DEFINITE INTEGRATION
EXERCISE # 1

Questions based on Definition of Definite Integration

Q.1 $\int_0^{\pi/4} \tan^2 x\,dx$ equals-

(A) π/4 (B) 1 + (π/4)
(C) 1– (π/4) (D)1– (π/2)

Sol. **[C]**

$I = \int_0^{\pi/4} (\sec^2 x - 1)\,dx = [\tan x - x]_0^{\pi/4}$

$= 1 - \frac{\pi}{4}$

Q.2 $\int_0^{\pi/4} \frac{\sec^2 x}{(1+\tan x)(2+\tan x)}dx$ equals -

(A) $\log_e \frac{2}{3}$ (B) $\log_e 3$

(C) $\frac{1}{2}\log_e \frac{4}{3}$ (D) $\log_e \frac{4}{3}$

Sol. **[D]**

Let tan x = t ⇒ $\sec^2 x\,dx = dt$

$I = \int_0^1 \frac{dt}{(1+t)(2+t)} = [\lambda n(1+t) - \lambda n(2+t)]_0^1$

$= \lambda n\, 2 - \lambda n\, 3 + \lambda n\, 2 = \lambda n \frac{4}{3}$

Questions based on Properties of Definite Integral

Q.3 If $f(x) = \begin{cases} x^2, & \text{when } 0 \le x < 1 \\ \sqrt{x}, & \text{when } 1 \le x < 2 \end{cases}$, then $\int_0^2 f(x)\,dx$ equals -

(A) $\frac{1}{3}(4\sqrt{2} - 1)$ (B) $\frac{1}{3}(4\sqrt{2} + 1)$

(C) 0 (D) does not exist

Sol. **[A]**

$I = \int_0^1 x^2 dx + \int_1^2 \sqrt{x}\,dx = \left[\frac{x^3}{3}\right]_0^1 + \frac{2}{3}[x^{3/2}]_1^2$

$= \frac{1}{3} + \frac{2}{3}(2^{3/2} - 1) =$

$\frac{1+4\sqrt{2}-2}{3} = \frac{1}{3}(4\sqrt{2} - 1)$

Q.4 $\int_0^1 |3x - 1|\,dx$ equals -

(A) 5/6 (B) 5/3 (C) 10/3 (D) 5

Sol. **[A]**

$I = -\int_0^{1/3} (3x-1)\,dx + \int_{1/3}^1 (3x-1)\,dx$

$= \left[x - \frac{3x^2}{2}\right]_0^{1/3} + \left[\frac{3x^2}{2} - x\right]_{1/3}^1$

$= \frac{1}{3} - \frac{1}{6} + \frac{3}{2} - 1 - \frac{1}{6} + \frac{1}{3} = \frac{5}{6}$

Q.5 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ equals -

(A) 0 (B) π/4 (C) $\pi^2/4$ (D) $\pi^2/2$

Sol. **[C]**

$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$

$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx$

$I = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$ [cos x = t]

$I = \frac{\pi}{2}[\tan^{-1} t]_{-1}^{1} = \frac{\pi}{2}.\frac{\pi}{2} = \frac{\pi^2}{4}$

Q.6 $\int_0^{\pi/2} \frac{dx}{1+\cot x}$ equals -

(A) 1 (B) $\frac{\pi}{4}$ (C) $\frac{1}{2}$ (D) $\frac{\pi}{2}$

Sol. **[B]**

$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx$

$I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}dx$

$$2I = \int_0^{\pi/2} dx$$

$$I = \frac{1}{2}[x]_0^{\pi/2} = \frac{\pi}{4}$$

Q.7 $\int_0^1 \frac{dx}{(x^2-2x+2)^3} =$

(A) $\frac{3\pi+8}{32}$ (B) $\frac{\pi+1}{4}$

(C) 0 (D) None of these

Sol. **[A]**

$$I = \int_0^1 \frac{dx}{((x-1)^2+1)^3} = \int_0^1 \frac{dx}{(x^2+1)^3}$$

Let x = tan θ ⇒ dx = $\sec^2\theta\, d\theta$

$$\int_0^{\pi/4} \frac{\sec^2\theta}{\sec^6\theta} d\theta = \int_0^{\pi/4} \cos^4\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} (1+\cos 2\theta)^2\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} (1+\cos 2\theta+\cos^2 2\theta)\, d\theta$$

$$= \frac{1}{4}\int_{0`}^{\pi/4} (1+\cos 2\theta)\, d\theta + \frac{1}{8}\int (1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} + \frac{1}{8}\left(\theta + \frac{\sin 4\theta}{4}\right)_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\pi}{4}+\frac{1}{2}\right] + \frac{1}{8}\left[\frac{\pi}{4}\right] = \frac{3\pi+8}{32}$$

Q.8 $\int_{-3}^{3} \frac{x^2 \sin x}{1+x^6} dx$ equals-

(A) 4 (B) 2

(C) 0 (D) None of these

Sol. **[C]**

$$I = \int_{-3}^{3} \frac{x^2\sin x}{1+x^6} dx = 0 \text{ (odd function)}$$

Q.9 $\int_{-1}^{1} (\sqrt{1+x+x^2} - \sqrt{1-x+x^2})\, dx$ equals-

(A) 1 (B) 0 (C) $\sqrt{2}$ (D) 2

Sol. **[B]**

$$I = \int_{-1}^{1} \left(\sqrt{1+x+x^2} - \sqrt{1-x+x^2}\right) dx = 0$$

odd function

Q.10 $\int_0^{2\pi} \cos^4 x\, dx$ equals-

(A) 3π/8 (B) 3π/4 (C) 3π/2 (D) 3π

Sol. **[B]**

$$I = \int_0^{2\pi} \cos^4 x\, dx = 2\int_0^{\pi} \cos^4 x\, dx$$

$$= 4\int_0^{\pi/2} \cos^4 x\, dx$$

$$= \int_0^{\pi/2} (1+\cos 2x)^2\, dx$$

Solving we get I = $\frac{3\pi}{4}$

Q.11 $\int_0^{2\pi} \frac{\sin 2\theta}{a-b\cos\theta} d\theta$ equals -

(A) 1 (B) 2 (C) π/4 (D) 0

Sol. **[D]**

$$I = \int_0^{2\pi} \frac{\sin 2\theta}{a-b\cos\theta} d\theta$$

$$I = \int_0^{2\pi} \frac{\sin(4\pi-2\theta)}{a-b\cos(2\pi-\theta)} d\theta = -\int_0^{2\pi} \frac{\sin 2\theta}{a-b\cos\theta}$$

⇒ I = 0

Q.12 $\int_0^{400\pi} \sqrt{1-\cos 2x}\, dx$ is equal to-

(A) 400 $\sqrt{2}$ (B) 800 $\sqrt{2}$

(C) 0 (D) None of these

Sol. **[B]**

$$I = \int_0^{400\pi} \sqrt{1-\cos 2x}\, dx$$

$$I = \sqrt{2}\int_0^{400\pi} |\sin x|\, dx$$

$$= 400\sqrt{2}\int_0^{\pi} |\sin x|\, dx$$

$$= 800\sqrt{2}\int_0^{\pi/2} \sin x\, dx$$

$$= 800\sqrt{2}\,[-\cos x]_0^{\pi/2} = 800\sqrt{2}$$

Q.13 The value of $\int_0^{1000} e^{x-[x]}\, dx$ is -

(A) $\frac{e^{1000}-1}{1000}$ (B) $\frac{e^{1000}-1}{e-1}$

(C) 1000(e – 1) (D) $\frac{e-1}{1000}$

Sol. **[C]**

$I = 1000\int_0^1 e^{x-[x]}\,dx$

$= 1000\int_0^1 e^x\,dx = 1000\,(e-1)$

Questions based on Some important Formulae

Q.14 $\int_0^{\pi/2} \log\cos x\,dx$ equals-

(A) (π/2) log (1/2) (B) πlog 2

(C) –πlog 2 (D) 2πlog 2

Sol. **[A]**

$\Theta \int_0^{\pi/2} \log\cos x = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$

Q.15 $\int_0^{\pi/2} \sin^7 x\cos x\,dx$ equals -

(A) 1/7 (B) 1/8

(C) π/16 (D) π/14

Sol. **[B]**

$\int_0^{\pi/2} \sin^7 x\cos x\,dx$

Put sin x = t ⇒ cos x dx = dt

$\int_0^1 t^7\,dt = \left[\frac{t^8}{8}\right]_0^1 = \frac{1}{8}$

Q.16 $\int_0^{\pi/2} \sin^5 x\,dx$ equals -

(A) 8/15 (B) 4/15

(C) $\frac{8\sqrt{\pi}}{15}$ (D) $\frac{8\pi}{15}$

Sol. **[A]**

$I = \int_0^{\pi/2} \sin^4 x\sin x\,dx$

$= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\,dx$

= Put cos x = t ⇒ – sin x dx = dt

$I = \int_0^1 (1-t^2)^2\,dt = \int_0^1 (1-2t^2+t^4)\,dt$

$= \left[t - \frac{2}{3}t^3 + \frac{t^5}{5}\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$

Questions based on Summation of series by Integration

Q.17 $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{3n}\right)$ equals-

(A) log 2 (B) log 4

(C) 0 (D) $\log_e 3$

Sol. **[D]**

$\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{1+\frac{r}{n}}.\frac{1}{n}$

$= \int_0^2 \frac{1}{1+x}\,dx = [\lambda n(1+x)]_0^2 = \lambda n\,3$

Q.18 $\lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+.....+\frac{r^2}{r^3+n^3}+.....+\frac{1}{2n}\right] =$

(A) (1/2) log 3 (B) (1/3) log 2

(C) 3 log 2 (D) (1/2) log 2

Sol. **[B]**

$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2/n^2}{\frac{r^3}{n^3}+1}.\frac{1}{n}$

$= \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\left[\lambda n(1+x^3)\right]_0^1 = \frac{1}{3}\lambda n\,2$

Q.19 $\lim_{n\to\infty}$

$\left\{\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2}....\left(1+\frac{n^2}{n^2}\right)^{2n/n^2}\right\}$

is equal to -

(A) e/4 (B) 4/e

(C) 1 (D) none of these

Sol.[B] Taking log both side we gety

$\log s = \frac{2}{n^2}\log\left(1+\frac{1^2}{n^2}\right) + \frac{4}{n^2}\log\left(1+\frac{2^2}{n^2}\right) +$

$......+\frac{2n}{n^2}\log\left(1+\frac{n^2}{n^2}\right)$

$$= \sum_{r=1}^{n} \frac{2r}{n^2} \log\left(1+\frac{r^2}{n^2}\right) = \int_0^1 2x \log(1+x^2)dx$$

put $1 + x^2 = t \Rightarrow 2xdx = dt$

$$= \int_1^2 \log t\, dt = [t\log t - t]_1^2 = 2\log 2 - 2 + 1$$

$$\log s = \log 4 - \log e \Rightarrow s = \frac{4}{e}$$

Questions based on Estimating an integral

Q.20 If $a < \int_0^{2\pi} \frac{dx}{10+3\cos x} < b$ then the ordered pair (a, b) =

(A) $\left(\frac{2\pi}{13}, \frac{2\pi}{7}\right)$ (B) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

(C) $(0, \pi)$ (D) None of these

Sol. **[A]**

we know that

$-1 < \cos x < 1$

$-3 < 3\cos x < 3$

$7 < 10 + 3\cos x < 13$

$$\frac{1}{13} < \frac{1}{10+3\cos x} < \frac{1}{7}$$

$$\int_0^{2\pi} \frac{1}{13}dx < \int_0^{2\pi} \frac{1}{10+3\cos x}dx < \int_0^{2\pi} \frac{1}{7}$$

$$\frac{2\pi}{13} < \int_0^{2\pi} \frac{1}{10+3\cos x}dx < \frac{2\pi}{7}$$

ordered pair $(a, b) = \left(\frac{2\pi}{13}, \frac{2\pi}{7}\right)$

Q.21 If $\int_0^1 e^{x^2}(x-\alpha)dx = 0$ then-

(A) $1 < \alpha < 2$ (B) $\alpha < 0$

(C) $0 < \alpha < 1$ (D) $\alpha = 0$

Sol. **[C]**

$$\int_0^1 xe^{x^2}dx = \alpha \int_0^1 e^{x^2}dx$$

$$\because 0 < x < 1 \Rightarrow 0 < xe^{x^2} < e^{x^2}$$

$$\Rightarrow 0 < \int_0^1 xe^{x^2}dx < \int_0^1 e^{x^2}dx$$

$$\Rightarrow 0 < \alpha \int_0^1 e^{x^2}dx < \int_0^1 e^{x^2}dx$$

$\Rightarrow 0 < \alpha < 1$ dividing by $\int_0^1 e^{x^2}dx$

Q.22 Let $I_1 = \int_1^2 \frac{dx}{\sqrt{1+x^2}}$ and $I_2 = \int_1^2 \frac{dx}{x}$ then-

(A) $I_1 > I_2$ (B) $I_2 > I_1$

(C) $I_1 = I_2$ (D) $I_1 > 2I_2$

Sol.[B] $\because x < 1 + x^2 \Rightarrow x < \sqrt{1+x^2}$

$$\Rightarrow \frac{1}{x} > \frac{1}{\sqrt{1+x^2}} \Rightarrow \int_1^2 \frac{1}{x}dx > \int_1^2 \frac{1}{\sqrt{1+x^2}}dx$$

Questions based on Miscellaneous

Q.23 Let f(x), g(x) & h(x) be continuous function on [0, a] such that $f(x) = f(a-x)$, $g(x) = -g(a-x)$, $3h(x) - 4h(a-x) = 5$. then $\int_0^a f(x)\,g(x)\,h(x)\,dx$ =

(A) 1 (B) 0

(C) a (D) None of these

Sol. **[B]**

$\because f(x) = f(a-x)$, $g(x) = -g(a-x)$

$\because 3h(x) - 4h(a-x) = 5$

$\Rightarrow 3h(a-x) - 4h(x) = 5$ [replace x by a – x]

Subtracting we get $h(x) = h(a-x)$

Now

$$I = \int_0^a f(x)\,g(x)\,h(x)\,dx$$

$$= \int_0^a f(a-x)\,g(a-x)\,h(a-x)\,dx$$

$$= -\int_0^a f(x)\,g(x)\,h(x)\,dx$$

$\Rightarrow \quad I = 0$

Q.24 If $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}dt$ then $\frac{d^2y}{dx^2}$ =

(A) 2y (B) 4y

(C) 8y (D) 6y

Sol. **[B]**

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}dt$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{8y}{2\sqrt{1+4y^2}}\cdot\frac{dy}{dx} = 4y$$

Q.25 $\int_a^b \frac{\sin(x-a)-\cos(x-a)}{\sin(b-x)-\cos(b-x)}\,dx$ is equal to -

(A) $\int_a^b \frac{\sin(x-a)+\cos(x-a)}{\sin(b-x)+\cos(b-x)}\,dx$

(B) $\int_a^b \frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}\,dx$

(C) $\int_a^b \frac{\sin(b-x)+\cos(b-x)}{\sin(x-a)+\cos(x-a)}\,dx$

(D) None of these

Sol. **[B]**

$$\int_a^b \frac{\sin(x-a)-\cos(x-a)}{\sin(b-x)-\cos(b-x)}\,dx$$

$$= \int_a^b \frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}\,dx$$

Q.26 If $I_n = \int_0^1 x^n e^{-x}\,dx$ for $n \in N$, then $I_7 - 7I_6 =$

(A) $-\frac{1}{2e}$ (B) $-\frac{1}{e}$ (C) $\frac{1}{e}$ (D) e

Sol. **[B]**

$$I_n = \int_0^1 x^n e^{-x}\,dx$$

$$I_{n+1} = \int_0^1 x^{n+1}e^{-x}\,dx$$

$$= \left[-x^{x+1}e^{-x}\right]_0^1 + \int_0^1 (x+1)x^n e^{-x}\,dx$$

$$I_{n+1} = -\frac{1}{e} + (n+1)\,I_n$$

put n = 6 we get

$$I_7 - 7I_6 = -\frac{1}{e}$$

Q.27 The value of $\int_{-1}^{3}(|x-2|+[x])\,dx$ is ([x] stands for greatest integer less than or equal to x)

(A) 7 (B) 5 (C) 4 (D) 3

Sol. **[A]**

$$I = \int_{-1}^{3}|x-2|\,dx + \int_{-1}^{3}[x]\,dx$$

$$= -\int_{-1}^{2}(x-2)\,dx + \int_{2}^{3}(x-2)\,dx + \int_{-1}^{0}(-1)\,dx$$

$$+ \int_0^1 0\,dx + \int_1^2 dx + \int_2^3 2\,dx$$

$$= \left[2x-\frac{x^2}{2}\right]_{-1}^{2} + \left[\frac{x^2}{2}-2x\right]_2^3 - [x]_{-1}^{0} + [x]_1^2 + 2[x]_2^3$$

$= 7$

Q.28 Let $f : R \to R$, $g : R \to R$ be continuous functions. Then the value of integral

$$\int_{\lambda n\,\lambda}^{\lambda n 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}\,dx \text{ is -}$$

(A) dependent on λ

(B) a non- zero constant

(C) zero

(D) None of these

Sol. **[C]**

$$I = \int_{\lambda n\lambda}^{\lambda n\frac{1}{\lambda}} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}\,dx$$

put x = – x

$$I = -\int_{\lambda n\lambda}^{\lambda n\frac{1}{\lambda}} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}\,dx$$

$\Rightarrow I = 0$

Q.29 If $\frac{d}{dx}f(x) = g(x)$ for $a \le x \le b$ then,

$\int_a^b f(x)g(x)\,dx$ equals -

(A) f(b) – f(a) (B) g(b) – g(a)

(C) $\frac{[f(b)]^2 - [f(a)]^2}{2}$ (D) $\frac{[g(b)]^2 - [g(a)]^2}{2}$

Sol. **[C]**

$I = \int_a^b f(x)\, g(x)\, dx$

Let $f(x) = t \Rightarrow \frac{d}{dx} f(x)\, dx = dt$

$\Rightarrow g(x)\, dx = dt$

$I = \int_{f(a)}^{f(b)} t\, dt = \left[\frac{t^2}{2}\right]_{f(a)}^{f(b)}$

$= \frac{[f(b)]^2 - [f(a)]^2}{2}$

Q.30 If f(0) = 1, f(2) = 3, f ′(2) = 5 and f ′ (0) is finite, then $\int_0^1 x.f''(2x)dx$ is equal to -

(A) zero (B) 1

(C) 2 (D) None of these

Sol. **[C]**

$I = \int_0^1 x\, f''(2x)\, dx$

$= \left[\frac{x f'(2x)}{2}\right]_0^1 - \frac{1}{2}\int_0^1 f'(2x)\, dx$

$= \frac{f'(2)}{2} - \frac{1}{4}\, [f(x)]_0^1$

$= \frac{5}{2} - \frac{1}{4}\, [3 - 1] = \frac{5}{2} - \frac{1}{2} = 2$

Q.31 Let $I_1 = \int_0^{3\pi} f(\cos^2 x)\,dx$, $I_2 = \int_0^{2\pi} f(\cos^2 x)\,dx$ and $I_3 = \int_0^{\pi} f(\cos^2 x)\,dx$, then -

(A) $I_1 + 2I_3 = 3I_2$ (B) $I_1 = 2I_2 + I_3$

(C) $I_2 + I_3 = I_1$ (D) $I_1 = 2I_3$

Sol. **[C]**

$I_1 = 3\int_0^{\pi} f(\cos^2 x)\, dx$, $I_2 = 2\int_0^{\pi} f(\cos^2 x)\, dx$

and $I_3 = \int_0^{\pi} f(\cos^2 x)\, dx$

Clearly $I_1 = I_2 + I_3$

Q.32 The value of the function

$f(x) = 1 + x + \int_1^x (\lambda n^2 t + 2\lambda nt)\,dt$ where f ′(x) vanishes is -

(A) e^{-1} (B) 0

(C) $2e^{-1}$ (D) $1 + 2e^{-1}$

Sol. **[D]**

$f(x) = 1 + x + \int_1^x (\lambda n^2 t + 2\lambda n\, t)\, dt$

$f'(x) = 1 + \lambda n^2 x + 2\lambda n\, x$

when $f'(x) = 0 \Rightarrow x = \frac{1}{e}$

then $f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{1/e} (\lambda n^2 t + 2\lambda n\, t)\, dt$

put $\lambda n\, t = y \Rightarrow dt = e^y dy$

$f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_0^{-1} e^y\, (y^2 + 2y)\, dy$

$= 1 + \frac{1}{e} + \left[e^y y^2\right]_0^{-1}$

$= 1 + \frac{1}{e} + \frac{1}{e}$

$= 1 + 2\, e^{-1}$

Q.33 $\lim_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{n}{r^2 - n^2}$ is equal to -

(A) $\log\sqrt{\frac{2}{3}}$ (B) $\log\sqrt{\frac{3}{2}}$

(C) $\log\frac{2}{3}$ (D) $\log\frac{3}{2}$

Sol. **[B]**

$\lim_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{1}{\frac{r^2}{n^2} - 1} . \frac{1}{n}$

$$= \int_2^3 \frac{1}{x^2-1}dx$$

$$= \frac{1}{2}\left[\lambda n\frac{x-1}{x+1}\right]_2^3 = \frac{1}{2}\left[\lambda n\frac{1}{2} - \lambda n\frac{1}{3}\right]$$

$$= \frac{1}{2}\ \lambda n\ \frac{3}{2} = \lambda n\sqrt{\frac{3}{2}}$$

Q.34 $\displaystyle\int_{\log\pi-\log 2}^{\log\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}dx$ is equal to -

(A) $\sqrt{3}$ (B) $-\sqrt{3}$

(C) $\frac{1}{\sqrt{3}}$ (D) $-\frac{1}{\sqrt{3}}$

Sol. **[A]**

Q.35 If $\displaystyle\int_0^{11}\frac{11^x}{11^{[x]}}dx = \frac{k}{\log 11}$, (where [ ] denotes greatest integer function) then value of k is -

(A) 11 (B) 101

(C) 110 (D) none of these

Sol. **[C]**

➢ True or false type questions

Q.36 The value of $\displaystyle\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^3}\sqrt{\cos t}\,dt}{1-\sqrt{\cos x}}$ is 12.

