DETERMINANTS
EXERCISE # 1

Questions based on Minor cofactors & expansion of determinant

Q.1 If ΔABC is a scalene triangle, then the value of $\begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ 1 & 1 & 1 \end{vmatrix}$ is

(A) $= 0$ (B) $\neq 0$
(C) can not say (D) None of these

Sol. **[B]**

Applying $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} \sin A - \sin B & \sin B - \sin C & \sin C \\ \cos A - \cos B & \cos B - \sin C & \cos C \\ 0 & 0 & 1 \end{vmatrix}$$

$= (\sin A - \sin B)(\cos B - \cos C) - (\sin B - \sin C)(\cos A - \cos B)$

$= \sin A \cos B - \sin A \cos C - \sin B \cos B + \sin B \cos C - \sin B \cos A + \sin B \cos B + \sin C \cos A - \sin C \cos B$

$= (\sin A \cos B - \cos A \sin B) + (\sin C \cos A - \sin A \cos C) + (\sin B \cos C - \sin C \cos B)$

$= \sin (A - B) + \sin (C - A) + \sin (B - C)$

$\because \Delta ABC$ is a scalene triangle

$\therefore A \neq B \neq C$

$\therefore \sin (A - B) + \sin (C - A) + \sin (B - C) \neq 0$

$\therefore$ value of given determinant is $\neq 0$

Q.2 Co-factors of element of the second row of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ are-

(A) 39, 3, 11 (B) –39, 3, 11
(C) 39, –3, 11 (D) 39, 3, –11

Sol. **[B]**

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

Cofactors of second rows elements i.e. – 4, 3 and 6

Cofactors of – 4 is given by = – [18 + 21] = –39

= –39

Cofactors of 3 is (9 – 6) = 3

Cofactors of 6 is – (–7 –4) = 11

$\therefore$ cofactors are –39, 3, 11

Q.3 Without expanding value of the determinant $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$ is-

(A) $(p-r)(q-r)$ (B) $(q-r)(p-q)$
(C) 0 (D) None of these

Sol. **[C]**

$$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$$

Above given determinant is a skew symmetric determinants of odd order therefore its value is equal to zero.

Questions based on Properties of determinant

Q.4 If a, b,c are positive and are the pth, qth and rth terms respectively of a G.P., then the value of $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$ is -

(A) 0 (B) p
(C) q (D) r

Sol. **[A]**

Let A be the first term and R be the common ratio of the G.P., then

$a = AR^{p-1} \Rightarrow \log a = \log A + (p - 1) \log R$ ….(i)

$b = AR^{q-1} \Rightarrow \log b = \log A + (q - 1) \log R$ ....(ii)

$c = AR^{r-1} \Rightarrow \log c = \log A + (r - 1) \log R$ ...(iii)

Now, multiplying (i), (ii) and (iii) by (q –r), (r – p) and (p – q) respectively and adding, we get

$\log a (q - r) + \log b (r - p) + \log c (p - q) = 0$

$\Rightarrow \Delta = 0$

Alternative

As above from (i), (ii) and (iii)

Now $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$

$= \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$

$\therefore$ Applying $C_2 \to C_2 - C_3$

$$= \log R \begin{vmatrix} p-1 & p-1 & 1 \\ q-1 & q-1 & 1 \\ r-1 & r-1 & 1 \end{vmatrix}$$

($\because$ C_1 & C_2 are identical)

= 0

Q.5 $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} =$

(A) a + b + c (B) 2a + b + c
(C) ab + bc + ca (D) 4abc.

Sol. **[D]**

$$\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 2(b+c) & c & b \\ 2(c+a) & c+a & a \\ 2(a+b) & a & a+b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$2\begin{vmatrix} b & c & b \\ 0 & c+a & a \\ b & a & a+b \end{vmatrix} = 2b\begin{vmatrix} 1 & c & b \\ 0 & c+a & a \\ 1 & a & a+b \end{vmatrix}$$

$$= 2b\begin{vmatrix} 0 & c-a & -a \\ 0 & c+a & a \\ 1 & a & a+b \end{vmatrix}$$ Applying $R_1 \to R_1 - R_3$

= 2b {a (c – a) + a(c + a)}

= 2b {ac – a^2 + ac + a^2} = 4abc

Q.6 If $ax^4 + bx^3 + cx^2 + dx + e =$

$\begin{vmatrix} 2x & x-1 & x+1 \\ x+1 & x^2-x & x-1 \\ x-1 & x+1 & 3x \end{vmatrix}$, then the value of e, is

(A) 0 (B) –2
(C) 3 (D) –1

Sol. **[A]**

$ax^4 + bx^3 + cx^2 + dx + e =$

$$\begin{vmatrix} 2x & x-1 & x+1 \\ x+1 & x^2-x & x-1 \\ x-1 & x+1 & 3x \end{vmatrix}$$

Put x = 0 both sides

$$e = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

= 1 (–1) + 1(1)

= –1 + 1 = 0

$\therefore$ e = 0

Questions based on Multiplication of determinants

Q.7 If A, B and C are the angles of a triangle, then $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} =$

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **[A]**

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$$

We know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

and $\cos A = \frac{b^2+c^2-a^2}{2bc} = a\frac{(b^2+c^2-a^2)}{2abc}$

$\therefore$ sin 2A = 2 sin A cos A

$$= 2.\ \frac{a}{2R}\ .\ \frac{(b^2+c^2-a^2)}{2bc}$$

Hence the elements of R_1 in Δ are

$$\frac{a(b^2+c^2-a^2)}{2R\,bc} \quad \frac{c}{2R} \quad \frac{b}{2R}$$

or $\frac{1}{2Rbc}$ {a(b^2 + c^2 –a^2) bc^2 b^2c}

$$\therefore \Delta = \frac{1}{(2Rbc)(2Rca)(2Rab)}$$

$$\begin{vmatrix} a(b^2+c^2-a^2) & bc^2 & b^2c \\ c^2a & b(c^2+a^2-b^2) & a^2c \\ b^2a & ba^2 & c(a^2+b^2-c^2) \end{vmatrix}$$

Take a, b, c common from C_1, C_2 and C_3 respectively.

$$\therefore \Delta = \frac{1}{8R^3abc}$$

$$\begin{vmatrix} b^2+c^2-a^2 & c^2 & b^2 \\ c^2 & c^2+a^2-b^2 & a^2 \\ b^2 & a^2 & a^2+b^2-c^2 \end{vmatrix}$$

Now apply $R_1 - R_2$ and $R_2 - R_3$ and take $(b^2 - a^2)$ and $(c^2 - b^2)$ common from R_1 and R_2.

$\therefore \Delta =$

$$\frac{(b^2-a^2)(c^2-b^2)}{8R^3abc}\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ b^2 & a^2 & a^2+b^2-c^2\end{vmatrix}$$

$\therefore \Delta = 0$ (as two rows are Identical)

Q.8 The value of the determinant

$$\begin{vmatrix}1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha)\\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta)\\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1\end{vmatrix}$$

is equal to

(A) $\cos\alpha + \cos\beta + \cos\gamma$

(B) $\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha$

(C) –1

(D) 0

Sol. **[D]**

$$\Delta = \begin{vmatrix}1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha)\\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta)\\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1\end{vmatrix}$$

The above determinant is obtained by multiplying two zero determinants:

$$\Delta = \begin{vmatrix}\cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\\ \cos\gamma & \sin\gamma & 0\end{vmatrix}\begin{vmatrix}\cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\\ \cos\gamma & \sin\gamma & 0\end{vmatrix} = 0$$

