$ **(C)** $k(x)$ **(D)** none of these

Comprehension # 2

$$\text{Let } f(x)=\begin{cases}e^{\{x^2\}}-1, & x>0\\ \dfrac{\sin x-\tan x+\cos x-1}{2x^2+\ln(2+x)+\tan x}, & x<0\\ 0, & x=0\end{cases}$$

where { } represents fractional part function. Suppose lines L_1 and L_2 represent tangent and normal to curve $y = f(x)$ at $x = 0$. Consider the family of circles touching both the lines L_1 and L_2.

1. Ratio of radii of two circles belonging to this family cutting each other orthogonally is

(A) $2+\sqrt{3}$ **(B)** $\sqrt{3}$ **(C)** $2+\sqrt{2}$ **(D)** $2-\sqrt{2}$

2. A circle having radius unity is inscribed in the triangle formed by L_1 and L_2 and a tangent to it. Then the minimum area of the triangle possible is

(A) $3+\sqrt{2}$ **(B)** $2+\sqrt{3}$ **(C)** $3+2\sqrt{2}$ **(D)** $3-2\sqrt{2}$

3. If centers of circles belonging to family having equal radii 'r' are joined, the area of figure formed is

(A) $2r^2$ **(B)** $4r^2$ **(C)** $8r^2$ **(D)** r^2

Comprehension # 3

Let $f : R \to R$ be a function defined as, $f(x) = \begin{cases} 1 - |x| & , \quad |x| \le 1 \\ 0 & , \quad |x| > 1 \end{cases}$ and $g(x) = f(x-1) + f(x+1), \forall x \in R$. Then

1. The value of g(x) is :

(A) $g(x) = \begin{cases} 0 & , \quad x \le -3 \\ 2+x & , \quad -3 \le x \le -1 \\ -x & , \quad -1 < x \le 0 \\ x & , \quad 0 < x \le 1 \\ 2-x & , \quad 1 < x \le 3 \\ 0 & , \quad x > 3 \end{cases}$

(B) $g(x) = \begin{cases} 0 & , \quad x \le -2 \\ 2+x & , \quad -2 \le x \le -1 \\ -x & , \quad -1 < x \le 0 \\ x & , \quad 0 < x \le 1 \\ 2-x & , \quad 1 < x \le 2 \\ 0 & , \quad x > 2 \end{cases}$

(C) $g(x) = \begin{cases} 0 & , \quad x \le 0 \\ 2+x & , \quad 0 < x < 1 \\ -x & , \quad 1 \le x \le 2 \\ x & , \quad 2 < x < 3 \\ 2-x & , \quad 3 \le x < 4 \\ 0 & , \quad 4 \le x \end{cases}$

(D) none of these

2. The function g(x) is continuous for, $x \in$

(A) $R - \{0, 1, 2, 3, 4\}$ **(B)** $R - \{-2, -1, 0, 1, 2\}$ **(C)** R **(D)** none of these

3. The function g(x) is differentiable for, $x \in$

(A) R **(B)** $R - \{-2, -1, 0, 1, 2\}$ **(C)** $R - \{0, 1, 2, 3, 4\}$ **(D)** none of these

Comprehension # 4

Let 'f' be a function that is differentiable every where and that has the following properties :

(i) $f(x) > 0$ (ii) $f'(0) = -1$ (iii) $f(-x) = \dfrac{1}{f(x)}$ & $f(x+h) = f(x).f(h)$

A standard result : $\int \dfrac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

On the basis of above information, answer the following questions :

1. Range of f(x) is-

(A) R **(B)** $R - \{0\}$ **(C)** R^+ **(D)** (0, e)

2. The range of the function $\Delta = f(|x|)$ is -

(A) [0, 1] **(B)** [0, 1) **(C)** (0, 1] **(D)** none of these

3. The function y = f(x) is -

(A) odd **(B)** even **(C)** increasing **(D)** decreasing

4. If h(x) = f'(x). then h(x) is given by -

(A) –f(x) **(B)** $\dfrac{1}{f(x)}$ **(C)** f(x) **(D)** $e^{f(x)}$

Exercise # 4 [Subjective Type Questions]

1. A function f is defined as follows : $f(x)=\begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+\sin x & \text{for } 0 \le x < \dfrac{\pi}{2} \\ 2+\left(x-\dfrac{\pi}{2}\right)^2 & \text{for } \dfrac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at $x = 0$ & $x = \pi/2$.