Sol. **[True]**

$$\lim_{x\to 0}\frac{3x^2\sqrt{\cos x^3}}{1-\sqrt{\cos x}}\ \frac{0}{0}\ \text{form}$$

$$\Rightarrow \lim_{x\to 0}\frac{3\sqrt{\cos x^3}(1+\sqrt{\cos x})}{\frac{1-\cos x}{x^2}}$$

$$= \frac{3.1.2}{1/2} = 12$$

Q.37 If f(x) = min. [x + 1, $\sqrt{1-x}$], then the value of $\displaystyle\int_{-1}^{1}\frac{12}{7}f(x)\,dx$ is 5.

Sol. **[False]**

$$I = \int_{-1}^{0}(x+1)\,dx + \int_0^1\sqrt{1-x}\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_{-1}^{0} + \frac{3}{2}\left[-(1-x)^{3/2}\right]_0^1$$

$$= -\frac{1}{2} + 1 + \frac{3}{2} = 2$$

Q.38 The value of $\displaystyle\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$ is $\frac{\pi}{2}-\frac{1}{2}$

Sol. **[False]**

$$I = \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$$

Let $x^2 = t \Rightarrow x\,dx = \frac{1}{2}\,dt$

$$I = \frac{1}{2}\int_0^1\sqrt{\frac{1-t}{1+t}}\,dt = \frac{1}{2}\int_0^1\frac{1-t}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\left[\sin^{-1}t+\sqrt{1-t^2}\right]_0^1 = \frac{\pi}{4}-\frac{1}{2}$$

Q.39 $\displaystyle\int_{-4}^{-5}e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3}e^{9(x-2/3)^2}\,dx = 0$

Sol. **[True]**

$$\int_{-4}^{-5}e^{(x+5)^2}dx + 3\int_{1/3}^{2/3}e^{9(x-2/3)^2}dx$$

↓ ↓

put x + 5 = t put $3\left(x-\frac{2}{3}\right) = -z$

$$= \int_1^0 e^{t^2}dt - \int_1^0 e^{z^2}dz$$

= 0

➢ Fill in the blanks type questions

Q.40 The value of $\displaystyle\int_0^{14}\frac{[x^2]}{[x^2-28x+196]+[x^2]}$ where [x] is the integral part of real x is……….

Sol. $I = \int_0^{14} \frac{[x^2]}{[(14-x^2)]+[x^2]}$

$I = \int_0^{14} \frac{[(14-x)^2]}{[x^2]+[(14-x)^2]}dx$

$2\,I = \int_0^{14} dx \Rightarrow I = 7$

Q.41 The least value of $\phi(x) = \int_x^2 \log_{1/3} t \; dt$ for

$x \in \left(\frac{1}{10}, 4\right)$ is at x = ……………

Sol. $\phi(x) = \int_2^x \log_3 t \; dt$

$= \frac{1}{\lambda n 3} \int_2^x \lambda n\, t \; dt$

$\phi'(x) = \frac{1}{\lambda n 3}(\lambda n\, x)$

$\phi_{min.}$ at x = 1.

Q.42 The value of $\int_0^{2\pi} \cos^{-1}(\cos x)\,dx$ is…………

Sol. $I = 2\int_0^{\pi} \cos^{-1}(\cos x)\,dx$

$= 2\int_0^{\pi} x\,dx = [x^2]_0^{\pi} = \pi^2$

EXERCISE # 2

Part-A **(Only single correct answer type questions)**

Q.1 If $\beta + 2\int_0^1 x^2 e^{-x^2} dx = \int_0^1 e^{-x^2} dx$, then the value of β is -

(A) e (B) 1 (C) 0 (D) 1/e

Sol. **[D]**

$$\beta - \int_0^1 x e^{-x^2}(-2x)dx = \int_0^1 e^{-x^2} dx$$

$$\Rightarrow \beta - \int_0^1 \underset{I}{x}\, d\underset{II}{(e^{-x^2})} = \int_0^1 e^{-x^2} dx$$

$$\Rightarrow \beta - \left[\left(xe^{-x^2}\right)_0^1 - \int_0^1 e^{-x^2} dx\right] = \int_0^1 e^{-x^2} dx$$

$$\Rightarrow \beta - \frac{1}{e} + \int_0^1 e^{-x^2} dx = \int_0^1 e^{-x^2} dx \Rightarrow \beta = \frac{1}{e}$$

Q.2 The value of $\int_{\lambda n3}^{\lambda n4} \frac{e^x\sqrt{e^x - 3}}{e^x - 2} dx$ is-

(A) $\frac{4-\pi}{2}$ (B) $4 - \frac{\pi}{2}$ (C) $2 - \pi$ (D) $\frac{2-\pi}{2}$

Sol. **[A]**

put $e^x - 3 = t^2 \Rightarrow e^x\, dx = 2t\, dt$

$$I = 2\int_0^1 \frac{t^2}{t^2+1} dt = 2\int_0^1 dt - 2\int_0^1 \frac{1}{t^2+1} dt$$

$$= 2[t]_0^1 - 2\left[\tan^{-1} t\right]_0^1 = 2 - \frac{2\pi}{4} = \frac{4-\pi}{2}$$

Q.3 If $\int_{-1}^{4} f(x)\,dx = 4$ and $\int_2^4 [3 - f(x)]\,dx = 7$, then the value of $\int_2^{-1} f(x)\,dx$ is-

(A) 2 (B) – 3

(C) – 5 (D) none of these

Sol. **[C]**

Given

$$\int_{-1}^{4} f(x)\,dx = 4 \text{ and } \int_2^4 [3 - f(x)]\,dx = 7$$

taking $\int_{-1}^{4} f(x)\,dx = 4$ we have

$$\int_{-1}^{2} f(x)dx + \int_2^4 f(x)dx = 4 \qquad \text{......(1)}$$

and we have

$$\int_2^4 3dx - \int_2^4 f(x)dx = 7$$

$$\Rightarrow \int_2^4 f(x)dx = -1 \qquad \text{.....(2)}$$

from (1) & (2)

$$\int_{-1}^{2} f(x)dx - 1 = 4 \Rightarrow \int_{-1}^{2} f(x)dx = 5$$

$$\int_2^{-1} f(x)dx = -5$$

Q.4 If $y = (x)^{[x]}$ where $x = [x] + (x)$, then $\int_0^3 y\,dx =$

(A) 2/3 (B) 5/6 (C) 1 (D) 11/6

Sol. **[D]**

$$\int_0^3 y\,dx = \int_0^3 (x)^{[x]}dx$$

$$= \int_0^3 (x - [x])^{[x]}dx$$

$$= \int_0^1 dx + \int_1^2 (x-1)dx + \int_2^3 (x-2)^2 dx$$

$$= \int_0^1 dx + \int_1^2 (x-1)\,dx \int_2^3 (x^2 - 4x + 4)\,dx$$

solving we get $= \frac{11}{6}$

Q.5 If the tangent to the graph of the function y = f (x) makes angles of π/4 and π/3 with the x-axis at the points x = 2 and x = 4 respectively, then the value of $\int_2^4 f'(x)\, f''(x)\,dx =$

(A) 0 (B) 2 (C) 3 (D) 1

Sol. **[D]**

y = f(x)

$$f'(x) = \frac{dy}{dx} \tan\theta$$

$$f'(2) = \tan\frac{\pi}{4} = 1 \text{ and } f'(4) = \tan\frac{\pi}{3} = \sqrt{3}$$

then $\int_2^4 f'(x)f''(x)dx$

put $f'(x) = t \Rightarrow f''(x)dx = dt$

$$\int_{f'(2)}^{f'(4)} t\,dt = \int_1^{\sqrt{3}} t\,dt = \left[\frac{t^2}{2}\right]_1^{\sqrt{3}} = \frac{3}{2} - \frac{1}{2} = 1$$

Q.6 If $\int_0^\infty e^{-x^2}.dx = \frac{\sqrt{\pi}}{2}$, then $\int_0^\infty e^{-ax^2}dx$, $a > 0$ is-

(A) $\frac{\sqrt{\pi}}{2}$ (B) $\frac{\sqrt{\pi}}{2a}$

(C) $\frac{2\sqrt{\pi}}{a}$ (D) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

Sol. **[D]**

$\because \int_0^\infty e^{-ax^2}dx$

Let $\sqrt{a}\ x = t \Rightarrow dx = \frac{1}{\sqrt{a}}dt$

$$= \frac{1}{\sqrt{a}}\int_0^\infty e^{-t^2}dt = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Q.7 Let a, b, c be non zero real number such that

$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)dx$

$= \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)dx$, then the

quadratic equation $ax^2 + bx + c = 0$ has -

(A) No root in (0, 2)

(B) At least one root in (0, 2)

(C) A double root in (0, 2)

(D) Two imaginary roots

Sol. **[B]**

Let $\int_0^x (1 + \cos^8 x)(ax^2 + bx + c)dx$

$\Rightarrow f'(x) = (1 + \cos^8 x)(ax^2 + bx + c)$

clearly f(x) is continuous on [1,2] and derivable on (1, 2)

Also Given that f(1) = f(2)

$\therefore$ By rolle's theorem there exist a point $t \in (1, 2)$ such that

$f'(t) = 0$

$\Rightarrow (1 + \cos^8 t)(at^2 + bt + c) = 0$

But $1 + \cos^8 t \neq 0$

$\Rightarrow at^2 + bt + c = 0$

The quadratic equation $ax^2 + bx + c = 0$ has at least one root in (1, 2)

By option equation has at least one root in (0, 2)

Q.8 The value of

$$\int_{-1}^{-2} \frac{\ell n\ \sin(2x+3)^2}{x}dx - 3\int_{7/3}^{4/3} \frac{\ell n\ \sin\left(\frac{11}{3} - 2x\right)^2}{(1-3x)}dx \text{ is -}$$

(A) 1 (B) 0

(C) 2 (D) none of these

Sol. **[B]**

$\because I = I_1 - I_2$

$$I_2 = 3\int_{7/3}^{4/3} \frac{\ell n \sin\left(\frac{11}{3} - 2x\right)^2}{(1-3x)}$$

let $\frac{1}{3} - x = t \Rightarrow dx = -dt$

$$I_2 = -\int_{-2}^{-1} \frac{\ell n \sin(2t+3)}{t}dt$$

$$= \int_{-1}^{2} \frac{\ell n \sin(2x+3)}{t}dx$$

$\Rightarrow I = I_1 - I_1 = 0$

Q.9 $\int_0^\infty \frac{x \log x}{(1+x^2)^2}dx$ -

(A) 0 (B) 1

(C) 1/2 (C) none of these

Sol. **[A]**

$$I = \int_0^\infty \frac{x \log x}{(1+x^2)^2}dx$$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$I = \int_0^{\pi/2} \sin\theta\cos\theta(\log\sin\theta - \log\cos\theta)d\theta$$

$$= \int_0^{\pi/2} \sin\theta\cos\theta\log\sin\theta\, d\theta - \int_0^{\pi/2} \sin\theta\cos\theta\log\cos\theta\, d\theta$$

$$= \int_0^1 t\log t\,dt - \int_0^1 t\log t\,dt = 0$$

Q.10 If $I_n = \int_0^{\infty} e^{-x}.x^{n-1}.dx$ then $\int_0^{\infty} e^{-\lambda x}\, x^{n-1}\, dx$ is equal to -

(A) $\lambda\, I_n$ (B) $\frac{1}{\lambda}.I_n$ (C) $\frac{I_n}{\lambda^n}$ (D) $\lambda^n.I_n$

Sol. **[C]**

Let $p = \int_0^{\infty} e^{-\lambda x} x^{n-1} dx$

Let $\lambda x = t \Rightarrow dx = \frac{1}{\lambda} dt$

$p = \frac{1}{\lambda}\int_0^{\infty} e^{-t}\, \frac{t^{n-1}}{\lambda^{n-1}} dt$

$= \frac{1}{\lambda^n}\int_0^{\infty} e^{-t} t^{n-1}\, dt \quad = \frac{1}{\lambda^n}\, I_n$

Q.11 $\int_0^{\pi} \frac{(2x+3)\sin x}{1+\cos^2 x}\, dx$ -

(A) $(\pi + 3)\frac{\pi}{2}$ (B) $(\pi - 3)\,\pi$

(C) $(\pi + 3)\,\pi$ (D) none of these

Sol. **[A]**

$I = \int_0^{\pi} \frac{(2x+3)\sin x}{1+\cos^2 x} dx$

$I = \int_0^{\pi} \frac{(2\pi - 2x+3)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$

$I = \int_0^{\pi} \frac{(2\pi - 2x+3)\sin x}{1+\cos^2 x}\, dx$

$\Rightarrow 2I = 2\int_0^{\pi} (\pi + 3)\frac{\sin x}{1+\cos^2 x} dx$

Let $\cos x = t \Rightarrow \sin x\, dx = -\,dt$

$I = -\int_1^{-1} \frac{(\pi+3)}{1+t^2} dt \;\Rightarrow I = 2(\pi + 3)\int_0^1 \frac{1}{1+t^2} dt$

$I = 2(\pi + 3)\left[\tan^{-1} t\right]_0^1 = (\pi + 3)\,\frac{\pi}{2}$

Q.12 The value of $\int_0^{16\pi/3} |\sin x|\, dx$ is -

(A) 21 (B) 16/3

(C) 32/3 (D) 21/2

Sol. **[D]**

$I = \int_0^{16\pi/3} |\sin x|\, dx$

$= \int_0^{\pi/3} |\sin x|\, dx + \int_{\pi/3}^{5\pi + \frac{\pi}{3}} |\sin x|\, dx$

$= \int_0^{\pi/3} \sin x\, dx + 5\int_0^{\pi} \sin x\, dx$

$= \left[-\cos x\right]_0^{\pi/3} + 5\left[-\cos x\right]_0^{\pi}$

$= \frac{1}{2} + 10 = \frac{21}{2}$

Q.13 The tangent to the curve $y = f(x)$ at the point with abscissa $x = 1$ form an angle of $\pi/6$ and at the point $x = 2$ an angle of $\pi/3$ and at the point $x = 3$ an angle of $\pi/4$. If $f''(x)$ is continuous, then the value of $\int_1^3 f''(x)\, f'(x)\, dx + \int_2^3 f''(x)\, dx$ is

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$

(C) $\frac{4-3\sqrt{3}}{3}$ (D) none of these

Sol. **[C]**

Given $f'(x) = \frac{dy}{dx} = \tan\theta$

$f'(x) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $f'(2) = \tan\frac{\pi}{3} = \sqrt{3}$,

$f'(3) = \tan\frac{\pi}{4} = 1$

$I = \int_1^3 f''(x) f'(x)\, dx + \int_2^3 f''(x)\, dx$

$I = \int_{1/\sqrt{3}}^{1} t\,dt + [f'(x)]_2^3$

$= \frac{1}{2} - \frac{1}{6} + f'(3) - f'(2)$

$= \frac{1}{3} + 1 - \sqrt{3} = \frac{4-3\sqrt{3}}{3}$

Q.14 If $p = \int_0^1 \tan^{-1}x\, dx$ and

$q = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$ then the value of $\frac{p}{q}$ is

(A) 1/4 (B) 1/2 (C) 1/8 (D) 1

Sol. **[B]**

Given P = $\int_0^1 \tan^{-1}x\,dx$,

q = $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

Taking q = $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$= \int_0^1 \tan^{-1}\frac{1}{1-x(1-x)}dx$

$= \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-(1-x))\,dx$

$= 2\int_0^1 \tan^{-1}x\,dx \Rightarrow \frac{p}{q} = \frac{1}{2}$

Q.15 $\int_{-\pi/2}^{\pi/2} \sqrt{\cos^{2n-1}x - \cos^{2n+1}x}\,dx$, n ∈ N -

(A) $\frac{2}{n+1}$ (B) $\frac{4}{2n+1}$

(C) 0 (D) none of these

Sol. **[B]**

I = $\int_{-\pi/2}^{\pi/2} |\cos^n x| \sqrt{\frac{1-\cos^2 x}{\cos x}}dx$

$= \int_{-\pi/2}^{\pi/2} |\cos^{n-1/2}x|\,|\sin x|dx$

$= 2\int_0^{\pi/2} \cos^{n-1/2}x\sin x\,dx$

$\int_0^{\pi/2} \cos^{n-1/2}x\sin x\,dx$

let cos x = t ⇒ sin x dx = – dt

$= -2\int_0^{\pi/2} t^{n-1/2}dt$

$= 2\int_0^{\pi/2} t^{n-1/2}dt = \left[\frac{t^{n+1/2}}{n+1/2}\right]_0^1 = \frac{4}{2n+1}$

Q.16 The value of $\int_{-1}^{1}$ [x [1 + sin πx] + 1] dx is, where [ ] denotes greatest integer function -

(A) 3 (B) 2 (C) 8 (D) 1

Sol. **[B]**

I = $\int_{-1}^{1} ([x[1+\sin\pi x]]+1)\,dx$

$= \int_{-1}^{1} ([x[1+\sin\pi x]])dx + \int_{-1}^{1} dx$

$= \int_{-1}^{0} [x[1+\sin\pi x]]dx \int_0^1 [x[1+\sin\pi x]]dx + (x)_{-1}^{1}$

$= \int_{-1}^{0} [x\times 0]dx + \int_0^1 [x\times 1]dx + 2$

$= \int_{-1}^{0} 0\,dx + \int_0^1 0\,dx + 2 = 2$

Q.17 Let f(x) and g(x) be two functions satisfying $f(x^2) + g(4-x) = 2x^3$; g(4 – x) + g(x) = 0, then the value of $\int_{-4}^{4} f(x^2)\,dx$ =

(A) 128 (B) 64

(C) 256 (D) none of these

Sol. **[C]**

Θ $\int_{-4}^{4} f(x^2)\,dx = 2\int_0^4 f(x^2)dx$, $f(x^2)$ is even

$\Rightarrow 2\int_0^4 \left[2x^3 - g(4-x)\right]dx$

[Θ $f(x^2) = 2x^3 - g(4-x)$]

I = $2\int_0^4 2x^3\,dx - \int_0^4 g(4-x)\,dx$

using property in $\int_0^4 g(4-x)\,dx$

$\Rightarrow \int_0^4 [g(4-(4-x)]\,dx = \int_0^4 g(x)dx$

$2I = 8\int_0^4 x^3dx - 2\int_0^4 [g(4-x)+g(x)]\,dx$

I = $4\int_0^4 x^3dx = 256$

Q.18 The value of $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\, dx$ is equal to -

(A) 0 (B) 1

(C) 2 (D) none of these

Sol. **[B]**

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\, dx$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(-x)}{1+e^{-x}}\, dx$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{e^x \cos x}{1+e^x}\, dx$$

$$2I = \int_{\pi/2}^{\pi/2} \cos x\, dx$$

$$I = \int_{0}^{\pi/2} \cos x\, dx$$

$$= [\sin x]_0^{\pi/2} = 1$$

Q.19 If $\int_0^{\pi} \frac{dx}{25\cos^2 x + 36\sin^2 x} = m\pi$, then m is -

(A) $\frac{1}{30}$ (B) $\frac{1}{15}$ (C) $\frac{1}{10}$ (D) $-\frac{1}{30}$

Sol. **[A]**

$$I = \int_0^{\pi} \frac{\sec^2 x}{25 + 36\tan^2 x}\, dx$$

$$\Theta \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(2a-x) = f(x)$$

$$\text{so } I = 2\int_0^{\pi/2} \frac{\sec^2 x}{25 + 36\tan^2 x}\, dx$$

put $6\tan x = t \Rightarrow \sec^2 x\, dx = \frac{1}{6}\, dt$

$$I = \frac{2}{6}\int_0^{\infty} \frac{1}{25+t^2}\, dt = \frac{1}{15}\left[\tan^{-1}\frac{t}{5}\right]_0^{\infty}$$

$$= \frac{1}{15}\cdot\frac{\pi}{2} = \frac{\pi}{30} \Rightarrow m = \frac{1}{30}$$

Q.20 Evaluate :

$$\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}.\tan\frac{2\pi}{2n}.\tan\frac{3\pi}{2n}.......\tan\frac{n\pi}{2n}\right]^{1/n}$$

(A) 0 (B) 1 (C) 2 (C) $2/\pi$

Sol. **[B]**

Taking log both side we get

let A = log s =

$$\frac{1}{n}\left[\log\tan\frac{\pi}{2n} + \log\tan\frac{2\pi}{2n} + ... + \log\tan\frac{n\pi}{2n}\right]$$

$$= \sum_{r=1}^{n} \log\tan\frac{r\pi}{2n}.\frac{1}{n} = \int_0^1 \log\tan\frac{\pi x}{2}\, dx$$

$$A = \int_0^{\pi/2} \log\tan t\, dt$$

$$A = \int_0^{\pi/2} \log\cot t\, dt$$

$$2A = \int_0^{\pi/2} \log 1\, dt = 0$$

$A = 0 \Rightarrow \log s = 0 \Rightarrow s = 1$

Q.21 The value of $\lim_{x\to\infty} \frac{d}{dx}\int_{\sqrt{3}}^{\sqrt{x}} \frac{r^3}{(r+1)(r-1)}\, dr$ is -

(A) 0 (B) 1

(C) 1/2 (D) non existent

Sol. **[C]**

Q.22 $\lim_{n\to\infty} \frac{\pi}{6n}\left[\sec^2\left(\frac{\pi}{6n}\right) + \sec^2\left(2.\frac{\pi}{6n}\right) + ... + \sec^2(n-1)\frac{\pi}{6n} + \frac{4}{3}\right]$

has the value equal to -

(A) $\frac{\sqrt{3}}{3}$ (B) $\sqrt{3}$

(C) 2 (D) $\frac{2}{\sqrt{3}}$

Sol. **[A]**

Q.23 $\lim_{n\to\infty} \frac{1}{n^3}\left(\sqrt{n^2+1} + 2\sqrt{n^2+2^2} + ... + n\sqrt{n^2+n^2}\right)$

$= \frac{\sqrt{a}-1}{\sqrt{b}}$ where a, b $\in$ N then (a + b) equals -

(A) 11 (B) 13

(C) 45 (D) 17

Sol. **[D]**

Q.24 Let f(x) = maximum $\{x\,|\,x\,|, x^2\,|\,x\,|\}$,

g(x) = minimum $\{x\,|\,x\,|, x^2\,|\,x\,|\}$, then

$$\int_{-1}^{1} (f(x) - g(x))\,dx =$$

(A) $\frac{1}{12}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{7}{12}$

Sol. **[C]**

Part-B **(One or more then one correct answer type questions)**

Q.25 If x satisfies the equation

$$x^2\left(\int_0^{\pi/2}(2\sin t+3\cos t)\,dt\right)-x\left(\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}\right)-2=0,$$

then the value of x is -

(A) – 1 (B) 1 (C) $-\sqrt{2/5}$ (D) $\sqrt{2/5}$

Sol. **[C,D]**

Θ $\int_0^{\pi/2}(2\sin t+3\cos t)\,dt$

$= \left[-2\cos t+3\sin t\right]_0^{\pi/2} = 5$

and $\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}\,dt = 0$ ∴ function is odd

put these values in given equation we get

$5x^2 - 2 = 0$

$x^2 = \frac{2}{5} \Rightarrow x = \pm\sqrt{\frac{2}{5}}$

Q.26 Which of the following is/are true-

(A) $\int_0^{n\pi} f(\cos^2 x)dx = n\int_0^{\pi} f(\cos^2 x)\,dx$ $n \in N$

(B) $\int_0^{b-c} f(x+c)\,dx = \int_b^c f(x)\,dx$

(C) $\int_a^{\pi-a} x f(\sin x)\,dx = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)\,dx$