$\therefore \Delta = 0$

Questions based on **Application of determinant**

Q.9 If the following equations

$x + y - 3 = 0$

$(1 + \lambda)x + (2 + \lambda)y - 8 = 0$

$x - (1 + \lambda)y + (2 + \lambda) = 0$

are consistent then the value of λ is

(A) 1 (B) –1 (C) 0 (D) 2

Sol. **[A]**

$x + y - 3 = 0$

$(1 + \lambda)x + (2 + \lambda)y - 8 = 0$

$x - (1 + \lambda)y + (2 + \lambda) = 0$

since, here the equation are in two variables x and y. If they are consistent then the value of x and y, obtained from first two equations should satisfy the third equation and hence D = 0, i.e.

$$\Rightarrow \begin{vmatrix}1 & 1 & -3\\ 1+\lambda & 2+\lambda & -8\\ 1 & -1-\lambda & 2+\lambda\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}1 & 0 & 0\\ 1+\lambda & 1 & -5+3\lambda\\ 1 & -2-\lambda & 5+\lambda\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 + 3C_1$

$\Rightarrow (5 + \lambda) + (2 + \lambda)(-5 + 3\lambda) = 0$

$\Rightarrow 3\lambda^2 + 2\lambda - 5 = 0$

$\Rightarrow (\lambda - 1)(3\lambda + 5) = 0$

$\Rightarrow \lambda = 1, -5/3$

$\therefore \lambda = 1$ given in option (A)

Q.10 If $a + b + c \neq 0$ and the system of equations

$ax + by + cz = 0$

$bx + cy + az = 0$

$cx + ay + bz = 0$

has a non-trivial solution, then the roots of the equation $at^2 + bt + c = 0$, are

(A) imaginary

(B) real and distinct

(C) real and of opposite sign

(D) real and equal

Sol. **[A]**

Since $a + b + c \neq 0$ and given system of equations has a non- trivial solution, therefore

$$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix}a+b+c & b & c\\ a+b+c & c & a\\ a+b+c & a & b\end{vmatrix} = 0 \Rightarrow (a+b+c)\begin{vmatrix}1 & b & c\\ 1 & c & a\\ 1 & a & b\end{vmatrix} = 0$$

Since $a + b + c \neq 0$ given

$$\therefore \begin{vmatrix}1 & b & c\\ 1 & c & a\\ 1 & a & b\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}0 & b-c & c-a\\ 0 & c-a & a-b\\ 1 & a & b\end{vmatrix} = 0$$

$\Rightarrow (a - b)(b - c) - (c - a)^2 = 0$

$\Rightarrow ab - ac - b^2 + bc - a^2 - c^2 + 2ac = 0$

$\Rightarrow ab + bc + ca - a^2 - b^2 - c^2 = 0$ …(1)

given equation is

$at^2 + bt + c = 0$

$D = b^2 - 4ac$

From (1)

$a^2 + b^2 + c^2 - ab - bc - ca = 0$

$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow$ either a = b
and b = c
and c = a
$\Rightarrow$ a = b = c
$\therefore$ $D = b^2 - 4a.c$
$D = b^2 - 4b^2$
$D = -3b^2$
$D < 0$
$\therefore$ roots are imaginary.

True or false type questions

Q.11 If a, b, c are sides of a scalene triangle, then value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$

$= (a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$= (a + b + c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$

$= (a + b + c) \{(a - b)(b - c) - (c - a)^2\}$

Since a, b, c are positive and unequal

$\therefore$ $a + b + c \neq 0$ and $a + b + c > 0$ always

and $(ab + bc + ca - a^2 - b^2 - c^2)$

$= -(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

Clearly negative.

EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 Solution of the system of equations $a^2x + ay + z = -a^3$, $b^2x + by + z = -b^3$, $c^2x + cy + z = -c^3$ is-

(A) $x = -(a + b + c)$, $y = ab + bc + ca$, $z = -abc$
(B) $x = (a + b + c)$, $y = ab + bc + ca$, $z = -abc$
(C) $x = -(a - b - c)$, $y = ab + bc + ca$, $z = -abc$
(D) None of these

Sol. **[A]**

$z + ay + a^2x + a^3 = 0$

$z + by + b^2x + b^3 = 0$

$z + cy + c^2x + c^3 = 0$

$$\Delta = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \begin{vmatrix} a^2-b^2 & a-b & 0 \\ b^2-c^2 & b-c & 0 \\ c^2 & c & 1 \end{vmatrix}$$

$\Delta = (a^2 - b^2)(b - c) - (a - b)(b^2 - c^2)$

$= (a - b)(b - c)\{a + b - b - c\}$

$= (a - b)(b - c)(a - c) = -(a - b)(b - c)(c - a)$

$$\Delta_x = \begin{vmatrix} -a^3 & a & 1 \\ -b^3 & b & 1 \\ -c^3 & c & 1 \end{vmatrix} = -\begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix}$$

$= (a - b)(b - c)(c - a)(a + b + c)$

$$\Delta_y = \begin{vmatrix} a^2 & -a^3 & 1 \\ b^2 & -b^3 & 1 \\ c^2 & -c^3 & 1 \end{vmatrix} = +\begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}$$

$= -(a - b)(b - c)(c - a)(ab + bc + ca)$

$$\Delta_z = \begin{vmatrix} a^2 & a & -a^3 \\ b^2 & b & -b^3 \\ c^2 & c & -c^3 \end{vmatrix} = -abc\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc\,(a - b)(b - c)(c - a)$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{(a-b)(b-c)(c-a)(a+b+c)}{-(a-b)(b-c)(c-a)}$$

$= -(a + b + c)$

$$\text{and } y = \frac{\Delta_y}{\Delta} = \frac{-(a-b)(b-c)(c-a)(ab+bc+ca)}{-(a-b)(b-c)(c-a)}$$

$= (ab + bc + ca)$

$$z = \frac{\Delta_z}{\Delta} = \frac{abc(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)} = -abc$$

$$\therefore \left.\begin{aligned} x &= -(a+b+c) \\ y &= (ab+bc+ca) \\ z &= -abc \end{aligned}\right\}$$

Q.2 The value of the determinant

$$\begin{vmatrix} {}^{n-1}C_{r-1} & {}^{n-1}C_r & {}^{n-1}C_{r+1} \\ {}^{n-1}C_r & {}^{n-1}C_{r+1} & {}^{n-1}C_{r+2} \\ {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \end{vmatrix} \text{ is -}$$

(A) 0 (B) 1 (C) –1 (D) None

Sol. **[A]**

$R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \\ {}^{n-1}C_r & {}^{n-1}C_{r+1} & {}^{n-1}C_{r+2} \\ {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \end{vmatrix}$$

Since R_1 and R_3 are identical

$\therefore \Delta = 0$

Q.3 If $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$

$= ax^5 + bx^4 + cx^3 + dx^2 + \lambda x + \mu$ be an identity in x, where a, b, c, d, λ, μ are independent of x. Then the value of λ is

(A) 3 (B) 2 (C) 4 (D) None

Sol. **[A]**

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} (1+x)^2 & x & x^2 \\ (1+x)^2 & 1+x & x^2 \\ (1+x)^2 & x & 1+x \end{vmatrix}$$

$$\Delta = (1+x)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & x^2 \\ 1 & x & 1+x \end{vmatrix}$$

Differentiating both sides of the given equality w.r.t. x, we get

$$2(1+x)\begin{vmatrix}1 & x & x^2\\ 1 & 1+x & x^2\\ 1 & x & 1+x\end{vmatrix}+(1+x)^2$$

$$\left\{\begin{vmatrix}1 & 1 & x^2\\ 1 & 1 & x^2\\ 1 & 1 & 1+x\end{vmatrix}+\begin{vmatrix}1 & x & 2x\\ 1 & 1+x & 2x\\ 1 & x & 1\end{vmatrix}\right\}$$

$= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + \lambda$

Now putting x = 0

$$2\begin{vmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{vmatrix}+\begin{vmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{vmatrix}=\lambda$$