2. Examine the origin for continuity & derrivability in case of the function f defined by $f(x) = x \tan^{-1}(1/x)$, $x \neq 0$ and $f(0) = 0$.

3. Draw a graph of the function, $y = [x] + |1 - x|$, $-1 \le x \le 3$. Determine the points, if any, where this function is not differentiable, where [·] denotes the greatest integer function.

4. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin|x|$, $x \in R$. Draw a rough sketch of the graph of f(x).

5. If $f(x)=\begin{cases} ax^2-b & \text{if } |x|<1 \\ -\dfrac{1}{|x|} & \text{if } |x|\ge 1 \end{cases}$ is derivable at x =1. Find the values of a & b

6. The function $f(x)=\begin{cases} ax(x-1)+b & \text{when } x<1 \\ x-1 & \text{when } 1\le x\le 3 \\ px^2+qx+2 & \text{when } x>3 \end{cases}$. Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x **(ii)** f'(1) does not exist **(iii)** f'(x) is continuous at x = 3

7. Let a function $f : R \to R$ be given by $f(x + y) = f(x) f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for any $x \in R$. If the function f (x) is differentiable at x = 0, show that $f'(x) = f'(0) f(x)$ for all $x \in R$. Also, determine f(x).

8. If $f(x) = -1 + |x-1|, -1 \le x \le 3$; $g(x) = 2 - |x+1|, -2 \le x \le 2$, then calculate (fog) (x) & (gof)(x). Draw their graph. Discuss the continuity of (fog) (x) at x = –1 & the differentiability of (gof)(x) at x =1.

9. Find the set of values of m for which $f(x)=\begin{cases} x^m \sin\left(\dfrac{1}{x}\right) & x>0 \\ 0 & x=0 \end{cases}$

(A) is derivable but its derivative is discontinuous at x = 0

(B) is derivable and has a continuous derivative at x = 0

10. Discuss the continuity & differentiability of the function $f(x) = |\sin x| + \sin |x|$, $x \in R$. Draw a rough sketch of the graph of f (x) . Also comment on periodicity of function f(x).

11. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{cases}$

where [ ] denote greatest integer function.

12. Given a function 'g' which has a derivative $g'(x)$ for every real 'x' and which satisfy $g'(0) = 2$ and $g(x+y) = e^y \cdot g(x) + e^x \cdot g(y)$ for all x & y. Find g(x)

13. Discuss the continuity & derivability of $f(x) = \begin{cases} \left|x - \dfrac{1}{2}\right| & ; \ 0 \le x < 1 \\ x \cdot [x] & ; \ 1 \le x \le 2 \end{cases}$

where [x] indicates the greatest integer not greater than x.

14. Let f(0) = 0 and f'(0) = 1. For a positive integer k, show that

$$\lim_{x \to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \ldots\ldots f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{k}$$

15. Let f(x) be a real value function not identically zero satisfies the equation, $f(x+y^n) = f(x) + (f(y))^n$ for all real x & y and $f'(0) \ge 0$ where n (>1) is an odd natural number. Find f(10).

16. Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.

$$f(x) = x \sin\frac{1}{x} \sin\frac{1}{x\sin\frac{1}{x}} \text{ where } x \ne 0,\ x \ne \frac{1}{r\pi} \ \& \ f(0) = f(1/r\pi) = 0,\ r = 1, 2, 3, \ldots\ldots$$

17. The function f is defined by $y = f(x)$, where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$. Draw the graph of f for the interval $-1 \le x \le 1$. Also discuss its continuity & differentiability at $x = 0$

18. Given $f(x) = \cos^{-1}\left(\text{sgn}\left(\dfrac{2[x]}{3x-[x]}\right)\right)$ where sgn (.) denotes the signum function & [.] denotes the greatest integer funcion. Discuss the continuity & differentiability at $x = \pm 1$.

19. If $f(x) = \begin{cases} \dfrac{\sin[x^2]\pi}{x^2-3x+8} + ax^3 + b & , 0 \le x \le 1 \\ 2\cos\pi x + \tan^{-1} x & , 1 < x \le 2 \end{cases}$ is differentiable in [0, 2], find 'a' and 'b'. Here [.] stands for the greatest integer function .