(D) $\int_0^{\pi} x f(\sin^3 x+\cos^2 x)\,dx$

$= 2\pi\int_0^{\pi/2} f(\sin^3 x+\cos^2 x)\,dx$

Sol. **[A,C]**

(A) $\int_0^{n\pi} f(\cos^2 x)dx$

$= n\int_0^{n\pi} f(\cos^2 x)\,dx$ ∴ π is period

A is correct

(B) $\int_0^{b-c} f(x+c)dx = \int_0^{b-c} f(b-x)\,dx$

$= -\int_b^c f(t)\,dt$ put b – x = t

$= \int_c^b f(x)\,dx$ B is wrong

(C) $I = \int_a^{\pi-a} x f(\sin x)dx$

$= \int_a^{\pi-a} (\pi-x)\, f(\sin x)\,dx$

$\Rightarrow 2I = \pi\int_a^{\pi-a} f(\sin x)dx$

$\Rightarrow I = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)dx$

C is correct

(D) $I = \int_0^{\pi} x f(\sin^3 x+\cos^2 x)\,dx$

$= \int_0^{\pi} (\pi-x) f(\sin^3 x+\cos^2 x)\,dx$

$\Rightarrow 2I = \pi\int_0^{\pi} f(\sin^3 x+\cos^2 x)\,dx$

$= 2\pi\int_0^{\pi/2} f(\sin^3 x+\cos^2 x)\,dx$

$I = \pi\int_0^{\pi/2} f(\sin^3 x+\cos^2 x)$

D is wrong

Q.27 The value of the integral

$\int_0^{\pi/4}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$ is -

(A) $\frac{1}{ab}\tan^{-1}\frac{b}{a}$ (a > 0, b > 0)

(B) $\frac{1}{ab}\tan^{-1}\frac{b}{a}$ (a < 0, b < 0)

(C) π/4 (a = 1, b = 1)

(D) none of these

Sol. **[A,B,C]**

$I = \int_0^{\pi/4}\frac{\sec^2 x dx}{a^2+b^2\tan^2 x}$

let b tanx = t ⇒ b $\sec^2 x$ dx = dt

$I = \frac{1}{b}\int_0^{b}\frac{dt}{a^2+t^2}$

$= \frac{1}{ab}\int_0^{b}\left[\tan^{-1}\frac{t}{a}\right]_0^{b}$

$= \frac{1}{ab}\tan^{-1}\frac{b}{a}$ a > 0, b > 0

or a < 0 , b < 0

clearly I = π/4 if a = 1, b = 1

option A, B, C are correct

Q.28 The points of extremum of $\int_0^{x^2}\frac{t^2-5t+4}{2+e^t}dt$ are-

(A) x = – 2 (B) x = 1

(C) x = 0 (D) x = – 1

Sol. **[A,B,C,D]**

$I = \int_0^{x^2}\frac{t^2-5t+4}{2+e^t}dt$

differentiate w.r.to. x we get

$\frac{dI}{dx} = \frac{x^4-5x^2+4}{2+e^{x^2}}.2x$

for point of extremum $\frac{dI}{dx} = 0$

$2x(x^4 - 5x^2 + 4) = 0$

$\Rightarrow x(x^2 - 4)(x^2 - 1) = 0$

$\Rightarrow x = 0$ or $x^2 - 4 = 0$ or $x^2 - 1 = 0$

$\Rightarrow x = 0, \pm 2, \pm 1$

option A, B, C, D all are correct

Q.29 If $I = \int_0^1\frac{dx}{1+x^{\pi/2}}$, then -

(A) I < π/4 (B) I > π/4

(C) I < λn2 (D) I > λn2

Sol. **[A, D]**

Q.30 If $I = \int_0^{2\pi}\sin^2 x\,dx$, then

(A) $I = 2\int_0^{\pi}\sin^2 x\,dx$ (B) $I = 4\int_0^{\pi/2}\sin^2 x\,dx$

(C) $I = \int_0^{2\pi}\cos^2 x\,dx$ (D) $I = 8\int_0^{\pi/4}\sin^2 x\,dx$

Sol. **[A, B, C]**

$I = \int_0^{2\pi}\sin^2 x\,dx = 2\int_0^{\pi}\sin^2 x\,dx = 4\int_0^{\pi/2}\sin^2 x\,dx$

$= 4\int_0^{\pi/2}\cos^2 x\,dx = \int_0^{2\pi}\cos^2 x\,dx$

⇒ A, B, C are correct

Q.31 $\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) is same as $\int_0^{\infty}\frac{dx}{(1+x)(1+x^2)}$

(D) cannot be evaluated

Sol.[A, C]

$I = \int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$

Let x = tan θ ⇒ dx = sec²θ dθ

$I = \int_0^{\pi/2}\frac{\tan\theta}{1+\tan\theta}d\theta$

$I = \int_0^{\pi/2}\frac{\cot\theta}{1+\cot\theta} = \int_0^{\pi/2}\frac{1}{1+\tan\theta}d\theta$

$2I = \int_0^{\pi/2}d\theta \Rightarrow I = \frac{\pi}{4}$

From option (C) we have

$I = \int_0^{\infty}\frac{dx}{(1+x)(1+x^2)}$

Let x = tan θ ⇒ dx = sec²θ dθ

$I = \int_0^{\pi/2}\frac{1}{1+\tan\theta}d\theta$

$I = \int_0^{\pi/2}\frac{1}{1+\cot\theta}d\theta = \int_0^{\pi/2}\frac{\tan\theta}{1+\tan\theta}d\theta$

$2I = \int_0^{\pi/2}d\theta \Rightarrow I = \frac{\pi}{4}$

⇒ option A, C are correct

Q.32 The value of integral $\int_a^b\frac{|x|}{x}dx$, a < b is

(A) $b - a$ if $a > 0$ (B) $a - b$ if $b < 0$

(C) $b + a$ if $a < 0 < b$ (D) $|b| - |a|$

Sol. **[A, B, C, D]**

$I = \int_a^b \frac{|x|}{x} dx \quad a < b$

If $a > 0$ $\frac{|x|}{x} = 1$

$I = \int_a^b dx = b - a$

If $b < 0$ $\frac{|x|}{x} = -1$

$I = \int_a^b (-1)\, dx = a - b$

If $a < 0 < b$ then

$I = \int_a^0 \frac{|x|}{x} dx + \int_0^b \frac{|x|}{x} dx$

$= \int_a^0 -dx + \int_0^b dx$

$= a + b$

Again $|x| = t \Rightarrow \frac{|x|}{x}\, dx = dt$

$I = \int_{|a|}^{|b|} dt = |b| - |a|$

$\Rightarrow$ option A, B, C, D all are correct

Q.33 If $f(x) = \int_0^x (\cos^4 t + \sin^4 t)\, dt$ then $f(x + \pi)$ will be equal to

(A) $f(x) + f(\pi)$ (B) $f(x) + 2f(\pi)$

(C) $f(x) + f\left(\frac{\pi}{2}\right)$ (D) $f(x) + 2f\left(\frac{\pi}{2}\right)$

Sol. **[A, D]**

$f(x) = \int_0^x (\cos^4 t + \sin^4 t)\, dt$

$f(x + \pi) = \int_0^{x+\pi} (\cos^4 t + \sin^4 t)\, dt$

$= \int_0^x (\cos^4 t + \sin^4 t)\, dt + \int_x^{x+\pi} (\cos^4 t + \sin^4 t)\, dt$

$= f(x) + \int_0^{\pi} (\cos^4 t + \sin^4 t)\, dt$

$f(x + \pi) = f(x) + f(\pi)$

Θ $f(x + \pi) = f(x) + \int_0^{\pi} (\cos^4 t + \sin^4 t)\, dt$

$= f(x) + 2\int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt$

$= f(x) + 2f\left(\frac{\pi}{2}\right)$

$\Rightarrow$ option A, D are correct

Q.34 $\lim_{x \to 0} \frac{x e^{x^2}}{\int_0^x e^{t^2} dt} =$

(A) 0 (B) 1

(C) –1 (D) does not exist

Sol. **[B]**

$\lim_{x \to 0} \frac{x e^{x^2}}{\int_0^x e^{t^2} dt}$

LH

$\lim_{x \to 0} \frac{e^{x^2}(1 + 2x)}{e^{x^2}} = 1$

Part-C Assertion-Reason type questions

The following questions consist of two statements each, printed as Statement-1 and Statement-2. While answering these questions you are to choose any one of the following four responses.

(A) If both Statement-1 and Statement-2 are true and the Statement-2 is correct explanation of the Statement-1.

(B) If both Statement-1 and Statement-2 are true but Statement-2 is not correct explanation of the Statement-1.

(C) If Statement-1 is true but the Statement-2 is false.

(D) If Statement-1 is false but Statement-2 is true.

Q.35 **Statement-1 :** Let f(x) be an even function which is periodic, then $g(x) = \int_a^x f(t)\,dt$ is also periodic.

Statement-2: If $\alpha(x)$ is a differentiable and periodic function, then $\alpha'(x)$ is also periodic.

Sol. **[D]**

Q.36 **Statement-1 :** $\int_0^{10\pi} |\cos x|\,dx = 20$

Statement-2 : $\int_a^b f(x)\,dx \ge 0$, then $f(x) \ge 0,\ \forall\ x \in (a, b)$

Sol. **[C]**

$$\int_0^{10\pi} |\cos x|\,dx = 10\int_0^{\pi} |\cos x|\,dx$$

$$= 20\int_0^{\pi/2} \cos x\,dx = 20\left[+\sin x\right]_0^{\pi/2} = 20$$

Statement-1 is true

If $\int_a^b f(x)\,dx \ge 0$ then it is not necessary that $f(x) \ge 0,\ \forall\ x \in (a, b)$

$\Rightarrow$ Statement-2 is false

$\Rightarrow$ option (C) is correct

Q.37 **Statement-1 :** $\int_0^{2\pi} \tan^2 x\,dx = 4\int_0^{\pi/2} \tan^2 x\,dx$

Statement-2 : $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$, where n is an integer and T is a period of f(x).

Sol. **[B]**

$$\because \int_0^{2\pi} \tan^2 x\,dx = 2\int_0^{\pi} \tan^2 x\,dx$$

$$= 4\int_0^{\pi/2} \tan^2 x\,dx$$

$$\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

Statement-1 is true

Also $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$ where n is an integer and T is a period of f(x)

$\Rightarrow$ Statement-2 is true but not correct explanation of Statement-1

$\Rightarrow$ option (B) is correct.

Part-D Column matching type questions

Q.38 **Column I** **Column II**

(A) $f(x) = \max\left(\sin x, \cos x, \frac{1}{\sqrt{2}}\right)$

(P) f(x) is continuous at x = 0

(B) $f(x) = \min(x, x^3)$

(Q) f(x) is differentiable at x = 0

(C) $f(x) = \max((x-1)^2, 2x - x^2)$

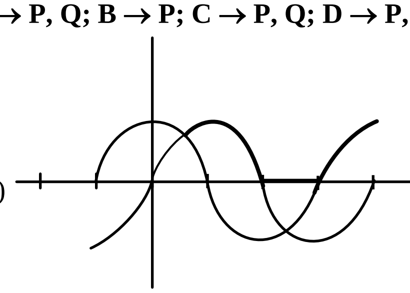
(R) $\int_0^{\pi} f(x)\,dx = 2$

(D) $f(x) = \min(|x|, \sin x)$

(S) $\int_{-1}^{1} f(x)\,dx = -\frac{1}{2}$

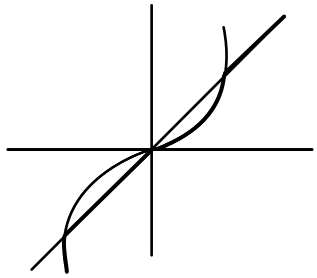
Sol. **A $\rightarrow$ P, Q; B $\rightarrow$ P; C $\rightarrow$ P, Q; D $\rightarrow$ P,Q,R**

(A)



$$\int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{5\pi/4} \sin x\,dx + \int_{5\pi/9}^{\pi} x.dx$$

(B)

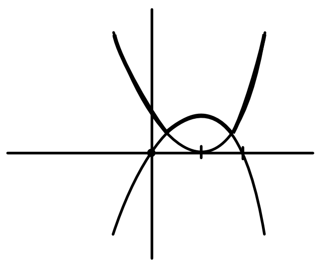


non diff. at x = 0

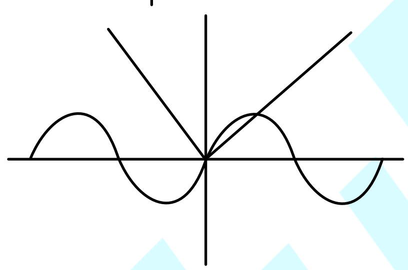
$$\int_{-1}^{0} x\,dx + \int_{0}^{1} x^3\,dx$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

(C)



(D)



Q.39 [x] and {x} are greatest and fractional part of real x.

Column I	**Column II**
(A) $\int_{1}^{2} \frac{dx}{x^2 - 2x[x] + 1 + [x]}$	(P) $\frac{\pi}{4}$
(B) $\int_{-(\pi/4)}^{0} \{\sin x + \cos x\}\,dx$	(Q) $\int_{1}^{2} \frac{dx}{x^2 - 2x + 2}$
(C) If $\sin^{-1}\frac{4x}{\sqrt{2}\pi} - \cos^{-1}\frac{4x}{\sqrt{2}\pi} = 0$ then x can be	
	(R) $\int_{-(\pi/4)}^{0} (\sin x + \cos x)\,dx$
(D) $\int_{0}^{\pi/4} \left\{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right\} dx + \int_{0}^{1/\sqrt{2}} \frac{dx}{\sqrt{1 - x^2}}$	
	(S) $\int_{0}^{\pi/4} \left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1\right) dx$

Sol. **A → P, Q; B → R; C → P, Q; D → S**

(A) $\int_{1}^{2} \frac{dx}{x^2 - 2x + 2}$

$$I = \int_{1}^{2} \frac{dx}{(x-1)^2 + 1} = \left[\tan^{-1}(x-1)\right]_1^2 = \frac{\pi}{4}$$

(B) $\int_{-\frac{\pi}{4}}^{0} \left\{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right\} dx$

$$I = \int_{-\frac{\pi}{4}}^{0} \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) dx - \int_{-\frac{\pi}{4}}^{0} \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) dx$$

↓ Zero

$$I = \int_{-\frac{\pi}{4}}^{0} \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) dx$$

$$= \sqrt{2} - \left[\cos\left(x + \frac{\pi}{4}\right)\right]_{-\frac{\pi}{4}}^{0}$$

$$= -\sqrt{2}\left[\frac{1}{\sqrt{2}} - 1\right] = \sqrt{2} - 1$$

(C) $x = \frac{\pi}{4}$

$$2\sin^{-1}\frac{4x}{\sqrt{2}\pi} = \frac{\pi}{2}$$

(D) $\int_{0}^{\pi/4} \left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1\right) dx + \left[\sin^{-1} x\right]_0^{1/\sqrt{2}}$

Q.40 Match the column -

Column I	**Column II**
(A) f(x) is a continuous function such that $f(x) \geq 0 \ \forall\ x \in [2, 10]$ and $\int_{4}^{8} f(x)\,dx = 0$, then f(6) is	(P) 2
(B) $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{4n^2 - r^2}}$ is	(Q) $\pi/2$
(C) f(x) is an odd function,	(R) $\pi/6$

then $\int_{-\pi/6}^{\pi/6} \frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)} dx$ is

(D) If $n \in I$, then $\int_{-\pi/6}^{\pi/6} \frac{\sin^{2n-1} x}{\cos^n x} dx$ is (S) 0

Sol. **A → S; B → R; C → S; D → S**

(A) f(r) = 0

(B) $\int_0^1 \frac{1}{\sqrt{4-x^2}} dx$

$\left[\sin^{-1}\frac{x}{2}\right]_0^1 = \frac{\pi}{2}$

(C) king & add

I = 0

(D) king & add

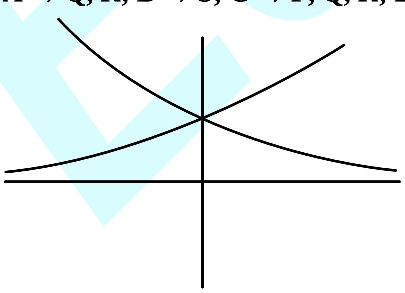
I = 0

Q.41 Let $I_1 = \int_{-1}^{1} f(x)\,dx$, $I_2 = \int_{-1}^{1} g(x)\,dx$, & $I_3 = \int_{-1}^{1} h(x)\,dx$, be three integrals, match the following columns accordingly.

Column I	Column II
(A) $f(x) = e^x$, $g(x) = e^{-x}$, $h(x) = e^{\|x\|}$	(P) $I_1 < I_2$
(B) $f(x) = \|x\|$, $g(x) = x^2$, $h(x) = x^3$	(Q) $I_2 < I_3$
(C) $f(x) = x$, $g(x) = \|x\|^3$, $h(x) = x^2$	(R) $I_1 < I_3$
(D) $f(x) = e^{-\|x\|}$, $g(x) = e^x$, $h(x) = e^{-x}$	(S) $I_1 > I_2$

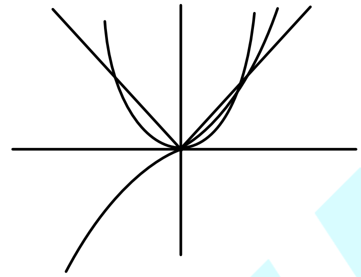
Sol. **A → Q, R; B → S; C → P, Q, R; D → P, R**

(A)



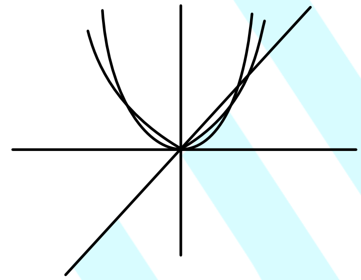
$I_1 = I_2 < I_3$

(B)



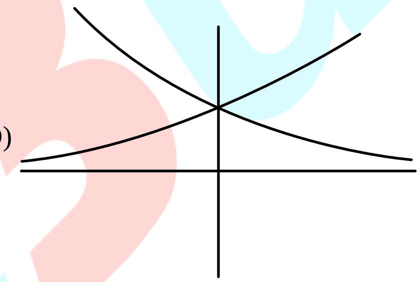
$I_1 > I_2 > I_3$

(C)



$I_1 < I_3 < I_2$

(D)



Q.42 Match the column :

Column-1	Column-2
(A) If [ ] denotes the greatest integer function and $f(x) = \begin{cases} 3[x] - \frac{5\|x\|}{x}; & x \neq 0 \\ 2 \quad ; & x = 0 \end{cases}$, then $\int_{-3/2}^{2} f(x)dx$ is equal to	(P) 1
(B) The value of definite $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$ is	(Q) $-\frac{11}{2}$
(C) If $I_1 = \int_{1}^{\sin\theta} \frac{x}{1+x^2} dx$ and $I_2 = \int_{1}^{\operatorname{cosec}\theta} \frac{1}{x(x^2+1)} dx$ then the value of	(R) $\frac{3}{2}$

$$\begin{vmatrix} I_1 & I_1^2 & I_2 \\ e^{I_1+I_2} & I_2^2 & -1 \\ 1 & I_1^2.I_2^2 & -1 \end{vmatrix}, \text{ is}$$

(D) If f(x) and g(x) are two continuous functions defined on R, then the value of

$$\int_{-a}^{a} \{f(x)+f(-x)\}\{g(x)-g(-x)\}dx,$$

is

(S) 0

Sol. **A → Q; B → P; C → R; D → S**

(A) $$I = \int_{-\frac{3}{2}}^{-1} -1.dx + \int_{-1}^{0} 2.dx + \int_{0}^{1} -5.dx + \int_{1}^{2} -2.dx$$

$$= -\frac{1}{2} + 2 - 5 - 2$$

$$= -\frac{11}{2}$$

(B) king & add

$$2I = \int_{-\pi/2}^{\pi/2} \cos x \, dx = 2$$

I = 1

(D) I = 0

$\therefore$ function is odd

EduBull

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Solve the equation $\int_{\ln 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$

Sol. $\int_{\ln 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$

$\Rightarrow \int_{\ln 2}^{x} \frac{e^x}{e^x\sqrt{e^x - 1}} dx = \frac{\pi}{6}$

put $e^x - 1 = t^2 \Rightarrow e^x\, dx = 2t\, dt$

$\Rightarrow \int_{1}^{\sqrt{e^x - 1}} \frac{2t\,dt}{(t^2+1)t} = \frac{\pi}{6}$

$\Rightarrow \int_{1}^{\sqrt{e^x - 1}} \frac{2}{t^2+1} dt = \frac{\pi}{6}$

$\Rightarrow 2\left[\tan^{-1} t\right]_1^{\sqrt{e^x-1}} = \frac{\pi}{6}$

$\Rightarrow \tan^{-1}\sqrt{e^x - 1} - \frac{\pi}{4} = \frac{\pi}{12}$

$\Rightarrow \tan^{-1}\sqrt{e^x - 1} = \frac{\pi}{3}$

$\Rightarrow \sqrt{e^x - 1} = \sqrt{3} \Rightarrow e^x = 4$

$x = \ln 4$

Q.2 $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x}\, dx$

Sol. $I = \int_0^{\pi} \sin^{3/2} x\, |\cos x|\, dx$

$= \int_0^{\pi/2} \sin^{3/2} x \cos x - \int_{\pi/2}^{\pi} \sin^{3/2} x \cos x\, dx$

in both let $\sin x = t \Rightarrow \cos x\, dx = dt$

$I = \int_0^1 t^{3/2} dt - \int_1^0 t^{3/2} dt$

$= 2\int_0^1 t^{3/2} dt = 2 . \frac{2}{5}\left[t^{5/2}\right]_0^1 = \frac{4}{5}$

Q.3 Find $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$.