$\therefore 2(1) + 1(1) = \lambda \Rightarrow \lambda = 3$

Q.4 The system of equations

$2x - y + z = 0$

$x - 2y + z = 0$

$\lambda x - y + 2z = 0$

has infinite number of nontrivial solutions for -

(A) $\lambda = 1$ (B) $\lambda = 5$

(C) $\lambda = -5$ (D) no real value of λ

Sol. **[B]**

$$\begin{vmatrix}2 & -1 & 1\\ 1 & -2 & 1\\ \lambda & -1 & 2\end{vmatrix}=0$$

$\Rightarrow 2(-4 + 1) + 1(2 - \lambda) + 1(-1 + 2\lambda) = 0$

$\Rightarrow -6 + 2 - \lambda + 2\lambda - 1 = 0 \Rightarrow \lambda - 5 = 0 \Rightarrow \lambda = 5$

Q.5 If $a \neq b \neq c$ such that

$$\begin{vmatrix}a^3-1 & b^3-1 & c^3-1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=0 \text{ then}$$

(A) $ab + bc + ca = 0$ (B) $a + b + c = 0$

(C) $abc = 1$ (D) $a + b + c = 1$

Sol. **[C]**

$$\Delta=\begin{vmatrix}a^3 & b^3 & c^3\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}+\begin{vmatrix}-1 & -1 & -1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$$= abc\begin{vmatrix}a^2 & b^2 & c^2\\ 1 & 1 & 1\\ a & b & c\end{vmatrix}-1\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$$= (abc-1)\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$= (abc - 1)(a - b)(b - c)(c - a)$

But $a \neq b \neq c$

So $abc = 1$, $(a \neq b \neq c)$

Q.6 $$\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}=$$

(A) $(1-a^2-b^2)^3$ (B) $(1+a^2+b^2)^3$

(C) $(1+a^2-b^2)^3$ (D) None of these

Sol. **[B]**

$$\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Apply $C_1 - bC_3$, $C_2 + aC_3$ and takeout $(1 + a^2 + b^2)$ each common from both new C_1 and C_2.

$$\therefore \Delta = (1+a^2+b^2)^2\begin{vmatrix}1 & 0 & -2b\\ 0 & 1 & 2a\\ b & -a & 1-a^2-b^2\end{vmatrix}$$

Again apply $R_3 - bR_1$

$$\therefore \Delta = (1+a^2+b^2)^2\begin{vmatrix}1 & 0 & -2b\\ 0 & 1 & 2a\\ 0 & -a & 1-a^2+b^2\end{vmatrix}$$

$= (1 + a^2 + b^2)^2 (1 - a^2 + b^2 + 2a^2)$

$= (1 + a^2 + b^2)^2 (1 + a^2 + b^2)$

$= (1 + a^2 + b^2)^3$

Q.7 If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of the determinant $\begin{vmatrix}\alpha & \beta & \gamma\\ \beta & \gamma & \alpha\\ \gamma & \alpha & \beta\end{vmatrix}$ is equal to

(A) –3 (B) 2

(C) 1 (D) none

Sol. **[D]**

$\because \alpha, \beta, \gamma$ are roots of $x^3 - 3x + 2 = 0$

$\therefore \alpha + \beta + \gamma = 0$

Now $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

Q.8 The value of the determinant

$$\begin{vmatrix} (2^x-2^{-x})^2 & (2^x+2^{-x})^2 & 1 \\ (3^x-3^{-x})^2 & (3^x+3^{-x})^2 & 1 \\ (4^x-4^{-x})^2 & (4^x+4^{-x})^2 & 1 \end{vmatrix}$$ is equal to

(A) 0 (B) –9

(C) 24 (D) 1

Sol. **[A]**

$$\begin{vmatrix} (2^x-2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x-3^{-x})^2 & (3^x+3^{-x})^2 & 1 \\ (4^x-4^{-x})^2 & (4^x+4^{-x})^2 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} (2^x-2^{-x})^2-(2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x-3^{-x})^2-(3^x+3^{-x})^2 & (3^x+3^{-x})^2 & 1 \\ (4^x-4^{-x})^2-(4^x+4^{-x})^2 & (4^x+4^{-x})^2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} -4 & (2^x-2^{-x})^2 & 1 \\ -4 & (3^x+3^{-x})^2 & 1 \\ -4 & (4^x+4^{-x})^2 & 1 \end{vmatrix}$$

$$= -4\begin{vmatrix} 1 & (2^x+2^{-x})^2 & 1 \\ 1 & (3^x+3^{-x})^2 & 1 \\ 1 & (4^x+4^{-x})^2 & 1 \end{vmatrix} = 0$$

Part-B One or more than one correct answer type questions

Q.9 If $\Delta_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$, then the value of $\sum_{r=1}^{n}\Delta_r$ is independent of -

(A) x (B) y

(C) z (D) n

Sol. **[A, B, C, D]**

$$\Delta_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$$

$$\therefore \sum_{r=1}^{n}\Delta_r = \begin{vmatrix} 2\sum_{r=1}^{n} r & x & n(n+1) \\ 6\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1 & y & n^2(2n+3) \\ 4\sum_{r=1}^{n} r^3 - 2n\sum_{r=1}^{n} r & z & n^3(n+1) \end{vmatrix}$$

$$= \begin{vmatrix} n(n+1) & x & n(n+1) \\ n^2(2n+3) & y & n^2(2n+3) \\ n^3(n+1) & z & n^3(n+1) \end{vmatrix} = 0$$

$\therefore \sum_{r=1}^{n}\Delta_r$ is independent of x, y, z, n.

Q.10 Let $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 \Delta(x)dx = -16$, where a, b, c, d are in A.P., then the common difference of the AP is -

(A) 1 (B) 2

(C) –2 (D) None

Sol. **[B, C]**

Let D be the common difference of A.P, a, b, c, d then a – c = –2D, d – b = 2D, 2b = a + c, 2c = b + d

Now apply $R_1 + R_3 - 2R_2$

$$\therefore \Delta(x) = \begin{vmatrix} 0 & 0 & 2 \\ x+b & x+c & (x-1) \\ x+c & x+d & x-b+d \end{vmatrix}$$

$= 2\,[(x + b)(x + d) - (x + c)^2]$

$= 2[x(b + d - 2c) + (bd - c^2)]$

$= 2[0 + (a + D)(a + 3D) - (a + 2D)^2]$

$= -2D^2$

$\because \int_0^2 \Delta(x)\,dx = -16$ given

$\therefore \int_0^2 -2D^2\,dx = [-2D^2x]_0^2 = -4D^2$

$\therefore -4D^2 = -16 \Rightarrow D^2 = 4 \Rightarrow D = \pm 2$

Q.11 Let $x \neq -1$ and let a, b, c nonzero real numbers. Then the determinant

$$\begin{vmatrix} a(1+x) & b & c \\ a & b(1+x) & c \\ a & b & c(1+x) \end{vmatrix}$$ is divisible by -

(A) abcx (B) $(1 + x)^2$

(C) $(1 + x)^3$ (D) $x(1 + x)^2$

Sol. **[A, B, C]**

$$\begin{vmatrix} a(1+x) & b & c \\ a & b(1+x) & c \\ a & b & c(1+x) \end{vmatrix}$$

$$\Rightarrow abc\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix}$$

$C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\Rightarrow abc\begin{vmatrix} x & 0 & 1 \\ -x & x & 1 \\ 0 & -x & 1+x \end{vmatrix}$$

$$\Rightarrow abc\, x^2\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1+x \end{vmatrix}$$

$\Rightarrow abc\, x^2 \{(1 + x + 1) + 1\}$

$\Rightarrow abc\, x^2 (3 + x)$ which is divisible by abc x & $(1 + x),^2$ $x(1 + x^2)$

Q.12 Let $\{\Delta_1, \Delta_2, \Delta_3, ..., \Delta_k\}$ be the set of third-order determinants that can be made with the distinct nonzero real numbers $a_1, a_2, a_3, ... a_9$. then -

(A) k = 9 ! (B) $\sum_{i=1}^{k} \Delta_i = 0$

(C) at least one $\Delta_i = 0$ (D) None of these

Sol. **[A, B]**

The number of third order determinant is equal to the number of arrangements of nine different numbers in nine places = 9!