20. A derivable function $f : R^+ \to R$ satisfies the condition $f(x) - f(y) \ge \ln\dfrac{x}{y} + x - y \ \forall\ x, y \in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\dfrac{1}{n}\right)$.

Exercise # 5 Part # I [Previous Year Questions] [AIEEE/JEE-MAIN]

1. If for all values of x & y ; $f(x+y) = f(x).f(y)$ and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is- **[AIEEE-2002]**

(1) 3 (2) 4 (3) 5 (4) 6

2. Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if

$\lim_{x\to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4$ then the value of k is- **[AIEEE - 2003]**

(1) 0 (2) 4 (3) 2 (4) 1

3. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then f(x) is- **[AIEEE 2003]**

(1) discontinuous everywhere
(2) continuous as well as differentiable for all x
(3) continuous for all x but not differentiable at x=0
(4) neither differentiable nor continuous at x = 0

4. Suppose f(x) is differentiable at x = 1; $\lim_{h\to 0} \frac{1}{h} f(1+h) = 5$, then $f'(1)$ equals- **[AIEEE-2005]**

(1) 3 (2) 4 (3) 5 (4) 6

5. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \leq (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then f(1) equals- **[AIEEE-2005]**

(1) –1 (2) 0 (3) 2 (4) 1

6. The set of points where $f(x) = \frac{x}{1+|x|}$ is differentiable- **[AIEEE-2006]**

(1) $(-\infty, -1) \cup (-1, \infty)$ (2) $(-\infty, \infty)$ (3) $(0, \infty)$ (4) $(-\infty, 0) \cup (0, \infty)$

7. Let $f(x) = x|x|$ and $g(x) = \sin x$. **[AIEEE-2009]**

Statement–1 : gof is differentiable at x = 0 and its derivative is continuous at that point.
Statement–2 : gof is twice differentiable at x = 0.
(1) Statement–1 is true, Statement–2 is false.
(2) Statement–1 is false, Statement–2 is true.
(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.
(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

8. If function f(x) is differentiable at x = a then $\lim_{x\to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ **[AIEEE-2011]**

(1) $2a\,f(a) + a^2 f'(a)$ (2) $-a^2 f'(a)$ (3) $af(a) - a^2 f'(a)$ (4) $2af(a) - a^2 f'(a)$

9. Consider the function,
$f(x) = |x-2| + |x-5|, x \in R$.
Statement–1 : $f'(4) = 0$.
Statement–2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5). **[AIEEE 2012]**
(1) Statement–1 is true, Statement–2 is false.
(2) Statement–1 is false, Statement–2 is true.
(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement1.
(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement1.

10. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$: **[Main 2014]**
(1) $2f'(c) = g'(c)$ **(2)** $2f'(c) = 3g'(c)$ **(3)** $f'(c) = g'(c)$ **(4)** $f'(c) = 2g'(c)$

11. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of k + m is : **[Main 2015]**
(1) $\frac{10}{3}$ **(2)** 4 **(3)** 2 **(4)** $\frac{16}{5}$

Part # II [Previous Year Questions][IIT-JEE ADVANCED]

1. (a) Let $f : R \to R$ be a function defined by, $f(x) = \max[x, x^3]$. The set of all points where f(x) is NOT differentiable is :
(A) {–1, 1} **(B)** {–1, 0} **(C)** {0, 1} **(D)** {–1, 0, 1}

(b) The left hand derivative of, $f(x) = [x]\sin(\pi x)$ at x = k, k an integer is :
where [ ] denotes the greatest function.
(A) $(-1)^k(k-1)\pi$ **(B)** $(-1)^{k-1}(k-1)\pi$ **(C)** $(-1)^k k\pi$ **(D)** $(-1)^{k-1}k\pi$

(c) Which of the following functions is differentiable at x = 0 ?
(A) $\cos(|x|) + |x|$ **(B)** $\cos(|x|) - |x|$ **(C)** $\sin(|x|) + |x|$ **(D)** $\sin(|x|) - |x|$
[JEE 2001]

2. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x-\alpha)$ for all $x \in R$. **[JEE 2001]**

3. The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$ is :-
(A) R – {0} **(B)** R – {1} **(C)** R – {–1} **(D)** R – {–1, 1}
[JEE 2002]

4. Let $f : R \to R$ be such that f(1) = 3 and f'(1) = 6. The $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals :-
(A) 1 **(B)** $e^{1/2}$ **(C)** e^2 **(D)** e^3
[JEE 2002]