Sol. $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$

$f'(x) = \sin^{-1}\sqrt{\sin^2 x}\ (2 \sin x \cos x) -$

$\cos^{-1}\sqrt{\cos^2 x}\ (2 \sin x \cos x)$

$f'(x) = 0 \Rightarrow f(x) = C$ constant

we take $x = \pi/4$

$f(x) = \int_0^{1/2} \sin^{-1}\sqrt{t} + dt + \int_0^{1/2} \cos^{-1}\sqrt{t} dt$

$= \int_0^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\, dt$

$= \int_0^{1/2} \pi/2\, dt = \frac{\pi}{4} = C \Rightarrow f(x) = \pi/4$

Q.4 Function f is continuous for all x, (and not every where zero), such that $\{f(x)\}^2 = \int_0^x \frac{f(t)\sin t}{2+\cos t} dt$, then prove that $f(x) = \frac{1}{2} \ln \frac{3}{2+\cos x}$

Sol. $2f\, f' = \frac{f(x)\sin x}{2+\cos x}$

$f'(x) = \frac{1}{2} \times \frac{\sin x}{2+\cos x}$

$f(x) = \frac{1}{2}\int \frac{-\sin x}{2+\cos x} dx$

$= -\frac{1}{2} \ln(2 + \cos x) + c$

$f(0) = 0$

$c = \frac{1}{2} \ln 3$

$\therefore f(x) = \frac{1}{2} \ln\left(\frac{3}{2+\cos x}\right)$

Q.5 Evaluate : $\int_1^3 \frac{dx}{x^2+[x]^2+1-2x[x]}$, where [.] denotes the greatest integer function.

Sol. $I = \int_1^3 \frac{dx}{x^2+[x]^2+1-2x[x]}$

$= \int_1^2 \frac{dx}{x^2+1+1-2x} + \int_2^3 \frac{dx}{x^2+4+1-4x}$

$= \int_1^2 \frac{dx}{(x^2-1)^2+1} + \int_2^3 \frac{dx}{(x^2-2)^2+1}$

$= \left[\tan^{-1}(x-1)\right]_1^2 + \left[\tan^{-1}(x-2)\right]_2^3$

$= \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$

Q.6 Prove that

$\int_0^\pi \frac{dx}{1-2a\cos x+a^2} = \frac{\pi}{a^2-1}$ if |a| > 1

$= \frac{\pi}{1-a^2}$ if |a| < 1

Sol. $I = \int_0^\pi \frac{1}{a^2-2a\cos x+1}dx$

$I = \int_0^\pi \frac{\sec^2 x/2}{(1-a)^2+(1+a)^2\tan^2 x/2}dx$

let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2 x/2\ dx = dt$

$= 2\int_0^\infty \frac{1}{(1+a)^2t^2+(1+a)^2}dt$

$= \frac{2}{(1+a)^2}\int_0^\infty \frac{1}{t^2+\left(\frac{1-a}{1+a}\right)^2}dt$

$= \frac{2}{(1+a)^2}\left|\frac{1+a}{1-a}\right|\left[\tan^{-}\left|\frac{1+a}{1-a}\right|t\right]_0^\infty$

$= \frac{\pi}{(1+a)|1-a|}$

$= \frac{\pi}{1-a^2}$ if 0 < a < 1

$= \frac{-\pi}{1-a^2}$ if a > 1

Q.7 $\int_0^1 |k-x|\cos\pi x\,dx$; where k is any real number.

Sol. **Case I :** k < 0

$I_1 = \int (x-k)\cos(\pi x)dx$

$= (x-k)\frac{\sin(\pi x)}{\pi} - \left(-\frac{\cos\pi x}{\pi^2}\right)$

$I = [I_1]_0^1$

Case II : k > 1

$I = -[I_1]_0^1$

Case III : 0 < k < 1

$I = (-I_1)_0^k + (I_1)_k^1$

Q.8 Show that $= \int_0^t e^{tx}.e^{-x^2}dx = e^{t^2/4}\int_0^t e^{-t^2/4}dt$

Sol. LHS

queen 1st case

$I = 2\int_0^{t/2} e^{tx-x^2} = 2\int_0^{t/2} e^{x[t-x]}dx$

king

$= 2\int_0^{t/2} e^{(t/2-x)(t/2+x)}dx = 2\int_0^{t/2} e^{t^2/4}e^{-x^2}dx$

$= 2e^{t^2/4}\int_0^{t/2} e^{-x^2}dx$

Put $x = \frac{u}{2} \Rightarrow 2e^{t^2/4}.\int_0^u e^{-u^2/4}\frac{du}{2}$

Q.9 Prove that

$\int_0^\pi x^3\ \lambda n \sin x\,dx = \frac{3\pi}{2}\int_0^\pi x^2\ \lambda n(\sqrt{2}\sin x)dx$

Sol. LHS

$\int_0^\pi x^3 \lambda n \sin x\,dx$

use **king**

$I = \int_0^\pi (\pi-x)^3 \lambda n \sin x\,dx$

add

$$2I = \int_0^{\pi} (\pi^3 - 3\pi^2 x + 3\pi x^2)\lambda n \sin x \, dx$$

$$2I = \pi^3 \int_0^{\pi} \lambda n \sin x \, dx - 3\pi^2 \int_0^{\pi} x \lambda n \sin x \, dx$$

$$+ 3\pi \int_0^{\pi} x^2 \lambda n\left(\frac{\sqrt{2}\sin x}{\sqrt{2}}\right)dx$$

$$= \pi^3 2\left(-\frac{\pi}{2}\lambda n\, 2\right) - 3\pi^2 (I_1)$$

$$+ \left[3\pi \int_0^{\pi} x^2 \lambda n(\sqrt{2}\sin x)dx\right] - 3\pi \int_0^{\pi} x^2 \lambda n\sqrt{2}\, dx \quad ...(1)$$

Now, $I_1 = \int_0^{\pi} x\lambda n \sin x \, dx$

$$= \int_0^{\pi} (\pi - x)\lambda n \sin x \, dx$$

$$2I_1 = \pi \int_0^{\pi} \lambda n \ \sin x \, dx = 2 \,.\, \pi \left(-\frac{\pi}{2}\lambda n\, 2\right)$$

$$I_1 = \frac{-\pi^2 \lambda n\, 2}{2}$$

use it in

$$2I = -\pi^4 \lambda n\, 2 - 3\pi^2\left(-\frac{\pi^2}{2}\lambda n 2\right)$$

$$+ \left[3\pi \int_0^{\pi} x^2 \lambda n\sqrt{2}\sin x \, dx\right] - 3\pi \ \lambda n\, \sqrt{2}\left(\frac{\pi^3}{3}\right)$$

Q.10 If $n > 1$, evaluate : $\int_0^{\infty} \frac{dx}{(x+\sqrt{1+x^2})^n}$

Sol. $I = \int_0^{\infty} \frac{dx}{(x+\sqrt{1+x^2})^n}$

Le $x + \sqrt{1+x^2} = t$

$\Rightarrow 1 + x^2 = (t - x)^2$

$\Rightarrow 1 = t^2 - 2tx \Rightarrow x = \frac{t^2 - 1}{2t} = \frac{1}{2}\left(t - \frac{1}{t}\right)$

$\Rightarrow dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right)dt$

$$I = \int_1^{\infty} \frac{1}{t^n} \cdot \frac{1}{2}\left(1 + \frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\int_1^{\infty}\left(t^{-n} + t^{-n-2}\right)dt$$

$$= \frac{1}{2}\left[\frac{t^{-n+1}}{-n+1} + \frac{t^{-n-1}}{-n-1}\right]_1^{\infty}$$

$$= \frac{1}{2}\left[\frac{t^{1-n}}{1-n} - \frac{t^{-(n+1)}}{n+1}\right]_1^{\infty}$$

$$= \frac{1}{2}\left[0 - \left\{\frac{1}{1-n} - \frac{1}{n+1}\right\}\right] = \frac{n}{n^2 - 1}$$

Q.11 Find the greatest value of the function: $F(x) = \int_{5\pi/3}^{x} (6\cos u - 2\sin u)\, du$ on the interval $[5\pi/3, 7\pi/4]$

Sol. $F(x) = \int_{5\pi/3}^{x} (6\cos u - 2\sin u)\, du$

$\Rightarrow F(x) = 6\cos x - 2\sin x$

For all $x \in \left[\frac{5\pi}{3}, \frac{7\pi}{4}\right]$ we have

$\cos x > 0$ and $\sin x < 0$

$\Rightarrow F'(x) = 6\cos x - 2\sin x > 0$

$\Rightarrow F(x)$ is an increasing function on $\left[\frac{5\pi}{3}, \frac{7\pi}{4}\right]$

Hence greatest value at $x = \frac{7\pi}{4}$

Greatest value $= F\left(\frac{7\pi}{4}\right)$

$$= \int_{\frac{5\pi}{3}}^{\frac{7\pi}{4}} (6\cos u - 2\sin u)\, du$$

$$= \left[6\sin u + 2\cos u\right]_{\frac{5\pi}{3}}^{\frac{7\pi}{4}} = 3\sqrt{3} - 2\sqrt{2} - 1$$

Q.12 Evaluate $\int_{-3/2}^{2} f(x)\,dx$, where f (x) is given by

$f(x) = \max.(|t-1| - |t| + |t+1|)\ ;\ -\frac{3}{2} \le t \le x$

Sol. $g(x) = t\ ;\ t \ge 1$

$-t + 1 - t + t + 1$

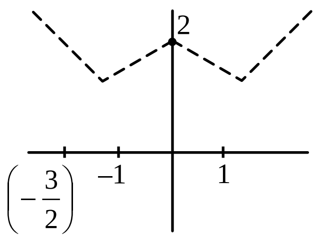
$= -t + 2\ ;\ 0 < t < 1$

$= -t + 1 + t + t + 1$

$= t + 2\ ;\ -1 \le t < 0$

$= -t + 1 + t - t - 1$

$= -t\ ;\ t < -1$



$f(x) = g_{max}(t) = \frac{3}{2}\ ;\ -\frac{3}{2} \le x \le -\frac{1}{2}$

$= g(x) = x + 2\ ;\ -\frac{1}{2} \le x < 0$

$= 2\ ;\ 0 \le x < 2$

$= g(x) = x\ ;\ x \ge 2$

Q.13 Evaluate : $\lim_{n\to\infty} [(1+1/n^2)\ (1+2^2/n^2)\ (1+3^2/n^2)$ ---------- $(1+n^2/n^2)]^{1/n}$

Sol.

$$A = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right).....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

Taking log both side

$$\log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right) + \log\left(1+\frac{2^2}{n^2}\right) + ...\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 \log(1+x^2)\,dx$$

$$= \left[x\log(1+x^2)\right]_0^1 - 2\int_0^1 \frac{x^2+1-1}{1+x^2}dx$$

$$= \log 2 - 2\left[x - \tan^{-1}x\right]_0^1$$

$$= \log 2 - 2 + \frac{\pi}{2}$$

$$\log\frac{A}{2} = \pi/2 - 2 \Rightarrow A = 2e^{\frac{1}{2}(\pi-4)}$$

Q.14 Prove the inequalities :

$$2e^{-1/4} < \int_0^2 e^{x^2-x}dx < 2e^2$$

Sol. $f(x) = e^{x^2} - x$

$f'(x) = e^{x^2} - x\,(2x-1)$

$f_{min} = f(1/2) = e^{-1/4}$

$f_{max} = f(0) = e$

$f(2) = e^2$

$$(2-0)e^{-1/4} < \int_0^2 e^{x^2-x}dx < (2-0)e^2$$

Q.15 If $I_{m,n} = \int_0^1 x^m(\lambda nx)^n\,dx$, then prove that

$$I_{m,n} = (-1)^n\frac{n!}{(m+1)^{n+1}}\ ;\ m, n \in N$$

Sol. $I_{m,n} = \int_0^1 x^m(\lambda nx)^n\,dx$,

$$= \left[\frac{x^{m+1}}{m+1}(\lambda nx)^n\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{n(\lambda n\,x)^{n-1}}{x}dx$$

$$= -\frac{n}{m+1}\int_0^1 x^m(\lambda nx)^{n-1}$$

$$I_{m,n} = -\frac{n}{m+1}\ I_{m,n-1}$$

$$\text{Again } I_{m,n} = -\frac{n}{m+1}\left[\left\{\frac{x^{m+1}}{m+1}(\lambda nx)^{n-1}\right\}_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{(n-1)(\lambda nx)^{n-2}}{x}dx\right]$$

$$= -\frac{n}{m+1}\left[-\frac{n-1}{m+1}\int_0^1 x^m(\lambda n\,x)^{n-2}dx\right]$$

$$= (-1)^2\frac{n(n-1)}{(m+1)^2}\ I_{m,n-2}$$

Continue we get

$$I_{m,n} = (-1)^n\frac{n(n-1).....3.2.1}{(m+1)^n}\ I_{m,o}$$

$= (-1)^n \dfrac{n!}{(m+1)^n} \int_0^1 x^m dx$

$= (-1)^n \dfrac{n!}{(m+1)^n} \cdot \left[\dfrac{x^{m+1}}{m+1}\right]_0^1$

$I_{m,n} = (-1)^n \dfrac{n!}{(m+1)^{n+1}}$ R.H.S. Hence proved

Q.16 Find all the values of α belonging to the interval [0, 2π] and satisfying the equation $\int_{\pi/2}^{\alpha} \sin x\, dx = \sin 2\alpha$

Sol. $\int_{\pi/2}^{\alpha} \sin x\, dx = \sin 2\alpha$

$\Rightarrow [\cos x]_{\pi/2}^{\alpha} = \sin 2\alpha$

$\Rightarrow -\cos\alpha = 2\sin\alpha\cos\alpha$

$\Rightarrow \cos\alpha\,(2\sin\alpha + 1) = 0$

$\Rightarrow \cos\alpha = 0$ or $\sin\alpha = -1/2$

in $[0, 2\pi]$ we get $\pi/2, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

Q.17 Evaluate : $\lim\limits_{x\to\infty} \dfrac{\left(\int_0^x e^{x^2} dx\right)^2}{\int_0^x e^{2x^2} dx}$

Sol. $\lim\limits_{x\to\infty} \dfrac{2\int_0^x e^{x^2} dx . e^{x^2}}{e^{2x^2} e^{x^2}}$

LH rule

$\lim\limits_{x\to\infty} \dfrac{2e^{x^2}}{e^{x^2} .2x} = 0$

Q.18 $\lim\limits_{x\to 0} \dfrac{\int_0^x \frac{t^2\, dt}{\sqrt{a+t}}}{bx - \sin x} = 1$ then find the value of a & b.

Sol. LH rule

$\lim\limits_{x\to 0} \dfrac{\frac{x^2}{\sqrt{a+x}}}{b - \cos x}$

b = 1

LH rule

$\lim\limits_{x\to 0} \dfrac{2x}{\sqrt{a+x}\sin x + \frac{(1-\cos x)}{\sqrt{a+x}}}$

$\lim\limits_{x\to 0} \dfrac{2x\sqrt{a+x}}{a\sin x + x\sin x + 1 - \cos x}$

LH rule

$\lim\limits_{x\to 0} \dfrac{2\left(\frac{x}{2\sqrt{a+x}} + \sqrt{a+x}\right)}{a\cos x + x\cos x + 2\sin x}$

Q.19 Evaluate the limits $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sin\sqrt{x}\, dx}{x^3}$

Sol. LH rule

$\lim\limits_{x\to 0} \dfrac{\sin x.2x}{3x^2} = \dfrac{2}{3}$

Q.20 Solve : $\int_{-1}^{x} (8t^2 + \frac{28}{3}t + 4)\, dt = \dfrac{1.5x+1}{\log_{x+1}\sqrt{x+1}}$

Sol. $\dfrac{8}{3}(x^3 + 1) + \dfrac{14}{3}(x^2 - 1) + 4(x + 1)$

$= \dfrac{\frac{3}{2}x + 1}{1/2}$

$\Rightarrow 8x^3 + 8 + 14x^2 - 14 + 12x + 12 = 9x + 6$

$\Rightarrow 8x^3 + 14x^2 + 3x = 0$

$x\,(8x^2 + 14x + 3) = 0$

$x\,(8x^2 + 2x + 12x + 3) = 0$

$x\,(2x + 3)(4x + 1) = 0$

$x = 0, \; -\dfrac{3}{2}, \; -\dfrac{1}{4}$

Q.21 Prove that $\int_0^x \left(\int_0^u f(x)\,dx\right) du = \int_0^x f(u)(x-u)\,du$

Sol. $\int_0^x 1.\left(\int_0^u f(t)\,dt\right) du$

$$\left[\int_0^u f(t)dt . u\right]_0^x = \int_0^x f(u)u\ du$$

$$\Rightarrow x\int_0^x f(t)dt - \int_0^x f(u)du$$

$$\int_0^x f(u)(x-u)du$$

Q.22 Let $y = \int_0^x \sin(2x-t)\,dt$. Find $\frac{dy}{dx}$?

Sol. $y' = 2\int_0^x \cos(2x-t)dt + \sin x$

$= -2\left[\sin(2x-t)\right]_0^x + \sin x$

$= -2[\sin x - \sin 2x] + \sin x$

$= 2\sin 2x - \sin x$

Q.23 Let f(x) be an odd periodic function in $\left[-\frac{T}{2}, \frac{T}{2}\right]$ where T be its period, then show that $g(x) = \int_a^x f(x)dx$ will also be periodic function with period T.

Sol. $g(x+T) = \int_a^{x+T} f(x)dx$

$$= \int_a^x f(x)dx + \underbrace{\int_x^{x+T} f(x)dx}_{I_1}$$

f(x) is odd, periodic

$f(-x) = -f(x)$

$f(x+T) = f(x)$

$f(-x+T) = f(-x) = -f(x)$

$I_1 = \int_0^T f(x)dx$

king

$I_1 = \int_0^T f(T-x)dx = \int_0^T -f(x)dx = -I_1$

$2I_1 = 0 \Rightarrow I_1 = 0$

$g(x+T) = g(x)$

Q.24 $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-7x^3-12x^2+x+1}{x^2+2}dx = ?$

Sol. **king**

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-7x^3-12x^2+x+1}{x^2+2}dx$$

$$2I = 2\int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}dx$$

$$I = 2\int_0^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}dx$$

$$= 2\int_0^{\sqrt{2}} \frac{3x^2(x^4-4)+1}{x^2+2}dx$$

$$= 2\int_0^{\sqrt{2}} \left(3x^2(x^2-2) + \frac{1}{x^2+2}\right)dx$$

$$= 6\int_0^{\sqrt{2}} (x^4-2x^2)\,dx + 2\int_0^{\sqrt{2}} \frac{1}{x^2+2}dx$$

$$= 6\ \frac{1}{5}(\sqrt{2})^5 - 4(\sqrt{2})^3 + \frac{2}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$= \frac{6}{5}(2^{5/2}) - 4(2^{3/2}) + \sqrt{2}\ \frac{\pi}{4}$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$$

Q.25 If m, n ∈ N, evaluate : $\int_a^b (x-a)^m\ (b-x)^n\ dx$.

Sol. $x = a\cos^2\theta + b\sin^2\theta$

$$I = \int_0^{\pi/2} \left[(b-a)\sin^2\theta\right]^m\ ((b-a)\cos^2\theta)^n$$

$$\times (b-a)\sin 2\theta\ d\theta$$

$$I = 2(b-a)^{m+n+1}\int_0^{\pi/2}\sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta$$

$$= 2(b-a)^{m+n+1}\frac{\lceil m+1\ \lceil n+1}{2\lceil m+n+2}$$

$$= (b-a)^{m+n+1}\frac{m!n!}{(m+n+1)!}$$

Q.26 Prove that $\int_0^{\infty}\frac{dx}{1+x^n} = \int_0^1\frac{dx}{(1-x^n)^{1/n}}$; $n > 1$

Q.27 Prove that $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\lambda n\, x}{x}dx = \lambda n\, a\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{dx}{x}$

Q.28 Find f(x), if $f(x) = \lambda\int_0^{\pi/2}\sin x.\cos t\,.\,f(t)\,dt + \sin x$

Sol. $f(x) = \lambda \sin x\int_0^{\pi/2}\cos t\, f(t)dt + \sin x$

f(x) = A sin x

where

$$A = \lambda\int_0^{\pi/2}\cos t\, f(t)dt + 1 = \frac{\lambda A}{2}\int_0^{\pi/2}2\cos t\sin t\,dt + 1$$

$$= -\frac{\lambda A}{4}[\cos 2t]_0^{\pi/2} + 1 \Rightarrow A = \frac{\lambda A}{2} + 1$$

$$A = \frac{2}{2-\lambda} \Rightarrow f(x) = \frac{2}{2-\lambda}\sin x$$

Q.29 $\int_1^{\frac{1+\sqrt5}{2}}\frac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\frac{1}{x}\right)dx$

Sol. $\int_1^{\frac{\sqrt5+1}{2}}\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2-1+\frac{1}{x^2}\right)}\lambda n\left(1+x-\frac{1}{x}\right)dx$

$$\int_1^{\frac{\sqrt5+1}{2}}\frac{\left(1+\frac{1}{x^2}\right)}{\left[\left(x-\frac{1}{x}\right)^2+1\right]}\lambda n\left(1+x-\frac{1}{x}\right)dx$$

$$x - \frac{1}{x} = t$$

$$\int_0^1\frac{\lambda n(t+1)dt}{[(t^2+1)]}$$

$$\left[\lambda n(t+1)\tan^{-1}(t)\right]_1^2$$

$$\int_0^1\frac{\tan^{-1}t}{t+1}dt$$

$\downarrow$

I_1

Put $t = \tan u$

$$\int_0^{\pi/4}\frac{u\sec^2 u}{(\tan u+1)}du$$

Part-B Passage based objective questions

Passage-1 (Question 30 to 32)

A polynomial $P(x) = ax^3 + bx^2 + cx + d$ vanishes at x = 1 and has relative maximum/minimum at x = 1 and x = – 2. If $\int_{-1}^{1}P(x)dx = 16$, then

On the basis of above information, answer the following questions:

Q.30 The nature of roots of P(x) = 0 is -

(A) One real and two complex

(B) All real and distinct

(C) All real but two of those are equal

(D) All real with two of those are negative

Sol. $P'(x) = \alpha(x-1)(x+2)$

$= \alpha(x^2 + x - 2)$

$$P(x) = \frac{\alpha}{3}x^3 + \frac{\alpha}{2}x^2 - 2\alpha x + k$$

$$\frac{\alpha}{3} + \frac{\alpha}{2} - 2\alpha + k = 0$$

$6k = 7\alpha$

$$I = \int_{-1}^{1}P(x)dx$$

$$2I = 2\int_{-1}^{1}\left(\frac{\alpha}{2}x^2 + k\right)dx$$

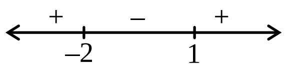
$16 = \frac{\alpha}{6}.2 + 2k$

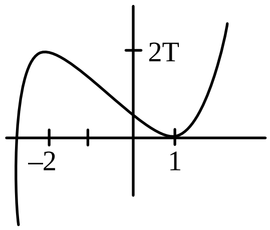
$\alpha + 6k = 48$

$\alpha = 6$ & $k = 7$

$P(x) = 2x^3 + 3x^2 - 12x + 7$

$P'(x) = 6(x - 1)(x + 2)$





Q.31 The set of values of a such that the equation $P(x) - a = 0$ has exactly three distinct real roots is -

(A) (0, 27) (B) (–2, 3)

(C) (0, 9) (D) (0, 8)

Sol. **[A]**

$a \in (0, 27)$

Q.32 $\int_{-2}^{2} P(x)\,dx$ is -

(A) 11 (B) 22 (C) 33 (D) 44

Sol. **[D]**

$$I = \int_{-2}^{2} P(x)\,dx$$

$$I = \int_{-2}^{2} (3x^2 + 7)\,dx = [x^3]_{-2}^{2} + 7[x]_{-2}^{2}$$

$= 16 + 28 = 44$

Passage-2 (Question 33 to 35)

Let $f(x) = ax^2 + bx + c$ is quadratic polynomial, where a, b, c $\in$ R and a > 0, f(x) = 0 has two real and different positive roots α and β ($\alpha < \beta$)

Now $\int_{0}^{\alpha} f(x)\,dx = A$ and $\int_{\alpha}^{\beta} f(x)\,dx = B$.