Corresponding to each determinant made, there is a determinant obtained by interchanging two consecutive rows (or columns), so, the sum of this pair will be zero.

$\therefore$ the sum of all the determinants = 0 + 0 + 0 ….. to $\frac{9!}{2}$ times = 0

$\therefore$ k = 9! and $\sum_{i=1}^{k} \Delta_i = 0$

Q.13 System of equation $x + 3y + 2z = 6$
$x + \lambda y + 2z = 7$
$x + 3y + 2z = \mu$ has

(A) unique solution if $\lambda = 2, \mu \neq 6$

(B) infinitely many solution if $\lambda = 4, \mu = 6$

(C) no solution if $\lambda = 5, \mu = 7$

(D) no solution if $\lambda = 3, \mu = 5$

Sol. **[B, C, D]**

$x + 3y + 2z = 6$

$x + \lambda y + 2z = 7$

$x + 3y + 2z = \mu$

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 1 & \lambda & 2 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$\therefore$ There is no unique solution.

$$\Delta_1 = \begin{vmatrix} 6 & 3 & 2 \\ 7 & \lambda & 2 \\ \mu & 3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3-\lambda & 0 \\ 7-\mu & \lambda-3 & 0 \\ \mu & 3 & 2 \end{vmatrix}$$

$\Delta_1 = 2\{(3 - \lambda) - (3 - \lambda)(7 - \mu)\}$

$\Delta_1 = 6 - 2\lambda - (21 - 3\mu - 7\lambda + \lambda\mu)$

$\Delta_1 = 6 - 2\lambda - 42 + 6\mu + 14\lambda - 2\lambda\mu$

$\Delta_1 = 12\lambda + 2\mu(3 - \lambda) - 36$

$\Delta_1 = 12(\lambda - 3) + 2\mu(3 - \lambda) = 0$

$\lambda = 3, \mu = -6$

Part-C Assertion-Reason type questions

The following questions consist of two statements each, printed as Assertion-1 and Reason-2. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion -1 and Reason-2 are true and the Reason-2 is correct explanation of the Assertion -1.
(B) If both Assertion -1 and Reason-2 are true but Reason -2 is not correct explanation of the Assertion -1.
(C) If Assertion-1 is true but the Reason-2 is false.
(D) If Assertion -1 is false but Reason-2 is true.

Q.14 **Assertion:** The system of equations possess a non trivial solution for the equations
$x + xy + 3z = 0,\ 3x + xy - 2z = 0$
& $2x + 3y - 4z = 0$
then value of k is $\frac{29}{2}$

Reason: for non trivial solution $\Delta = 0$

Sol. **[D]**
For non-trivial solution D = 0

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

Apply $R_2 - 3R_1,\ R_3 - 2R_1$

$$\therefore \Delta = \begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix} = 0$$

or $20k + 11(3 - 2k) = 0$
or $33 - 2k = 0$
$\therefore k = 33/2$

Q.15 **Assertion:**

$$\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix} \text{ is}$$

independent of θ

Reason: If $f(\theta) = c$, then $f(\theta)$ is independent of θ.

Sol. **[B]**

$$\Delta = \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

expanding with R_3, we get
$\therefore \Delta = \sin(\beta - \gamma)\,[\sin(\gamma - \beta)\ldots\ldots..]$
$= -\Sigma \sin^2(\beta - \gamma)$
i.e. independent of θ.

Part-D Column Matching type questions

Q.16 **Column-I** **Column-II**

(A) Let $|A| = |a_{ij}|_{3\times3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $|B|$ the resulting Determinant, where $k_1|A| + k_2|B| = 0$. Then $k_1 + k_2 =$ — (P) 0

(B) The maximum value of a third order determinant each of its entries are ± 1 equals — (Q) 4

(C) If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix}$ — (R) 1

$= \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\beta \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$

(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ — (S) 2

$= Ax + B$ where A and B are determinants of order 3. Then
$A + 2B =$

Sol. **A → P ; B → Q ; C → R ; D → P**

(A)
$|A| = |a_{ij}|_{3\times3} \neq 0$

$$\text{Let } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|B| = \begin{vmatrix} k^0.a_{11} & k^{-1}.a_{12} & k^{-2}.a_{13} \\ k^1.a_{21} & k^0.a_{22} & k^{-1}.a_{23} \\ k^2.a_{31} & k^1.a_{32} & k^0.a_{33} \end{vmatrix}$$

Multiply C_2 & C_3 by k & k^2 respectively.
We get

$$|B| = \frac{k}{k^2}.\frac{k^2}{k}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow |B| = |A|$$

Since $k_1|A| + k_2|A| = 0$ (given)
$\Rightarrow (k_1 + k_2)|A| = 0$

$\Rightarrow k_1 + k_2 = 0 \quad \Theta\ |A| \neq 0$ (Given)

(B) $\Theta \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 \underbrace{(b_2c_3 - c_2b_3)}_{\text{Max.2}} - b_1 \underbrace{(a_2c_3 - c_2b_3)}_{\text{zero}}$

$+ c_1 \underbrace{(a_2b_3 - b_2a_3)}_{\text{Max.2}}$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$

$= 1\,(1 + 1) - 1\,(1 - 1) + 1(1 + 1)$

$= 2 - 0 + 2$

$= 4$

(C) On expanding, we get

$1(1 - \cos^2\gamma) - \cos\alpha\,(\cos\alpha - \cos\beta\cos\gamma) + \cos\beta$ $\{(\cos\alpha\cos\gamma) - \cos\beta)\} = -\cos\alpha\,(-\cos\beta\cos\gamma)$ $+ \cos\beta\cos\alpha\cos\gamma$

$\Rightarrow (1 - \cos^2\gamma) - \cos^2\alpha + \cos\alpha\cos\beta\cos\gamma + \cos$ $\alpha\cos\beta\cos\gamma - \cos^2\beta = \cos\alpha\cos\beta\cos\gamma + \cos\alpha\cos\beta$ $\cos\gamma$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1 = 0$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(D) $\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_3 - R_2$, we get

$\begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ & $C_2 \to C_2 - C_3$

$\begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 2 & 3 & 3x - 3 \\ x^2 + 4 & 0 & 2x - 1 \end{vmatrix}$

$= 4\,(6x - 3)$

$= 24x - 12$

$\therefore A = 24, B = -12$

$\therefore A + 2B$

$= 24 - 24 = 0$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Prove that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (ab+bc+ca)(a-b)(b-c)(c-a)$$

Sol. Applying, $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

Now apply $C_1 \to C_1 - C_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 1 & b+c & c^2 \\ 1 & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 0 & b+c+a & c^2-ab \\ 1 & -a & ab \end{vmatrix}$$

$= (a-b)(b-c)(a-c) . 1 \{(c^2 - ab - c(b + c + a)\}$

$= (a-b)(b-c)(a-c)(c^2 - ab - bc - c^2 - ac)$

$= (a-b)(b-c)(a-c)(-ab - bc - ca)$

$= (a-b)(b-c)(c-a)(ab + bc + ca)$

Q.2 If a, b, c are all different and if

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0 \text{ then prove that } abc = -1$$

Sol. Since a, b, c all are different i.e. $a \neq b \neq c$

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}(1 + abc) = 0$$

$\Rightarrow (a-b)(b-c)(c-a)(1 + abc) = 0$

$\because a \neq b, b \neq c, c \neq a$

$\therefore 1 + abc = 0 \Rightarrow abc = -1$

Q.3 Prove that

$$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix} = 0$$

Sol. $\Delta = \Delta_1\Delta_2$

$$= \begin{vmatrix} 1 & 1 & 0 \\ a+b & c+d & 0 \\ ab & cd & 0 \end{vmatrix} . \begin{vmatrix} 1 & 1 & 0 \\ c+d & a+b & 0 \\ cd & ab & 0 \end{vmatrix} = 0$$

Q.4 Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Sol.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ 1 & 1 & 1+c \end{vmatrix}$$

$= a(b(1 + c) + c) + b(c)$

$= a(b + bc + c) + bc$

= ab + abc + ac + bc

= ab + bc + ca + abc

= abc + ab + bc + ca

= abc (1 + 1/a + 1/b + 1/c) (Hence proved).