5. $f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0 ? Justify your answer. **[JEE 2002]**

6. If a function f: [–2a, 2a] → R is an odd function such that f(x) = f(2a – x) for x ∈ [a, 2a] and the left hand derivative at x = a is 0, then find the left hand derivative at x = –a. **[JEE 2003]**

7. (A) The function given by y = | | x | –1 | is differentiable for all real numbers except the points

(A) {0, 1, –1} **(B)** ± 1 **(C)** 1 **(D)** –1 **[JEE 2005]**

(B) If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve y = f(x) at the point (1, 2). **[JEE 2005]**

8. If $f(x) = \min.(1, x^2, x^3)$, then

(A) f(x) is continuous ∀ x ∈ R **(B)** $f'(x) > 0, \forall x > 1$

(C) f(x) is not differentiable but continuous ∀ x ∈ R **(D)** f(x) is not differentiable for two values of x **[JEE 2006]**

9. Let $g(x) = \dfrac{(x-1)^n}{\ln \cos^m (x-1)}$; 0 < x < 2, m and n are integers, m ≠ 0, n > 0 and let p be the left hand derivative of | x – 1 | at x = 1. If $\lim\limits_{x \to 1^+} g(x) = p$, then :- **[JEE 2008]**

(A) n = 1, m = 1 **(B)** n = 1, m = –1 **(C)** n = 2, m = 2 **(D)** n > 2, m = n

10. Let *f* : R → R be a function such that

$$f(x + y) = f(x) + f(y),\ \forall x, y \in R.$$

If *f*(x) is differentiable at x = 0, then

(A) *f*(x) is differentiable only in a finite interval containing zero

(B) *f*(x) is continuous ∀ x ∈ R

(C) *f*'(x) is constant ∀ x ∈ R

(D) *f*(x) is differentiable except at finitely many points **[JEE 2011]**

11. If $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$ then - **[JEE 2011]**

(A) *f*(x) is continuous at $x = -\dfrac{\pi}{2}$ **(B)** *f*(x) is not differentiable at x = 0

(C) *f*(x) is differentiable at x = 1 **(D)** *f*(x) is differentiable at $x = -\dfrac{3}{2}$

12. Let $f(x) = \begin{cases} x^2 \left|\cos \dfrac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$, x ∈ ℝ, then *f* is - **[JEE 2012]**

(A) differentiable both at x = 0 and at x = 2 **(B)** differentiable at x = 0 but not differentiable at x = 2
(C) not differentiable at x = 0 but differentiable at x=2 **(D)** differentiable neither at x = 0 nor at x = 2

13. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)\,dt & \text{if } a \le x \le b \\ \int_a^b f(t)\,dt & \text{if } x > b \end{cases}$$

[JEE Ad. 2014]

Then

(A) g(x) is continuous but not differentiable at a

(B) g(x) is differentiable on R

(C) g(x) is continuous but not differentiable at b

(D) g(x) is continuous and differentiable at either a or b but not both

14. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$.

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0 \\ \min\{f(x), g(x)\} & \text{if } x > 0 \end{cases}$$

[JEE Ad. 2014]

The number of points at which h(x) is not differentiable is

15. Let $g : R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$.

Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let (foh) (x) denote f (h(x)),

Then which of the following is (are) true ? **[JEE Ad. 2015]**

(A) f is differentiable at $x = 0$

(B) h is differentiable at $x = 0$

(C) foh is differentiable at $x = 0$

(D) hof is differentiable at $x = 0$

16. Let $f, g : [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table : **[JEE Ad. 2015]**

	x = –1	x = 0	x = 2
f(x)	3	6	0
g(x)	0	1	–1

In each of the intervals (–1, 0) and (0, 2) the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solutions in (–1, 0)

(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in (0, 2)

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

Comprehension

Let $F : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$

17. The correct statement(s) is (are) **[JEE Ad. 2015]**

(A) $f'(1) < 0$ **(B)** $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$ **(D)** $f'(x) = 0$ for some $x \in (1, 3)$

18. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, then the correct expression(s) is(are) **[JEE Ad. 2015]**

(A) $9f'(3) + f'(1) - 32 = 0$ **(B)** $\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$ **(D)** $\int_1^3 f(x)dx = -12$