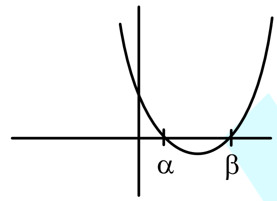
On the basis of above information, answer the following questions.

Q.33 Which statement is correct-

(A) A > 0, B > 0 (B) A < 0, B < 0

(C) A > 0, B < 0 (D) A < 0, B > 0

Sol. **[C]**



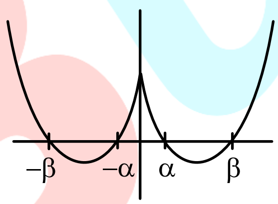
A > 0, B < 0

Q.34 Find $\int_{-\beta}^{0} f(|x|)\,dx$

(A) A + B (B) 2A – 2B

(C) A – B (D) none of these

Sol. **[A]**



$$\int_{-\beta}^{0} (f(|x|)dx = \int_{0}^{\beta} f(x)\,dx$$

$= A + B$

Q.35 Find $\int_{-\beta}^{\beta} (f(|x|) + |f(|x|)|)\,dx$ -

(A) 4A (B) 2B

(C) 2(A + B) (D) none of these

Sol. **[A]**

$$\int_{-\beta}^{\beta} f(|x|)dx + \int_{-\beta}^{\beta} |f(|x|)|\,dx$$

$(2A + 2B) + (2A - 2B)$

$= 4A$

Passage-3 (Question 36 to 38)

Let there are two function

$f(x) = \min(|x - 1|, |x|, |x + 1|)$ and

$g(x) = \max(\sin^{-1}(\sin x), \cos^{-1}(\cos x))$

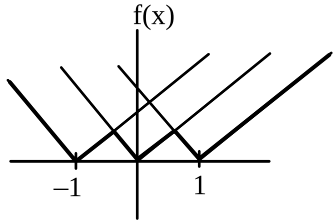
On the basis of above information, answer the following questions -

Q.36 Find the no. of non differential points of f(x) in $x \in (-1, 1)$

(A) 2 (B) 3

(C) 1 (D) none of these

Sol. **[B]**



3 non diff. point in (–1, 1)

Q.37 Which statement is correct -

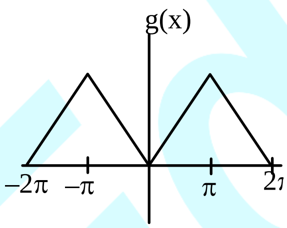
(A) $\int_0^1 f(x)\,dx = \int_0^2 g(x)\,dx$

(B) $\int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{1/2} g(x)dx$

(C) $\int_0^1 f(x)\,dx > \int_0^{\pi/2} g(x)dx$

(D) none of these

Sol. **[B]**



$$\int_0^1 f(x)\,dx = \frac{1}{2}$$

$$\int_0^2 g(x)\,dx = 1$$

Q.38 Find $\int_0^{2\pi} g(x)\,dx$ -

(A) π^2 (B) $2\pi^2$

(C) $\pi^2/2$ (D) none of these

Sol. **[A]**

$$\int_0^{2\pi} g(x)\,dx = \frac{1}{2} \times \pi \times 2\pi = \pi^2$$

Passage-4 (Question 39 to 41)

Let b be a real parameter, for which a real valued function, f(x) is defined as follows

$f(x) = \int_a^x (bt^2 + b + \cos t)\ dt$: here a is real

One the basis of above passage, answer the following questions.

Q.39 If f(x) is monotonic for all real values of x, then b belongs to -

(A) $(-\infty, 0]$ (B) $[0, \infty)$

(C) $(-\infty, -1] \cup [1, \infty)$ (D) $(-\infty, 0] \cup [1, \infty)$

Sol. **[C]**

$f'(x) = bx^2 + b + \cos x \geq 0$ or ≤ 0

$bx^2 + b \geq 1$ or $bx^2 + b \leq -1$

$-4b(b-1) < 0$ $-4b(b+1) < 0$

$b \in (-\infty, 0) \cup (1, \infty)$ $b \in (-\infty, -1) \cup (0, \infty)$

Hence $b \in (-\infty, -1] \cup [1, \infty)$

Q.40 If $\lim_{x \to 0} \frac{1}{x^3} \int_a^x (bx^2 + b + \cos x)\ dx$ exists finitely, then b is -

(A) 0 (B) 1 (C) 2 (D) –1

Sol. **[D]**

Q.41 With increasing values of b, function f (x) -

(A) remain constant

(B) increases for x > a

(C) decreases for x > a

(D) increases for x < a

Sol. **[B]**

EXERCISE # 4

Old IIT-JEE questions

Q.1 Let $f : (0, \infty) \to R$ and $F(x) = \int_0^x f(t)\,dt$.

If $F(x^2) = x^2(1 + x)$. Then f(4) equals- **[IIT-2001]**

(A) 5/4 (B) 7 (C) 4 (D) 2

Sol. **[C]**

$F(x) = \int_0^x f(t)dt$ (given)

By Leibnitz rule

$F'(x) = f(x)$

But $F(x^2) = x^2(1 + x) = x^2 + x^3$

$\Rightarrow F(x) = x + x^{3/2} \Rightarrow F'(x) = 1 + \frac{3}{2}x^{1/2}$

$\Rightarrow f(x) = F'(x) = 1 + \frac{3}{2}x^{1/2}$

$\Rightarrow f(4) = 1 + \frac{3}{2}(4)^{1/2}$

$\Rightarrow f(4) = 1 + \frac{3}{2} \times 2 = 1 + 3 = 4$

Q.2 The integral $\int_{-(1/2)}^{1/2}\left([x] + \lambda n\left(\frac{1+x}{1-x}\right)\right)dx$ equals - **[IIT Scr. 2002]**

(A) – 1/2 (B) 0 (C) 1 (D) 2 λn(1/2)

Sol. **[A]**

$I = \int_{-1/2}^{1/2}[x]dx + \int_{-1/2}^{1/2}\lambda n\left(\frac{1+x}{1-x}\right)dx$

$= \int_{-1/2}^{1/2}[x]dx + 0$

$= \int_{-1/2}^{0}(-1)\,dx + \int_{0}^{1/2}0\,dx$

$= -[x]_{-1/2}^{0} = -\frac{1}{2}$

Q.3 Let $F(x) = \int_1^x \sqrt{2 - t^2}\,dt$, then the real roots of the equation $x^2 - F'(x) = 0$ are- **[IIT-2002]**

(A) ± 1 (B) $\pm\frac{1}{\sqrt{2}}$ (C) $\pm\frac{1}{2}$ (D) 0 and 1

Sol. **[A]**

$F(x) = \int_1^x \sqrt{2 - t^2}\,dt$

$\Rightarrow F'(x) = \sqrt{2 - x^2}$ …(1)

$\therefore x^2 - F'(x) = 0$ …(2)

$\Rightarrow x^2 = \sqrt{2 - x^2} \Rightarrow x^4 = 2 - x^2$

$\Rightarrow x^4 + x^2 - 2 = 0 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow x = \pm 1$

Q.4 Let T > 0 be a fixed number. Suppose f is a continuous function such that for all x ∈ R; $f(x + T) = f(x)$ .If $I = \int_0^T f(x)\,dx$, then the value of $\int_3^{3+3T} f(2x)\,dx$ is- **[IIT-2002]**

(A) $\frac{3}{2}I$ (B) I (C) 3I (D) 6I

Sol. **[C]**

$\int_3^{3+3T} f(2x)dx$, put 2x = y, the given integral reduces to

$\frac{1}{2}\int_6^{6+6T} f(y)\,dy = \frac{6I}{2} = 3I$

Q.5 $I_{(m,\,n)} = \int_0^1 t^m(1 + t)^n\,dt$, then $I_{m,\,n} = ?$ **[IIT Scr.-2003]**

(A) $I_{(m,n)} = \frac{n}{m+1}.\frac{I_{(m+1,n-1)}}{m+1}$

(B) $I_{(m,n)} = \frac{1}{m+1}.\frac{I_{(m+1,\,n-1)}}{m+1}$

(C) $I_{(m,n)} = \frac{2^n}{1+m} - \frac{n.I_{(m+1,n-1)}}{m+1}$

(D) $I_{(m,n)} = \frac{2^n}{1+m} + \frac{n.I_{(m+1,n-1)}}{m+1}$

Sol. **[C]**

Integrating by parts we get

$$I_{m,n} = \left[(1+t)^n \frac{t^{m+1}}{m+1}\right]_0^1 - \int_0^1 \frac{t^{m+1}}{m+1} n(1+t)^{n-1} dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} \int_0^1 t^{m+1}(1+t)^{n-1} dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} I_{(m+1,\, n-1)}$$

Q.6 If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$. Then f(x) increase in - **[IIT-2003]**

(A) (–2, 2) (B) No value of x
(C) (0, ∞) (D) (–∞, 0)

Sol. **[D]**

Here, $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

Differentiating both sides using Newton's Leibnitz formula, we get,

$$f'(x) = e^{-(x^2+1)^2} \cdot \left\{\frac{d}{dx}(x^2+1)\right\} - e^{-(x^2)^2} \cdot \left\{\frac{d}{dx}(x^2)\right\}$$

$$= e^{-(x^2+1)^2} \cdot 2x - e^{-(x^2)^2} \cdot 2x$$

$$= 2xe^{-(x^4+2x^2+1)} \{1 - e^{2x^2+1}\}$$

{where, $e^{2x^2+1} > 1, \forall x$ and $e^{-(x^4+2x^2+1)} > 0\ \forall x$ }

$\therefore f(x) > 0$

which shows $2x < 0$ or $x < 0$

$\Rightarrow x \in (-\infty, 0)$

Q.7 If f(x) is an even function then prove that $\int_0^{\pi/2} f(\cos 2x) \cos x\, dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x\, dx$. **[IIT-2003]**

Sol. Let $I = \int_0^{\pi/2} f(\cos 2x) \cos x\, dx$ ….(1)

$$= \int_0^{\pi/2} f[\cos 2(\pi/2 - x)] \cos(\pi/2 - x)\, dx$$

$$= \int_0^{\pi/2} f(-\cos 2x) \sin x\, dx$$

$I = \int_0^{\pi/2} f(\cos 2x) \sin x\, dx$ (f is even) …(2)

adding (1) and (2)

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)\, dx$$

$$I = \frac{\sqrt{2}}{2} \int_0^{\pi/2} f(\cos 2x)\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right) dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} f(\cos 2x) \cos(x - \pi/4)\, dx$$

let $x - \pi/4 = t \Rightarrow dx = dt$

$$I = \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f\left(\cos 2\left(\frac{\pi}{4} + t\right)\right) \cos t\, dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f(-\sin 2t) \cos t\, dt$$

$$= \sqrt{2} \int_0^{\pi/4} f(\sin 2t) \cos t\, dt$$

f is even Hence proved

Q.8 If $\int_0^{t^2} x f(x) dx = \frac{2}{5} t^5$ for $t > 0$, then f (4/25) is equal to - **[IIT-Scr.2004]**

(A) $-\frac{2}{5}$ (B) 0 (C) $\frac{2}{5}$ (D) 1

Sol. **[C]**

$$\int_0^{t^2} x f(x) dx = \frac{2}{5} t^5$$

Differentiating both side we get

$t^2 f(t^2).\ 2t = 2t^4$

$f(t^2) = t$

put $t = \frac{2}{5}$ we get $f\left(\frac{4}{25}\right) = \frac{2}{5}$

Q.9 $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ equal to - **[IIT-Scr.2004]**

(A) $\frac{\pi}{2} + 1$ (B) $\frac{\pi}{2} - 1$ (C) 1 (D) π

Sol. **[B]**

$$I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$$

Put x = cos 2 θ dx = –2 sin 2θ

$$= -\int_{\pi/4}^{0} 2\sin 2\theta . \frac{\sin\theta}{\cos\theta} d\theta$$

$$= 2\int_0^{\pi/4} 2\sin^2\theta\, d\theta \;=\; 2\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

$$= 2\left[x - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi/2 - 1$$

Q.10 If $y(x) = \int_{\pi^2/10}^{x^2} \frac{\cos x . \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$, find $\frac{dy}{dx}$ at x = π. **[IIT-2004]**

Sol. $y(x) = \int_{\pi^2/10}^{x^2} \frac{\cos x . \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$

$$= \cos x \int_{\pi^2/10}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$$

$$\frac{dy}{dx} = -\sin x \int_{\pi^2/10}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x\left[\frac{\cos x}{1+\sin^2 x}.2x\right]$$

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 0\int_{\pi^2/10}^{\pi^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + (-1)\left[\frac{-1}{1+0}.2\pi\right]$$

$= 2\pi$

Q.11 Evaluate: $\int_{-\pi/3}^{\pi/3} \frac{\pi+4x^3}{2-\cos(|x|+\pi/3)} dx$ **[IIT-2004]**

Sol. $I = \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx$

The second integral becomes zero ; f(x) is odd.

$$I = 2\pi\int_0^{\pi/3} \frac{1}{2-\cos(x+\pi/3)} dx$$

Put x + π/3 = t ⇒ dx = dt

$$I = 2\pi\int_{\pi/3}^{2\pi/3} \frac{1}{2-\cos t} dt$$

Put $\cos t = \frac{1-\tan^2(t/2)}{1+\tan^2(t/2)}$ and

Solving we have

$$I = \frac{2\pi}{3}\int_{\pi/3}^{2\pi/3} \frac{\sec^2 t/2}{\tan^2 t/2 + \left(\frac{1}{\sqrt{3}}\right)^2} dt$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}(\sqrt{3}\tan t/2)\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{4\pi\sqrt{3}}{3}\left|\tan^{-1}3 - \pi/4)\right|$$

Q.12 $\int_{\sin x}^{1} t^2 f(t)\, dt = 1-\sin x$; $0 \le x \le \frac{\pi}{2}$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is - **[IIT-Scr.2005]**

(A) 3 (B) 1/3 (C) 1 (D) $\sqrt{3}$

Sol. **[A]**

Differentiable both side we get

– cos x sin^{2}x f(sin x) = – cos x

$$f(\sin x) = \frac{1}{\sin^2 x}$$

put $\sin x = \frac{1}{\sqrt{3}}$

$$f\left(\frac{1}{\sqrt{3}}\right) = 3$$

Q.13 Evaluate:

$\int_0^{\pi} e^{|\cos x|}[2\sin(1/2\cos x) + 3\cos(1/2\cos x)]\sin x\, dx$

[IIT-2005]

Sol. $I = \int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right)\right]\sin x\, dx + \int_0^{\pi} e^{|\cos x|}\left[3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\, dx$

$I = I_1 + I_2$

$I_1 \to (2a - x) = - f(x)$ So I = 0

$$I = 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

Put cos x = t ⇒ –sin x dx = dt

$$I = -6\int_1^0 e^t\cos\frac{t}{2}dt = 6\int_0^1 e^t\cos\frac{t}{2}dt$$

$$= 6\left[\left(e^t\cos\frac{t}{2}\right)_0^1 + \frac{1}{2}\int_0^1 e^t\sin\frac{t}{2}dt\right]$$

$$= 6\left[e\cos\frac{1}{2} - 1 + \frac{1}{2}\left\{\left(e^t\sin\frac{t}{2}\right)_0^1 - \frac{1}{2}\int_0^1 e^t\cos\frac{t}{2}dt\right\}\right]$$

$$= 6\left[e\cos\frac{1}{2} - 1 + \frac{1}{2}e\sin\frac{1}{2}\right] - \frac{1}{4}I$$

$$I + \frac{1}{4}I = 6\left[e\cos\frac{1}{2} + \frac{1}{2}e\sin\frac{1}{2} - 1\right]$$

$$I = \frac{24}{5}\left[e\cos\frac{1}{2} + \frac{1}{2}e\sin\frac{1}{2} - 1\right]$$

Q.14 The value $5050\dfrac{\int_0^1(1-x^{50})^{100}dx}{\int_0^1(1-x^{50})^{101}dx}$ is equal to …………………..... **[IIT-2006]**

Sol. 5051

Passage - (Q.15 to 17) **[IIT-2006]**

Let us define the definite integral using the formula $\int_a^b f(x)\,dx = \frac{b-a}{2}$ (f(a) + f(b)), and for c ∈ (a, b), when $c = \frac{a+b}{2}$, for more accurate result

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{4}\right)(f(a) + f(b) + 2f(c))$$

$$F(c) = \left[\frac{c-a}{2}(f(a)+f(c))\right] + \left[\frac{b-c}{2}(f(b)+f(c))\right]$$

Q.15 Evaluate $\int_0^{\pi/2}\sin x\,dx$ -

(A) $\frac{\pi}{8\sqrt{2}}(1+\sqrt{2})$ (B) $\frac{\pi}{8}(1+\sqrt{2})$

(C) $\frac{\pi}{4}(1+\sqrt{2})$ (D) $\frac{\pi}{4}(1+2\sqrt{2})$

Sol.[B] $\int_0^{\frac{\pi}{2}}\sin x\,dx = \dfrac{\frac{\pi}{2}-0}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

Q.16 If f ″(x) < 0, ∀ x ∈ (a, b) and c is a point such that a < c < b, and (c, f (c)) is the point lying on the curve for which F(c) is maximum, then f ′(c) is equal to -

(A) $\frac{f(b)-f(a)}{b-a}$ (B) $\frac{2(f(b)-f(a))}{b-a}$

(C) $\frac{2f(b)-f(a)}{2b-c}$ (D) 0

Sol.[A] f ″(x) < 0 ∀ x ∈ (a, b) for c ∈ (a, b)

$$F(c) = \frac{c-a}{2}(f(a) + f(c)) + \frac{b-c}{2}(f(b) + f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$$

$$= \frac{1}{2}[(b-a)f'(c) + f(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a)f''(c) < 0$$

Θ f ″(c) < 0

∴ F(c) is max. at the point (c, f(c))

Where F ′(c) = 0 ⇒ $f'(c) = \frac{1}{2}\left(\frac{f(b)-f(a)}{b-a}\right)$

Q.17 If f(x) is a polynomial and

$$\lim_{t\to a}\frac{\int_a^t f(x)dx - \left(\frac{t-a}{2}\right)(f(t)+f(a))}{(t-a)^3} = 0 \text{ for all}$$

a, then degree of f(x) can atmost be-

(A) 1 (B) 2 (C) 3 (D) 4

Sol.[D] $\displaystyle\lim_{h\to 0}\frac{\int_a^{a+h}f(x)dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = \frac{0}{0}$

using L.Hospital rule

$$\lim_{h\to 0}\frac{f(a+h) - \frac{1}{2}(f(a+h)+f(a)) - \frac{h}{2}f'(a+h)}{3h^2}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = \frac{0}{0}$$

Using L.Hospital rule

$$= \lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h}$$

$$= -\frac{f''(a+h)}{12}$$

For f(x) exist minimum degree 1

$\Rightarrow$ at most degree from option is 4

Q.18 Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and **[IIT-2008]**

$T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for n = 1, 2, 3, ... Then

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Sol. **[A,C]**

$$T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$$

$$\lim_{n\to 0} T_n = \int_0^1 \frac{dx}{1+x+x^2}$$

$$\text{Let } x = \frac{k}{n} = \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left.\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|_0^1$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left.\frac{2x+1}{\sqrt{3}}\right|_0^1$$

$$= \frac{2}{\sqrt{3}} [\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}) \frac{2}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$\lim_{n\to\infty} T_n = \frac{2}{\sqrt{3}} \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

This is for $n \to \infty$

$\therefore$ But $n < \infty$

$$\therefore T_n < \frac{\pi}{3\sqrt{3}}$$

$$S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$$

$$\lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{1+x+x^2}$$

$$\text{Let } x = \frac{k}{n} = \frac{\pi}{3\sqrt{3}}$$

Again $n \to \infty$ then $S_n = \frac{\pi}{3\sqrt{3}}$

$\therefore S_n < \frac{\pi}{3\sqrt{3}}$ as n is not exactly ∞.