Q.5 Find the value of k for which the following system of equations is consistent.

$(k + 1)^3x + (k + 2)^3 y = (k + 3)^3$

$(k + 1)x + (k + 2)y = k + 3$

$x + y = 1$

Sol. Given system of equation is

$(k + 1)^3x + (k + 2)^3 y = (k + 3)^3$

$(k + 1)x + (k + 2)y = k + 3$

$x + y = 1$

The system of equations will be consistent if D = 0

$$\begin{vmatrix} (k+1)^3 & (k+2)^3 & -(k+3)^3 \\ k+1 & (k+2) & -(k+3) \\ 1 & 1 & -1 \end{vmatrix} = 0$$

cancel minus from third column

Now put u = (k + 1), v = (k + 2), w = (k + 3)

Then u – v = –1, v –w = –1

w – u = 2 and u + v + w = 3k + 6 …(i)

Also, D = 0 reduces to

$$\begin{vmatrix} u^3 & v^3 & w^3 \\ u & v & w \\ 1 & 1 & -1 \end{vmatrix} = 0$$

or (u – v) (v – w) (w – u) (u + v + w) = 0

or (–1) (–1) (2) (3k + 6) = 0

or (k + 2) = 0 or k = –2

$\therefore$ k = –2

Part-B Passage based objective questions

Passage I (Question 6 to 8)

Consider the determinant $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

M_{ij} = Minor of the element of i^{th} row and j^{th} column

C_{ij} = Cofactor of the element of i^{th} row and j^{th} column

Q.6 Value of $b_1. C_{31} + b_2. C_{32} + b_3. C_{33}$ is

(A) 0 (B) Δ (C) 2Δ (D) Δ^2

Sol. **[A]**

$b_1. C_{31} + b_2. C_{32} + b_3. C_{33}$

Since b_1 belongs to first column and second row where C_{31}, is cofactors of 3^{rd} row and first column. Similarly b_2 belongs to second row and second column and C_{32} is cofactor of 3^{rd} row and 2^{nd} column and b_3 belongs second row and third column.

$\therefore$ sum of these product will be zero.

i.e. $b_1. C_{31} + b_2. C_{32} + b_3. C_{33} = 0$

Q.7 If all the elements of the determinant are multiplied by 2, then the value of new determinant is

(A) 0 (B) 8Δ (C) 2Δ (D) $2^9. \Delta$

Sol. **[B]**

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

All elements are multiplied by two, then

New determinant $\Delta' = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ 2c_1 & 2c_2 & 2c_3 \end{vmatrix}$

$\Delta' = 2 \times 2 \times 2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Delta' = 8\Delta$

Q.8 $a_3 M_{13} - b_3. M_{23} + d_3 . M_{33}$ is equal to

(A) 0 (B) 4Δ (C) 2Δ (D) Δ

Sol. [D] $a_3 M_{13} - b_3. M_{23} + d_3 . M_{33}$ is

Since a_3 belongs to first row and third column and M_{13} is the minor of 1^{st} row and 3^{rd} column element. Similarly we can say that

$a_3M_{13} - b_3. M_{23} + d_3. M_{33} = \Delta$

Passage II (Question 9 to 11)

Let x, y, z ∈ R^+ & $\Delta = \begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix}$

Q.9 If x ≠ y ≠ z & x, y, z are in GP and Δ = 0 then y is equal to -

(A) 1 (B) 2 (C) 4 (D) None

Sol. **[A]**

Q.10 If x, y, z are roots of $t^3 - 21t^2 + bt - 343 = 0$, $b \in R$ then $\Delta =$

(A) 1 (B) 0

(C) depends on x, y, z (D) data inadequate

Sol. **[B]**

Q.11 If x, y, z are in A.P. and $\Delta = 0$ then $2xy^2z + x^2z^2 =$

(A) 1 (B) 2 (C) 3 (D) None

Sol. **[C]**

EXERCISE # 4

Old IIT-JEE questions

Q.1 Let a , b , c be real no. with $a^2 + b^2 + c^2 = 1$ then show that

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

represent a straight line. **[IIT- 2001]**

Sol. Applying $C_1 \to aC_1$

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2x - aby - ac & bx + ay & cx + a \\ abx + a^2y & -ax + by - c & cy + b \\ acx + a^2 & cy + b & -ax - by + c \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Delta = \frac{1}{a}\begin{vmatrix} (a^2 + b^2 + c^2)x & ay + bx & cx + a \\ (a^2 + b^2 + c^2)y & by - c - ax & cy + b \\ (a^2 + b^2 + c^2) & b + cy & -ax - by + c \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} x & ay + bx & cx + a \\ y & by - c - ax & b + cy \\ 1 & b + cy & c - ax - by \end{vmatrix}$$

As $a^2 + b^2 + c^2 = 1$

$C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

Then $\Delta = \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix}$

$$= \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix}$$

Applying $R_1 \to R_1 + yR_2 + R_3$

$$\Delta = \frac{1}{ax}\begin{vmatrix} x^2 + y^2 + 1 & 0 & 0 \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix}$$

On expanding along R_1

$$\Delta = \frac{(x^2 + y^2 + 1)}{ax}\, ax\,(ax + by + c)$$

$= (x^2 + y^2 + 1)(ax + by + c)$

Given $\Delta = 0$,

$\Rightarrow ax + by + c = 0$

which represents a straight line.

Q.2 The number of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0 \text{ in the interval}$$

$-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is - **[IIT scr- 2001]**

(A) 0 (B) 2 (C) 1 (D) 3

Sol. **[C]**

To simplify the determinant let sin x = a, cos x = b then equation becomes

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0$$

operating $C_2 - C_1$ & $C_3 - C_2$, we get

$$\begin{vmatrix} a & b - a & 0 \\ b & a - b & b - a \\ b & 0 & a - b \end{vmatrix} = 0$$

$\Rightarrow a(a - b)^2 - (b - a)$

$[b(a - b) - b(b - a)] = 0$

$\Rightarrow a(a - b)^2 - 2b(b - a)(a - b) = 0$

$\Rightarrow (a - b)^2 (a - 2b) = 0$

$\Rightarrow (a = b)$ or $a = 2b$

$\Rightarrow \frac{a}{b} = 1$ or $\frac{a}{b} = 2$

$\Rightarrow \tan x = 1$ or $\tan x = 2$

But $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

$\Rightarrow \tan\left(-\frac{\pi}{4}\right) \le \tan x \le \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow -1 \le \tan x \le 1$

$\therefore \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

$\therefore$ only one real root is there,

Hence (C) option is correct.

Q.3 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is- **[IIT 2002S]**

(A) 3ω (B) $3\omega(\omega - 1)$
(C) $3\omega^2$ (D) $3\omega(1 - \omega)$

Sol. **[B]**

Given that $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\therefore \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Also $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

Now given determinant is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

($\because \omega = -1 - \omega^2$ and $\omega^3 = 1$)

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & \omega \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

Expanding along C_1, we get

$3(\omega^2 - \omega^4)$ $3(\omega^2 - \omega)$

$= 3\omega(\omega - 1)$

Q.4 The number of values of k for which the system of equations $(k + 1)x + 8y = 4k$; $kx + (k + 3)y = 3k - 1$ has infinitely many solutions is- **[IIT 2002S]**

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. **[B]**

For infinitely many solutions the two equations become identical

$$\Rightarrow \frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1}$$

$\Rightarrow k = 1$

$\therefore$ one value of k.