19. Let $a, b \in R$ and $f : R \to R$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is **[JEE Ad. 2016]**

(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$

(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$

(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$

(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$

20. Let $f : R \to (0,\infty)$ and $g: R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$. and $g''(2) \neq 0$. **[JEE Ad. 2016]**

if $\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

(A) f has a local minimum at $x = 2$ **(B)** f has a local maximum at $x = 2$

(B) $f''(2) > f(2)$ **(D)** $f(x) - f''(x) = 0$ for at least one $x \in R$

21. Let $f : \left[-\frac{1}{2}, 2\right] \to R$ and $g : \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by $f(x) = [x^2-3]$ and $g(x) = |x| f(x) + |4x-7| f(x)$, where [y] denotes the greatest integer less than or equal to y for $y \in R$. Then **[JEE Ad. 2016]**

(A) f is discountinous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discountinous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

MOCK TEST

SECTION - I : STRAIGHT OBJECTIVE TYPE

1. Number of points where the function $f(x) = \max(|\tan x|, \cos|x|)$ is non differentiable in the interval $(-\pi, \pi)$ is

(A) 4 **(B)** 6 **(C)** 3 **(D)** 2

2. The function $f(x) = \text{maximum}\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to :

(A) 1 **(B)** 0, 2 **(C)** 0, 1 **(D)** 1, 2

3. A function $f : R \to R$ satisfies $\sin x . \cos y . (f(2x+2y) - f(2x-2y)) = \cos x \sin y (f(2x+2y) + f(2x-2y))$

If $f'(0) = \frac{1}{2}$, then :

(A) $f''(x) = f(x) = 0$ **(B)** $4f''(x) + f(x) = 0$ **(C)** $f''(x) + f(x) = 0$ **(D)** $4f''(x) - f(x) = 0$

4. If $f(x) = [x^2] + \sqrt{\{x\}^2}$, where [.] and {.} denote the greatest integer and fractional part functions respectively, then-

(A) f(x) is continuous at all integral points except 0

(B) f(x) is continuous and differentiable at $x = 0$

(C) f(x) is discontinuous for all $x \in I - \{1\}$

(D) f(x) is not differentiable for all $x \in I$.

5. Let $f: R \to R$ be any function and $g(x) = \frac{1}{f(x)}$. Then g is

(A) onto if f is onto **(B)** one-one if f is one-one

(C) continuous if f is continuous **(D)** differentiable if f is differentiable

6. If $f(x) = \text{maximum}\left(\cos x, \frac{1}{2}, \{\sin x\}\right), 0 \le x \le 2\pi$, where { . } represents fractional part function, then number of points at which f(x) is continuous but not differentiable, is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

7. If $f(x) = \begin{cases} x^2\{e^{1/x}\} & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at $x = 0$, then

({ } denotes fractional part function)

(A) it is differentiable at $x = 0$ **(B)** $k = 1$

(C) continuous but not differentiable at $x = 0$ **(D)** continuous every where in its domain

8. The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at

(A) –1 **(B)** 0 **(C)** 1 **(D)** 2.

9. S_1 : If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} [f(x)]$ (where [ ] denotes greatest integer function) and f(x) is non constant continuous function then f(a) is an integer.

S_2 : $\cos|x| + |x|$ is differentiable at x = 0

S_3 : If a function has a tangent at x = a then it must be differentiable at x = a.

S_4 : If f(x) & g(x) both are discontinuous at any point, then there composition may be differentiable at that point.

(A) FTFT **(B)** TFFT **(C)** TFFF **(D)** FFFT

10. Consider the following statements :

S_1 : If $f'(a^+)$ and $f'(a^-)$ exist finitely at a point then f is continuous at x = a.

S_2 : The function f(x) = 3 tan 5x – 7 is differentiable at all points in its domain

S_3 : The existence of $\lim\limits_{x\to c} (f(x) + g(x))$ does not imply existance of $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$.

S_4 : If $f(x) < g(x)$ then $f'(x) < g'(x)$.