Q.19 Let f(x) be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then - **[IIT-2008]**

(A) $f''(x)$ vanishes at least twice on [0, 1]

(B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right) \sin x\, dx = 0$

(D) $\int_0^{1/2} f(t) e^{\sin \pi t} dt = \int_{1/2}^{1} f(1-t) e^{\sin \pi t} dt$

Sol. **[A,B,C,D]**

$f(x) = f(1-x)$

$f'(x) = -f'(1-x) \Rightarrow f'(1/4) = -f'(3/4) = 0$ (given)

put x = 1/2

$$\Rightarrow f'\left(\frac{1}{2}\right) = -f'\left(\frac{1}{2}\right)$$

$$\Rightarrow f'\left(\frac{1}{2}\right) = 0$$

Since $f'(1/2) = f'(1/4) = f'(3/4) = 0$ hence by Rolle's theorem statement A is true.

as given $f(x) = f(1-x)$

replace x by $x + \frac{1}{2}$

$$f\left(\frac{1}{2}+x\right) = f\left[1-\left(\frac{1}{2}+x\right)\right] = f\left(\frac{1}{2}-x\right) \quad ...(1)$$

$$\text{Now, } I = \int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right) \sin x\, dx$$

$I = -\int_{-1/2}^{1/2} f\left(-x+\frac{1}{2}\right)\sin x\,dx$ (put x = – x)

$I = -\int_{-1/2}^{1/2} f\left(\frac{1}{2}+x\right)\sin x\,dx$ [by (1)]

$I = -I$

So $I = 0$

Now, let $t = 1 - x$

$-\int_{1}^{1/2} f(1-x)e^{\sin \pi x}\,dx$

$= \int_{1/2}^{1} f(1-t)e^{\sin \pi t}\,dt$

Q.20 Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$, and $f(0) = 0$, then- **[IIT- 2009]**

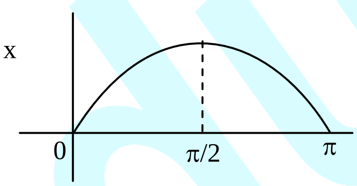
(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Sol. **[C]**



$\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$

On differentiating

$\sqrt{1-(f'(x))^2} = f(x)$

On solving $f'(x) = \sqrt{1-(f(x))^2}$

$f(x) = \sin x$

Θ $\sin x < x, \ \forall x > 0$

So $f\left(\frac{1}{2}\right) < \frac{1}{2}$

$f\left(\frac{1}{3}\right) < \frac{1}{3}$

Q.21 If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$, n = 0, 1, 2,........., then : **[IIT- 2009]**

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

Sol. **[A, B, C]**

$I_n = \int_{-\pi}^{\pi} \frac{\sin n\, x}{(1+\pi^x)\sin x}$

$\Rightarrow I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}dx$

Adding $2I_n = \int_{-\pi}^{\pi} \frac{\sin (nx)}{\sin x}dx$

or $I_n = \int_{0}^{\pi} \frac{\sin nx}{\sin x}dx$

$\Rightarrow I_{n+2} = \int_{0}^{\pi} \frac{\sin(n+2)x}{\sin x}dx$

Consider

$I_{n+2} - I_n = \int_{0}^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}dx$

$= \int_{0}^{\pi} \frac{2\cos(n+1)x\,\sin x}{\sin x}dx = 0$

$\Rightarrow I_{n+2} = I_n$ … (1)

again $\sum_{m=1}^{10} I_{2m+1} = I_3 + I_5 + I_7 + \ldots.. I_{21}$

using (1) $= 10I_3 = 10I_1$

$= 10\pi$

$\sum_{m=1}^{10} I_{2m} = 0$ because

$I_{2m} = \int_{0}^{\pi} \frac{\sin 2mx}{\sin x}dx = 0$ use $f(a - x) = -f(x)$

Also $I_{n+1} \ne I_n$

as by trial $I_1 \ne I_2$

Q.22 The value of $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\,\lambda n(1+t)}{t^4+4}dt$ is -

[IIT-2010]

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

Sol. **[B]**

Use L'hospital rule in

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\,\lambda n(1+t)}{t^4+4}dt$$

$$= \lim_{x\to 0} \frac{x\,\lambda n(1+x)}{(x^4+4)3x^2}$$

$$= \lim_{x\to 0} \frac{\lambda n(1+x)}{x.3(x^2+4)}$$

$$= \frac{1}{3.4} = \frac{1}{12}$$

Q.23 The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) -

[IIT-2010]

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

Sol. **[A]**

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}$$

$$= \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{1+x^2}dx$$

$$= \int_0^1 (x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2})dx$$

$$= \frac{22}{7}-\pi$$

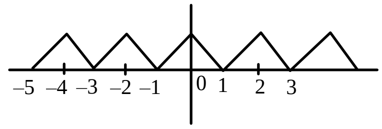
Q.24 For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by **[IIT 2010]**

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\ \cos \pi x\ dx$ is

Sol.[4] $f(x) = \begin{cases} \{x\} & \forall [x] \text{is odd} \\ 1=\{x\} & \forall [x] \text{is even} \end{cases}$

graph of y = f(x) is



Θ f(x) & cos πx both are even functions

$$\text{So, } I = \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\ dx$$

$$= \frac{\pi^2}{5}\int_0^{10} f(x)\cos(\pi x)dx$$

∴ f(x) & cos πx both are periodic then

$$I = \pi^2\int_0^2 f(x)\ \cos(\pi x)\ dx$$

$$= \pi^2\left[\int_0^1 (1-x)\cos x(\pi x)dx + \int_1^2 (x-1)\cos(\pi x)dx\right]$$

$$= \pi^2\left[\frac{2+2}{\pi^2}\right] = 4$$

Q.25 The value of $\int_{\sqrt{\lambda n2}}^{\sqrt{\lambda n3}} \frac{x\sin x^2}{\sin x^2+\sin(\lambda n\,6-x^2)}dx$ is

[IIT-2011]

(A) $\frac{1}{4}\lambda n\frac{3}{2}$ (B) $\frac{1}{2}\lambda n\frac{3}{2}$

(C) $\lambda n\frac{3}{2}$ (D) $\frac{1}{6}\lambda n\frac{3}{2}$

Sol.[A] Let $x^2 = t$ $\quad xdx = \frac{dt}{2}$

$$I = \frac{1}{2}\int_{\lambda n2}^{\lambda n3} \frac{\sin t}{\sin t+\sin(\lambda n6-t)}dt \quad \ldots\ldots (1)$$

$$I = \frac{1}{2}\int_{\lambda n2}^{\lambda n3} \frac{\sin(\lambda n6-t)}{\sin t+\sin(\lambda n6-t)}dt \quad \ldots\ldots (2)$$

Add (1) & (2)

$$2I = \frac{1}{2}\int_{\lambda n2}^{\lambda n3} dt$$

$$I = \frac{1}{4}(\lambda n3-\lambda n2) = \frac{1}{4}\lambda n\frac{3}{2}$$

Q.26 Let f : [1, ∞) → [2, ∞) be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3x f(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is. **[IIT-2011]**

Sol. **[6]**

$$6\int_1^x f(t)\,dt = 3xf(x) - x^3$$

$$6 f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$3f(x) = 3x f'(x) - 3x^2$$

$$3y = 3x\frac{dy}{dx} - 3x^2$$

$$x\frac{dy}{dx} - y = x^2$$

$$\frac{dy}{dx} - \frac{y}{x} = x$$

$$\text{I.F.} = e^{\int \frac{-1}{x}dx} = e^{-\lambda n\,x} = \frac{1}{x}$$

$$y\cdot\frac{1}{x} = \int x.\frac{1}{x}dx \Rightarrow \frac{y}{x} = x + c \Rightarrow y = x^2 + cx$$

$$\Theta\ f(1) = 2 \Rightarrow c = 1$$

$$y = x^2 + x$$

$$f(2) = 4 + 2 = 6$$

Q.27 The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ln\frac{\pi + x}{\pi - x}\right)\cos x\,dx$ is **[IIT-2012]**

(A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$

Sol. [B] $\int_{-\pi/2}^{\pi/2} x^2\cos x\,dx + \int_{-\pi/2}^{\pi/2}\lambda n\left(\frac{\pi - x}{\pi + x}\right)\cos x\,dx$

↓ Even function ↓ Odd function

$$= 2\int_0^{\pi/2} x^2\cos x\,dx + 0$$

$$= 2[x^2\sin x + 2x\cos x - 2\sin x]_0^{\pi/2} = 2\left[\frac{\pi^2}{4} - 2\right]$$

$$= \frac{\pi^2}{2} - 4$$

Q.28 If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then **[IIT-2012]**

(A) f has a local maximum at x = 2

(B) f is decreasing on (2, 3)

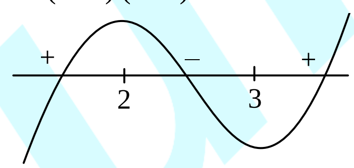
(C) there exists some c ∈ (0, ∞) such that $f''(c) = 0$

(D) f has a local minimum at x = 3

Sol. **[A, B, C, D]**

$$f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$$

$$f'(x) = e^{x^2}(x-2)(x-3)$$



$f'(x) < 0 \quad \forall\ x \in (2, 3)$

so f(x) is decreasing on (2, 3)

also at x = 2, $f'(x)$ changes its sign from +ve to –ve.

Hence x = 2 is point of maxima

At x = 3, $f'(x)$ changes its sign from –ve to +ve.

Hence x = 3 is point of minima.

Also $f'(2) = f'(3) = 0$

So from Rolle's Theorem there exist a point c such that $f''(c) = 0$

Passage (Q 29 to Q. 30)

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in$ IR, and let $g(x) = \int_1^x\left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt$ for all $x \in (1, \infty)$. **[IIT-2012]**

Q.29 Consider the statements :

P : There exists some x ∈ IR such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some x ∈ IR such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both **P** and **Q** are true

(B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true

(D) both **P** and **Q** are false

Sol. [C] (P): $(\sin^2 x)(1-x)^2 + x^2 + 2x = 2 + 2x^2$

$(\sin^2 x)(1-x)^2 = x^2 - 2x + 2$

$\sin^2 x = \dfrac{(1-x)^2+1}{(1-x)^2}$, which is greater than 1 $\Rightarrow$

no solution

$\Rightarrow$ P is false.

(Q) : $2\sin^2 x = \dfrac{x^2}{(1-x)^2} - 1$

$0 \le \dfrac{x^2}{(1-x)^2} - 1 \le 2$.

This inequality is satisfied by some value of x.

Q.30 Which of the following is true?

(A) g is increasing on $(1, \infty)$

(B) g is decreasing on $(1, \infty)$

(C) g is increasing on (1, 2) and decreasing on $(2, \infty)$

(D) g is decreasing on (1, 2) and increasing on $(2, \infty)$

Sol. [B] $g'(x) = \left(\underbrace{\dfrac{2(x-1)}{x+1} - \ln x}_{\substack{-\text{ve} \\ \text{for } x>1}}\right)\underbrace{f(x)}_{\text{always}+\text{ve}}$

$g'(x) < 0 \Rightarrow g(x) \downarrow$ on $(1, \infty)$

➤ Match the Column type questions

Q.31 Match the integrals in **Column I** with the values in **Column II** **[IIT-2007]**

Column I	**Column II**
(A) $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$	(P) $\dfrac{1}{2}\log\left(\dfrac{2}{3}\right)$
(B) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	(Q) $2\log\left(\dfrac{2}{3}\right)$
(C) $\displaystyle\int_{2}^{3} \frac{dx}{1-x^2}$	(R) $\dfrac{\pi}{3}$
(D) $\displaystyle\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	(S) $\dfrac{\pi}{2}$

Sol. **[A → S, B → S , C → P , D → R]**

(A) $\displaystyle\int_{-1}^{1}\frac{1}{1+x^2}dx = 2\int_{0}^{1}\frac{1}{1+x^2}dx = 2\tan^{-1}1 = \pi/2$

(B) $\displaystyle\int_{0}^{1}\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}1 - \sin^{-1}0 = \pi/2$

(C) $\displaystyle\int_{2}^{3}\frac{dx}{1-x^2} = \frac{1}{2}\left(\log\left|\frac{1+x}{1-x}\right|\right)_2^3 = \frac{1}{2}\log\frac{2}{3}$

(D) $\displaystyle\int_{1}^{2}\frac{dx}{x\sqrt{x^2-1}} = \left(\sec^{-1}x\right)_1^2$

$= \sec^{-1}2 - \sec^{-1}1 = \dfrac{\pi}{3}$

Q.32 Match the statements in **Column-I** with the values in **Column-II**. **[IIT-2010]**

Column-I	**Column-II**
(A) A line from the origin meets the lines $\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1}$ and $\dfrac{x-\frac{8}{3}}{2} = \dfrac{y+3}{-1} = \dfrac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(P) –4
(B) The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\dfrac{3}{5}\right)$ are	(Q) 0
(C) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b} = 0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ and $2\lvert\vec{b}+\vec{c}\rvert = \lvert\vec{b}-\vec{a}\rvert$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	(R) 4
(D) Let f be the function on $[-\pi, \pi]$ given by f(0) = 9 and $f(x) = \sin\left(\dfrac{9x}{2}\right)\Big/\sin\left(\dfrac{x}{2}\right)$ for $x \ne 0$. The value of $\dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ is	(S) 5
	(T) 6

Sol. **A → T ; B → P, R ; C → Q, S ; D → R**

(A) Let $P \equiv (\lambda + 2, 1 - 2\lambda, \lambda - 1)$

$Q \equiv (2\mu + \dfrac{2}{3}; -\mu - 3, \mu + 1)$

equation line PQ

$\vec{r} = (\lambda + 2)\,\hat{i} + (1 - 2\lambda)\,\hat{j} + (\lambda - 1)\,\hat{k}$

$+ \alpha\left((2\mu - \lambda + \frac{2}{3})\,\hat{i} + (2\lambda - \mu - 4)\,\hat{j} + (\mu + 2 - \lambda)\,\hat{k}\right)$

$\therefore$ This line passing through origin so.

$\lambda + 2 + \alpha\,(2\mu - \lambda + \frac{2}{3}) = 0$

$1 - 2\lambda + \alpha(2\lambda - \mu - 4) = 0$

$\lambda - 1 + \alpha(\mu - \lambda + 2) = 0$

on solving above three $\mu = \frac{1}{3}$ & $\lambda = 3$

So $P \equiv (5, -5, 2)$ & $Q \equiv (\frac{10}{3}, \frac{-10}{3}, \frac{4}{3})$

So $PQ = \sqrt{6} \Rightarrow (PQ)^2 = 6$

(B) $\tan^{-1}(x + 3) - \tan^{-1}(x - 3) = \sin^{-1}\frac{3}{5}$

$\tan^{-1}\frac{6}{x^2 - 8} = \tan^{-1}\frac{3}{4}$

$\Rightarrow x^2 - 8 = 8$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

(C) $|\vec{b}|^2 + \vec{b}.\vec{c} = \vec{a}.\vec{c}$ …. (1)

put $\vec{a} = \mu\vec{b} + 4\vec{c}\ \forall\ \vec{a}.\vec{b} = 0 \Rightarrow \vec{b}.\vec{c} = -\frac{\mu}{4}|\vec{b}|^2$ …(2)

from (1) and (2)

$\frac{b^2}{c^2} = \frac{16}{4 - \mu + \mu^2}$ … (3)

$\therefore 2|\vec{b} + \vec{c}| = |\vec{b} - \vec{a}|$ and $\vec{a} = \mu\vec{b} + 4\vec{c}$

$\frac{b^2}{c^2} = \frac{12}{3 - 2\mu + \mu^2}$ … (4)

from (3) and (4)

m = 0,5

(D) $f(x) = \frac{\sin\frac{9x}{2}}{\sin\frac{x}{2}} = \frac{\sin 5x}{\sin x} + \frac{\sin 4x}{\sin x}$

$I = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{4}{\pi}\int_{0}^{\pi} f(x)\,dx = \frac{4}{\pi}\int_{0}^{\pi}\frac{\sin 5x}{\sin x}$

$= \frac{8}{\pi}\int_{0}^{\pi/2}\frac{\sin 5x}{\sin x}\,dx$

$= \frac{8}{\pi}\int_{0}^{\pi/2}\frac{\sin(3x + 2x)}{\sin x}\,dx = \frac{8}{\pi}\int_{0}^{\pi/2}(1 + 2\cos 4x)\,dx$

$= 4$

EXERCISE # 5

Old IIT-JEE objective type questions

Q.1 For any integer n the integral

$\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\, dx$ has the value

[IIT-1985]

(A) π (B) 1

(C) 0 (D) None of these

Sol. **[C]**

$$I = \int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\, dx$$

using

$$\int_0^a f(x)\,dx = \begin{cases} 0, & f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)\,dx, & f(a-x) = f(x) \end{cases}$$

where, $f(x) = e^{\cos^2 x} \cdot \cos^3 \{(2n+1)x\}$

$\therefore f(\pi - x) = (e^{\cos^2 x})(-\cos^3 (2n+1)x) = -f(x)$

$\Rightarrow \quad I = 0$

Q.2 Let f : R → R be a differentiable function and f(1) = 4 then the value of $\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1}\,dt$ is -

[IIT-1990]

(A) 8f ′(1) (B) 4f ′(1)

(C) 2f ′(1) (D) f ′(1)

Sol. **[A]**

$$\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1}\,dt = \lim_{x\to 1} \frac{\int_4^{f(x)} 2t\,dt}{x-1}$$

{using, L-Hospital's rule}

$$= \lim_{x\to 1} \frac{2f(x).f'(x)}{1}$$

= 2f(1). f ′(1), where f (1) = 4.

= 8 f ′ (1)

Q.3 Let f : R → R and g : R → R be continuous functions then the value of the integral

$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)]\,[g(x) - g(-x)]\,dx$ is -

[IIT-1990]

(A) π (B) 1 (C) –1 (D) 0

Sol. **[D]**

$$I = \int_{-\pi/2}^{\pi/2} \{f(x) + f(-x)\}\,\{g(x) - g(-x)\}\,dx$$

using $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(-x) = f(x) \end{cases}$

$\therefore$ Let $\phi(x) = \{f(x) + f(-x)\}\,\{g(x) - g(-x)\}$

$\Rightarrow \phi(-x) = \{f(-x) + f(x)\}\,\{g(-x) - g(x)\}$

$\Rightarrow \phi(-x) = -\phi(x) \quad \therefore \quad \phi(x)$ is odd

$\Rightarrow \int_{-\pi/2}^{\pi/2} \phi(x)\,dx = 0$

Q.4 The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is: **[IIT-1993]**

(A) 0 (B) 1 (C) π/2 (D) π/4

Sol. **[D]**

$$\text{Let } I = \int_0^{\pi/2} \frac{1}{1+\tan^3 x}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{1+\frac{\sin^3}{\cos^3 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx \quad \ldots(1)$$

$$= \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\cos^3\left(\frac{\pi}{2} - x\right) + \sin^3\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} 1 dx$$

$$2I = [x]_0^{\pi/2} = \pi/2 \Rightarrow I = \pi/4$$

Q.5 The value of $\int_0^{2\pi} [2\sin x]\,dx$ where [.] represents the greatest integral function is : **[IIT-1995]**

(A) $-\frac{5\pi}{3}$ (B) $-\pi$ (C) $\frac{5\pi}{3}$ (D) -2π

Sol. **[B]**

Let $I = \int_0^{2\pi} [2\sin x]\,dx$

For $0 \le x < \pi/6 \Rightarrow 0 \le 2\sin x < 1$

$\Rightarrow [2\sin x] = 0$

$\pi/6 \le x < 5\pi/6 \Rightarrow 1 \le 2\sin x < 2$

$\Rightarrow [2\sin x] = 1$

$5\pi/6 \le x < \pi \Rightarrow 0 \le 2\sin x < 1$

$\Rightarrow [2\sin x] = 0$

$\pi \le x < 7\pi/6 \Rightarrow -1 \le 2\sin x < 0$

$\Rightarrow [2\sin x] = -1$

$7\pi/6 \le x < 11\pi/6 \Rightarrow -2 \le 2\sin x < -1$

$\Rightarrow [2\sin x] = -2$

$11\pi/6 \le x < 2\pi \Rightarrow -1 \le 2\sin x < 0$

$\Rightarrow [2\sin x] = -1$

$$\therefore I = \int_{\pi/6}^{5\pi/6} 1 dx + \int_{\pi}^{7\pi/6} -1 dx + \int_{7\pi/6}^{11\pi/6} -2 dx + \int_{11\pi/6}^{2\pi} -1 dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + \left(-\frac{7\pi}{6} + \pi\right) + 2\left(-\frac{11\pi}{6} + \frac{7\pi}{6}\right) + \left(-2\pi + \frac{11\pi}{6}\right)$$

$= -\pi$

Q.6 If $f(x) = A\sin(\pi x/2) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)\,dx = \frac{2A}{\pi}$, then the constants A and B are **[IIT-1995]**

(A) $\pi/2$ and $\pi/2$ (B) $2/\pi$ and 3π

(C) 0 and $-4/\pi$ (D) $4/\pi$ and 0

Sol. **[D]**

We have $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$

And $\int_0^1 f(x)\,dx = \frac{2A}{\pi}$

$f'(x) = \frac{A\pi}{2}\cos\frac{\pi x}{2} \Rightarrow f'\left(\frac{1}{2}\right) = \frac{A\pi}{2}\cos\frac{\pi}{4}$

$\Rightarrow f'(1/2) = \frac{A\pi}{2\sqrt{2}}$

But $f'\left(\frac{1}{2}\right) = \sqrt{2}$

So $\frac{A\pi}{2\sqrt{2}} = \sqrt{2} \Rightarrow A = \frac{4}{\pi}$

Now $\int_0^1 f(x)\,dx = \frac{2A}{\pi}$

$$\Rightarrow \int_0^1 \left[A\sin\left(\frac{\pi x}{2}\right) + B\right]dx = \frac{2A}{\pi}$$

$$\Rightarrow \left[-\frac{2A}{\pi}\cos\frac{\pi x}{2} + Bx\right]_0^1 = \frac{2A}{\pi}$$

$$\Rightarrow B + \frac{2A}{\pi} = \frac{2A}{\pi} \Rightarrow B = 0$$

Q.7 Let f be a positive function, let

$I_1 = \int_{1-k}^{k} x.\, f[x(1-x)]dx$ & $I_2 = \int_{1-k}^{k} f[x(1-x)]dx$,

where $(2k-1) > 0$, then $\frac{I_1}{I_2}$ is - **[IIT-1997]**

(A) 2 (B) k (C) 1/2 (D) 1

Sol. **[C]**

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

On I_1 we get

$I_1 = I_2 - I_1$

$2I_1 = I_2 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$

Q.8 If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals - **[IIT-1997]**

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$

(C) $g(x).\ g(\pi)$ (D) $\frac{g(x)}{g(\pi)}$

Sol. **[A]**

$g(x) = \int_0^x \cos^4 t\,dt$ (given)