Q.5 If $x + ay = 0$; $y + az = 0$; $z + ax = 0$, then value of 'a' for which system of equations will have infinite number of solutions is **[IIT scr- 2003]**

(A) a = 1 (B) a = 0
(C) a = –1 (D) no value of a

Sol. **[C]**

The given system is,

$x + ay = 0$

$az + y = 0$

$ax + z = 0$

it is system of homogeneous equations, therefore it will have infinite many solution, if determinant of coefficient matrix is zero,

i.e. $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow 1(1 - 0) - a(0 - a^2) = 0$

$\Rightarrow 1 + a^3 = 0$

$\Rightarrow a^3 = -1$

$\Rightarrow a = -1$

Q.6 If the system of equations

$2x - y - 2z = 2$

$x - 2y + z = -4$

$x + y + \lambda z = 4$

has no solutions then λ is equal to-**[IIT scr- 2004]**

(A) – 2 (B) 3
(C) 0 (D) – 3

Sol. **[D]**

Since the system has no solution

$$\begin{vmatrix} 2 & -1 & -4 \\ 1 & -2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 2(-2\lambda + 1) + 1(\lambda + 1) - 4(3) = 0$

$\Rightarrow -4\lambda + 2 + \lambda + 1 - 12 = 0$

$\Rightarrow -3\lambda = 9$

$\Rightarrow \lambda = -3$

Q.7 Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

Assertion: The system of equations has no solution for $k \neq 3$ and

Reason : The determinant

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0, \text{ for } k \neq 3$$ **[IIT 2008]**

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Sol. **[A]**

since $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$

$\therefore$ for having either $\Delta_x \neq 0$ or $\Delta_y \neq 0$ or $\Delta_z \neq 0$ no solution

$\therefore \ \Delta_x = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} \neq 0$

$\Rightarrow \ 3 - k \neq 0 \Rightarrow \ k \neq 3$

Now again

$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0 \Rightarrow k \neq 3$

Q.8 The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$(y + z) \cos 3\theta = (xyz) \sin 3\theta$

$x \sin 3\theta = \dfrac{2\cos 3\theta}{y} + \dfrac{2\sin 3\theta}{z}$

$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta)$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is

[IIT- 2010]

Sol. **[3]**

$(xyz) \sin 3\theta + y (- \cos 3\theta) + z (- \cos 3\theta) = 0$

$(xyz) \sin 3\theta + y (-2 \sin 3\theta) + z (-2 \cos 3\theta) = 0$

$(xyz) \sin 3\theta + y (- \cos 3\theta - \sin 3\theta) + z (- 2\cos 3\theta) = 0$

For $y_0 z_0 \neq 0 \Rightarrow$ Nontrivial solution

$$\begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\sin 3\theta & -2\cos 3\theta \\ \sin 3\theta & -\cos 3\theta - \sin 3\theta & -2\cos 3\theta \end{vmatrix} = 0$$

$$\sin 3\theta \cos 3\theta \begin{vmatrix} 1 & \cos 3\theta & 1 \\ 1 & 2\sin 3\theta & 2 \\ 1 & \cos 3\theta + \sin 3\theta & 2 \end{vmatrix} = 0$$

$\sin 3\theta \cos 3\theta \, [(4\sin 3\theta - 2\cos 3\theta - 2\sin 3\theta) - (2\cos 3\theta - \cos 3\theta - \sin 3\theta) + 2\cos 3\theta - 2\sin 3\theta] = 0$

$\Rightarrow (\sin 3\theta \cos 3\theta) \, [2\sin 3\theta - 2\cos 3\theta - \cos 3\theta + \sin 3\theta + 2\cos 3\theta - 2\sin 3\theta] = 0$

$\Rightarrow (\sin 3\theta \cos 3\theta) (\sin 3\theta - \cos 3\theta) = 0$

$$\left.\begin{array}{ll} \Rightarrow \sin 3\theta = 0 & \Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3} \\ \Rightarrow \cos 3\theta = 0 & \Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6} \end{array}\right\}$$

These two donot satisfy system of equations

$\Rightarrow \sin 3\theta = \cos 3\theta \ \Rightarrow 3\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$

$\Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4} = 3$

No. of solutions = 3

EXERCISE # 5

Q.1 If $f_r(x)$, $g_r(x)$, $h_r(x)$, r = 1, 2, 3 are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, r = 1, 2, 3 and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} \text{ then}$$

F′(x) at x = a is .... **[IIT - 85]**

Sol. $$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$\therefore F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$$\therefore F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$\because f_r(a) = g_r(a) = h_r(a)$, r = 1, 2, 3

$\therefore F'(a) = 0$

Q.2 Consider the system of linear equation in x, y, z

$(\sin 3\theta)\, x - y + z = 0$

$(\cos 2\theta)\, x + 4y + 3z = 0$

$2x + 7y + 7z = 0$

Find the values of θ for which this system has non-trivial solution **[IIT - 86]**

Sol. The system will have a non trivial solution if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1, we get

$\Rightarrow (28 - 21)\sin 3\theta - (-7-7)\cos 2\theta + 2(-3-4) = 0$

$\Rightarrow 7\sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0$ or $\sin\theta = \dfrac{1}{2}$

$\left(\sin\theta = -\dfrac{3}{2}\right.$ not possible)

$\Rightarrow \theta = n\pi$ or $n\pi + (-1)^n \dfrac{\pi}{6}$, $n \in Z$

Q.3 Let

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

then prove that $\displaystyle\int_0^{\pi/2} f(x)dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$

[IIT - 87]

Sol. $$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 - \sec x\, R_3$, we get

$$\begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1, we get

$= (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(\cos^4 x - \cos^2 x)$

$$= \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x\right)\cos^2 (\cos^2 x - 1)$$

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\cos^2 x \sin^2 x}\right]\cos^2 x \sin^2 x$$

$= -\sin^2x - \cos^3x\,(1-\sin^2x) = -\sin^2x - \cos^5x$

$$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} (\sin^2 x + \cos^5 x)\,dx$$

$$= -\left[\frac{1}{2}.\frac{\pi}{2} + \frac{4}{5}.\frac{2}{3}\right] = -\left[\frac{\pi}{4} + \frac{8}{15}\right]$$

Use

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{(n-1)\,(n-3).....2 \text{ or } 1}{(n)(n-2)......2}$$

And multiply above by $\pi/2$ when n is even.

Q.4 The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is…….. **[IIT - 88]**

Sol. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$= (a-b)\,(b-c)\,(c-a) - (a-b)\,(b-c)\,(c-a)$

$= 0$

Q.5 The value of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation **[IIT - 88]**

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ are -}$$

(A) $\frac{7\pi}{24}$ (B) $\frac{5\pi}{24}$ (C) $\frac{11\pi}{24}$ (D) $\frac{\pi}{24}$

Sol. **[A, C]**

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}$$

Apply $R_3 - R_2$ & $R_2 - R_1$

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

Apply $C_1 + C_2$, we get

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2 + 4\sin 4\theta = 0$

$\Rightarrow \sin 4\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\Rightarrow \theta = [6n - (-1)^n]\frac{\pi}{24}$

$\Rightarrow$ for n = 1, 2

$\theta = \frac{7\pi}{24}$ and $\frac{11\pi}{24} \in \left(0, \frac{\pi}{2}\right)$

Q.6 Let $\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$ show that $\sum_{a=1}^{n} \Delta_a = c$, a constant. **[IIT - 89]**

Sol. $\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

Then $\sum_{a=1}^{n} \Delta_a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} +$

$\begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \ldots$

$\ldots + \begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$= \begin{vmatrix} 1+2+3+.....+(n-1) & n & 6 \\ 1^2+2^2+3^2+......+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+.....+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{12}\begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

= 0 [$\because$ C_1 and C_3 are identical]

Q.7 Let the three digit numbers A28, 3B9, 62C where A, B and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

[IIT - 90]

Sol. Given that A, B, C are integers between 0 and 9 and the three digit numbers A 28, 3B9 and 62C are divisible by a fixed integer k.