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TTTF **(B)** FFTF **(C)** TFTF **(D)** TFTT

SECTION - II : MULTIPLE CORRECT ANSWER TYPE

11. Which of the following function(s) has/have removable discontinuity at x = 1.

(A) $f(x) = \frac{1}{\ln|x|}$ **(B)** $f(x) = \frac{x^2-1}{x^3-1}$ **(C)** $f(x) = 2^{-2^{\frac{1}{1-x}}}$ **(D)** $f(x) = \frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

12. A function f(x) satisfies the relation f(x + y) = f(x) + f(y) + xy (x + y) ∀ x, y ∈ R. If f′(0) = –1, then

(A) f(x) is a polynomial function **(B)** f(x) is an exponential function

(C) f(x) is twice differentiable for all x ∈ R **(D)** f′(3) = 8

13. If f : R → R be a differentiable function and f(0) = 0 and f′(0) = 1, then

$\lim\limits_{x\to 0} \frac{1}{x}\left[f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{3}\right) + ... + f\left(\frac{x}{n}\right)\right]$, where n ∈ N, equals

(A) 0 **(B)** $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

(C) ${}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - .... + (-1)^{n-1}\frac{{}^nC_n}{n}$ **(D)** does not exist

14. $f(x) = \frac{[x]+1}{\{x\}+1}$ for $f : [0, \frac{5}{2}) \to (\frac{1}{2}, 3]$, where [.] represents greatest integer function and { . } represents fractional part of x , then which of the following is true.

(A) f (x) is injective discontinuous function **(B)** f (x) is surjective non differentiable function

(C) $\min\left(\lim\limits_{x\to 1-} f(x), \lim\limits_{x\to 1+} f(x)\right) = f(1)$ **(D)** max (x values of point of discoutinuity) = f (1)

15. If f(x) = 0 for x < 0 and f(x) is differentiable at x = 0, then for x > 0, f(x) may be

(A) x^2 **(B)** x **(C)** sin x **(D)** $-x^{3/2}$

SECTION - III : ASSERTION AND REASON TYPE

16. **Statement-I :** $|x^3|$ is differentiable at $x = 0$

Statement-II : If f(x) is differentiable at x = a then |f(x)| is also differentiable at x = a.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.

(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) Statement-I is True, Statement-II is False

(D) Statement-I is False, Statement-II is True

17. **Statement-I :** $f(x) = |x|. \sin x$ is differentiable at $x = 0$.

Statement-II : If f(x) is not differentiable and g(x) is differentiable at x = a, then f(x). g(x) can still be differentiable at x = a.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.

(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) Statement-I is True, Statement-II is False

(D) Statement-I is False, Statement-II is True

18. **Statement-I :** $f(x) = |[x]\, x|$ in $x \in [-1, 2]$, where [.] represents greatest integer function, is non differentiable at $x = 2$

Statement-II : Discontinuous function is always non differentiable.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.

(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) Statement-I is True, Statement-II is False

(D) Statement-I is False, Statement-II is True

19. **Statement-II :** Sum of left hand derivative and right hand derivative of $f(x) = |x^2 - 5x + 6|$ at $x = 2$ is equal to zero.

Statement-II : Sum of left hand derivative and right hand derivative of $f(x) = |(x - a)(x - b)|$ at $x = a$ $(a < b)$ is equal to zero, (where $a, b \in R$)

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.

(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) Statement-I is True, Statement-II is False

(D) Statement-I is False, Statement-II is True

20. **Statement-I :** If f is continuous and differentiable in $(a - \delta, a + \delta)$, where $a, \delta \in R$ and $\delta > 0$, then $f'(x)$ is continuous at $x = a$.

Statement-II : Every differentiable function at x = a is continuous at x = a

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.

(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) Statement-I is True, Statement-II is False

(D) Statement-I is False, Statement-II is True

SECTION - IV : MATRIX - MATCH TYPE

21. **Column - I** **Column - II**

(A) Number of points where the function

$$f(x)=\begin{cases}1+\left[\cos\frac{\pi x}{2}\right] & 1<x\le 2\\ 1-\{x\} & 0\le x<1\\ |\sin \pi x| & -1\le x<0\end{cases} \text{ and } f(1)=0 \text{ is}$$

continuous but non-differentiable

(B) $f(x)=\begin{cases}x^2e^{1/x} & , \; x\ne 0\\ 0 & , \; x=0\end{cases}$, then $f'(0^-)=$

(C) The number of points at which $g(x)=\dfrac{1}{1+\dfrac{2}{f(x)}}$ is not differentiable where $f(x)=\dfrac{1}{1+\dfrac{1}{x}}$, is

(D) Number of points where tangent does not exist for the curve $y=\text{sgn}\,(x^2-1)$