$g(x+\pi) = \int_0^{\pi+x} \cos^4 t\,dt$

$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{\pi+x} \cos^4 t\,dt$

$= I_1 + I_2$

$\Rightarrow$ $I_1 = g(\pi)$

$I_2 = \int_{\pi}^{\pi+x} \cos^4 t\,dt$

Put $t = \pi + y \Rightarrow dt = dy$

$I_2 = \int_0^x \cos^4 (y+\pi)\,dy = \int_0^x [\cos(\pi+y)]^4\,dy$

$= \int_0^x (-\cos y)^4\,dy = \int_0^x \cos^4 y\,dy = g(x)$

$\therefore\ g(x+\pi) = g(x) + g(\pi)$

Q.9 $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals - **[IIT-1997]**

(A) $1+\sqrt{5}$ (B) $-1+\sqrt{5}$

(C) $-1+\sqrt{2}$ (D) $1+\sqrt{2}$

Sol. **[B]**

$I = \frac{1}{2}\int_0^2 \frac{2x}{\sqrt{1+x^2}}\,dx$

$= \left[\sqrt{1+x^2}\right]_0^2$

$= \sqrt{5} - 1$

Q.10 If $\int_0^x f(t)\,dt = x + \int_x^1 t f(t)\,dt$, then the value of f(1) is - **[IIT-1998]**

(A) 1/2 (B) 0 (C) 1 (D) – ½

Sol. **[A]**

$\int_0^x f(t)\,dt = x + \int_x^1 t f(t)\,dt$

Differentiating both side with respect to x

We get

$f(x) = 1 - x f(x) \Rightarrow (1+x) f(x) = 1$

$\Rightarrow f(x) = \frac{1}{1+x} \Rightarrow f(1) = \frac{1}{2}$

Q.11 Let f(x) = x –[x] for every real number x, where [x] is the integral part of x. then $\int_{-1}^{1} f(x)\,dx$ is: **[IIT-1998]**

(A) 1 (B) 2 (C) 0 (D) –1/2

Sol. **[A]**

$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x-[x])\,dx$

$= \int_{-1}^{1} x\,dx - \int_{-1}^{1} [x]\,dx$

$= 0 - \int_{-1}^{1} [x]\,dx$ [$\because$ x is an odd function]

$\therefore \int_{-1}^{1} [x]\,dx = \int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx$

$= \int_{-1}^{0} (-1)\,dx + \int_0^1 0\,dx = [x]_{-1}^{0} + 0 = 1$

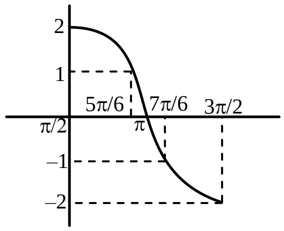
Q.12 If for a real number y, [y] is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$ is - **[IIT-1999]**

(A) $-\pi$ (B) 0

(C) $-\pi/2$ (D) $\pi/2$

Sol. **[C]**

The graph of $y = 2 \sin x$ for $\pi/2 \le x \le \dfrac{3\pi}{2}$ is



from graph

$$f(x) = [2 \sin x] = \begin{cases} 2\,, & x = \pi/2 \\ 1\,, & \pi/2 < x \le \dfrac{5\pi}{6} \\ 0\,, & \dfrac{5\pi}{6} < x \le \pi \\ -1\,, & \pi < x \le 7\pi/6 \\ -2\,, & \dfrac{7\pi}{6} < x \le \dfrac{3\pi}{2} \end{cases}$$

There

$$\int_{\pi/2}^{3\pi/2} [2\sin x]dx$$

$$= \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6} (-1)\,dx$$

$$+ \int_{7\pi/6}^{3\pi/2} (-2)\,dx$$

$$= [x]_{\pi/2}^{\frac{5\pi}{6}} - [x]_{\pi}^{7\pi/6} - 2[x]_{7\pi/6}^{3\pi/2}$$

Solving we get $= -\dfrac{\pi}{2}$

Q.13 $\displaystyle\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to: **[IIT-1999]**

(A) 2 (B) –2 (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$

Sol. **[A]**

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} \quad \ldots(1)$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)}$$

$$(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\ )$$

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x} \quad \ldots(2)$$

Adding (1) and (2)

$$2I = \int_{\pi/4}^{3\pi/4} \frac{2}{1-\cos^2 x} dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \text{cosec}^2 x\,dx = [-\cot x]_{\pi/4}^{3\pi/4}$$

$$= \left[-\cot\frac{3\pi}{4} + \cot\frac{\pi}{4}\right] = 2$$

Q.14 $\displaystyle\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx,\ a > 0$ **[IIT Scr. 2000]**

(A) π (B) πa (C) $\pi/2$ (D) 2π

Sol. **[C]**

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$$

$$\Rightarrow I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx$$

$$\Rightarrow 2I = \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \cos^2 x\,dx$$

$$= \int_0^{\pi} (1+\cos 2x)\,dx = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi$$

$$I = \frac{\pi}{2}$$

Q.15 Let $g(x) = \displaystyle\int_0^x f(t)\,dt$ where t is such that $\dfrac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \dfrac{1}{2}$ for $t \in [1, 2]$. Then g(2) satisfies the inequality - **[IIT-2000]**

(A) $-\dfrac{3}{2} \le g(2) < \dfrac{1}{2}$ (B) $0 \le g(2) < 2$

(C) $\dfrac{3}{2} < g(2) \le \dfrac{5}{2}$ (D) $2 < g(2) < 4$

Sol. **[B]**

$$g(x) = \int_0^x f(t)\,dt \Rightarrow g(2) = \int_0^2 f(t)\,dt$$

$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

Now, $\dfrac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$

We get $\displaystyle\int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)\,dt \le \int_0^1 1\,dt$

(apply line integral on inequality)

$$\frac{1}{2} \le \int_0^1 f(t)\,dt \le 1 \ \ldots(1)$$

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$ …(2)

$$\int_1^2 0\,dt \le \int_1^2 f(t)\,dt \le \int_1^2 \frac{1}{2}\,dt$$

(apply line integral on inequality)

$\Rightarrow$ $0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$

From (1) and (2), we get

$$\frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le 3/2$$

or $\frac{1}{2} \le g(2) \le 3/2$

$\Rightarrow$ $0 \le g(2) \le 2$

Q.16 The value of integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is -

[IIT-2000]

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) 3 (D) 5

Sol. **[B]**

$$\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx = \int_{e^{-1}}^{1} \left|\frac{\log_e x}{x}\right| dx + \int_{1}^{e^2} \left|\frac{\log_e x}{x}\right| dx$$

Since 1 is turning point for

$\left|\frac{\log_e x}{x}\right|$ for + ve and – ve values

$$= -\int_{e^{-1}}^{1} \frac{\log_e x}{x}\,dx + \int_1^{e^2} \left|\frac{\log_e x}{x}\right| dx$$

$$= -\frac{1}{2}\,[(\log_e x)^2]_{e^{-1}}^{1} + \frac{1}{2}\,[(\log_e x)^2]_1^{e^2}$$

Solving we have

$$= \frac{1}{2}\{0-(-1)^2\} + \frac{1}{2}(2^2-0) = 5/2$$

Q.17 If $F(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$.

Then $\int_{-2}^{3} F(x)\,dx =$ **[IIT-2000]**

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **[C]**

If $F(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$

$$\int_{-2}^{3} F(x)\,dx = \int_{-2}^{2} F(x)\,dx + \int_2^3 F(x)$$

$$= \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2dx$$

$= 0 + 2[x]_2^3$ ($\because e^{\cos x}\sin x$ is an odd function)

$= 2[3 – 2] = 2$

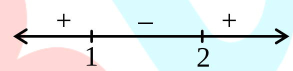
Q.18 Let $f(x) = \int e^x (x-1)(x-2)\,dx$. Then f decreases in the interval - **[IIT-2000 Scr]**

(A) (–∞, –2) (B) (–2, –1)

(C) (1, 2) (D) (2, +∞)

Sol. **[C]**

$f'(x) = e^x (x – 1) (x – 2)$



➤ Fill in the blanks type questions

Q.19 $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

then $\int_0^{\pi/2} f(x)\,dx$ .... **[IIT-1987]**

Sol. $f(x) = \begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x.\cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

$$R_3 \to \frac{1}{\cos x} R_3$$

$$\therefore f(x) = \cos x \begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x.\cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

$$R_1 \to R_1 - R_3$$

$\Rightarrow f(x)$

$$= \cos x \begin{vmatrix} 0 & 0 & \operatorname{cosec} x.\cot x + \sec^2 x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

= (cosecx . cot x + $\sec^2$ x– cos x). ($\cos^3$ x – cos x). cos x

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x.\sin^2 x}{\sin^2 x.\cos^2 x}\right].\cos^2 x.\ \sin^2 x.$$

$= -\sin^2 x - \cos^3 x\,(1 - \sin^2 x)$

$= -\sin^2 x - \cos^5 x$

$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} (\sin^2 x + \cos^5 x)dx$

{using Gamma function,}

$$\int_0^{\pi/2} \sin^m x.\cos^n x\,dx = \left\{\frac{\overline{\left|\frac{m+1}{2}\right.}\overline{\left|\frac{n+1}{2}\right.}}{2\overline{\left|\frac{m+n+2}{2}\right.}}\right\}$$

$$= -\left\{\frac{\overline{\left|\frac{3}{2}\right.}.\overline{\left|\frac{1}{2}\right.}}{2\overline{|2}} + \frac{\overline{\left|\frac{6}{2}\right.}.\overline{\left|\frac{1}{2}\right.}}{2\overline{\left|\frac{7}{2}\right.}}\right\}$$

$$= -\left\{\frac{\frac{1}{2}.\pi}{2} + \frac{2\sqrt{\pi}}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}\right\}$$

$$= -\left\{\frac{\pi}{4} + \frac{8}{15}\right\} = -\left\{\frac{\pi}{4} + \frac{8}{15}\right\} = -\left(\frac{15\pi + 32}{60}\right)$$

Q.20 The Integral $\int_0^{1.5} [x^2]dx$, where [ ] denotes the greatest function equal to...... **[IIT-1988]**

Sol. $\int_0^{1.5} [x^2]dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$

$= 0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{1.5}$

$= (\sqrt{2} - 1) + 2\,(1.5 - \sqrt{2})$

$= \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}$

Q.21 The value of $\int_{-2}^{2} |1 - x^2|\,dx$ is .... **[IIT-1989]**

Sol. $\int_{-2}^{2} |1 - x^2|\,dx = \int_{-2}^{-1} (x^2 - 1)\,dx + \int_{-1}^{1} (1 - x^2)\,dx + \int_1^2 (x^2 - 1)\,dx$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_1^2$$

$$= \left(-\frac{1}{3} + 1 + \frac{8}{3} - 2\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$= 4$

Q.22 The value of $\int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\,dx$ is..... **[IIT-1993]**

Sol. $I = \int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}dx$ …(1)

$$I = \int_{\pi/4}^{3\pi/4} \frac{(\pi/4 + 3\pi/4 - x)}{1 + \sin\left(\frac{\pi}{4} + \frac{3\pi}{4} - x\right)}\,dx$$

$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx\right)$

$$I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1 + \sin(\pi - x)}dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi}{1 + \sin x}dx - \int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}dx$$

$\Rightarrow I = \pi\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \sin x} - I$ (from (1))

$$\Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \sin x}$$

$$\Rightarrow I = \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{dx}{(1 + \sin x)}$$

$$\Rightarrow I = \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)}\,dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{(1 - \sin x)}{1 - \sin^2 x}dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} (\sec^2 x - \sec x.\tan x).\ dx$$

$$= \frac{\pi}{2}\left[\tan x - \sec\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{\pi}{2}[-1 - 1 - (-\sqrt{2} - \sqrt{2})]$$

$$= \frac{\pi}{2}[-2 + 2\sqrt{2}] = \pi[\sqrt{2} - 1]$$

Q.23 The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is.........

[IIT-1994]

Sol. $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ ...(1)

Since $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\Rightarrow I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ...(2)

Adding (1) and (2), we get

$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$

$\Rightarrow 2I = \int_2^3 1dx = 1 \Rightarrow I = \frac{1}{2}$

Q.24 If for non zero x,

$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then

$\int_1^2 f(x)\,dx =$.........

[IIT-1996]

Sol. $af(x) + bf(1/x) = \frac{1}{x} - 5$ (given) ...(1)

replace x by 1/x in (1), we get

$af(1/x) + bf(x) = x - 5$ ...(2)

multiply (1) by a and (2) by b and substract we get

$(a^2 - b^2) f(x) = \frac{a}{x} - bx - 5a + 5b$

$\Rightarrow f(x) = \frac{1}{(a^2-b^2)}\left[\frac{a}{x} - bx - 5a + 5b\right]$

Now,

$\int_1^2 f(x)\,dx = \frac{1}{(a^2-b^2)}\int_1^2\left[\frac{a}{x} - bx - 5a + 5b\right]$

$= \frac{1}{(a^2-b^2)}\left[a\log|x| - \frac{b}{2}x^2 - 5(a-b)x\right]_1^2$

$= \frac{1}{(a^2-b^2)}\Big[a\log 2 - 2b - 10(a-b) - a\log 1$

$\left.+\frac{b}{2} + 5(a-b)\right]$

$= \frac{1}{(a^2-b^2)}\left[a\log 2 - 5a + \frac{7}{2}b\right]$

Q.25 For $n > 0$, $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx =$.......

[IIT-1996]

Sol. **king**

$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$2I = 2\pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$= 2\pi\int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$= 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$= 4\pi \times \frac{\pi}{4} = \pi^2$

Q.26 The value of $\int_1^{e^{37}} \frac{\pi\sin(\pi\log x)}{x}dx$ is.........

[IIT-1997]

Sol. $I = \int_1^{e^{37}} \frac{\pi\sin(\pi\log x)}{x}dx$

$\pi\log x = t$

$I = \int_0^{37\pi}\sin t\,dt = \int_0^{36\pi}\sin t\,dt + \int_{36\pi}^{37\pi}\sin t\,dt$

$I = -[\cos t]_{36\pi}^{37\pi} = 2$

Q.27 $\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2 dt}{x\sin x} =$ .... **[IIT-1997]**

Sol. LH rule

$$\lim_{x\to 0} \frac{\cos x^4 . 2x}{x\cos x + \sin x}$$

$$\lim_{x\to 0} \frac{2\cos x^4}{\cos x + \left(\frac{\sin x}{x}\right)} = 1$$

Q.28 Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k) - F(1)$, then one of the possible value of k is............. **[IIT-1997]**

Sol. $I = \int_1^4 \frac{2x\,e^{\sin x^2}}{x^2}dx$

$x^2 = t$

$I = \int_1^{16} \frac{e^{\sin t}}{t}dt = F(16) - F(1)$

$\Rightarrow k = 16$

➢ Old IIT-JEE subjective type questions

Q.29 Evaluate the following **[IIT-1985]**

$$\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x + \sin^4 x}dx$$

Sol. $I = \int_0^{\pi/2} \frac{x\sin x.\cos x}{\cos^4 x + \sin^4 x}dx$

$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right).\sin x\cos x}{\cos^4 x + \sin^4 x}$

$\therefore\ 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x.\sec^2 x}{\tan^4 x + 1}dx$

$\Rightarrow 2I = \frac{\pi}{2}.\frac{1}{2}\int_0^{\pi/2} \frac{1}{1+(\tan^2 x)^2}.d(\tan^2 x)$,

where $t = \tan^2 x$

$\therefore\ 2I = \frac{\pi}{4}.\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0)$

$\Rightarrow I = \frac{\pi^2}{16}$

Q.30 Evaluate $\int_0^{\pi} \frac{x\,dx}{1-\cos\alpha\sin x}$; $0 < \alpha < \pi$

[IIT -1986]

Sol. Let $I = \int_0^{\pi} \frac{x}{1-\cos\alpha\sin x}dx$ …(1)

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1-\cos\alpha\sin x}dx$ …(2)

Adding (1) and (2), we get

$2I = \pi\int_0^{\pi} \frac{dx}{1-\cos\alpha\sin x}$

$\therefore\ 2I = \pi\int_0^{\pi} \frac{1+\tan^2\frac{x}{2}dx}{(1+\tan^2\frac{x}{2}) - 2\cos\alpha\tan\frac{x}{2}}$

$\Rightarrow 2I = \pi\int_0^{\infty} \frac{2dt}{1+t^2+2t\cos\alpha}$, where $t = \tan\frac{x}{2}$

$2I = 2\pi\int_0^{\infty} \frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$

$I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$

$= \frac{\pi}{\sin\alpha}(\tan^{-1}(\infty) - \tan^{-1}(\cot\alpha))$

$= \frac{\pi}{\sin\alpha}\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right) = \frac{\alpha\pi}{\sin\alpha}$

Q.31 Investigate for maxima and minima the function

$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

[IIT-1988]

Sol. $f(x) = \int_1^x \{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\}\,dt$

$\therefore\ f'(x) = \{2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2\}.1$

$= (x-1)(x-2)^2\{2(x-2) + 3(x-1)\}$

$= (x-1)(x-2)^2\{5x-7\}$

+ – +

1 7/5

$\therefore$ f(x) attains maximum at x = 1

and f(x) attains minimum at $x = \frac{7}{5}$.

Q.32 Evaluate

$\int_0^1 \log[\sqrt{(1-x)} + \sqrt{(1+x)}]dx$ **[IIT-1988]**

Sol. $I = \int_0^1 \log(\sqrt{1-x} + \sqrt{1+x})dx$

Put, x = cos 2θ, then dx = – 2 sin 2θ dθ

$\therefore I = -2\int_{\pi/4}^{0} \log(\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta})(\sin 2\theta)\,d\theta$

$= 2\int_{\pi/4}^{0} \log\{\sqrt{2}(\sin\theta + \cos\theta)\}\sin 2\theta\,d\theta$

$= 2\int_{\pi/4}^{0} \{(\log\sqrt{2})\sin 2\theta + \log(\sin\theta + \cos\theta).\sin 2\theta\}$ dθ

$= 2\log\sqrt{2}\left(\frac{-\cos 2\theta}{2}\right)_{\pi/4}^{0}$

$-2\int_{\pi/4}^{0} \log(\sin\theta + \cos\theta).\sin 2\theta\,d\theta$

$= \log\sqrt{2} - 2\left[-\left\{\log(\sin\theta+\cos\theta).\frac{\cos 2\theta}{2}\right\}_{\pi/4}^{0} - \int_{\pi/4}^{0}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \times \frac{-\cos 2\theta}{2}\right)d\theta\right]$

$= \log(\sqrt{2}) - 2\left[0 + \frac{1}{2}\int_{\pi/4}^{0}(\cos\theta - \sin\theta)^2 d\theta\right]$

$= \frac{1}{2}\log 2 - \int_{\pi/4}^{0}(1 - \sin 2\theta)\,d\theta$

$= \frac{1}{2}\log 2 - \left[\theta + \frac{\cos 2\theta}{2}\right]_{\pi/4}^{0}$

$= \frac{1}{2}\log 2 - \left(\frac{1}{2} - \frac{\pi}{4}\right) = \frac{1}{2}\log 2 - \frac{1}{2} + \frac{\pi}{4}$

Q.33 Prove that the value of integral

$\int_0^{2a}\left[\frac{f(x)}{f(x)+f(2a-x)}\right]dx$ is equal to a.

[IIT-1988]

Sol. $I = \int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ …(i)

$I = \int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}dx$ …(ii)

Adding (i) and (ii), we get

$2I = \int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx = 2a$

$\Rightarrow I = a$

Q.34 Prove that for any positive integer k,

$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos(2k-1)x]$

Hence prove that

$\int_0^{\pi/2}\sin 2kx \cot x\,dx = \frac{\pi}{2}$

Sol. We know,

2 sin x {cos x + cos 3x + cos 5x + …+ cos (2k –1) x}

= 2 sin x cos x + 2 sin x cos 3x + 2 sin x cos 5x +

… + 2 sin x cos (2k – 1) x

= sin 2x + (sin 4x – sin 2x) + (sin 6x – sin 4x) +

…. + {sin 2kx – sin (2k – 2)x} = sin 2kx

∵ 2 {cos x + cos 3x + cos 5x + … cos(2k – 1) x}

$= \frac{\sin 2kx}{\sin x}$ …(1)

Now,

$\sin 2kx.\cot x = \frac{\sin 2kx}{\sin x}.\cos x$

= 2 cos x {cos x + cos 3x + cos 5x + … + cos (2k–1)x}

{using (1)}

= {2 cos^2 x + 2 cos x cos 3x + 2 cosx cos 5x +

… +2 cos x cos (2k – 1) x}

= (1+ cos 2x)+ (cos 4x + cos 2x) + (cos 6x + cos 4x)

+…. +(cos2kx + cos(2k – 2)x)

= 1 + 2 {cos 2x + cos 4x + cos 6x

+…cos (2k – 2)x} + cos 2kx

$\therefore \int_0^{\pi/2}(\sin 2kx).\cot x\,dx$

$= \int_0^{\pi/2} 1.dx$

$+ 2\int_0^{\pi/2}(\cos 2x + \cos 4x \ldots \cos(2k-2)x)dx$

$+ \int_0^{\pi/2}\cos(2k)x\,.\,dx$

$$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}.$$

Q.35 Show that

$$\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

[IIT-1990]

Sol. Let, $I = \int_0^{\pi/2} f(\sin 2x).\sin x dx$ ....(1)

Then, $I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right).\sin\left(\frac{\pi}{2} - x\right)\right\}dx$

$= \int_0^{\pi/2} f\{\sin 2x\}.\cos x\, dx$ ...(2)

adding (1) and (2), we get

$2I = \int_0^{\pi/2} f(\sin 2x).(\sin x + \cos x)dx$

$= 2\int_0^{\pi/4} f(\sin 2x).(\sin x + \cos x)dx$

$= 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right)dx$

$= 2\sqrt{2}\int_0^{\pi/4} f\left(\sin 2\left(\frac{\pi}{4} - x\right)\right).\sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right)dx$

$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x).\cos x\, dx$

Q.36 $\int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi}dx$ **[IIT-1991]**

Sol. $I = \int_0^{\pi} \frac{x\sin(2x).\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$ ...(1)

Then,

$= \int_0^{\pi} \frac{(\pi - x).\sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)}{\pi - 2x}dx$ ....(2)

$= \int_0^{\pi} \frac{(x-\pi)\sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$ ...(3)

adding (1) and (3), we get

$2I = \int_0^{\pi} \sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)dx$

$\Rightarrow I = \int_0^{\pi} \sin x\cos x.\sin\left(\frac{\pi}{2}\cos x\right)dx$

{put, $\frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin x\, dx = dt$

$\Rightarrow \sin x\, dx = -\frac{2}{\pi}dt$}

$\Rightarrow I = \frac{-2}{\pi}\int_{\pi/2}^{-\pi/2} \frac{2t}{\pi}.\sin t.dt$

$= \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t\, dt$

$\Rightarrow I = \frac{4}{\pi^2}\left[-t\cos t + \sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2} \times 2 = \frac{8}{\pi^2}$

Q.37 A cubic f(x) vanishes at x = –2 and has relative minimum/maximum at x = –1 and x = 1/3. If $\int_{-1}^{1} f(x)dx = 14/3$. Find the cubic f(x). **[IIT-1992]**

Sol. f(x) is a cubic polynomial.
Therefore, $f'(x)$ is a quadratic polynomial and f(x) has relative maximum and minimum at $x = \frac{1}{3}$ and $x = -1$ respectively, therefore, –1 and 1/3 are the roots of $f'(x) = 0$. Therefore,

$f'(x) = a(x+1)(x - 1/3)$

$= a\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$

Now, integrating w.r.t. x, we get

$f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + c$

where c is constant of integration.
Again f(–2) = 0 (given). Therefore,

$f(-2) = a\left(-\frac{8}{3} + \frac{4}{3} + \frac{2}{3}\right) + c$

$\Rightarrow 0 = \frac{-2a}{3} + c \Rightarrow c = \frac{2a}{3}$

$\Rightarrow f(x) = a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}+\frac{2}{3}\right)$

Again $\int_{-1}^{1} f(x)\,dx = 14/3$ (given)

$\Rightarrow \int_{-1}^{1}\frac{a}{3}(x^3+x^2-x+2)dx = \frac{14}{3}$

$\Rightarrow \int_{-1}^{1}\frac{a}{3}(0+x^2+0+2)\,dx = \frac{14}{3}$

($\because$ $y = x^3$ and $y = -x$ are odd functions)

$\Rightarrow \frac{a}{3}\left[2\int_0^1 x^2dx + 4\int_0^1 1dx\right] = \frac{14}{3}$

($\because$ $y = x^2$ and $y = x^0$ are even functions.)