Now, $D = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

Applying $R_2 \to R_2 + 10R_3 + 100R_1$, we get

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ kn_1 & kn_2 & kn_3 \\ 2 & B & 2 \end{vmatrix}$$

As A28, 3B9 and 62C are divisible by k.

$\Rightarrow$ A28 = kn_1

3B9 = kn_2

62C = kn_3

$$= k\begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix}$$

= kx. Some integral value

$\Rightarrow$ D is divisible by k.

Q.8 If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$

[IIT - 91]

Sol. Given $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Apply $R_1 - R_2$ and $R_2 - R_3$, we get

$$\begin{vmatrix} p-a & -(q-b) & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

Taking (p – a), (q – b) and (r – c) common from C_1, C_2 and C_3 respectively, we get

$$(p-a)(q-b)(r-c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

Expanding along R_1,

$\Rightarrow$ (p – a) (q – b) (r – c)

$$\left[1\left(\frac{r}{r-c} + \frac{b}{q-b}\right) + \frac{a}{p-a}\right] = 0$$

$\because$ $p \neq a$, $q \neq b$, $r \neq c$ therefore

$$\frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

$$\Rightarrow \frac{r}{r-c} + \frac{q-(q-b)}{q-b} + \frac{p-(p-a)}{p-a} = 0$$

$$\Rightarrow \frac{r}{r-c} + \frac{q}{q-b} - 1 + \frac{p}{p-a} - 1 = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Q.9 For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

then show that $[(D/(n!)^3] - 4]$ is divisible by n.

[IIT - 92]

Sol. Given that,

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$= n!\,(n+1)!\,(n+2)! \begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Apply $R_2 - R_1$ and $R_3 - R_1$, we get

$= (n!)^3 (n+1)^2 (n+2)$

$$\begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Apply $R_3 - R_2$,

$$(n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 0 & 2 \end{vmatrix}$$

$= (n!)^3 (n+1)^2 (n+2)\, 1(2)$

$\Rightarrow \dfrac{D}{(n!)^3} = 2(n+1)^2 (n+2)$

$\Rightarrow \dfrac{D}{(n!)^3} - 4 = 2(n^3 + 4n^2 + 5n + 2) - 4$

$= 2(n^3 + 4n^2 + 5n) = 2n(n^2 + 4n + 5)$

$\Rightarrow \dfrac{D}{(n!)^3} - 4$ is divisible by n.

Q.10 Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$; $x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$; $-x + (\sin\alpha)\, y - (\cos\alpha)\, z = 0$ has a non - trivial solution. For $\lambda = 1$, find all values of α. **[IIT - 93]**

Sol. Given that, $\lambda,\ \alpha \in R$ and system of linear equations

$\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$

$x + (\cos\alpha)\, y + (\sin\alpha) z = 0$

$-x + (\sin\alpha) y - (\cos\alpha) z = 0$

Has a non-trivial solution $\Rightarrow D = 0$

$$\Rightarrow \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \lambda\,(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha\,(-\cos\alpha + \sin\alpha) + \cos\alpha\,(\sin\alpha + \cos\alpha) = 0$

$\Rightarrow -\lambda + \sin\alpha\cos\alpha - \sin^2\alpha + \sin\alpha\cos\alpha + \cos^2\alpha = 0$

$\Rightarrow \lambda = \cos^2\alpha - \sin^2\alpha + 2\sin\alpha\cos\alpha$

$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$

For $\lambda = 1$

$\cos 2\alpha + \sin 2\alpha = 1$

$\Rightarrow \dfrac{1}{\sqrt{2}}\cos 2\alpha + \dfrac{1}{\sqrt{2}}\sin 2\alpha = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos 2\alpha \cos\dfrac{\pi}{4} + \sin 2\alpha \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(2\alpha - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4}$

$\Rightarrow 2\alpha - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4}$

$\Rightarrow 2\alpha = 2n\pi \pm \dfrac{\pi}{4} + \dfrac{\pi}{4}$

$\Rightarrow \alpha = n\pi + \dfrac{\pi}{4}$ or $n\pi$

Q.11 For positive numbers x, y and z, the numerical value of the determinant

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$ is……… **[IIT - 93]**

Sol.
$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

Taking $\dfrac{1}{\log x}, \dfrac{1}{\log y}, \dfrac{1}{\log z}$ common from R_1, R_2, R_3 respectively

$$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

Q.12 If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$ then the two triangles with vertices (x_1, y_1) $(x_2. y_2)$, (x_3, y_3) and (a_1,b_1), (a_2, b_2), (a_3, b_3) must be congruent. **[T/F]**

[IIT - 95]

Sol. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$\Rightarrow$ Area of Δ_1 = Area of Δ_2

But two Δ's of same area may not be congruent.

$\therefore$ Given statement is false.

Q.13 If ω $(\neq 1)$ is a cube root of unity, then $\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$ equals **[IIT - 95]**

(A) 0 (B) 1

(C) i (D) ω

Sol. **[A]**

$R_1 \to R_1 - R_2 + R_3$

$\begin{vmatrix} 0 & 1+\omega+\omega^2 & 0 \\ 1-i & -1 & \omega^2-1 \\ -1 & -i+\omega-1 & -1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 \\ 1-i & -1 & \omega^2-1 \\ -1 & -i+\omega-1 & -1 \end{vmatrix}$

$= 0$

Q.14 Let a, b, c be the real numbers. Then following system of equations in x, y and z

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,

$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ has **[IIT - 95]**

(A) no solution

(B) unique solution

(C) infinitely many solutions

(D) finitely many solutions

Sol. **[B]**

Let $\frac{x^2}{a^2}$ = X, $\frac{y^2}{b^2}$ = Y and $\frac{z^2}{c^2}$ = Z

Then given system of equation is

X + Y – Z = 1

X – Y + Z = 1

–X + Y + Z = 1

Coefficient matrix A is given by

$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$|A| = 1(-1-1) - 1(1+1) - 1(1-1) = -4 \neq 0$

$\therefore$ Given system of equation has a unique solution.

Q.15 Let a > 0, d > 0. Find the value of the determinant **[IIT - 96]**

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Sol. Let us denote the given determinant by Δ and taking $\frac{1}{a(a+d)(a+2d)}$ as common from R_1 and

$\frac{1}{(a+d)(a+2d)(a+3d)}$ from R_2 and

$\frac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3, we get

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta_1$$

where $\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta_1 = (2d^2)\begin{vmatrix} a+2d & a \\ d & d \end{vmatrix} = (2d^2)(d)(a + 2d - a) = 4d^4$$

$$\Rightarrow \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

Q.16 Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ where a, b and c are respectively the p^{th}, q^{th} and r^{th} terms of a harmonic progression. **[IIT - 97]**

Sol. Given that a, b, c are p^{th}, q^{th} and r^{th} terms of a H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are p^{th}, q^{th}, r^{th} terms of an A.P.

$$\left.\begin{aligned} \frac{1}{a} &= A+(p-1)D \\ \frac{1}{b} &= A+(q-1)D \\ \frac{1}{c} &= A+(r-1)D \end{aligned}\right\} \quad \dots (1)$$

Now given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Substituting the values of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ from (1), we get

$$= abc \begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $R_1 \to R_1 - (A - D) R_3 - DR_2$, we get

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\Delta = 0$

Q.17 The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ if **[IIT - 97C]**

(A) x, y, z are in A.P.
(B) x, y, z are in G.P.
(C) x, y, z are in H.P.
(D) xy, yz, zx are in A.P.

Sol. **[B]**

$$\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$$

operating $C_1 - pC_2 - C_3$, we get

$$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+2py+z) & xp+y & yp+z \end{vmatrix} = 0$$

$\Rightarrow (xz - y^2)(xp^2 + 2py + z) = 0$

$\Rightarrow xz - y^2 = 0$

$\Rightarrow y^2 = xz$

$\Rightarrow$ x, y, z are in G.P.