(p) 0

(q) 1

(r) 2

(s) 3

(t) 4

22. **Column – I** **Column – II**

(A) $f(x)=|x^3|$ is

(B) $f(x)=\sqrt{|x|}$ is

(C) $f(x)=|\sin^{-1} x|$ is

(D) $f(x)=\cos^{-1}|x|$ is

(p) continuous in $(-1, 1)$

(q) discontinuous in $(-1, 1)$

(r) differentiable in $(0, 1)$

(s) not differentiable atleast at one point in $(-1, 1)$

(t) differentiable in $(-1, 1)$

SECTION - V : COMPREHENSION TYPE

23. **Read the following comprehension carefully and answer the questions.**

Let a function is defined as $f(x)=\begin{cases}[x] & , \; -2\le x\le -\frac{1}{2}\\ 2x^2-1 & , \; -\frac{1}{2}<x\le 2\end{cases}$, where [.] denotes greatest integer function.

Answer the following question by using the above information.

1. The number of points of discontinuity of f(x) is

(A) 1 **(B)** 2 **(C)** 3 **(D)** None of these

2. The function f(x – 1) is discontinuous at the points

(A) $-1, -\frac{1}{2}$ **(B)** $-\frac{1}{2}, 1$ **(C)** $0, \frac{1}{2}$ **(D)** 0, 1

3. Number of points where |f(x)| is not differentiable is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

24. **Read the following comprehension carefully and answer the questions.**

Consider two functions y = f(x) and y = g(x) defind as $f(x) = \begin{cases} ax^2+b & , \ 0 \le x \le 1 \\ 2bx+2b & , \ 1 < x \le 3 \\ (a-1)x+2a-3 & , \ 3 < x \le 4 \end{cases}$ and

$$g(x) = \begin{cases} cx^2+d & , \ 0 \le x \le 2 \\ dx+3-c & , \ 2 < x < 3 \\ x^2+b+1 & , \ 3 \le x \le 4 \end{cases}$$

1. f(x) is continuous at x = 1 but not differentiable at x = 1, if

(A) a = 1, b = 0 **(B)** a = 1, b = 2 **(C)** a = 3, b = 1 **(D)** a and b are integers

2. g(x) is continuous at x = 2, if

(A) c = 1, d = 2 **(B)** c = 2, d = 3 **(C)** c = 1, d = –1 **(D)** c = 1, d = 4

3. If f is continuous and differentiable at x = 3, then

(A) $a = -\frac{1}{3}, b = \frac{2}{3}$ **(B)** $a = \frac{2}{3}, b = -\frac{1}{3}$ **(C)** $a = \frac{1}{3}, b = -\frac{2}{3}$ **(D)** $a = 2, b = \frac{1}{2}$

25. **Read the following comprehension carefully and answer the questions.**

Left hand derivative and Right hand derivative of a function f(x) at a point x = a are defined as

$$f'(a^-) = \lim_{h\to 0^+} \frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and}$$

$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a)-f(a-h)}{h} = \lim_{x\to a+} \frac{f(a)-f(x)}{a-x} \text{ respectively.}$$

Let f be a twice differentiable function.

1. If f is odd, which of the following is Left hand derivative of f at x = – a

(A) $\lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h}$ **(B)** $\lim_{h\to 0^-} \frac{f(h-a)-f(a)}{h}$

(C) $\lim_{h\to 0^+} \frac{f(a)+f(a-h)}{-h}$ **(D)** $\lim_{h\to 0^-} \frac{f(-a)-f(-a-h)}{-h}$

2. If f is even which of the following is Right hand derivative of f' at $x = a$.

(A) $\lim_{h\to 0^-} \frac{f'(a)+f'(-a+h)}{h}$ **(B)** $\lim_{h\to 0^+} \frac{f'(a)+f'(-a-h)}{h}$

(C) $\lim_{h\to 0^-} \frac{-f'(-a)+f'(-a-h)}{-h}$ **(D)** $\lim_{h\to 0^+} \frac{f'(a)+f'(-a+h)}{-h}$

3. The statement $\lim_{h\to 0} \frac{f(-x)-f(-x-h)}{h} = \lim_{h\to 0} \frac{f(x)-f(x-h)}{-h}$ implies that

(A) f is odd **(B)** f is even

(C) f is neither odd nor even **(D)** nothing can be concluded

SECTION - VI : INTEGER TYPE

26. If $f(x) = \begin{cases} \frac{\tan[x^2]\pi}{ax^2} + ax^3 + b & , \quad 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x & , \quad 1 < x \le 2 \end{cases}$ is differentiable in [0, 2], then $a = \frac{1}{k_1}$ and $b = \frac{\pi}{4} - \frac{26}{k_2}$. Find $k_1^2 + k_2^2$ {where [.] denotes greatest integer function}.