$\Rightarrow \frac{a}{3}\left[\left(\frac{2x^3}{3}+4x\right)\right]_0^1 = \frac{14}{3}$

$\Rightarrow \frac{a}{3}\left(\frac{2}{3}+4\right) = \frac{14}{3} \Rightarrow \frac{a}{3}\left(\frac{14}{3}\right) = \frac{14}{3} \Rightarrow a = 3$

Hence , $f(x) = x^3 + x^2 - x + 2$

Q.38 Determine a positive integer $n \le 5$ such that

$\int_0^1 e^x(x-1)^n dx = 16 - 6e.$ **[IIT-1992]**

Sol. Let $I_n = \int_0^1 e^x(x-1)^n\,dx$.

Put $x - 1 = t \Rightarrow dx = dt$, then

$I_n = \int_{-1}^{0} e^{t+1}.t^n\,dt = e\int_{-1}^{0} t^n e^t dt$

$= e\left([t^n e^t]_{-1}^{0} - n\int_{-1}^{0} t^{n-1}e^t dt\right)$

Integrating by parts taking t^n as first function

$= e\left(0-(-1)^{n-1}e^{-1} - n\int_{-1}^{0} t^{n-1}e^t\,dt\right)$

$= (-1)^{n+1} - ne\int_{-1}^{0} t^{n-1}e^t\,dt$

$\Rightarrow I_n = (-1)^{n+1} - nI_{n-1}$ ...(1)

for $n = 1$,

$I_1 = \int_0^1 e^x(x-1)dx = [e^x(x-1)]_0^1 - \int_0^1 e^x dx$

$= e^1(1-1) - e^0(0-1) - [e^x]_0^1 = 1 - (e-1)$

$= 2 - e$

Therefore, from equation (1), we get

$I_2 = (-1)^{2+1} - 2I_1 = -1 - 2(2-e) = 2e - 5$

and $I_3 = (-1)^{3+1} - 3I_2 = 1 - 3(2e-5) = 16 - 6e$

Hence, n = 3 is the answer.

Q.39 Evaluate

$\int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$ **[IIT-1993]**

Sol. $I = \int_2^3 \frac{2x^5-2x^3+x^4+1+2x^2}{(x^2+1)(x^2-1)(x^2+1)}dx$

$= \int_2^3 \frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx$

$= \int_2^3 \frac{2x^3}{(x^2+1)^2}dx + \int_2^3 \frac{1}{(x^2-1)}dx$

$= I_1 + I_2$

Now, $I_1 = \int_2^3 \frac{2x^3}{(x^2+1)^2}dx$

Put $x^2 + 1 = t \Rightarrow 2x\,dx = dt$

$= \int_5^{10}\frac{(t-1)}{t^2}dt = \int_5^{10}\frac{1}{t}dt - \int_5^{10}\frac{1}{t^2}dt$

$= [\ln t]_5^{10} + \left[\frac{1}{t}\right]_5^{10}$

$= \ln 10 - \ln 5 + \frac{1}{10} - \frac{1}{5} = \ln 2 - \frac{1}{10}$

Again $I_2 = \int_2^3 \frac{1}{(x^2-1)}dx$

$= \int_2^3 \frac{1}{(x-1)(x+1)}dx$

$= \frac{1}{2}\int_2^3 \frac{2}{(x-1)(x+1)}dx$

$= \frac{1}{2}\int_2^3 \frac{1}{(x-1)}dx - \frac{1}{2}\int_2^3 \frac{1}{(x+1)}dx$

$\Rightarrow I_2 = \left[\frac{1}{2}\ln(x-1)\right]_2^3 - \frac{1}{2}[\ln(x+1)]_2^3$

$= \frac{1}{2}\ln\frac{2}{1} - \frac{1}{2}\ln\frac{4}{3}$

Hence , $I = I_1 + I_2$

$= \ln(2) - \frac{1}{10} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln\frac{4}{3}$

$= \frac{1}{2}\ln 6 - \frac{1}{10}$

Q.40 Show that $\int_0^{n\pi+v}|\sin x|\,dx = 2n + 1 - \cos v$

Where n is a positive integer and $0 \le v < \pi$

[IIT-1994]

Sol. $\int_0^{n\pi+v}|\sin x|dx = \int_0^{\pi}|\sin x|dx + \int_{\pi}^{2\pi}|\sin x|dx$

$+\ldots.. + \int_{(n-1)\pi}^{n\pi}|\sin x|dx + \int_{n\pi}^{n\pi+v}|\sin x|dx$

$= \sum_{r=1}^{\pi}\int_{(r-1)\pi}^{r\pi}|\sin x|\,dx + \int_{n\pi}^{n\pi+v}|\sin x|dx$

Now to solve, $\int_{(r-1)\pi}^{r\pi}|\sin x|dx$, we have

$x = (r-1)\pi + t$

$\Rightarrow \sin x = \sin[(r-1)\pi + t] = (-1)^{r-1}\sin t$

and when $x = (r-1)\pi$, $t = 0$ and when

$x = r\pi$, $t = \pi$

$\Rightarrow \int_{(r-1)\pi}^{r\pi}|\sin x|dx = \int_0^{\pi}|(-1)^{r-1}\sin t|dt$

$= \int_0^{\pi}|\sin t|dt$

$= \int_0^{\pi}\sin t\,dt$

$= [-\cos t]_0^{\pi} = -\cos\pi + \cos 0 = 2$

Again $\int_{n\pi}^{n\pi+v}|\sin x|dx$, putting $x = n\pi + t$

Then $\int_{n\pi}^{n\pi+v}|\sin x|dx$

$= \int_0^{v}|(-1)^n\sin t|dt = \int_0^{v}\sin t\,dt$

$= [-\cos t]_0^{v} = -\cos v + \cos 0 = 1 - \cos v$

Therefore

$\int_0^{n\pi+v}|\sin x|dx = \sum_{r=1}^{n}\int_{(r-1)\pi}^{n}|\sin x|\,dx$

$+ \int_{n\pi}^{n\pi+v}|\sin x|dx$

$= \sum_{r=1}^{n}2 + \int_{n\pi}^{n\pi+v}|\sin x|\,dx$

$= 2n + 1 - \cos v$

Q.41 $\int_0^{\pi/2}\frac{\sin 8x\log\cot x}{\cos 2x}$ **[REE-1995]**

Sol. $I = \int_0^{\pi/2}\frac{\sin 8x\log\cot x}{\cos 2x}dx$

$I = \int_0^{\pi/2}\frac{-\sin 8x(-\log\cot x)}{-\cos 2x}dx$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

Q.42 $\int_{-(1/\sqrt{3})}^{(1/\sqrt{3})}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$ **[IIT-1995]**

Sol. Let $x = -y \Rightarrow dx = -dy$

$\Rightarrow I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}\left[\pi - \cos^{-1}\frac{2y}{1+y^2}\right]dy$

$\Rightarrow 2I = 2\pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx$ put $y = x$

$\Rightarrow I = \pi\left[\int_0^{1/\sqrt{3}} -1dx + \frac{1}{2}\int_0^{1/\sqrt{3}}\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)dx\right]$

$= \pi\left[-\frac{1}{\sqrt{3}} + \frac{1}{4}\left(\lambda n\frac{1+x}{1-x}\right)_0^{1/\sqrt{3}} + \frac{1}{2}\left(\tan^{-1}x\right]_0^{1/\sqrt{3}}\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{6} + \frac{1}{2}\lambda n\frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{2}{\sqrt{3}}\right]$

Q.43 Evaluate $\int_0^{\pi/4}\lambda n(1+\tan x)dx$ **[IIT-1997]**

Sol. $I = \int_0^{\pi/4}\lambda n(1+\tan x)\,dx$ .....(1)

$= \int_0^{\pi/4}\lambda n\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$= \int_0^{\pi/4}\lambda n\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4}\lambda n\left(\frac{2}{1+\tan x}\right)dx$

$I = \int_0^{\pi/4}(\lambda n2 - \lambda n(1+\tan x))dx$ ....(2)

adding (1) & (2)

$2I = \int_0^{\pi/4}\lambda n\,2\,dx = \lambda n2[x]_0^{\pi/4}$

$I = \frac{\pi}{8}\lambda n2$

Q.44 Determine the value of $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$ **[IIT-1997]**

Sol. $I = \int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$

$I = 0 + 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ....(1)

$I = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ ....(2)

adding (1) & (2)

$2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

put cos x = t $\Rightarrow$ sin x dx = – dt

$I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2} = 2\pi \int_{-1}^{1} \frac{1}{1+t^2} dt$

$= 4\pi \left[\tan^{-1} t\right]_0^1 = 4\pi . \frac{\pi}{4} = \pi^2$

Q.45 Let a + b = 4, where a < 2 and let g(x) be a differentiable function, If $\frac{dg}{dx} > 0$ for all x. prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as (b – a) increases **[IIT-1997]**

Sol. let b – a = t where a + b = 4

$\Rightarrow a = \frac{4-t}{2}$, $b = \frac{t+4}{2}$

As given a < 2 , b > 2 $\Rightarrow$ t > 0

Now $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

Let $\phi(x) = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{t+4}{2}} g(x)\,dx$

$\phi'(x) = g\left(\frac{4-t}{2}\right)\left(\frac{-1}{2}\right) + g\left(\frac{t+4}{2}\right)\frac{1}{2}$

$= \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$

Since g(x) increasing function so

So $\frac{4+t}{2} > \frac{4-t}{2} \Rightarrow g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right)$

$\Rightarrow \phi'(t) > 0$

Hence ϕ (t) increasing as t increases

$\Rightarrow \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as b – a increases

Q.46 Prove that

$\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2 . \int_0^1 \tan^{-1} x\,dx$

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1 - x + x^2)\,dx.$ **[IIT-1998]**

Sol. $\int_0^1 \tan^{-1}\left(\frac{1}{1-x(1-x)}\right) dx$

$= \int_0^1 \tan^{-1}\left(\frac{1-x+x}{1-x(1-x)}\right) dx$

$= \int_0^1 \tan^{-1}(1-x)dx + \int_0^1 \tan^{-1} x\,dx$

$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx = 2\int_0^1 \tan^{-1} x\,dx$

Hence proved

$\Theta \int_0^1 \tan^{-1}(1-x+x^2)\,dx$

$= \int_0^1 \left[\pi/2 - \tan^{-1}\left(\frac{1}{1-x+x^2}\right)\right] dx$

$= \int_0^1 \frac{\pi}{2} dx - \int_0^1 \tan^{-1}\left(\frac{1}{1-x(1-x)}\right) dx$

$= \frac{\pi}{2}[x]_0^1 - 2\int_0^1 \tan^{-1} x\,dx$

$= \frac{\pi}{2} - 2\left[x \tan^{-1} x\right]_0^1 + 2\int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{2} - \frac{2\pi}{4} + \left[\lambda n(1+x^2)\right]_0^1 = \lambda n2$

Q.47 Evaluate the integral $\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x} dx$ **[REE-1999]**

Sol. $\sqrt{6}\,\frac{\pi}{3} - 2\left(\cot^{-1}\frac{1}{\sqrt{2}}\right)$

Q.48 For x > 0, let $f(x) = \int_1^x \frac{\lambda nt}{1+t} dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$. **[IIT-2000]**

Sol. $f(x) = \int_1^x \frac{\lambda nt}{1+t} dt$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\lambda nt}{1+t}$$

Let $t = \frac{1}{p} \Rightarrow dt = -\frac{1}{p^2}dp$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{x} \frac{\log(1/p)}{\left(1+\frac{1}{p}\right)}\left(-\frac{1}{p^2}\right)dp$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{x} \frac{\lambda np}{(1+p)p}dp = \int_1^{x} \frac{\lambda nt}{(1+t)t}dt$$

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^{x} \frac{\lambda nt}{1+t}\left(\frac{1+t}{t}\right)dt = \int_1^{x} \frac{\lambda nt}{t}dt$$

$$= \left(\frac{(\lambda nt)^2}{2}\right)_1^x = \frac{(\lambda nx)^2}{2}$$

$$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = \frac{(\lambda nx)^2}{2}$$

$$f(e) + f\left(\frac{1}{e}\right) = \frac{(\lambda ne)^2}{2} = \frac{1}{2}$$

Q.49 Given $\int_0^1 \frac{\sin t}{1+t}dt = \alpha$, find the value of $\int_{4\pi-2}^{4\pi} \frac{\sin(t/2)}{4\pi+2-t}dt$ in terms of α. **[REE-2000]**

Sol. $I = \int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}dt = \frac{1}{2}\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{1+(2\pi-t/2)}dt$

Let $2\pi - t/2 = u \Rightarrow dt = -2\,du$

$$I = \frac{1}{2}\int_1^0 \frac{\sin(2\pi-u)}{1+u}(-2\,du)$$

$$= -\int_0^1 \frac{\sin u}{1+u}du = -\alpha$$

Q.50 Evaluate : $\int_0^{\pi/2} \frac{\cos^9 x}{\cos^3 x+\sin^3 x}dx$ **[REE-2001]**

Sol. $I = \int_0^{\pi/2} \frac{\cos^9 x}{\cos^3 x+\sin^3 x}dx$ ....(1)

$I = \int_0^{\pi/2} \frac{\sin^9 x}{\cos^3 x+\sin^3 x}dx$ ....(2)

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\cos^9 x+\sin^9 x}{\cos^3 x+\sin^3 x}dx$$

$$= \int_0^{\pi/2} (\cos^6 x+\sin^6 x-\sin^3 x\cos^3 x)\,dx$$

$$= \int_0^{\pi/2} (1-3\sin^2 x\cos^2 x-\sin^3 x\cos^3 x)\,dx$$

Solving we get

$$2I = \int_0^{\pi/2} \left(\frac{5}{8}+\frac{3}{8}\cos 4x-\frac{3}{32}\sin 2x+\frac{1}{32}\sin 6x\right)dx$$

$$= \left(\frac{5}{8}x+\frac{3}{32}\sin 4x+\frac{3}{64}\cos 2x-\frac{1}{192}\cos 6x\right)_0^{\pi/2}$$

$$= \left[\frac{5\pi}{16}-\frac{3}{64}+\frac{1}{192}-\frac{3}{64}+\frac{1}{192}\right]$$

$$= \frac{5\pi}{16} - \frac{1}{12}$$

$$I = \frac{5\pi}{32} - \frac{1}{24}$$

Q.51 If $\int_0^{\pi}\sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}\,dx$ has the value equal to $\left(\frac{\pi}{k}+\sqrt{w}\right)$ where k and w are positive integers find the value of (k^2+w^2)

Sol. 153

Q.52 Match the column :

	Column-1		Column-2
(A)	$\int_5^{10}\left[\frac{x-5}{5}\right]dx =$	(P)	1
(B)	$\int_{-\tan 1}^{0}[-\tan^{-1}x]dx =$	(Q)	2
(C)	$\frac{6}{\pi}\int_{\pi/6}^{\pi/3}[2\sin x]dx =$	(R)	0
(D)	$\int_{-1}^{1}\frac{\cot^{-1}x}{\pi}dx =$	(S)	3

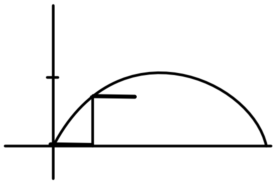
where [.] represents greatest integer function.

Sol. **A → R; B → R; C → P; D → P**

(A) $\int_5^{10}\left[\frac{x}{5}\right]-1\ dx = 0$

(B) $-x = t$

$$I = \int_{\tan i}^{0}[\tan^{-1}t](-dt) = 0$$

(C) 

$$A = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \times 1 = \frac{\pi}{6}$$

$$\therefore \quad \frac{6}{\pi} \int_{\pi/6}^{\pi/3} [2\sin x]\,dx = 1$$

(D) **king**

$$I = \int_{-1}^{1} \frac{\pi - \cot^{-1} x}{\pi}$$

add

$$2I = \int_{-1}^{1} 1\,dx \Rightarrow I = 1$$

ANSWER KEY

EXERCISE # 1

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	D	A	A	C	B	A	C	B	B	D	B	C	A	B
Qus.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	A	D	B	B	A	C	B	B	B	B	B	A	C	C	C
Qus.	31	32	33	34	35										
Ans.	C	D	B	A	C										

36. True **37.** False **38.** False **39.** True **40.** 7 **41.** 1 **42.** π^2

EXERCISE # 2

PART-A

Qus.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	D	A	C	D	D	D	B	B	A	C	A	D	C	B	B
Qus.	16	17	18	19	20	21	22	23	24						
Ans.	B	C	B	A	B	C	A	D	C						

PART-B

Qus.	25	26	27	28	29	30	31	32	33	34
Ans.	C,D	A, C	A,B,C	A,B,C,D	A,D	A,B,C	A,C	A,B,C,D	A,D	B

PART-C

35. D **36.** C **37.** B

PART-D

38. A $\to$ P,Q; B $\to$ P; C $\to$ P,Q; D $\to$ P,Q,R

39. A $\to$ P,Q; B $\to$ R; C $\to$ P,Q ; D $\to$ S

40. A $\to$ S; B $\to$ R; C $\to$ S ; D $\to$ S

41. A $\to$ Q, R; B $\to$ S; C $\to$ P,Q, R ; D $\to$ P, R

EXERCISE # 3

1. $\lambda n\ 4$ **2.** $\frac{4}{5}$ **3.** $\frac{\pi}{4}$ **5.** $\frac{\pi}{2}$ **7.** $-\frac{2}{\pi^2}\cos\pi k\ ;\ 0 < k < 1,\ \frac{2}{\pi^2}\ ;\ k \ge 1,\ -\frac{2}{\pi^2}\ ;\ k \le 0$

10. $\frac{n}{n^2-1}$ **11.** $3\sqrt{3} - 2\sqrt{2} - 1$ **12.** $\frac{51}{8}$ **13.** $2e^{(1/2)(\pi-4)}$ **16.** $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$ **17.** 0

18. a = 4, b = 1 **19.** $\frac{2}{3}$ **20.** –1/4 **22.** 2 sin 2x – sin x **24.** $\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$

25. $(b-a)^{m+n+1}\frac{m!n!}{(m+n+1)!}$ **28.** $\frac{2}{2-\lambda}\sin x$ **29.** π/8 ln 2

Qus.	30	31	32	33	34	35	36	37	38	39	40	41
Ans.	C	A	D	C	A	A	B	B	A	C	D	B

EXERCISE # 4

Qus.	1	2	3	4	5	6
Ans.	C	B	A	C	C	D

8. C **9.** B **10.** 2π **11.** $\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\frac{\pi}{4}\right]$ **12.** A

13. $\frac{12}{\sqrt{5}}[e\cos(\frac{1}{2}-\tan^{-1}\frac{1}{2})-\frac{2}{\sqrt{5}}]$ **14.** [5051] **15.** B

16. A **17.** A **18.** A,D **19.** A,B,C,D **20.** C

21. A,B,C **22.** B **23.** A **24.** 4 **25.** A

26. 6 **27.** B **28.** A, B, C, D **29.** C **30.** B

31. $A\to S$; $B\to S$; $C\to P$; $D\to R$ **32.** $A\to T$; $B\to P, R$; $C\to Q, S$; $D\to R$

EXERCISE # 5

Qus.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	D	D	B	D	C	A	B	A
Qus.	11	12	13	14	15	16	17	18		
Ans.	A	C	A	C	B	B	C	C		

19. $\frac{-(15\pi+32)}{60}$ **20.** $(2-\sqrt{2})$ **21.** 4 **22.** $p(\sqrt{2}-1)$ **23.** 1/2 **24.** $\frac{1}{a^2-b^2}\left[a\log 2-5a+\frac{7}{2}b\right]$

25. π^2 **26.** 2 **27.** 1 **28.** 16

29. $\frac{\pi^2}{16}$ **30.** $\frac{\alpha\pi}{\sin\alpha}$ **31.** f(x) attains maximum at x = 1; f(x) attains minimum at x = 7/5

32. $\frac{1}{2}\log 2-\frac{1}{2}+\frac{\pi}{4}$ **36.** $\frac{8}{\pi^2}$ **37.** $f(x)=x^3+x^2-x+2$ **39.** $\frac{1}{2}\log(6)-\frac{1}{10}$

41. 0 **42.** $\frac{\pi}{2}\left[\frac{\pi}{6}+\frac{1}{2}\lambda n\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|-\frac{2}{\sqrt{3}}\right]$ **43.** $\frac{\pi}{8}\lambda n\, 2$ **44.** π^2

46. $\lambda n\, 2$ **47.** $\sqrt{6}\,\frac{\pi}{3}-2\left(\cot^{-1}\frac{1}{\sqrt{2}}\right)$ **48.** $\frac{1}{2}(\lambda nx)^2$

49. $-\alpha$ **50.** $\frac{5\pi}{32}-\frac{1}{24}$ **51.** 153

52. $A\to R$; $B\to R$; $C\to P$; $D\to P$