Q.18 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x = 0$ is- **[IIT - 97]**

(A) p (B) $p + p^2$
(C) $p + p^3$ (D) independent of p

Sol. **[D]**

We are given $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is constant. Now keeping in mind that

$$\frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} +$$

$$\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

We get $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0 \ (\because R_1 \equiv R_2)$

= Independent of p.

Q.19 The parameter on which the value of the determinant

$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not

depend upon is - **[IIT - 97]**

(A) a (B) p (C) d (D) x

Sol. **[B]**

$\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2$

$\Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x+\cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x+\sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px\ \cos dx & \cos px & \cos(p+d)x \\ 2\sin px\ \cos dx & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 - 2\cos dx\, C_2$

$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$

expanding along C_1, we get

$\Delta = (1 + a^2 - 2a \cos dx)$

$[\sin (p + d) x \cos px - \sin px \cos (p + d) x]$

$\Rightarrow \Delta = (1 + a^2 - 2a \cos dx) [\sin\{ (p + d) x - px\}]$

$\Rightarrow \Delta = (1 + a^2 - 2a \cos dx) [\sin dx]$

Which is independent of p.

Q.20 Suppose f(x) is function satisfying the following conditions

(a) f(0) = 2. f(1) = 1

(b) f has minimum value at x = 5/2 and

(c) for all x

$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

where a, b are some constants. Determine the constants a, b and the function f(x). **[IIT - 98]**

Sol. Applying $R_3 \to R_3 - R_1 - 2R_2$, we get

$f'(x) = \begin{vmatrix} 2ax & 2ax-a & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix}$

$= \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix}$ $[C_2 \to C_2 - C_1]$

$\Rightarrow f'(x) = 2ax + b$

Integrating, we get, $f(x) = ax^2 + bx + c$

Where c is an arbitrary constant since f has maximum at x = 5/2

$f'(5/2) = 0 \Rightarrow 5a + b = 0$ … (i)

Also $f(0) = 2 \Rightarrow c = 2$ and $f(1) = 1$

$\Rightarrow a + b + c = 1$

$\therefore a + b = -1$ …(ii)

From (i) and (ii) we get $a = \frac{1}{4}$, $b = -\frac{5}{4}$

Thus, $f(x) = \frac{x^2}{4} - \frac{5x}{4} + 2$

Q.21 Let a, b, c, d be real numbers in G.P. If u , v , w satisfy the system of equations, u + 2v + 3w = 6 ; 4u + 5v + 6w = 12 ; 6u + 9v = 4,then show that the roots of the equation $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 +$ $[(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$ and $20x^2 + 10 (a-d)^2 x - 9 = 0$ are reciprocals of each other. **[IIT - 99]**

Sol. System of equations is

u + 2v + 3w = 6

$4u + 5v + 6w = 12$

$6u + 9v = 4$

On solving the above system of equations, we get

$u = -\frac{1}{3}, v = \frac{2}{3}, w = \frac{5}{3}$

$\therefore u + v + w = 2, \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = -\frac{9}{10}$

Let r be the common ratio of the G.P. a, b, c, d. Then $b = ar, c = ar^2, d = ar^3$

Then the first equation

$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (a-b^2)]\, x + (u + v + w) = 0$

Becomes

$\frac{-9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]\, x + 2 = 0$

i.e $9x^2 - 10a^2(1-r)^2\, [r^2 + (r+1)^2 + r^2 (r+1)^2]x - 20 = 0$

i.e. $9x^2 - 10a^2(1-r)^2\, (r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$

$\Rightarrow 9x^2 - 10a^2\, (1-r^3)^2\, x - 20 = 0 \quad \ldots(i)$

The second equation is, $20x^2 + 10\,(a - ar^3)^2\, x - 9 = 0$

i.e. $20x^2 + 10a^2(1-r^3)^2\, x - 9 = 0 \quad \ldots.(ii)$

Since (ii) can be obtained by the substitution $x \to \frac{1}{x}$

Equation (i) and (ii) have reciprocal roots.

Q.22 If f(x) =

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix},$$

then f(100) is equal to - **[IIT - 99]**

(A) 0 (B) 1

(C) 100 (D) – 100

Sol. **[A]**

f(x) =

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

Apply $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - (x-1)\, R_2$

$$= \begin{vmatrix} 1 & x & x+1 \\ 2 & x(x-1)-x^2 & 0 \\ x(x-1) & x(x-1)(x-2)-x(x-1)^2 & 0 \end{vmatrix}$$

$$\Rightarrow f(x) = x\,(x-1) \begin{vmatrix} 1 & x & x+1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 0$$

$\therefore f(100) = 0$

Q.23 Prove that for all values of θ ; **[IIT - 2000]**

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin(\theta+2\pi/3) & \cos(\theta+2\pi/3) & \sin(2\theta+4\pi/3) \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix} = 0$$

Sol. Applying $R_2 \to R_2 + R_3$, we get

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos\frac{2\pi}{3} & 2\sin\theta\cos\frac{2\pi}{3} & 2\sin 2\theta\cos\frac{4\pi}{3} \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $2 \sin\theta \cos\frac{2\pi}{3} = 2\sin\theta\cos\left(\pi-\frac{\pi}{3}\right)$

$= -2\sin\theta\cos\frac{\pi}{3} = -\sin\theta$

$2\cos\theta\cos\frac{2\pi}{3} = 2\cos\theta\left(\frac{-1}{2}\right) = -\cos\theta$

$2\sin 2\theta\cos\frac{4\pi}{3} = 2\sin 2\theta\cos\left(\pi+\frac{\pi}{3}\right)$

$= 2\sin 2\theta\cos\frac{\pi}{3}$

$= -\sin 2\theta$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$= 0$

$(\because R_1 = -R_2)$ proved.

Q.24 If the system of equations

$x - ky - z = 0,\ kx - y - z = 0,\ x + y - z = 0$ has a non-zero solution, then the possible values of k are- **[IIT 2000]**

(A) –1, 2 (B) 1, 2

(C) 0, 1 (D) –1, 1

Sol. **[D]**

For the given homogeneous system to have non-zero solution, determinant of coefficient matrix should be zero, i.e. $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$= 1\,(1 + 1) + k\,(-k + 1) - 1\,(k + 1) = 0$

$\Rightarrow 2 - k^2 + k - k - 1 = 0$

$\Rightarrow k^2 = 1$

$\Rightarrow k = \pm 1$

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10
Ans.	B	B	C	A	D	A	A	D	A	A

11. True

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8
Ans.	A	A	A	B	C	B	D	A

(PART-B)

Q.No.	9	10	11	12	13
Ans.	A,B,C,D	B,C	A	A,B	B,C,D

(PART-C)

14. D **15.** B

(PART-D)

16. A $\to$ P; B $\to$ Q; C $\to$ R; D $\to$ P

EXERCISE # 3

5. k = – 2 **6.** A **7.** B **8.** D **9.** A **10.** B **11.** C

EXERCISE # 4

1. 1 **2.** C **3. B** **4.** B **5.** C **6.** D **7.** A **8.** 3

EXERCISE # 5

1. 0 **2.** $\theta = m\pi$, or $\theta = n\pi + (-1)^n \frac{\pi}{6}\ \forall\ m, n \in I$ **4.** 0 **5.** A,C **8.** 2

10. $\lambda = \sqrt{2}\ \sin(2\alpha = \pi/4)$; $\alpha = \pi/8, 7\pi/8, 9\pi/8$ **11.** 0 **12.** False **13.** A

14. D **15.** $\frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$ s **16.** zero **17.** B **18.** D

19. B **20.** $a = \frac{1}{4}, b = -\frac{5}{4}; f(x) = \frac{1}{4}(x^2 - 5x + 8)$ **22.** A **24.** D