27. Let $f(x) = \begin{cases} |x|^p \sin\frac{1}{x} + x\,|\tan x|^q & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$ be differentiable at x = 0, then find the least possible value of [p + q], (where [.] represents greatest integer function).

28. Let f(x) be a real valued function not identically zero such that $f(x + y^3) = f(x) + [f(y)]^3 \;\forall\; x, y \in R$ and $f'(0) \ge 0$, then find f(10)

29. If $f(x) = \sin^{-1} 2x\sqrt{1-x^2}$ and the values of $f'(1/2) + f'(-1/2)$ is $\frac{\lambda}{\sqrt{3}}$, then find λ

30. Let f(x) is differentiable function & f(0) = 1. Also if f(x) satisfies $f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$ $\forall$ x,y $\in$ R and $f(x) = a(x + 1)^b$, then find (a + b)

ANSWER KEY

EXERCISE - 1

1. C **2.** B **3.** A **4.** B **5.** C **6.** C **7.** B **8.** B **9.** B **10.** C **11.** B **12.** A **13.** C
14. A **15.** C **16.** C **17.** D **18.** B **19.** B **20.** D **21.** C **22.** B **23.** A **24.** C **25.** C **26.** D
27. A **28.** D **29.** C **30.** C

EXERCISE - 2 : PART # I

1. AB **2.** ACD **3.** BD **4.** AB **5.** ABC **6.** BCD **7.** ABD **8.** AD **9.** AC
10. ABC **11.** ABD **12.** AC **13.** BD **14.** BD **15.** ABD **16.** ACD **17.** AB **18.** AC
19. BC **20.** AD

PART - II

1. B **2.** A **3.** C **4.** C **5.** C **6.** B **7.** C **8.** C

EXERCISE - 3 : PART # I

1. $A \to r$ $B \to p$ $C \to s$ $D \to q$
2. $A \to p,r,s$ $B \to p,r,s$ $C \to q,r,s$ $D \to r,s$
3. $A \to r,s$ $B \to p,s$ $C \to p,q$ $D \to p,q$
4. $A \to p,q,r$ $B \to p,r,s$ $C \to prs$ $D \to p,r,s$
5. $A \to p,s$ $B \to p,q$ $C \to r,s$ $D \to p,q$
6. $A \to q$ $B \to p$ $C \to s$ $D \to p$

PART - II

Comprehension # 1 : 1. D **2.** B **3.** B
Comprehension # 2 : 1. A **2.** C **3.** B
Comprehension # 3 : 1. B **2.** C **3.** B
Comprehension # 4 : 1. C **2.** C **3.** D **4.** A

EXERCISE - 5 : PART # I

1. 4 **2.** 2 **3.** 3 **4.** 3 **5.** 2 **6.** 2 **7.** 1 **8.** 4 **9.** 4
10. 4 **11.** 3

PART - II

1. **(a)** D **(b)** A **(c)** D **3.** D **4.** C **5.** $a = 1$; $b = 0$ $(gof)'(0) = 0$ **6.** $f'(a^-) = 0$
7. **(a)** A, **(b)** $y - 2 = 0$ **8.** AC **9.** C **10.** BC **11.** ABCD
12. B **13.** AC **14.** 3 **15.** AD **16.** BC
17. AC **18.** CD **19.** AB **20.** AD **21.** BC

MOCK TEST

1. A **2.** D **3.** B **4.** A **5.** B **6.** D **7.** A **8.** D **9.** B
10. A **11.** BD **12.** ACD **13.** BC **14.** ABD **15.** AD **16.** C **17.** A **18.** A
19. A **20.** D
21. $A \to q$ $B \to p$ $C \to s$ $D \to p$ **22.** $A \to p,t,r$ $B \to p,r,s$ $C \to p,r,s$ $D \to p,r,s$
23. **1.** B **2.** C **3.** C **24.** **1.** C **2.** A **3** D **25.** **1.** A **2.** A **3.** B
26. 180 **27.** 1 **28.** 10 **29.** 8 **30.** 3